Mathematics 100 Midterm Review Package

UBC Engineering Undergraduate Society

Problems are ranked in difficulty as (•) for easy, (••) for medium, and (•••) for difficult. Note that sometimes difficulty can be subjective, so do not be discouraged if you are stuck on a (•) problem.

Solutions posted at: [http://ubcengineers.ca/tutoring/](http://ubcengineers.ca/tutoring/)

If you believe that there is an error in these solutions, or have any questions, comments, or suggestions regarding EUS Tutoring sessions, please e-mail us at: tutoring@ubcengineers.ca. If you are interested in helping with EUS tutoring sessions in the future or other academic events run by the EUS, please e-mail vpacademic@ubcengineers.ca.

The first 7 problems are review of high school material and are highly optional. They cover the basics of the different functions covered in high school.

Some of the problems in this package were not created by the EUS. Those problems originated from one of the following sources:

- Schuam’s Outline of Calculus 2 ed; Ayres Jr., Frank
- Calculus – Early Transcendentals 7 ed; Stewart, James
- Calculus – 3 ed; Spivak, Michael
- Calculus Volume 1 2 ed; Apostol, Tom

Want a warm up? These are the easier problems
1, 4, 7, 8

Short on study time? These cover most of the material
3, 10, 15, 18, 21

Want a challenge? These are some tougher questions
15, 21, 25, 26

EUS Health and Wellness Study Tips

- **Eat Healthy**—Your body needs fuel to get through all of your long hours studying. You should eat a variety of food (not just a variety of ramen) and get all of your food groups in.

- **Take Breaks**—Your brain needs a chance to rest: take a fifteen minute study break every couple of hours. Staring at the same physics problem until your eyes go numb won’t help you understand the material.

- **Sleep**—We have all been told we need 8 hours of sleep a night, university should not change this. Get to know how much sleep you need and set up a regular sleep schedule.
(∗) 1. Evaluate the limit. \( \lim_{x \to -1} \frac{x^2 + 3x + 2}{x^2 + 4x + 3} \)

(∗∗) 2. Suppose that

\[ \lim_{x \to a} (f(x) + g(x)) = 2 \]
\[ \lim_{x \to a} (f(x) - g(x)) = 1 \]

find the value of \( \lim_{x \to a} f(x) \cdot g(x) \)
3. Find one pair of values $a, b$ such that $f(x)$ is continuous at $x = 0$.

$$f(x) = \begin{cases} (e^x - 1) \cos \left( \frac{1}{x} \right), & x < 0 \\ \sin(x + a) + b, & x \geq 0 \end{cases}$$
4. Differentiate the following function: \( y = \left( \frac{x}{1 + x} \right)^3 \)

5. Differentiate the following function. \( f(x) = (x + x^{-1})^3 \)

6. Differentiate the following function. \( f(x) = \frac{1-xe^x}{x+e^x} \)
7. Using the Squeeze Theorem, show that

\[ \lim_{x \to 0} \sqrt{x^3 + x^2 \sin \sqrt{\pi/x}} = 0 \]

8. Using the formal definition of the derivative, find the derivative of the function \( f(x) = ax^2 + bx + c \)

9. Using the definition of the derivative, find the derivative of the function \( f(x) = \sqrt{4-x} \)
(**) 10. Show that there exist at least 2 real solutions to the equation $8 \sin x = x^2 + 1$

(**) 11. Show that there exists at least 1 real solution to $\cos x = \sqrt{x}$
12. Evaluate the limit \( \lim_{x \to -4} \frac{x^{1001} + 4^{1001}}{x + 4} \)

13. Suppose \( f(x) \) is differentiable at \( x = a \). Find the value of the limit in terms of \( f'(a) \):
\[
\lim_{x \to a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}}
\]

14. Let \( t(x) = \frac{u(x)}{\sin x} \) and suppose \( u(\pi/3) = 4 \) and \( u'(\pi/3) = -1 \). Find \( t'(\pi/3) \)
15. For the given function $f(x)$, determine if

(a) $f(x)$ is continuous at $x = 2$, and

(b) if $f(x)$ is differentiable at $x = 2$.

$$f(x) = \frac{|x - 2| - 7}{x + 4}$$
16. Compute a linear approximation to $z = z(w)$ at the point $w = 0$ if

$$z = \frac{w}{\sqrt{1 - w^2}}$$

17. Show that

$$\lim_{x \to 0^+} \sqrt{x} e^{\sin \sqrt{\frac{\pi}{x}}} = 0$$

18. Find the equation of the line tangent to $f(x) = \frac{4\pi}{x^2}$ at the point $x = 1$. 
19. Suppose \( f(x) \) is defined as

\[
f(x) = \lim_{t \to x} \frac{\sec t - \sec x}{t - x}
\]

Compute \( f'(\pi/4) \).
(***) 20. Compute $f'(x)$ using the definition of the derivative if $f(x) = \frac{\sqrt{x + 1}}{x}$. 
21. Suppose $f$ is a function that is everywhere positive.

$$f(x) = \begin{cases} 
(sin x)^2 \sin \left( \frac{1}{x} \right) + 2 & x < 0 \\
ax^2 + bx + c & x \geq 0 
\end{cases}$$

Find the values of $a, b, c$ such that the following conditions all hold:

(a) $f(x)$ is continuous at $x = 0$.
(b) $f(x)$ is differentiable at $x = 0$.
(c) $g''(0) = 3$, with $g(x) = f(f(x))$.

Remark 1. The following information may be useful, and you may use the results without proof:

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1, \quad \lim_{x \to 0} \frac{e^x - 1}{x} = 1$$
22. Suppose $h$ is a function such that $h'(x) = \sin^2(\sin(x + 1))$, and $h(0) = 3$. Find

(a) $(h^{-1})'(3)$.

(b) $(\beta^{-1})'(3)$, where $\beta(x) = h(x + 1)$. 


23. Suppose that $f$ is a differentiable and invertible function, and that $f = F'$. Let

$$G(x) = xf^{-1}(x) - F(f^{-1}(x)).$$

Show that $G'(x) = f^{-1}(x)$. 

24. (**) A body moves along a horizontal line according to the law \( s = f(t) = t^3 - 9t^2 + 24t \).

(a) When is \( s \) increasing and when decreasing?
(b) When is \( v \) increasing and when decreasing?
(c) When is the speed of the body increasing and when decreasing?
(d) Compute the total distance travelled in the first 5 seconds of motion.
25. Two sides of a triangle are 15 and 20 m long, respectively.

(a) How fast is the third side increasing if the angle between the given sides is \( \pi/3 \) and is increasing at a rate of \( \pi/45 \) radians per second?

(b) How fast is the area increasing?
26. Two ships sail from the origin at the same time. One sails south at 15 km/hr; the other sails east at 25 km/hr for 1 hour and then turns north. Find the rate of rotation in rad/s of the line joining them after 3 hours.