Derived brackets = hamiltonian flow

Yaël Frégier

Abstract

We show that T. Voronov’s derived bracket construction, a widely used method to build $L_\infty$ algebras, can be understood as a hamiltonian flow for a suitable shifted symplectic structure, answering a conjecture of B. Vallette. As an application, we extend Voronov’s derived brackets to a more general setting, replacing the dgLa’s by $L_\infty$-algebras as the input data of the construction. Applications to homotopy derivations studied by Lada are also given.

1 Introduction

One knows from the pioneering work of Gerstenhaber [8] and Kodaira-Spencer [11] that differential graded Lie algebras can govern deformation problems. Lurie has shown that actually, in the derived sense, every deformation problem is govern by dgLa’s. However, there are concrete situations in which, if not in the derived framework, $L_\infty$-algebras are the actual controlling structures [14], [4], [7], [10],[9]. We recall that $L_\infty$ algebras generalise dgLa’s and were introduced by Stasheff and co-authors in [16] and [12]. Building such $L_\infty$-algebras is a difficult task that can be approached by operad theory ([6], [13]) in an algebraic setting. However, certain geometric situations ([4], [7], [10],[9]) require another technique, the derived bracket construction, introduced by Theodore Voronov in [17] and [18]. His technique builds an $L_\infty$ algebra out of $(L, a, P, \Delta)$, a graded Lie algebra $L$, an abelian Lie sub-algebra $a$, a projection $P$ onto $a$ and an element $\Delta$ of degree one in $L$. This construction is now widely used and has been generalised to a non abelian setting in [3] and [2].

A geometric formulation of a problem often provides a clearer understanding and enables generalisations [1], [15]. There has been recent progress [5] in giving a geometric interpretation of many constructions of homotopical algebra in the operadic language as actions under suitable gauge groups. B. Vallette has conjectured that this approach should encompass the derived bracket construction.

The aim of this note is to show that this indeed the case, given that one is willing to consider and additional structure of a 1-shifted symplectic form.

The idea goes as follows. One views the graded Lie algebra used in Voronov’s construction as a Poisson bivector $\pi$ on $L^*$. We then consider $\pi$ as a function $h_\pi \in C^\infty(T^*[1]L^*)$. 
Exponentiating the associated hamiltonian vector field $Q_\pi = (h_\pi, -)$ with respect to the canonical 1-shifted Poisson structure $(-, -)$ on $C^\infty(T^*[1]L^*)$, one obtains a new function $e^{Q_\pi}$. Explicit expressions are given by lemma 1 and lemma 3.

We then restrict this function to $T^*[1]\alpha^*$, which can be embedded in $T^*[1]L^*$, see proposition 1, thanks to the projection $P$. The associated hamiltonian vector field on $T^*[1]\alpha^*$, when restricted to the fiber $\alpha \simeq \alpha^{**}$, gives back (corollary 1 of proposition 2) the homological vector field encoding the $L_\infty$-algebra produced by Voronov’s construction.

A first spin-off of our geometric point of view is that it enables to generalise Voronov’s construction for an input data consisting of a $L_\infty$-algebra instead of a graded Lie algebra. However, we can not rely on Voronov’s proof that the the obtained brackets form an $L_\infty$-algebra in this new setting. This is why we need our own proof of this fact in section 4. It is based on an alternative proof of the fact, due to Bordemann [3], that Voronov’s construction is a morphism of graded Lie algebras.

## 2 Main lemma

Let $(L, \alpha, P, \Delta)$ a V-data. The Kostant-Kirilov-Souriau (KKS) Poisson structure on $M = L^*$ is defined by

$$\{f, g\}_L(x) := [df_x, dg_x]$$

whith $df_x$ and $dg_x$ elements of $(T^*_xM) \simeq L$. Let $\pi$ be the corresponding bi-vector.

Recall that $T[1]^*M$ has a one-shifted symplectic structure that reads in coordinates $x^i, x^*_i$ (base/fiber),

$$\omega := dx^*_i dx^i.$$

The isomorphism $\chi^*(M) \simeq C^\infty(T^*[1]M)$ identifies $\pi = C^{ij}_k x^k \partial_{x^i} \wedge \partial_{x^j}$ with $h_\pi := C^{ij}_k x^k x^*_i x^*_j$. Consider $Q_\pi$ defined by

$$i_{Q_\pi} \omega = dh_\pi. \quad (1)$$

**Lemma 1.**

$$Q_\pi = (-1)^{(x_i + x_j + x_k + 1)} 2 C^{ij}_k x^k x^*_j dx^*_i \frac{\partial}{\partial x^i} + (-1)^{(x_k + x_i)(x_i + 1)} C^{kl}_i x^*_k x^*_l \frac{\partial}{\partial x^*_i}.$$ 

**Proof.** We want to deduce from (1) the coefficients $A^i$ and $B^i$ of $Q_\pi = A^i \frac{\partial}{\partial x^i} + B^i \frac{\partial}{\partial x^*_i}$. But

$$i_{A^i} \frac{\partial}{\partial x^*_i} \omega = A^i (i \frac{\partial}{\partial x^*_j} (dx^*_j) dx^i) + (-1)^{(x_i + 1)(x_j + 1)} dx^*_i i \frac{\partial}{\partial x^*_i} (dx^j) = (-1)^{(x_i + 1)(x_j + 1)} A^i dx^*_i,$$

$$i_{B^i} \frac{\partial}{\partial x^*_i} \omega = B^i dx^i.$$
and on the other hand
\[ dh_\pi = d(C_k^{ij} x^k x^*_i x^*_j) \]
\[ = C_k^{ij} dx^k x^*_i x^*_j + (-1)^{x_k} C_k^{ij} x^k x^*_i x^*_j + (-1)^{x_k+x_i+1} C_k^{ij} x^k x^*_i dx^*_j. \]

Therefore, (1) reads
\[
(-1)^{(x_i+1)(x_j+1)} A_i dx^*_i = (-1)^{x_k} C_k^{ij} x^k x^*_i x^*_j + (-1)^{x_k+x_i+1} C_k^{ij} x^k x^*_i dx^*_j \\
= (-1)^{x_k+(x_i+2)(x_j+1)} C_k^{ij} x^k x^*_j dx^*_i + (-1)^{x_k+x_j+1} C_k^{ji} x^k x^*_j dx^*_i \\
= (-1)^{x_k} ((-1)^{x_i(x_j+1)} C_k^{ij} + (-1)^{x_j+1} C_k^{ji}) x^k x^*_j dx^*_i \\
= (-1)^{x_k} (-1)^{x_k} C_k^{ij} x^k x^*_j dx^*_i + (-1)^x E^x I^x I^x I^x I^x C_k^{ij} x^k x^*_j dx^*_i \\
= (-1)^{x_k+x_2(x_j+1)} 2 C_k^{ij} x^k x^*_j dx^*_i.
\]

i.e.
\[
A_i dx^*_i = (-1)^{(x_i+1)(x_j+1)} (-1)^{x_k+x_2(x_j+1)} 2 C_k^{ij} x^k x^*_j dx^*_i \\
= (-1)^{(x_i+x_j+x_k+1)} 2 C_k^{ij} x^k x^*_j dx^*_i
\]

while
\[
B_i dx^i = C_k^{ij} dx^k x^*_i x^*_j \\
= (-1)^{(x_i+1+x_j+1)(x_k+1)} C_k^{ij} x^*_i x^*_j dx^k \\
= (-1)^{(x_k+x_i)(x_j+1)} C_k^{ij} x^*_i x^*_j dx^i.
\]

\[ \square \]

Lemma 2. \( Q_\pi g = [g, -] + \sum g \circ_i [-, -] \)

Notation 1. \( e^f = f + [f, -] + \frac{1}{2} [[f, -], -] + \ldots \)

In particular, one recognises the derived brackets formulas as the Taylor coefficients of \( e^\Delta \).

The previous lemma enables us to understand \( e^\Delta \) in terms of a hamiltonian flow as follows:

Lemma 3. If \( f \in \mathcal{C}^\infty(T^*[1]M) \) is of polynomial degree one in the \( x^i \),

\[ e^{Q_\pi} f = e^f + \rho \circ h_\pi \quad (2) \]
Proof. One first remarks that $Q_\pi$ is a derivation of $C^\infty(T^*[1]M)$ that preserves the degree in the $x^i$. 

Our task in the next section is to manage to get rid of the second summand in (3) to justify the claim of the title.

3 Restriction to $a$

Our guiding idea is that $h_{\Delta} = e^{Q_\pi \Delta}$ is supposed to encode the $L_\infty$ structure on $a$ that we are looking for. Therefore, we wish to associate to $h_{\Delta}$ a homological vector field on $a$.

The trick is to embed $T^*[1]a^*$ in $T^*[1]M$. One can therefore consider $h_{\Delta}|$ the restriction of $h_{\Delta}$ to $T^*[1]a^*$. It suffices then to show that $Q_{\Delta} := (h_{\Delta}|, - )$ gives back the derived brackets $??$ as its Taylor coefficients. This relies on identifying $a^{**}[1]$, the fiber of $T^*[1]a^*$, with $a[1]$ and show that it is preserved by $Q_{\Delta}$.

**Proposition 1.** A projection $P : L \to a$ induces an embedding

$$T^*[1]a^* \hookrightarrow T^*[1]L^*.$$ 

Proof. Since $T^*[1]L^* = L^{**}[1] \oplus L^*$, one can apply lemma 4 to $L = a \oplus \text{Ker}(P)$ and get

$$T^*[1]L^* \simeq (a \oplus \text{Ker}(P))[1] \oplus a^* \oplus \text{Ker}(P)^*$$

$$\simeq T^*[1]a^* \oplus T^*[1]\text{Ker}(P)^*.$$ 

Given a sub-vector space $E \subset F$, one uses the notation $F^\circ = \{ l \in E^* / l|_F = 0 \}$.

**Lemma 4.** Consider a direct sum decomposition of a vector space $V = S \oplus T$. Then

$$V^* = S^\circ \oplus T^\circ.$$ 

Moreover the maps $i_T^* : S^\circ \to T^*$ and $P_T^* : T^* \to S^\circ$ defined by $i_T^*(l) = l \circ i_T$ and $P_T^*(l) = l \circ P_T$ are inverse to each other. Similarly with $i_S^*$ and $P_S^*$. In particular

$$V^* = S^* \oplus T^*.$$ 

Proof. Exercise in linear algebra.

The algebra of polynomial functions on $T^*[1]L^*$ is generated by the coordinate functions $x_i^*$ and $x^i$. Lemmas 5 and 6 below describe the restriction of these generators to $T^*[1]a^*$ and proposition 2 gives the expression of $h_{\Delta}|$ that we are looking for.

**Lemma 5.** $x_i^*_{|a^*} = P_a(x_i^*)$. Here $x_i^*$, an element of $L^{**}$, is seen as an element of $L$ and hence can pass in $P_a$. The result is then itself interpreted as an element of $a^{**}$. 

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Proof. Let \( t \in a^*[1] \),

\[
x^*_i(t) = x^*_i(P_a^*(t)) = P_a^*(t)(x^*_i) = t(P_a(x^*_i)) = P_a(x^*_i)(t).
\]

Lemma 6. \( x^i|_a = i^*_a x^i \).

Proof.

Proposition 2.

\[
h_{\Delta} = P \Delta + P[\Delta, -]|_a + \frac{1}{2} P[[\Delta, -], -]|_a + \ldots
\]

(3)

Proof. By proposition 3,

\[
e^{Q_\pi} \Delta = e^\Delta + \rho \circ h_\pi,
\]

but \( \rho \circ h_\pi|_a = 0 \) since \( a \) is an abelian sub-Lie algebra of \( L \). Hence

\[
h_{\Delta}|_{T^*[1]a^*} = e^{\Delta}|_{T^*[1]a^*}.
\]

With notation 1, lemmas 6 and 5 mean that

\[
h_{\Delta} = P \Delta + P[\Delta, -]|_a + \frac{1}{2} P[[\Delta, -], -]|_a + \ldots
\]

Corollary 1. \( Q_\Delta := (h_{\Delta}|, -) \) gives back the derived brackets ?? as its Taylor coefficients.

Proof.

4 Proof of the \( L_\infty \) structure

Lemma 7. \( e^{Q_\pi} [f, g] = ((\pi, e^{Q_\pi} f), e^{Q_\pi} g) \)

Proof.

\[
e^{Q_\pi} [f, g] = e^{Q_\pi}(Q_\pi f, g) = (e^{Q_\pi} Q_\pi f, e^{Q_\pi} g) = (Q_\pi e^{Q_\pi} f, e^{Q_\pi} g) = ((\pi, e^{Q_\pi} f), e^{Q_\pi} g)
\]

\( \square \)
5 Sanity check and recap

6 Applications

6.1 "higher" higher derived brackets

6.2 Homotopy derivations

References


