Physics 158 Final Exam Review Package

UBC Engineering Undergraduate Society

Problems are ranked in difficulty as (*) for easy, (**) for medium, and (***) for difficult. Note that sometimes difficulty can be subjective, so do not be discouraged if you are stuck on a (*) problem.

Solutions posted at: [http://ubcengineers.ca/tutoring/](http://ubcengineers.ca/tutoring/)

If you believe that there is an error in these solutions, or have any questions, comments, or suggestions regarding EUS Tutoring sessions, please e-mail us at: tutoring@ubcengineers.ca. If you are interested in helping with EUS tutoring sessions in the future or other academic events run by the EUS, please e-mail vpacademic@ubcengineers.ca.

Want a warm up? Short on study time? Want a challenge?
These are the easier problems These cover most of the material These are some tougher questions
1,2,3 4,5,8,9 14,15,16,17

Some of the problems in this package were not created by the EUS. Those problems originated from one of the following sources:

- Introduction to Electrodynamics 3 ed. / David J. Griffths
- Electricity, Magnetism, and Light / Wayne Saslow
- Exercises for the Feynman Lectures on Physics / Matthew Sands, Richard Feynman, Robert Leighton.

All solutions prepared by the EUS.

EUS Health and Wellness Study Tips

- **Eat Healthy**—Your body needs fuel to get through all of your long hours studying. You should eat a variety of food (not just a variety of ramen) and get all of your food groups in.
- **Take Breaks**—Your brain needs a chance to rest: take a fifteen minute study break every couple of hours. Staring at the same physics problem until your eyes go numb wont help you understand the material.
- **Sleep**—We have all been told we need 8 hours of sleep a night, university should not change this. Get to know how much sleep you need and set up a regular sleep schedule.

Good Luck!
1. A wire of the shape shown in the figure carries a current $I$. What is the magnetic field $B$ at the centre of the semicircle arising from

(a) Each straight segment of length $l$?

(b) The semicircular segment of length $\pi r$?

(c) The entire wire?

Solution:

(a) The magnetic field contribution due to either of the straight wires is 0 because the current is parallel to the position vector that points from the current elements to the centre of the semicircle.

(b) The magnetic field contribution due to an element of the semicircle is

$$|dB| = \frac{\mu_0 I}{4\pi} \frac{|dl \times \hat{r}|}{r^2}$$

So the total magnetic field is then

$$|B| = \int_0^{\pi} \frac{\mu_0 I r d\theta}{4\pi} \frac{r}{r^2} = \frac{\mu_0 I}{4r}$$

(c) The magnetic field due to the entire wire is then

$$B = \frac{\mu_0 I}{4r}$$

pointing into the page.
2. Suppose conducting spherical shell possess a net charge $Q$. Let a point charge $q$ be within the shell.

(a) Find the total charge $Q_{\text{inner}}$ on the inner surface of the shell.

(b) Find the total charge $Q_{\text{outer}}$ on the outer surface of the shell.

(c) Determine if $\sigma_{\text{outer}}$ is distributed uniformly or not.

(d) If $q$ is on-center, determine if $\sigma_{\text{inner}}$ is distributed uniformly or not.

Solution:

(a) In order for the electric field inside the conducting sphere to be 0, the total enclosed charge for any imaginary sphere within the shell must be 0. Thus, the charge on the inside surface of the shell must be equal and opposite to the point charge and we have $Q_{\text{inner}} = -q$

(b) $Q_{\text{outer}} = Q - Q_{\text{inner}} = Q + q$

(c) The charge on the outer surface is distributed evenly regardless of the placement of the point charge $q$. The charge on the inner surface completely negates the point charge $q$ so it is not “felt” by any charge on the outer surface.

(d) If the point charge is in the centre of the sphere, the charge on the inner surface must be evenly distributed due to symmetry.
3. (a) Find the radial component of the electric field $E_r$ as a function of the radial distance.
(b) Find the potential relative to infinity as a function of the radial distance.
(c) If the centre sphere is moved slightly off the centre of the outer sphere, explain what happens to the electric field for $r_b < r < r_c$, and what happens to the electric flux through the surface of the outer sphere.

Solution:
(a) Electric field is always 0 inside a conductor. These spheres are made of metal, so the electric field inside each of them is 0. The electric field for $r_a < r < r_b$ is given by $E_r = \frac{q'}{4\pi \epsilon_0 r^2}$. The electric field for $r > r_c$ is given by $E_r = \frac{q + q'}{4\pi \epsilon_0 r^2}$. Thus

$$E_r = \begin{cases} 
0 & 0 < r < r_a \\
\frac{q'}{4\pi \epsilon_0 r^2} & r_a < r < r_b \\
\frac{q + q'}{4\pi \epsilon_0 r^2} & r_b < r < r_c \\
0 & r_c < r 
\end{cases}$$

(b) Since we have $E_r = -\frac{d}{dr}V$, and we know that the potential is 0 at infinity, we can integrate to find the potential function $V$. We have

$$V(r) = \begin{cases} 
C_1 & 0 < r < r_a \\
\frac{q'}{4\pi \epsilon_0 r^2} + C_2 & r_a < r < r_b \\
C_3 & r_b < r < r_c \\
\frac{q + q'}{4\pi \epsilon_0 r^2} + C_4 & r_c < r 
\end{cases}$$

Since $V(\infty) = 0$, we have $C_4 = 0$. Since the potential must be continuous, plugging in the appropriate values (due to continuity) yields

$$V(r) = \begin{cases} 
\frac{1}{4\pi \epsilon_0} \left( \frac{q'}{r_a} + \frac{q + q'}{r_c} - \frac{q'}{r_b} \right) & 0 < r < r_a \\
\frac{q'}{4\pi \epsilon_0 r^2} + \frac{q + q'}{4\pi \epsilon_0 r_c} & r_a < r < r_b \\
\frac{q + q'}{4\pi \epsilon_0 r^2} - \frac{q'}{4\pi \epsilon_0 r_b} & r_b < r < r_c \\
\frac{q + q'}{4\pi \epsilon_0 r^2} & r_c < r 
\end{cases}$$
(c) The electric field for $r_b < r < r_c$ will remain 0 because it is still inside a conductor. The electric flux through the outer surface will not change, because the total charge enclosed does not change.
4. A source emitting light of wavelength $\lambda = 400$ nm lies a distance $h = 1$ mm above a smooth mirror, which is located at $y = 0$. There is a screen $L = 50$ m away from the source.

(a) Find the height of the second dark band above $y = 0$.

(b) Suppose the mirror is removed, and another light source emitting the same wavelength (in phase) is placed a distance of 2 mm below the first source. Additionally, there is a flat lens of with index of refraction $n = 2$ and thickness $t = 3 \mu$m placed in front of this new source. Find the first location above $y = 0$ at which a bright spot is observed.

Solution:

(a) The source-mirror system is equivalent to two sources located 2 mm apart. Since the distance between the sources and screen is much farther than the distance between the two sources, we can apply the formula $d \sin \theta = m \lambda$. Note that we aren’t using $m + 1/2$ because we are looking for a dark spot, but there is also a phase shift upon hitting the mirror. There is a dark spot at $y = 0$, but we want the second dark spot above that. Then we plug in $m = 2$, $\lambda = 400$ nm, and $d = 2$ mm to obtain

$$\sin \theta = 0.0004$$

For small angles, $\sin \theta \approx \tan \theta$, so

$$\tan \theta = 0.0004$$

thus we have that $y = 0.0004L = 0.02$ m.

(b) Adding this lens will “increase” the distance travelled by an amount $t(n - 1)$. Thus we have the modified equation

$$d \sin \theta + t(n - 1) = m \lambda$$

but we no longer know if $m = 1$. We instead need to try various $m$ and find the resulting $\theta$. Plugging in $d = 2$ mm, $\lambda = 400$ nm, $t = 3 \mu$m, and $n = 2$, we obtain

$$\sin \theta = \frac{0.4m - 3}{2} \times 10^{-3}$$

which we notice doesn’t give a positive $\theta$ until $m = 8$. Thus letting $m = 8$, we get

$$\sin \theta = 0.0001$$

and thus $y = 5$ mm.
5. Consider an insulating wire \( l_1 + l_2 \) metres long, which contains a charge density of \( \lambda \) coulombs/meter.

(a) Find the potential \( V \) at the point \( P \) a distance \( r \) from the line of charge.

(b) Compare your answer to part (a) with the expected potential if \( r \gg (l_1 + l_2) \).

(c) Check your answer to part (a) in the limit \( r \ll (l_1 + l_2) \) by comparing the electric field \( E \) derived from \( V \) with the field derived using Gauss’s law.

**Hint.** The antiderivative \( \int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left| x + \sqrt{x^2 + a^2} \right| \) may be useful. Also, recall linear approximations \( f(x) \approx f(0) + xf'(0) \)

Solution:

(a) The potential \( dV \) at the point \( P \) due to a small length \( dx \) of the line is given by

\[
dV = \frac{dq}{4\pi \varepsilon_0 R} = \frac{\lambda dx}{4\pi \varepsilon_0 \sqrt{x^2 + r^2}}
\]

Thus

\[
V = \int_{-l_1}^{l_2} \frac{\lambda dx}{4\pi \varepsilon_0 \sqrt{x^2 + r^2}} = \frac{\lambda}{4\pi \varepsilon_0} \ln \left| \frac{l_2 + \sqrt{l_2^2 + r^2}}{-l_1 + \sqrt{l_1^2 + r^2}} \right|
\]

(b) If \( r \gg l_1 + l_2 \), then we expect the line of charge to behave as a point charge with magnitude \( Q = \lambda(l_1 + l_2) \). That would be \( V \approx \frac{\lambda(l_1 + l_2)}{4\pi \varepsilon_0 r} \). Recall \( \ln(1 + x) \approx x \) for small \( x \). For the formula from part (a) we have

\[
V = \frac{\lambda}{4\pi \varepsilon_0} \ln \left| \frac{l_2 + \sqrt{l_2^2 + r^2}}{-l_1 + \sqrt{l_1^2 + r^2}} \right|
\]

\[
\approx \frac{\lambda}{4\pi \varepsilon_0} \ln \left| \frac{l_2 + r}{r - l_1} \right|
\]

\[
= \frac{\lambda}{4\pi \varepsilon_0} \ln \left| \frac{l_2 + l_1 + r - l_1}{r - l_1} \right|
\]

\[
= \frac{\lambda}{4\pi \varepsilon_0} \ln \left| \frac{l_2 + l_1}{r - l_1} \right|
\]

\[
\approx \frac{\lambda}{4\pi \varepsilon_0} \frac{l_2 + l_2}{r - l_1}
\]

\[
= \frac{\lambda}{4\pi \varepsilon_0} \frac{l_2}{r}
\]
Which agrees with what we have from the potential formula as if the length of charge was a point charge.

(c) Recall that \((x + 1)^{1/2} \approx 1 + x/2\) for small \(x\). If \(r \ll l_1 + l_2\), then

\[
V = \frac{\lambda}{4\pi\epsilon_0} \ln \left| \frac{l_2 + \sqrt{l_2^2 + r^2}}{-l_1 + \sqrt{l_1^2 + r^2}} \right|
\]

\[
= \frac{\lambda}{4\pi\epsilon_0} \ln \left| \frac{l_2(1 + \sqrt{1 + r^2/l_2^2})}{l_1(-1 + \sqrt{1 + r^2/l_1^2})} \right|
\]

\[
\approx \frac{\lambda}{4\pi\epsilon_0} \ln \left| \frac{l_2(1 + 1 + r^2/2l_2^2)}{l_1(-1 + 1 + r^2/2l_1^2)} \right|
\]

\[
= \frac{\lambda}{4\pi\epsilon_0} \ln \left| \frac{l_2(1 + r^2/2l_2^2)}{l_1(r^2/2l_1^2)} \right|
\]

\[
\approx \frac{\lambda}{4\pi\epsilon_0} \ln \left| \frac{l_2(2)}{r^2/2l_1^2} \right|
\]

\[
= \frac{\lambda}{4\pi\epsilon_0} \ln \left| \frac{4l_1l_2}{r^2} \right|
\]

Differentiating,

\[
E = \left( -\frac{d}{dr}V \right) \hat{r}
\]

\[
= \left( -\frac{d}{dr} \left( \frac{\lambda}{4\pi\epsilon_0} \ln \left| \frac{4l_1l_2}{r^2} \right| \right) \right) \hat{r}
\]

\[
= \frac{\lambda}{2\pi\epsilon_0 r} \hat{r}
\]

Using Gauss's law with a cylindrical gaussian surface, with axis along the line gives us

\[
E = \frac{\lambda}{2\pi\epsilon_0 r} \hat{r}
\]

Which agrees with the above result.
6. Two long concentric conducting cylinders are insulated from each other and charged. Far from the ends, the inner cylinder has a charge density of $+\lambda_1$, and the outer one a charge density of $+\lambda_2$ coulombs per unit length. The inner cylinder has inner and outer radii $r_1$ and $r_2$, while the outer cylinder has radii $r_3$ and $r_4$.

(a) Find $E_r$
   i. at a point $a_1 < r < a_2$
   ii. just outside the outer cylinder

(b) Find the potential difference $|\Delta V|$ between the two cylinders.

(c) Describe qualitatively any changes in the fields and potentials
   i. if $r_1$ is decreased
   ii. if $r_2$ is increased
   iii. if the outside cross section of the inner cylinder is made square with sides $2r_2$ (assuming $\sqrt{2}r_2 < r_3$).

Solution:

(a) i. The electric field in the middle of the cylinder is 0 since the cylinder is conducting.
   ii. We can use Gauss’ law to find the electric field outside both cylinders.

\[
\int \mathbf{E} \cdot d\mathbf{A} = \frac{q_{enc}}{\varepsilon_0}
\]

\[
EA = \frac{\lambda_1 + \lambda_2}{\varepsilon_0}
\]

\[
E = \frac{\lambda_1 + \lambda_2}{2\pi r_4 \varepsilon_0}
\]

(b) \[
V_{r_2} - V_{r_3} = \int_{r_2}^{r_3} \mathbf{E} \cdot d\mathbf{r}
\]

\[
V_{r_2} - V_{r_3} = \int_{r_2}^{r_3} \frac{\lambda_1}{2\pi r \varepsilon_0} \cdot dr = \frac{\lambda_1}{2\pi \varepsilon_0} \ln \left( \frac{r_3}{r_2} \right)
\]

(c) i. Changing $r_1$ has no effect. Since both cylinders are conductors the charge will be concentrated on the outer surface of the cylinder.
ii. Increasing $r_2$ will decrease the potential difference between the two cylinders as can be shown using the equation in (b). This change will not affect the electric field outside since the total enclosed charge used in the Gauss law calculation does not change.

iii. This transformation results in the gap between the two cylinders becoming smaller on average. Although this electric field would be complicated to calculate, the potential difference between the two cylinders still decreases according to the integral in (b).
7. Two optically flat glass plates \((n = 1.5)\) of width 6.4 cm are in contact at one end, and at the other end are separated by a distance \(d = 0.02\) mm, forming a wedge of variable thickness. See the figure. Light of wavelength 620 nm is normally incident on the top plate. Consider interference between light reflected off the bottom of the top plate and off the top of the bottom plate.

(a) Over the length of the wedge, find how many bright fringes are observed.

(b) Repeat if oil with \(n = 1.2\) is between the plates.

(c) Repeat if oil with \(n = 1.6\) is between the plates.

![Diagram of the wedge](image)

**Solution:** Interference results from the phase difference between light reflected off the bottom surface of the top mirror and light reflected off the top surface of the bottom mirror. This means that the total phase shift is due to the extra distance traveled through the air gap as well as any phase shift that occurs at boundaries.

(a) In the initial configuration we note that the light reflecting off the bottom of the top glass plate will have no phase shift \((n = 1.5 < n = 1.0)\) while the light reflecting off the lower glass plate will have a \(\lambda/2\) phase shift \((n = 1.0 < n = 1.5)\). The condition for constructive interference is therefore satisfied when the path length difference between the two paths is:

\[
\Delta d = (k - 0.5)\lambda
\]

The path length difference is equal to twice the distance between the plates, so:

\[
\Delta d = 2 \cdot \frac{0.02x}{64} \text{ mm}
\]

Where \(x = 64\) mm is the horizontal distance from the point where the plates touch. We are only concerned with the total number of fringes and not the distance between fringes. This is given by:

\[
k = 2 \frac{\Delta d}{\lambda} + 0.5 = 2 \cdot \frac{0.02 \cdot 10^{-3}}{620 \cdot 10^{-9}} + 0.5 = 65.0
\]

There are 65 fringes of constructive interference.

(b) The gap between the plates is now filled with oil having refractive index \(n = 1.2\). This is still less than the refractive index of the plates so the behaviour at the reflective surfaces is the same as in (a). Our criteria for constructive interference is now:

\[
\Delta d = (k - 0.5)\frac{\lambda}{n}
\]

\[
k = 2 \frac{\Delta d n}{\lambda} + 0.5 = 2 \cdot \frac{1.2 \cdot 0.02 \cdot 10^{-3}}{620 \cdot 10^{-9}} + 0.5 = 77.9
\]

There are now 77 fringes of constructive interference.
(c) With a refractive index of 1.6, the oil has a higher refractive index than the glass. We now have a $\lambda/2$ phase shift ($n = 1.6 > n = 1.5$) at the upper reflective surface and no phase shift ($n = 1.5 < n = 1.6$) at the lower surface. The total phase shift from reflections is still $\lambda/2$ however so our criteria for constructive interference does not change.

$$k = 2 \frac{\Delta n}{\lambda} + 0.5 = 2 \cdot \frac{1.6 \cdot 0.02 \cdot 10^{-3}}{620 \cdot 10^{-9}} + 0.5 = 103.7$$

There are now 103 fringes of constructive interference.
8. A charge $Q$ is uniformly distributed over a *arc* (no charge along the $x$ axis or radius) of radius $a$ and angle $\alpha$ that extends from the $x$-axis counterclockwise. Find the force (as a function of $\alpha$) on the charge $q$.

**Hint.** Pretend that the arc extends from $-\alpha/2$ to $\alpha/2$ so that you can exploit symmetry when performing the integration.

![Diagram of a charge $q$ on an arc with radius $a$ and angle $\alpha$]

**Solution:** To find the force on charge $q$ we will integrate the force over the circular arc. Linear charge density is

$$\lambda = \frac{Q}{a\alpha}$$

Consider the force $d\mathbf{F}$ due to a small length of charge $dQ$, which resides at an angle $\theta$. In order to simplify the problem, pretend like the arc extends along $[-\alpha/2, \alpha/2]$ instead of $[0, \alpha]$. This will cause the $y$ components of the force to cancel out, so we would only need to find the force in the $x$ direction. The force will then be

$$dF_x = \frac{1}{4\pi\epsilon_0} \frac{qdQ \cos \theta}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{q \cos \theta dQ}{a^2}$$

Since $dQ = \lambda d\theta$, we can plug that in to the resulting integral:

$$F_x = \frac{1}{4\pi\epsilon_0} \int_{-\alpha/2}^{\alpha/2} \frac{q\lambda}{a} \cos \theta d\theta = \frac{1}{4\pi\epsilon_0} \frac{qQ}{a^2} \int_{0}^{\alpha/2} \frac{1}{2} \frac{\cos \theta d\theta}{\alpha^2}$$

Thus we have for the magnitude of the force in the $x$ direction (with the rotated arc) to be

$$F_x = \frac{1}{4\pi\epsilon_0} \frac{qQ}{a^2} \sin \left(\frac{\alpha}{2}\right) = \frac{1}{2\pi\epsilon_0} \frac{qQ}{a^2} \sin \left(\frac{\alpha}{2}\right)$$

If we want to find the force on the charge $q$ when the arc is in its true position, we can deduce from symmetry that resultant force will be at an angle of $\alpha/2$, and pointing towards the origin. Thus we can multiply the magnitude of the force by a unit vector that points towards the origin. That unit vector will be

$$\mathbf{u} = \left( -\cos \left(\frac{\alpha}{2}\right), -\sin \left(\frac{\alpha}{2}\right) \right)$$

Thus the force on the charge $q$ due to the wire which arcs from $[0, \alpha]$ will be

$$\mathbf{F} = \frac{1}{2\pi\epsilon_0} \frac{qQ}{a^2} \sin \left(\frac{\alpha}{2}\right) \left( -\cos \left(\frac{\alpha}{2}\right), -\sin \left(\frac{\alpha}{2}\right) \right)$$
9. Consider the circuit shown in the figure.

(a) Find an expression for the potential difference from point $a$ to point $b$ as a function of time. Is the amplitude dependent on $\omega$?

(b) Draw a sketch of the phase angle of this potential difference as a function of $\omega$.

**Hint.** The following trigonometric identity may be useful: $a \sin \theta + b \cos \theta = \sqrt{a^2 + b^2} \cos(\theta - \arctan(a/b))$.

Solution:

(a) The current through either of the two branches is given by

$$I(t) = \frac{V_0 \cos(\omega t - \phi)}{\sqrt{X_C^2 + R^2}}$$

where $\tan \phi = -X_C/R$. Thus we have from Ohm’s law:

$$V_a = \frac{V_0 R}{\sqrt{X_C^2 + R^2}} \cos(\omega t - \phi), \quad V_b = \frac{V_0 X_C}{\sqrt{X_C^2 + R^2}} \sin(\omega t - \phi)$$

We then have

$$|V_a - V_b| = \left| \frac{V_0 R}{\sqrt{X_C^2 + R^2}} \cos(\omega t - \phi) - \frac{V_0 X_C}{\sqrt{X_C^2 + R^2}} \sin(\omega t - \phi) \right| = V_0 |\cos(\omega t - 2\phi)|$$

Thus the amplitude of the voltage $|V_a - V_b|$ is independent of $\omega$. Note that we used the identity in the hint to arrive at this formula.

(b) The phase angle of the potential difference is

$$2\phi = 2 \arctan(-X_C/R) = -2 \arctan(1/\omega CR) = -2 \arccot(\omega CR)$$

Note that there are asymptotes are 0 and $-2\pi$. 

\[\text{Diagram of phase angle as a function of } \omega.\]
10. The circuit shown in the figure is arranged so that the connection to point \( P_3 \) could be made to any of the points \( P_0, P_1, P_2, \ldots, P_n \).

(a) Find an expression for the average power \( P_{avg} \) dissipated in \( R \) if the connection is made to point \( P_m \), where \( 0 < m < n \).

(b) If \( R = 1000 \Omega, L = 10 \) H, \( C = 20 \) \( \mu \)F, \( \omega = 100 \) rad/sec:
   i. For what value of \( m \) is the power a maximum?
   ii. If \( m = 2 \), what are the maximum instantaneous voltages \( (V_{P_0P_2})_{max} \) between points \( P_0 \) and \( P_2 \), and \( (V_R)_{max} \) across \( R \)?

\[
\text{Solution:}
\]

(a) The equivalent capacitance of \( m \) capacitors of equal capacitance connected in series is \( C/m \). Thus we can calculate the impedance of the circuit to be

\[
Z = \sqrt{(X_L - X_C)^2 + R^2} = \sqrt{(\omega L - m/\omega C)^2 + R^2}
\]

We then have

\[
P_{avg} = V_{rms}I_{rms} \cos \phi
\]

\[
= \frac{V_0I_0}{2} \frac{R}{Z}
\]

\[
= \frac{V_0^2R}{2Z^2}
\]

\[
= \frac{V_0^2R}{2((\omega L - m/\omega C)^2 + R^2)}
\]

(b) i. Since \( m \) only appears in the denominator, the value of \( m \) that will make the mean power \( P_{avg} \) a maximum is the value that makes the denominator the smallest. Thus we wish to minimize the quantity \( \omega L - m/\omega C \). Plugging in the values gives

\[
\omega L - m/\omega C = 1000 - 1000m/2 = 1000(1 - m/2)
\]

Thus the minimizing value for \( m \) is \( m = 2 \).

ii. The maximum voltage across the two capacitors will be \( (V_{P_0P_2})_{max} = V_0X_C/Z = 2V_0/\omega CR \) Volts, and the maximum voltage across the resistor will be \( (V_R)_{max} = V_0 \)
11. A stiff wire bent into a semicircle of radius $r$ is rotated with a frequency $f$ in a uniform magnetic field $B$ as shown in the figure. If the internal resistance of the meter $M$ is $R_M$, and the remainder of the circle has negligible resistance,

(a) What is the amplitude $V_0$ and angular frequency $\omega_V$ of the induced voltage?

(b) What is the amplitude $I_0$ and angular frequency $\omega_I$ of the induced current?

Solution:

(a) The area of the semicircle is $\pi r^2/2$, and when it rotates the angle $\theta$ between its normal vector and the magnetic field will change as $\theta = \omega t = 2\pi ft$. Thus the flux through the loop changes as $\Phi_B = B(\pi r^2/2) \cos(2\pi ft)$, so then

$$\frac{d\Phi_B}{dt} = -B\pi^2 r^2 f \sin(2\pi ft)$$

From Faraday’s law, we have that the emf is then $\mathcal{E}(t) = B\pi^2 r^2 f \sin(2\pi ft)$ Thus the magnitude of the induced voltage is $V_0 = \pi^2 r^2 f$, and the angular frequency of the induced voltage is $\omega_V = 2\pi f$.

(b) The amplitude of the current is $I_0 = V_0/R_M = B\pi^2 r^2 f / R_M$, and the angular frequency of the induced current is $\omega_I = 2\pi f$. 
(**) 12. The RLC series circuit shown in the figure contains a generator supplying an alternating voltage of fixed frequency $\omega$. The capacitor is variable.

(a) For a value $C = C_1$, the current $I_1$ is found to be in phase with the applied voltage. What is $C_1$ in terms of $L$ and $\omega$?

(b) The capacitance is then changed to $C = C_2$, so that the voltage is observed to lead the current $I_2$ by a phase angle of $45^\circ$. What is $C_2$ in terms of $C_1$, $R$, and $\omega$?

Solution:

(a) If the voltage and current are in phase, then the circuit is at resonance, so $\omega^2 = 1/LC_1$. Thus $C_1 = 1/L\omega^2$.

(b) With the formula

$$\tan \phi = \frac{X_L - X_C}{R}$$

If the phase angle is $45^\circ$, then $\tan(45^\circ) = 1$, so then $R = X_L - X_C$. Thus we have

$$R = \omega L - 1/\omega C_2$$

Rearranging yields

$$C_2 = \frac{C_1}{1 - R\omega C_1}$$
13. (**) A large parallel-plate capacitor with uniform surface charge $\sigma$ on the upper plate and $-\sigma$ on the lower is moving with constant speed $v$, as shown in the figure.

(a) Find the magnetic field between the plates, and above and below the plates.

(b) Find the magnetic force per unit area on the upper plate, including its direction.

(c) At what speed $v$ would the magnetic force balance the electrical force?

Solution:

(a) Between the plates, both plates produce a magnetic field pointing into the page. The magnetic field will be uniform because the plates are large. Use Ampere’s law to find the magnitude. We have

$$\int B \cdot dl = \mu_0 I$$

and consider a square loop of side length $s$ whose normal vector is parallel to the capacitor plates. The integral evaluates to $2Bs$, because the net magnetic field above the plates and below the plates is 0 since they cancel each other out. Thus between the plates we have

$$2Bs = \mu_0 I = \mu_0 \frac{dq}{dt} = \mu_0 \frac{\sigma dx}{dt} = \mu_0 \sigma sv$$

Thus we have

$$B = \frac{\mu_0 \sigma v}{2}$$

to be the magnitude of the magnetic field between the plates due to one of the plates, and it points into the page. Thus the total magnetic field between the plates due to both plates is

$$B = \mu_0 \sigma v$$

The magnetic field is 0 above and below the plates.

(b) Consider a small square area in the top plate, with area $A = s^2$. The current due to that small square is $I = \sigma sv$, and the magnetic field there is $B = \sigma v \mu_0/2$, because there is no contribution due to the top plate. Thus we have

$$\frac{F}{A} = \frac{IsB}{A} = \frac{(\sigma sv)s(\mu_0 \sigma v/2)}{s^2} = \frac{\sigma^2 v^2 \mu_0}{2}$$

Thus force points upwards because the current is rightwards, and the magnetic field is into the page. The bottom plate is repelled downwards by the same magnitude.

(c) The electric field between the plates is given by $E = \sigma/\epsilon_0$ due to Gauss’s Law (prove as an exercise) and the force per unit area due to the electric field is then

$$\frac{F}{A} = \frac{qE}{A} = \frac{(s^2 \sigma)(\sigma/\epsilon_0)}{s^2} = \frac{\sigma^2}{\epsilon_0}$$
Equating this to the force per unit area from the magnetic field (which must be doubled because the plates are pushing each other away) we have

\[ \frac{\sigma^2}{\varepsilon_0} = \sigma^2 v^2 \mu_0 \]

Thus we have \( v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \), so \( v = c \), the speed of light. Thus the two plates can never have balanced forces.
14. Consider an infinite (in the $x$ and $z$ directions) insulating slab of thickness $2L$. There is a cylindrical hole bored through the slab, as shown in the figure. The cylindrical hole has diameter $L$. The slab has uniform charge density $\rho$.

(a) Find the electric field anywhere inside the cylinder.

(b) Where in the cylindrical cavity is the magnitude of the electric field minimum?

Solution:

(a) We need to apply superposition. First, consider an infinite solid insulating slab of thickness $2L$. The electric field due to the infinite slab will be called $E_1$. Applying Gauss's law on a cylindrical surface inside the slab gives us

$$|E_1| = \frac{\rho y}{\epsilon_0}$$

And translating this into vector notation gives

$$E_1 = \frac{\rho}{\epsilon_0} (0, y)$$

Second, consider a solid cylinder of charge density $-\rho$ that is located in the position as shown in the figure. Let a cylindrical surface of length $l$ coaxial with the cylinder solid be the Gaussian surface. The magnitude of the electric field $E_2$ is then

$$|E_2| = -\frac{\rho r}{2\epsilon_0}$$

where $r$ is measured from the centre of the cylinder. The electric field points radially inwards to the axis of the cylinder. Now, translating this to vector notation we have

$$E_2 = -\frac{\rho}{2\epsilon_0} \left( x, y - \frac{L}{2} \right)$$

Thus we have the electric field in the cylinder to be a superposition of these two fields.

$$E = E_1 + E_2 = \frac{\rho}{\epsilon_0} (0, y) - \frac{\rho}{2\epsilon_0} \left( x, y - \frac{L}{2} \right)$$

Thus

$$E = \frac{\rho}{\epsilon_0} \left( \frac{-x}{2}, \frac{y}{2} + \frac{L}{4} \right)$$

is the electric field inside the cylinder.

(b) The minimum electric field inside the cylinder is at $(x, y) = (0, 0)$.

**Remark.** One may think that one could plug in $(0, -L/2)$ to make the electric field 0. However, the formula found in part (a) only applies inside the cylinder, and $(0, -L/2)$ is not inside the cylinder.
15. (*** A thin disk of radius \(a\) rotates at an angular velocity \(\omega\). It has a uniform surface charge density \(\sigma\). (This can be thought of as a set of infinitesimally thin coils).

(a) Find the current per unit radial length, \(dI/dr\).

(b) Find \(I\).

(c) Find the magnetic field a distance \(x\) along its axis, away from the surface of the disk.

**Hint.** Note that \(dI\) is the current through an infinitesimal annular region on the disk. Also the following integral may be useful:

\[
\int \frac{u^3}{(u^2 + \alpha^2)^{3/2}} du = \sqrt{u^2 + \alpha^2} + \frac{\alpha^2}{\sqrt{u^2 + \alpha^2}}
\]

Solution:

(a) Consider the amount of charge \(dQ\) in a small annular region of width \(dr\). Then \(dQ = 2\pi r\sigma dr\).

To find what current this corresponds to, divide by the period \(T = 2\pi/\omega\). Thus \(dI = \omega\sigma dr\), so \(dI/dr = \omega\sigma\).

(b) Thus

\[
I = \int_0^a \omega\sigma r dr = \frac{\omega\sigma a^2}{2}
\]

(c) We need to apply the Biot Savart Law. Think of the disk as infinitely many little wires going around in a circle. We will first find the magnetic field due to one of these wires. Consider one of these tiny wires, call a segment of it \(dl\), where the vector points in the direction of angular velocity (same direction as current) \(\omega\). Let \(\rho\) be the distance from an annulus of radius \(r\) on the disk to the point \(x\). Then \(\rho = \sqrt{x^2 + r^2}\). The Biot Savart Law then tells us that the magnetic field \(B\) due to one of these tiny wires is

\[
dB = \frac{\mu_0}{4\pi} \frac{I dl \times \hat{\rho}}{\rho^2} = \frac{\mu_0}{4\pi} \frac{I dl \times \rho}{\rho^3}
\]

We need to compute this cross product directly. In Cartesian coordinates, if we look at the point \((0, r, 0)\) on the disk we can write

\[
\rho = (x, -r, 0), \quad dl = (0, 0, -1) dl
\]
then computing the cross product we have
\[ dl \times \rho = r\hat{x} + x\hat{y} \]

The magnetic field in the \( y \) direction will be end up being 0 (after integration) due to symmetry, and the magnetic field in the \( x \) direction will be given by (substituting \( dl = rd\theta \))

\[ dB_x = \frac{\mu_0 I dl}{4\pi \rho^2} \cdot \frac{r}{\rho} = \frac{\mu_0 r Ird\theta}{4\pi \rho \rho^2} = \frac{\mu_0 r^2 I d\theta}{4\pi \rho^3} \]

Thus the magnetic field \( B_x \) due to one of these tiny wires is

\[ B_x = \frac{\mu_0 r^2 I}{4\pi \rho^3} = \frac{\mu_0 r^2 I}{2\rho^3} \]

However, this magnetic field is due only to one of these tiny wires. This infinitesimally small wire can be thought of as having an infinitesimal current \( dI \) through it. We thus need to add up the magnetic fields due to all of these tiny wires. Replacing \( I \) by \( dI \) we have

\[ dB_x = \frac{\mu_0 r^2 dI}{2\rho^3} = \frac{\mu_0 r^2 (\omega \sigma r dr)}{2\rho^3} = \frac{\mu_0 \sigma r^3 dr}{2\rho^3} \]

Thus the total magnetic field is given by

\[ B_x = \frac{\mu_0 \sigma}{2} \left[ \int_0^a \frac{r^3}{\rho^3} dr \right] = \frac{\mu_0 \sigma}{2} \int_0^a \frac{x^3}{(r^2 + x^2)^{3/2}} dr = \frac{\mu_0 \sigma}{2} \left( \sqrt{r^2 + x^2} + \frac{x^2}{\sqrt{r^2 + x^2}} \right) \bigg|_0^a \]

The total magnetic field is then

\[ B_x = \frac{\mu_0 \omega \sigma}{2} \left( \sqrt{a^2 + x^2} + \frac{x^2}{\sqrt{a^2 + x^2}} - 2x \right) \]
16. Two large, flat metal plates are held parallel to each other and separated by a distance \( d \). They are connected together at their edge by a metal strip. A thin plastic sheet carrying a surface charge \( \sigma \) per unit area is placed between the plates at a distance \( d/3 \) from the upper plate, as shown in the figure. Calculate \( E_a \) and \( E_b \) the electric field near the upper and lower plates, respectively.

Solution: The charge in the metals plates is free to move from one plate to another since they are connected by a conductor. Call the surface charge density on the bottom surface of the top plate \( \sigma_a \), and the surface charge density on the top surface of the bottom plate \( \sigma_b \). Since the keyword “large” is given we can assume that charge density is even over each plate.

- We can then consider a Gaussian surface enclosing \( \sigma_a \) and \( \sigma \). Applying Gauss’s law to this surface yields
  \[ E_b = -\left( \frac{\sigma_a + \sigma}{\varepsilon_0} \right) \hat{y} \]

- Applying Gauss’s law to a Gaussian surface enclosing \( \sigma_b \) and \( \sigma \) gives us
  \[ E_a = \left( \frac{\sigma_b + \sigma}{\varepsilon_0} \right) \hat{y} \]

Now we are only left to determine what \( \sigma_a \) and \( \sigma_b \) are. To do this, we note the following.

- The potential between the top and bottom plates must be 0 because they’re connected by a metal strip. If we integrate from one plate to another,
  \[
  \int_a^b E \cdot dy = \int_0^{2d/3} E_b \cdot dy + \int_{2d/3}^d E_a \cdot dy
  = -\frac{2d}{3} \left( \frac{\sigma_a + \sigma}{\varepsilon_0} \right) + \frac{d}{3} \left( \frac{\sigma_b + \sigma}{\varepsilon_0} \right)
  = 0
  \]

  This gives us the relationship
  \[ 2(\sigma_a + \sigma) = \sigma_b + \sigma \]

- We also observe that if we take a Gaussian surface enclosing all three of \( \sigma_a \), \( \sigma \), and \( \sigma_b \), this goes through the metal, and thus the electric field flux is zero. Therefore
  \[ \sigma_a + \sigma_b + \sigma = 0 \]

Combining these two equations gives us that
  \[ \sigma_a = -\frac{2\sigma}{3}, \quad \sigma_b = -\frac{\sigma}{3}. \]

Now we can plug these charge densities into our formulas for the electric field:
  \[ E_a = \frac{2\sigma}{3\varepsilon_0} \hat{y}, \quad E_b = -\frac{\sigma}{3\varepsilon_0} \hat{y} \]
17. In practical magnetic structures, uniform magnetic fields are frequently necessary. The uniformity of the field produced by Helmholtz coils, or two co-axial loops which carry currents in the same direction, is one of their most important characteristics. Assume that the coils have a radius $a$, have axes on the $x$-axis, carry a current $I$ each, and are separated by a distance $b$, as shown in the figure.

(a) Find the magnetic field at a point $P$ on the axis of the loops and a distance $x$ from the mid-point $O$.

(b) Expand the expression for the field in a Taylor Polynomial of order $x^2$.

(c) What relationship must exist between $a$ and $b$ such that the $x^2$ terms vanish? What is the significance of this?

(d) Find the field created by the coils to the order and under the condition established in part (c).

**Hint.** If $f(x) = \frac{1}{(ax^2 + bx + c)^{3/2}}$, then its derivatives are

$$f'(x) = -\frac{3}{2} \frac{2ax + b}{(ax^2 + bx + c)^{5/2}}, \quad f''(x) = \frac{48ax^2 + 48axb - 12ac + 15b^2}{4(ax^2 + bx + c)^{7/2}}$$

Solution:

(a) Call the left hand side coil (1), and the right hand side coil (2). Then the contribution to the magnetic field by (1) is given by

$$dB_1 = \frac{\mu_0 I}{4\pi} \frac{dl \times \hat{r}}{r^2}$$

Due to symmetry, the $y$ and $z$ components of the field will cancel out, and we will be left with only the $x$ component, which is given by

$$(dB_x)_1(x) = \frac{\mu_0 I}{4\pi} \frac{dl \times \hat{r}}{r^2} \sin \theta$$

$$= \frac{\mu_0 I}{4\pi} \frac{da}{r^2}$$

$$= \frac{\mu_0 Ia}{4\pi} \frac{d\theta}{(a^2 + (b/2 + x)^2)^{3/2}}$$

Thus the magnetic field is

$$(B_x)_1(x) = \int_0^{2\pi} \frac{\mu_0 Ia^2}{4\pi} \frac{d\theta}{(a^2 + (b/2 + x)^2)^{3/2}}$$

$$(B_x)_1(x) = \frac{\mu_0 Ia^2}{2} \frac{1}{(a^2 + (b/2 + x)^2)^{3/2}}$$

By symmetry, it is easy to see that the magnetic field due to (2) is

$$(B_x)_2(x) = \frac{\mu_0 Ia^2}{2} \frac{1}{(a^2 + (b/2 - x)^2)^{3/2}}$$
The total magnetic field due to both fields is then given by

\[ B_x(x) = \frac{(B_x)_1 + (B_x)_2(x)}{2} = \frac{\mu_0 I a^2}{2} \left[ \frac{1}{(a^2 + (b/2 + x)^2)^{3/2}} + \frac{1}{(a^2 + (b/2 - x)^2)^{3/2}} \right] \]

(b) The approximation to \( B \) centred around \( x = 0 \) is given by

\[ B(x) \approx B(0) + B'(0)x + \frac{B''(0)x^2}{2} \]

Define \( k^2 = a^2 + b^2/4 \). Computing the derivatives using the formulas given in the hint yields

\[ B'(x) = \frac{-3 \mu_0 I a^2}{2} \left[ \frac{2x + b}{(k^2 + bx + x^2)^{3/2}} + \frac{2x - b}{(k^2 - bx + x^2)^{3/2}} \right] \]

\[ B''(x) = \frac{\mu_0 I a^2}{2} \left[ \frac{48x^2 + 48xb - 12k^2 + 15b^2}{4(x^2 + bx + k^2)^{7/2}} + \frac{48x^2 - 48xb - 12k^2 + 15b^2}{4(x^2 - bx + k^2)^{7/2}} \right] \]

This gives \( B(0) = \frac{\mu_0 I a^2}{k^3}, B'(0) = 0, \) and \( B''(0) = \mu_0 I a^2 \left( \frac{15b^2}{4k^7} - \frac{3}{k^5} \right) \)

Thus the magnetic field is approximated by

\[ B(x) \approx \frac{\mu_0 I a^2}{k^3} \left[ \frac{1}{k^3} + \frac{x^2}{2} \left( \frac{15b^2}{4k^7} - \frac{3}{k^5} \right) \right] \]

(c) In order for the \( x^2 \) terms to vanish, we must have \( 15b^2/4k^7 - 3/k^5 = 0 \). Thus we have \( 15b^2/4k^7 = 3/k^5 \). Plugging back in \( k^2 = a^2 + b^2/4 \) gives \( a = b \). The significance of this is that the field will be very close to uniform if \( a = b \) because then \( B \) is independent of \( x \), except for \( x^3 \) terms and higher. Those terms will be very close to 0 for small \( x \).

(d) Evaluating the magnetic field approximation (up to second order) for \( a = b \) gives

\[ B_x \approx \frac{\mu_0 I a^2}{k^3} = \frac{\mu_0 I a^2}{(5b^2/4)^{3/2}} = \frac{8\mu_0 I}{5^{3/2}a} \]
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**Standing Waves**

\[ f = \frac{v}{\lambda}, \quad k = \frac{2\pi}{\lambda}, \quad L = \frac{1}{2} \mu \omega^2 A^2 v, \quad p_o = \rho \omega s_0 \]

**Beats:** \[ \Delta f = f_2 - f_1, \quad y = A \cos(kx + \omega t + \phi) \]

**Interference:** \[ k\Delta x + \Delta \phi = 2\pi n \text{ or } \pi(2n + 1), \quad n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots \]

**Standing Waves** \[ f_m = \frac{mv}{2L}, \quad m = 1, 2, 3, \ldots \]

**Electromagnetic Equations:**

- Maxwell's Equations:
  
  \[ F = \frac{kq_1 q_2}{r^2}, \quad |E| = \frac{kq}{r} \mu \nu + \text{Constant} \]
  
  Electric potential and potential energy \[ \Delta V = V_a - V_b = \int_E^b E \cdot dl = - \int_b^a E \cdot dl \]
  
  \[ E_x = -\frac{dV}{dx}, \quad E = -\nabla V, \quad \Delta U = U_a - U_b = q(V_a - V_b) \]

**Additional Equations:**

- \[ \mu_0 = 4\pi \times 10^{-7} \text{ Tm/A}, \quad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 299,792, 458 \text{ m/s} \]

- \[ \text{Point Charge:} \quad |F| = \frac{k|q_1 q_2|}{r^2}, \quad |E| = \frac{kq}{r} \mu \nu + \text{Constant} \]

**Maxwell's Equations:**

\[ \int_S E \cdot dA = \frac{Q_{enc}}{\epsilon_0} \quad \text{and} \quad \int_S B \cdot dA = 0 \]

\[ \int_C B \cdot dl = \mu_0 (I_{enclosed}) + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt} \quad \int_C E \cdot dl = -\frac{d\Phi_B}{dt} \]

Where S is a closed surface and C is a closed curve. \[ \Phi_E = \int E \cdot dA \text{ and } \Phi_B = \int B \cdot dA \]

**Energy Density:**

\[ u_E = \frac{1}{2} \epsilon_0 E^2 \text{ and } u_B = \frac{1}{2\mu_0} B^2 \text{ (energy per volume)} \]

**Forces:**

\[ \text{Forces: } F = qE + qv \times B, \quad \text{F} = IL \times B \]

**Capacitors:**

\[ q = CV, \quad U_C = \frac{1}{2} \frac{q^2}{C}, \quad \text{For parallel plate capacitor with vacuum (air): } C = \epsilon_0 A \text{, } C_{\text{dielectric}} = KC_{\text{vacuum}} \]

**Inductors:**

\[ E_L = -L \frac{dI}{dt}, \quad U_L = \frac{1}{2} LI^2, \quad \text{where } L = N \Phi_B / I \text{ and } N \text{ is the number of turns.} \]

For a solenoid \( B = \mu_0 n I \) where \( n \) is the number of turns per unit length.

**DC Circuits:**

\[ V_R = IR, \quad P = VI, \quad P = I^2 R \]

(For RC circuits) \( q = ae^{-t/\tau} + b, \quad \tau = RC, \quad a \text{ and } b \) are constants

(For LR circuits) \( I = ae^{-t/\tau} + b, \quad \tau = L/R, \quad a \text{ and } b \) are constants

**AC circuits:**

\[ X_L = \omega L, \quad X_C = 1/(\omega C), \quad V_C = X_C I, \quad V_L = X_L I \]

\[ V = ZI, \quad Z = \sqrt{(X_L - X_C)^2 + R^2}, \quad P_{\text{average}} = I_{\text{rms}}^2 R, \quad I_{\text{rms}} = \frac{I_{\text{max}}}{\sqrt{2}} \]

If \( V = V_0 \cos(\omega t) \), then \( I = I_{\text{max}} \cos(\omega t - \phi) \), where \( \tan \phi = \frac{X_L - X_C}{R} \), \( P_{\text{av}} = V_{\text{rms}} I_{\text{rms}} \cos \phi \)

**Additional Equations:**

\[ dB = \frac{\mu_0}{4\pi} \left( \frac{I}{r} \right) \times r \]

LRC Oscillations: \[ q = A_0 e^{-\frac{R}{2L} \cos(\omega t + \phi)}, \quad \text{where } \omega = \sqrt{\omega_0^2 - \left( \frac{R}{2L} \right)^2} \text{ and } \omega_0 = \frac{1}{LC} \]