Obstructions to factorizations of differential operators on the algebra of densities on the line

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(a Happy Birthday talk to Murray Gerstenhaber & Jim Stasheff
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Joint work with Th. Voronov
Factorization of differential operators in classical situation in 1D case

\( \partial^2 = \partial \circ \partial = \left( \partial + \frac{1}{x + c} \right) \circ \left( \partial - \frac{1}{x + c} \right) \)
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Frobenius Theorem

For \( L = a_n \partial^n + a_{n-1} \partial^{n-1} + \cdots + a_1 \partial + a_0 \) factorizations into first-order factors \( \iff \) maximal flags in \( \ker L \).

(Dimension of the manifold of flags is \( \frac{n(n-1)}{2} \).)
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\[ \{x\} \subseteq \{x, 1\} = \ker L \iff L = \left( \partial + \frac{1}{x} \right) \circ \left( \partial - \frac{1}{x} \right) \]

which can be computed as \( L = L_2 \circ L_1 \), where \( L_1 = x \circ \partial \circ (x)^{-1} \) and then \( \varphi_2 = L_1(1) = -1/x \), and \( L_2 = \varphi_2 \circ \partial \circ \varphi_2^{-1} \).
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Already in 2D the situation is different, as there is

E. Landau example

For $P = \partial_x + x\partial_y$, $Q = \partial_x + 1$, $R = \partial_x^2 + x\partial_x\partial_y + \partial_x + (2 + x)\partial_y$ (irreducible in any reasonable extension) we have $QQP = RQ$. So different number of irreducible factors.
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In general, in the multidimensional case there is no general theory but only some results for some particular situations are available.

✓ ...  
✓ For geometrical objects: Li, ES, Th. V, 2017, Analogue of Frobenius theorem for non-degenerate operators for the super 1|1 case.
Densities

A density of weight \( \lambda \in \mathbb{R} \) (on a manifold with local coordinates \( x^a \)) is

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\rho = \rho(x) |dx|^{\lambda}
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The Sturm–Liouville operator is

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There is non-degenerate pairing between $\mathcal{F}_\lambda$ and $\mathcal{F}_{1-\lambda}$ and

$$\langle \psi, \varphi \rangle = \int_M \psi(x) \varphi(x)|dx|$$
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The algebra of densities $\mathcal{S}(M)$ was discovered in 2004 by Khudaveridian and Th. Voronov as the “correct” framework for the BV $\Delta$-operator (before ... functions and semidensities and works of Schwarz and Khudaverdian).
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The viewpoint in this work: the algebra of densities is a useful tool when work on problems for differential operators acting on densities.
Algebra of densities

Duval, Ovsienko + a series of works by different authors considered spaces of differential operators of order two,

\[ D^2_\lambda : \mathcal{F}_\lambda \rightarrow \mathcal{F}_\lambda \]

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**Theorem**

\[ L^2 = a_{2}^{ij} \partial_i \partial_j + a_{1}^{i} \partial_i + a_{0}. \] Equivariant \( L^2_{\lambda \mu} : \mathcal{D}^2_{\mu} \to \mathcal{D}^2_{\lambda} : \)

\[ \tilde{a}_{2}^{ij} = a_{2}^{ij}, \]

\[ \tilde{a}_{1}^{i} = \frac{2 \lambda + 1}{2 \mu + 1} a_{1}^{i} + 2 \frac{\mu - \lambda}{2 \mu + 1} \partial_i a_{2}^{ij}, \]

\[ \tilde{a}_{0} = \frac{\lambda(\lambda + 1)}{\mu(\mu + 1)} a_{0} + \frac{\lambda(\mu - \lambda)}{(2 \mu + 1)(\mu + 1)} \left( \partial_i a_{1}^{i} - \partial_i \partial_j a_{2}^{ij} \right). \]
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$t = |dx| \to$ extended $\hat{M}$ and $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial t}$ ($\Rightarrow \hat{w} = t \frac{\partial}{\partial t}$)
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Now differential operators on $\mathfrak{F}$ restricted onto $\mathfrak{F}_\lambda$ are of the needed type.
Factorization of operators on $\mathcal{F}(\mathbb{R})$

\[
L = t^2 \left( \partial^2 + (p_1 \hat{w} + p_0) \partial + q_2 \hat{w}^2 + q_1 \hat{w} + q_0 \right)
= t \left( \partial - \alpha_1 \hat{w} - \alpha_0 \right) \cdot t \left( \partial - \beta_1 \hat{w} - \beta_0 \right)
= t^2 \left( \partial - \alpha_1 (\hat{w} + 1) - \alpha_0 \right) \cdot \left( \partial - \beta_1 \hat{w} - \beta_0 \right)
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or in the condensed notation:

\[ L = t^2 \left( \partial^2 + \hat{p} \partial + \hat{q} \right) = t^2 \left( \partial - \hat{\alpha} \right) \cdot \left( \partial - \hat{\beta} \right) \]

So we have $\hat{\alpha} = -\hat{p} - \hat{\beta}$ and familiar Riccati equation for $\hat{\beta}$:

\[ -\hat{\beta}' = \hat{\beta}^2 + \hat{p} \hat{\beta} + \hat{q} \]

In the classical case, we make the substitution $\hat{\beta} = \partial (\ln \varphi)$, which transforms the Riccati into

\[ \varphi'' + \hat{p} \varphi' + \hat{q} \varphi = 0 \]
Factorization of operators on $\mathcal{F}(\mathbb{R})$

So operator is factorizable IF(!) there is a solution $\varphi$ to $\varphi'' + \hat{p}\varphi' + \hat{q}\varphi = 0$ such that $\hat{\beta}$ computed as $\hat{\beta} = \partial (\ln \varphi)$ will be linear in $\hat{w}$. 

Example of an operator that is not factorizable

$L = t^2 \cdot (\partial^2 - \hat{w}^2 - \hat{w}) \neq t^2 \cdot (\partial - \alpha_1 (\hat{w} + 1) - \alpha_0) \cdot (\partial - \beta_1 \hat{w} - \beta_0)$
So operator is factorizable if there is a solution $\varphi$ to $\varphi'' + \hat{p}\varphi' + \hat{q}\varphi = 0$ such that $\hat{\beta}$ computed as $\hat{\beta} = \partial (\ln \varphi)$ will be linear in $\hat{\omega}$.

Example of an operator that is not factorizable

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Generalized Sturm-Liouville operator

The generalized Sturm–Liouville operator,

\[ L = t^2 \left( \partial^2 + \gamma(2\hat{w} + 1)\partial + \theta\hat{w}(\hat{w} + 1) + \gamma'\hat{w} \right) \]  \hspace{1cm} (1)

parametrized by functions \( \gamma \) and \( \theta \). When specialized on \( \hat{F}_{-1/2} \), it becomes the classical Sturm-Liouville operator \( t^2 (\partial^2 + u) \) with potential

\[ u = -\frac{1}{2} \left( \gamma' + \frac{\theta}{2} \right). \]
The necessary and sufficient condition for the generalized Sturm-Liouville operator \( L \) to be factorizable is that \( \psi = (\gamma^2 - \theta)^{-1/4} \) satisfies the classical Sturm-Liouville equation

\[
(\partial^2 + u)\psi = 0.
\]
Generalized Sturm-Liouville operator

**Theorem**

Density $\psi = (\gamma^2 - \theta)^{-1/4} |dx|^{-1/2}$ is a (density) invariant (under a change of coordinates) of the generalized Sturm-Liouville operator.
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Theorem

Density \( \psi = (\gamma^2 - \theta)^{-1/4} |dx|^{-1/2} \) is a (density) invariant (under a change of coordinates) of the generalized Sturm-Liouville operator.

By the properties of the classical Sturm–Liouville operator, the condition that it is a solution does not depend on a choice of coordinate. This establishes the invariance of the factorization criterion.
Theorem

An incomplete factorization

\[ L = t^2 \left( (\partial - \hat{\alpha})(\partial - \hat{\beta}) + f \right), \]

where \( f = f(x) \) does not contain \( \hat{\psi} \), of the generalized Sturm–Liouville operator \( L \) is always possible and it is unique. It is given by the formulas

\[ b_0 = \partial \ln \psi, \]
\[ b_1 = -\gamma \pm \frac{1}{\psi^2}, \]
\[ f = \frac{1}{\psi} (\partial^2 + u) \psi. \]

Here \( \psi \) is as above.
Happy birthday Murray and Jim!!!