BFV and Poisson

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Outline

1. Differential Graded Symplectic Manifolds
2. BFV
3. Relaxed BF\textsuperscript{nV} Structures
4. Examples
A graded manifold is “like a manifold,” but we also allow odd local coordinates.

The odd coordinates anticommute among themselves and commute with the even coordinates.

We also assign a \( \mathbb{Z} \)-degree to coordinates (in physics ghost number). In this talk, I will assume

\[
\text{parity} = \mathbb{Z} \text{-degree modulo } 2
\]

If \( M \) is a graded manifold, then \( C^\infty(M) \) is a graded commutative algebra.

Example: \( M = T[1]N \), \( N \) an ordinary manifolds:

- Coordinates \( q^i \) on \( N \) degree 0; fiber coordinates \( v^i \) degree +1.
- \( C^\infty(M) = \Omega^\bullet(N) \).
Cohomological vector fields

- If $M$ is a graded manifold, a differential on $\mathcal{C}^\infty(M)$ has been called by Jim a cohomological vector field (cvf). 

  - Explanation:
    - Derivation on $\mathcal{C}^\infty(M) =$ vector field $Q$ on $M$. 
    - Differential $= \deg Q = 1$ and $[Q, Q] = 0$. 

  The de Rham differential on $N$ is a cvf on $\mathcal{C}^\infty(M)$: $Q = \sum_i v^i \frac{\partial}{\partial q^i}$

- Example: $M = g[1]$, $g$ a Lie algebra:
  - All coordinates $c_i$ have degree 1.
  - $\mathcal{C}^\infty(M) = \Lambda^\bullet g^*$. 
  - The Chevalley–Eilenberg differential on $g$ is a cvf:

    $$Q = \frac{1}{2} \sum_{ijk} f^{ij}_k c_i c_j \frac{\partial}{\partial c_k}$$

    with $f^{ij}_k$ the structure constants. (In physics the BRST operator.)
A graded symplectic form $\omega$ of degree $n$ is a closed, nondegenerate 2-form with internal degree equal to $n$.

Example: $M = T^*[1]N$, $N$ an ordinary manifolds:
- Coordinates $q^i$ on $N$ degree 0; fiber coordinates $p_i$ degree 1.
- $C^\infty(M) = \mathfrak{X}^*(N)$ (multivector fields).
- $\omega = \sum_i dp_i \, dq^i$ is a graded symplectic form of degree 1.
- The Schouten–Nijenhuis bracket on multivector fields is the associated Poisson bracket.

This may be generalized to $M = T^*[n]N$, $N$ an ordinary manifolds:
- Coordinates $q^i$ on $N$ degree 0; fiber coordinates $p_i$ degree $n$.
- $\omega = \sum_i dp_i \, dq^i$ is a graded symplectic form of degree $n$. 
Differential graded symplectic structure

- A differential graded symplectic structure of degree $n$ on a graded manifold $M$ is a pair $(\omega, Q)$ where:
  - $\omega$ is a graded symplectic form of degree $n$, and
  - $Q$ is a symplectic cvf, i.e.,

$$\text{deg } Q = 1, \quad [Q, Q] = 0, \quad \text{and } L_Q \omega = 0$$

- A stronger version is when $Q$ is Hamiltonian, i.e., there is a function $S$ (necessarily of degree $n + 1$) such that

$$\iota_Q \omega = dS \quad \text{and} \quad \{S, S\} = 0 \quad \text{(classical master equation)}$$

- As observed by Roytenberg, if $n \neq -1$, a symplectic cvf is always Hamiltonian (with a unique Hamiltonian function):

$$S = \frac{1}{n+1} \iota_E \iota_Q \omega$$

with $E$ the “graded Euler vector field”: $E(f) = \text{deg } f f$. 
Examples of differential graded symplectic structures

- **Example:** $M = T^*[1]N$, $N$ an ordinary manifolds:
  - A function $S$ of degree 2 on $M$ is then the same as a bivector field $\pi$ on $N$.
  - The master equation $\{S, S\} = 0$ translates to $[\pi, \pi] = 0$; i.e., $\pi$ is a Poisson bivector field.
  - $Q$ is then the Poisson–Lichnerowicz differential.

- **Example:** $M = g[1]$, $g$ a Lie algebra:
  - A nondegenerate symmetric bilinear form on $g$ can be viewed as a constant symplectic form of degree 2 on $g[1]$.
  - If $Q$ corresponds to the Chevalley–Eilenberg differential, it is symplectic iff the pairing is invariant.
  - The corresponding Hamiltonian function turns out to be
    \[
    S = \frac{1}{6} \sum f^{ijk} c_i c_j c_k
    \]
  - with $f^{ijk}$ the structure constants with one index raised by using the pairing.
BF$^n$V structures

There are three important particular cases:

$n = -1$ This is the Batalin–Vilkovisky (BV) formalism used in QFT. The Hamiltonian function is required to exist as an extra assumption (I will not talk about it).

$n = 0$ This is the Batalin–Fradkin–Vilkovisky (BFV) formalism used to give a cohomological resolution of symplectic reduction (see next).

$n = 1$ If there are only coordinates of nonnegative degree, this is just the example of $M = T^*[1]N$ with a Poisson structure on $N$. More generally, it describes Poisson structures up to homotopy (i.e., the Poisson bracket is an $L_\infty$-structure).

We may call the general case of degree $n$ a BF$^{n+1}$V structure.

$\text{Poisson}_\infty = \text{BF}^2V$
Differential Graded Symplectic Manifolds

BFV

Relaxed BF/V Structures

Examples

**Symplectic reduction in codimension one**

- Let \((N, \omega_N)\) be a symplectic manifold, \(\phi\) a function and \(C := \phi^{-1}(0)\) a submanifold.
- The restriction of \(\omega_N\) to \(C\) is degenerate. Its kernel is generated by the Hamiltonian vector field \(X_\phi\) of \(\phi\):
  \[
  \iota_{X_\phi} \omega_N = d\phi \approx 0
  \]
- We define \(\underline{\mathcal{C}} = C / X_\phi\). Algebraically,
  \[
  C^\infty(\underline{\mathcal{C}}) = (C^\infty(N) / \langle \phi \rangle)^{X_\phi} = N(\langle \phi \rangle) / \langle \phi \rangle,
  \]
  with \(N(\langle \phi \rangle) = \{ f \in C^\infty(N) : \{ \phi, f \} = g\phi, \; g \in C^\infty(N) \}\).
- Define \(M = N \times T^*\mathbb{R}[1], \omega = \omega_N + dbdc, S = c\phi\) a dgs manifold of degree 0. Then
  \[
  Qb = \phi, \quad Qf = \{ \phi, f \} c, \quad Qc = 0.
  \]
  In particular, in degree 0 and \(-1:\)
  \[
  Q(f + gcb) = \{ \phi, f \} c - g\phi c, \quad Q(hb) = h\phi + \{ \phi, h \} cb.
  \]
  Hence \(H_Q^0(M) = C^\infty(\underline{\mathcal{C}})\).
Symplectic reduction of coisotropic submanifolds

- The previous slide may be generalized to the case when we have a set of $r$ functions $\phi^i$ and a submanifold
  \[ C := \{ x \in N : \phi^i(x) = 0, \ i = 1, \ldots, r \} \]

- Assume that $C$ is coisotropic, i.e., $\{\phi^i, \phi^j\}|_C = 0 \ \forall i, j$.

- The kernel of the restriction of $\omega_N$ to $C$ is generated by the Hamiltonian vector fields of the $\phi^i$s. We denote by $\hat{C}$ the quotient of $C$ by this kernel. (The reduced phase space.)

- The main result by BFV and Jim’s paper is that (under some conditions)
  \[ C^\infty(\hat{C}) = H^0_Q(M) \]
  as Poisson algebras

with: $M = N \times T^*\mathbb{R}^r[1]$, $\omega = \omega_N + \sum_{i=1}^r db^i dc^i$,

\[ S = \sum_{i=1}^r c^i \phi^i + \cdots \]

where the dots contain higher powers of the $b^i$s and are obtained by cohomological perturbation theory.
Equivariant momentum map

A special case is when the $\phi^i$'s are the components of an equivariant momentum map $\phi: N \rightarrow g^*$. In this case we have

$$S = \sum_{i=1}^{r} c_i \phi^i + \frac{1}{2} \sum f_{ij}^k b^k c_i c_j$$

$Q$ in this case is also called the BRS operator.
Relaxed structures

Suppose we have a graded manifold $M$ with a cohomological vector field $Q$ and a closed 2-form $\omega$ of degree $n$. We set

$$\tilde{\alpha} := \iota_Q \omega - dS$$

and

$$\tilde{\omega} := d\tilde{\alpha} = -L_Q \omega_P$$

It turns out that $\tilde{\omega}$ is a closed, $Q$-invariant 2-form $\omega$ of degree $n+1$.

We denote by $M$ the quotient of $M$ by the kernel of $\tilde{\omega}$ (assume it is smooth). We denote by $\omega$ its symplectic form of degree $n + 1$.

It turns out that $Q$ is projectable to a cohomological vector field $Q$. So $M$ becomes a dgs manifold of degree $n + 1$. 
Field theory

- Suppose that $M$ is a space of fields on some closed manifold $\Sigma$.
- Suppose we have a BF$^{n-1}V$ structure on $M$ with $\omega$, $Q$, and $S$ local.
- This allows us to write $\omega$, $Q$, and $S$ also on some compact $\Sigma$ with boundary.
- If $S$ contains derivatives of the fields, there will be boundary terms that spoil the structure.
- This relaxed structure will however induce a BF$^nV$ structure on the fields on $\partial \Sigma$ (the kernel of $\tilde{\omega}$ contains in particular fields in the bulk).
- Some related results can be obtained in the context of derived geometry (à la Pantev–Toën–Vaquié–Vezzosi) as shown by Safronov.
An application

- Suppose \((M, \omega, Q, S)\) is a BFV structure on a space of fields on \(\Sigma\), describing the reduced phase space of some field theory.

- Then on the space of fields \(M\) on \(\partial \Sigma\) we get a BF\(^2\)V structure, i.e., a Poisson structure (possibly up to homotopy).

- If \(\partial \Sigma = \emptyset\), one expects to quantize \(M\) to some graded vector space \(\mathcal{H}\) (and \(S\) to some coboundary operator on \(\mathcal{H}\)).

- If \(\partial \Sigma \neq \emptyset\), we expect \(\mathcal{H}\) to be a representation of a quantization of \(M\).

- For example, we may consider the deformation quantization of the Poisson structure described by \(M\).

- By Kontsevich, deformation quantization of a (f.d.) Poisson manifold \(P\) is always possible. It is related to a quasi-isomorphism between

  - multivector fields on \(P\) with Schouten–Nijenhuis bracket
  - multidifferential operators on \(P\) with Gerstenhaber bracket
Chern–Simons

- Let $\Sigma$ be a 2-manifold and $\mathfrak{g}$ a quadratic Lie algebra.
- Let $N$ be the space of $\mathfrak{g}$-valued 1-forms $A$ (connections) on $\Sigma$ with the Atiyah–Bott symplectic structure $\omega = \frac{1}{2} \int_{\Sigma} \delta A \delta A$.
- We let $C$ denote the space of flat connections. Then $C$ turns out to be the quotient by gauge transformations.
- BFV: $M = N \times T^* \Omega^0(\Sigma, \mathfrak{g})[1]$ and
  \[ S = \int_{\Sigma} (c, F_A) + \frac{1}{2} (b, [c, c]) \]
- On $\partial \Sigma$ we get $\omega = \int_{\partial \Sigma} \delta A \delta c$,
  \[ S = \frac{1}{2} \int_{\partial \Sigma} c d_A c \]
- We can interpret this as an affine Poisson structure on $\Omega^1(\partial \Sigma, \mathfrak{g})$, which we may regard as the dual of the affine Lie algebra $\hat{\mathfrak{g}} = \Omega^0(\partial \Sigma, \mathfrak{g}) \oplus \mathbb{R}$. 


This procedure may be applied to other field theories, assuming the reduction $\tilde{\omega} \to \omega$ is smooth.

4d gravity can be analyzed this way using Cartan’s coframe formalism.

The constraints yield a BFV structure on the fields on a 3-manifold $\Sigma$.

On the fields on $\partial \Sigma$ we get a Poisson structure. Its quantization is some sort of current algebra for 4d gravity.
Cartan’s coframes

- Let $\Sigma$ be a 3-manifold and $V \to \Sigma$ a vector bundle isomorphic to $T\Sigma \oplus \mathbb{R}$. We fix a fiberwise Minkowski metric $\eta$.
- The first field (coframe) is an injective bundle map $e : T\Sigma \to V$. We assume that $e^*\eta$ is nondegenerate.
- The second field is a connection $\varpi$ for the orthogonal bundle associated to $(V, \eta)$.
- We work modulo the equivalence relation $\varpi \sim \varpi' \text{ if } \varpi - \varpi' = u \in \Omega^1(\Sigma, \Lambda^2 V) : e u = 0$.

The symplectic form is $\omega = \int_\Sigma e \delta e \delta \varpi$.

The constraints defining $C$ are

\[ e d \varpi e = 0 \quad e F_{\varpi} = 0 \]

One can show that $C$ is coisotropic and that its reduction is the same as the reduced phase space of general relativity in its Einstein–Hilbert formulation.

The BFV action has nonlinear terms in the $b$ variables (i.e., non BRS type).
The Poisson structure

- On $\partial \Sigma$ the fields include the restriction of $e$ and the restriction of the ghost $\alpha \in \Omega^0(\partial \Sigma, \Lambda^2 V)[1]$.  
- The presymplectic form is $\int_{\partial \Sigma} \frac{1}{2} e \delta e \delta \alpha + \cdots$.
- The action (before reduction) is $S = \int_{\partial \Sigma} \frac{1}{2} [\alpha, \alpha] e e + \cdots$.
- This part of the theory may be interpreted as a Poisson structure on the space of 0-forms on $\partial \Sigma$ taking values in the pure tensors $(E := ee)$ in $\Lambda^2 V$.
- Restricting to constant fields, we have $\{\text{pure tensors}\}$ as a Poisson submanifold of $\mathfrak{so}(3,1)^*$. It is actually determined by the vanishing of a certain quadratic Casimir.
- A representation of the quantization of this Poisson manifold is then a representation of $\mathfrak{so}(3,1) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ where the two quadratic Casimirs are equal: integral spin representations.