Mathematics 152 Midterm 2 Review Package – **Solutions**

UBC Engineering Undergraduate Society

Attempt questions to the best of your ability. Problems are ranked in difficulty as (∗) for easy, (∗∗) for medium, and (∗∗∗) for difficult.

Solutions posted at: [https://ubcengineers.ca/tutoring](https://ubcengineers.ca/tutoring)

If you believe that there is an error in these solutions, or have any questions, comments, or suggestions regarding EUS Tutoring sessions, please e-mail us at: tutoring@ubcengineers.ca. If you are interested in helping with EUS tutoring sessions in the future or other academic events run by the EUS, please e-mail vpacademic@ubcengineers.ca.

Some of the problems in this package were not created by the EUS. Those problems originated from one of the following sources (All solutions prepared by the EUS.):

- Schuam’s Outline of Matrix Operations; Richard Bronson
- Calculus 7th ed; James Stewart
- Linear Algebra; Sterling K. Berberian
- Linear Algebra and Its Applications 3rd ed; Gilbert Strang
- Linear Algebra and Matrix Theory; Robert Stoll

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**EUS Health and Wellness Study Tips**

- **Eat Healthy**—Your body needs fuel to get through all of your long hours studying. You should eat a variety of food (not just a variety of ramen) and get all of your food groups in.

- **Take Breaks**—Your brain needs a chance to rest: take a fifteen minute study break every couple of hours. Staring at the same physics problem until your eyes go numb won’t help you understand the material.

- **Sleep**—We have all been told we need 8 hours of sleep a night, university shouldn’t change this. Get to know how much sleep you need and set up a regular sleep schedule.
1. Consider the linear system
\[
\begin{align*}
&x + 2y + z = 1 \\
&-x + 3z = 1 \\
&x - y - 3z = 0
\end{align*}
\]
(a) Write this system as an augmented matrix.
(b) Write the system to row echelon form
(c) Write the system in reduced row echelon form
(d) Find the solution to the system

**Solution:**
(a) Taking the coefficients and putting them in the matrix, and constants on the right of the line,
\[
\begin{pmatrix}
1 & 2 & 1 & | & 1 \\
-1 & 0 & 3 & | & 1 \\
1 & -1 & -3 & | & 0
\end{pmatrix}
\]
(b) Performing elimination until all of the elements below the diagonal, we obtain
\[
\begin{pmatrix}
1 & 2 & 1 & | & 1 \\
0 & 1 & 2 & | & 1 \\
0 & 0 & 1 & | & 1
\end{pmatrix}
\]
(c) Continuing elimination until there are only ones on the diagonal, we obtain
\[
\begin{pmatrix}
1 & 0 & 0 & | & 2 \\
0 & 1 & 0 & | & -1 \\
0 & 0 & 1 & | & 1
\end{pmatrix}
\]
(d) Thus, \(x = 2, y = -1, z = 1\).
2. Compute the rank of \( A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 4 & 0 & 2 \end{pmatrix} \)

**Solution:** Performing elimination, we obtain
\[
\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 4 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]
and see that there are now only two independent rows. Note that there are also only two independent columns. Thus \( \text{rank}(A) = 2 \)

3. (a) Find the work done in moving an object along a vector \( \mathbf{r} = 3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k} \) if the applied force is \( \mathbf{F} = 2\mathbf{i} - \mathbf{j} - \mathbf{k} \)

(b) Find the angle between the applied force and the displacement.

**Solution:**
(a) \( W = \mathbf{r} \cdot \mathbf{F} = 6 - 2 + 5 = 9 \) J

(b) We can find the angle using the dot product: \( \mathbf{r} \cdot \mathbf{F} = 9 = |\mathbf{r}| \cdot |\mathbf{F}| \cos \theta \). Dividing through by \( |\mathbf{r}| \) and \( |\mathbf{F}| \), then inverting the cosine, we have
\[
\theta = \arccos \left( \frac{9}{\sqrt{6} \cdot \sqrt{38}} \right)
\]
4. Consider the following lines of Matlab code:
   \[
   x = 1:7;
y = 1:0.3:1.7;
   \]
   (a) What is \(x\)?
   (b) What is \(y\)?
   (c) If you call \(\sin(y)\), what will the output be? If this operation is defined, you may leave your answers in terms of trigonometric functions.
   (d) Is \(\text{cross}(x,y)\) defined?

Solution:
(a) The notation means the vector starting with 1, then incrementing by 1 until 7. That is,
\[
x = [1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7]
\]
(b) The notation means the vector starting with 1, then incrementing by 0.3 until 1.7. Note that it will not surpass 1.7
\[
y = [1 \ 1.3 \ 1.6]
\]
(c) Calling \(\sin(y)\) will apply the \(\sin\) function to every entry in the vector. That is,
\[
\sin(y) = [\sin(1) \ \sin(1.3) \ \sin(1.6)]
\]
(d) The operation \(\text{cross}(x,y)\) is not defined because only \(y\) has three components. Since \(x\) has 7 components, it cannot enter into the cross product.

5. What matrix \(A : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) represents projection onto the \(x\) axis followed by projection onto the \(y\) axis?

Solution: The zero matrix \(
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\) performs this operation, because every vector will be mapped into the zero vector. Projecting a vector onto the \(x\) - axis will remove its \(y\) - component, and when projecting onto the \(y\) - axis, there is already no \(y\) - component when the \(x\) component is removed the zero vector is produced.
6. \( \ast \) If \( A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ -2 & -1 & 1 \end{pmatrix} \), \( B = \begin{pmatrix} 2 & 3 & 1 \\ 2 & -2 & -2 \\ -1 & 2 & 1 \end{pmatrix} \), \( C = \begin{pmatrix} 3 & 1 & -3 \\ 0 & 2 & 6 \\ -1 & 2 & 1 \end{pmatrix} \)

Compute

(a) \( AB \)

(b) \( AC \)

What can you say about \( AB \) and \( AC \)? What does it say about cancellation of matrices? Does \( AB = AC \) imply that \( B = C \)?

**Solution:**

(a)

\[
AB = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 2 & -2 & -2 \\ -1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 4(2) + 2(2) + 0(-1) & 4(3) + 2(-2) + 0(2) & 4(1) + 2(-2) + 0(1) \\ 2(2) + 1(2) + 0(-1) & 2(3) + 1(-2) + 0(2) & 2(1) + 1(-2) + 0(1) \\ -2(2) + (-1)(2) + 1(-1) & -2(3) + (-1)(-2) + 1(2) & -2(1) + (-1)(-2) + 1(1) \end{pmatrix} = \begin{pmatrix} 12 & 8 & 0 \\ 6 & 4 & 0 \\ -7 & -2 & 1 \end{pmatrix}
\]

(b)

\[
AC = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & -3 \\ 0 & 2 & 6 \\ -1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 4(3) + 2(0) + 0(-1) & 4(1) + 2(2) + 0(2) & 4(-3) + 2(6) + 0(1) \\ 2(3) + 1(0) + 0(-1) & 2(1) + 1(2) + 0(2) & 2(-3) + 1(6) + 0(1) \\ -2(3) + (-1)(0) + 1(-1) & -2(1) + (-1)(2) + 1(2) & -2(-3) + (-1)(6) + 1(1) \end{pmatrix} = \begin{pmatrix} 12 & 8 & 0 \\ 6 & 4 & 0 \\ -7 & -2 & 1 \end{pmatrix}
\]

**Remark 1.** \( AC = AB \), but \( B \neq C \). This shows that the cancellation law is not always valid for matrices. The cancellation law in this case is not valid because \( \det A = 0 \). This means that one could **not** multiply both sides of \( AC = AB \) by \( A^{-1} \) to obtain \( B = C \).
7. Given \( T(x) = \begin{pmatrix} -1 & 3 \\ 9 & 4 \end{pmatrix} x \), and \( S(x) = \begin{pmatrix} 3 & -2 & 6 \\ -4 & 6 & 2 \end{pmatrix} x \), compute the following (if defined)

(a) \( T \circ S \)
(b) \( S \circ T \)
(c) \( T \left( \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) \)
(d) \( S \left( \begin{pmatrix} -2 \\ 4 \end{pmatrix} \right) \)
(e) \( S \left( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right) \)

Solution:

(a) \[ T \circ S = \begin{pmatrix} -1 & 3 \\ 9 & 4 \end{pmatrix} \begin{pmatrix} 3 & -2 & 6 \\ -4 & 6 & 2 \end{pmatrix} \]
\[ = \begin{pmatrix} -1(3) + 3(-4) & -1(-2) + 3(6) & -1(6) + 3(2) \\ 9(3) + 4(-4) & 9(-2) + 4(6) & 9(6) + 4(2) \end{pmatrix} \]
\[ = \begin{pmatrix} -15 & 20 & 0 \\ 11 & 6 & 62 \end{pmatrix} \]

(b) \( S \circ T = \text{undefined} \)

(c) \[ T \left( \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} -1 & 3 \\ 9 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \]
\[ = \begin{pmatrix} -1(2) + 3(1) \\ 9(2) + 4(1) \end{pmatrix} \]
\[ = \begin{pmatrix} 1 \\ 22 \end{pmatrix} \]

(d) \( S \left( \begin{pmatrix} -2 \\ 4 \end{pmatrix} \right) = \text{undefined} \)

(e) \[ S \left( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right) = \begin{pmatrix} 3 & -2 & 6 \\ -4 & 6 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \]
\[ = \begin{pmatrix} 3(1) + (-2)(2) + 6(3) \\ -4(1) + 6(2) + 2(3) \end{pmatrix} \]
\[ = \begin{pmatrix} 17 \\ 14 \end{pmatrix} \]
8. (a) Find the matrix \( R : \mathbb{R}^2 \to \mathbb{R}^2 \) that rotates vectors by 225° counterclockwise.

(b) Find the image of (2, 5) under this linear transformation.

**Solution:**

(a) The general rotation matrix is given by

\[
R = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

and we need to evaluate this rotation matrix for \( \theta = \frac{5\pi}{4} \). Thus

\[
R = \begin{pmatrix}
\cos(\frac{5\pi}{4}) & -\sin(\frac{5\pi}{4}) \\
\sin(\frac{5\pi}{4}) & \cos(\frac{5\pi}{4})
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-1/\sqrt{2} & 1/\sqrt{2} \\
-1/\sqrt{2} & -1/\sqrt{2}
\end{pmatrix}
\]

\[
= \frac{-1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\
1 & 1 \end{pmatrix}
\]

(b) Applying this matrix to the given vector:

\[
R \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \frac{-1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\
1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix}
\]

\[
= \frac{-1}{\sqrt{2}} \begin{pmatrix} 1(2) + (-1)(5) \\
1(2) + 1(5) \end{pmatrix}
\]

\[
= \frac{-1}{\sqrt{2}} \begin{pmatrix} -3 \\
7 \end{pmatrix}
\]

9. (** What matrix has the effect of rotating a vector \( v \in \mathbb{R}^2 \) through 90° clockwise, and then projecting the result onto the \( x \) axis?**

**Solution:** The rotation matrix will be

\[
R = \begin{pmatrix}
\cos(-90^\circ) & -\sin(-90^\circ) \\
\sin(-90^\circ) & \cos(-90^\circ)
\end{pmatrix} = \begin{pmatrix} 0 & 1 \\
-1 & 0 \end{pmatrix}
\]

and the projection matrix will be

\[
P = \begin{pmatrix} 1 \\ 0 \\
0 \\ 0 \end{pmatrix}
\]

Multiplying the two matrices together has the combined effect. Note that we multiply \( PR \) because the rotation is applied first, and the projection second.

\[
PR = \begin{pmatrix} 1 \\ 0 \\
0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\
-1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\
0 \\ 0 \end{pmatrix}
\]
10. (**) (a) Find the matrix $R : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects vectors across the line $y = -2x$.

(b) Show that $R^2 = I$.

(c) Reflect the vector $(-2, 3)$ across the line $y = -2x$.

Solution:

(a) Let $\mathbf{x}$ be the direction of the line. Then,

$$\mathbf{x} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

The formula for the reflection matrix is as follows:

$$R = 2 \frac{\mathbf{x} \mathbf{x}^T}{\mathbf{x}^T \mathbf{x}} - I$$

$$= \frac{1}{2} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{2}{5} \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2/5 & -4/5 \\ -4/5 & 8/5 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{pmatrix}$$

(b) We perform the straightforward computation

$$R^2 = \begin{pmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{pmatrix} \begin{pmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{pmatrix}$$

$$= \frac{1}{25} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

$$= \frac{1}{25} \left( 3(3) + 4(4) \quad 3(4) + 4(-3) \\ 4(3) + (-3)(4) \quad 4(4) + (-3)(-3) \right)$$

$$= \frac{1}{25} \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Intuitively, $R^2 = I$ because if you reflect twice, you get back to where you started, so there is no net change.
(c) Now reflecting the given vector across the given line,

\[
R \left( \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right) = \begin{pmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -3 & -4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -3(-2) + (-4)(3) \\ -4(-2) + 3(3) \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -6 \\ 17 \end{pmatrix}
\]
11. Set up the augmented matrix $A$ corresponding to this resistor network, with the loop currents in the first three columns.

**Solution:** To find the equations from each loop, go in the direction of the arrow (same for all loops) and sum up voltage drops (hence why going counter-clockwise through $E$ is negative) and using a difference of currents when going through resistors bounded by two current loops.

The four equations are

\[
\begin{align*}
-6 + 1(i_1) + E &= 0 \\
-E + 2(i_2) + 4(i_2 - i_3) + 7 + 3(i_2) &= 0 \\
4(i_3 - i_2) + 5(i_3) + 8 &= 0 \\
i_2 - i_1 &= 9
\end{align*}
\]

Rearranging and simplifying:

\[
\begin{align*}
i_1 + E &= 6 \\
9i_2 - 4i_3 - E &= -7 \\
-4i_2 + 9i_3 &= -8 \\
i_1 + i_2 &= 9
\end{align*}
\]

We thus have the matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 6 \\
0 & 9 & -4 & -1 & -7 \\
0 & -4 & 9 & 0 & -8 \\
-1 & 1 & 0 & 0 & 9
\end{pmatrix}
\]
Consider the linear system for the unknowns $x$, $y$, and $z$.

\[
\begin{align*}
4x + 2y - 3z - 6 &= 0 \\
x - 4y + z + 4 &= 0 \\
-x + 2z - 2 &= 0
\end{align*}
\]

(a) Write the system in an augmented matrix.

(b) Perform row operations on the augmented matrix to change it to upper triangular form.

(c) Find the solution to the problem from above.

**Solution:**

(a) Move constants to the right side of the equations.

\[
A = \begin{pmatrix}
4 & 2 & -3 & 6 \\
1 & -4 & 1 & -4 \\
-1 & 0 & 2 & 2
\end{pmatrix}
\]

(b) Perform row operations.

\[
\begin{align*}
&A \\ &\sim \begin{pmatrix}
4 & 2 & -3 & 6 \\
0 & -\frac{9}{2} & \frac{7}{2} & \frac{11}{2} \\
0 & 1 & \frac{5}{2} & \frac{7}{2}
\end{pmatrix} \\
&\sim \begin{pmatrix}
4 & 2 & -3 & 6 \\
0 & -18 & 7 & -22 \\
0 & 2 & 5 & 14
\end{pmatrix} \\
&\sim \begin{pmatrix}
4 & 2 & -3 & 6 \\
0 & -18 & 7 & -22 \\
0 & 0 & \frac{52}{9} & \frac{104}{9}
\end{pmatrix} \\
&\sim \begin{pmatrix}
4 & 2 & -3 & 6 \\
0 & -18 & 7 & -22 \\
0 & 0 & 52 & 104
\end{pmatrix}
\end{align*}
\]

(c) We can now apply back substitution to solve the system. We start with the last equation

\[52z = 104\]

to find

\[z = 2\]

Now plugging this result in to

\[-18y + 7z = -22\]

we can solve

\[y = 2\]
Now taking the results for \( y \) and \( z \) and plugging into the first equation,

\[ 4x + 2y - 3z = 6 \]

we can find

\[ x = 2 \]

This yields the final result of

\[ (x, y, z) = (2, 2, 2) \]

A unique solution to this linear system of equations.

(\* \*) 13. Consider the resistor network below:

(a) Set up the augmented matrix \( A \) corresponding to this resistor network

(b) Solve the augmented matrix for the currents 1, 2, and 3

(c) Find the voltage \( V_0 \)

\[ \text{Solution:} \]

(a) The equations from the circuit are:

\[ \begin{align*}
\text{Loop 1} & : \quad I_1 = 2 \\
\text{Loop 2} & : \quad -12 + (I_2 - I_1) + (I_2 - I_3) = 0 \\
\quad & \Rightarrow -I_1 + 2I_2 - I_3 = 12 \\
\text{Loop 3} & : \quad (I_3 - I_2) + (I_3 - I_1) + 2I_3 = 0 \\
\quad & \Rightarrow -I_1 - I_2 + 4I_3 = 0
\end{align*} \]

Note: Loop 1 has a single current source inline with the direction of loop 1, hence the current of the loop must equal the current source 2A.

The first loop shows the current of loop 1 while the second and third loops use the theorem which states the voltages around a closed loop must sum to 0.

Then set up the augmented matrix:

\[ A = \begin{pmatrix}
1 & 0 & 0 & | & 2 \\
-1 & 2 & -1 & | & 12 \\
-1 & -1 & 4 & | & 0
\end{pmatrix} \]
(b) Then solve the augmented matrix using reduced row echelon form.

\[
A = \begin{pmatrix}
1 & 0 & 0 & | & 2 \\
-1 & 2 & -1 & | & 12 \\
-1 & -1 & 4 & | & 0
\end{pmatrix}
\sim \begin{pmatrix}
1 & 0 & 0 & | & 2 \\
0 & 2 & -1 & | & 14 \\
0 & -1 & 4 & | & 2
\end{pmatrix}
\sim \begin{pmatrix}
1 & 0 & 0 & | & 2 \\
0 & 2 & -1 & | & 14 \\
0 & 0 & 7 & | & 9
\end{pmatrix}
\sim \begin{pmatrix}
1 & 0 & 0 & | & 2 \\
0 & 2 & -1 & | & 14 \\
0 & 0 & 1 & | & 18
\end{pmatrix}
\sim \begin{pmatrix}
1 & 0 & 0 & | & 2 \\
0 & 2 & 0 & | & 116 \\
0 & 0 & 1 & | & 18
\end{pmatrix}
\sim \begin{pmatrix}
1 & 0 & 0 & | & 2 \\
0 & 0 & 1 & | & 38 \\
0 & 0 & 1 & | & 18
\end{pmatrix}
\]

\[
(I_1, I_2, I_3) = (2, \frac{38}{7}, \frac{18}{7})
\]

(c) \( V_0 \) is simply \( 2I_3 = \frac{36}{7}V \)