Mathematics 253 Midterm 1 Review Package

UBC Engineering Undergraduate Society

Attempt questions to the best of your ability. This review package consists of 11 pages, including 1 cover page and 24 questions. The questions are meant to be the level of a real examination or slightly above, in order to prepare you for the real exam. Material from lectures and from the relevant textbook sections is examinable, and the problems for this package were chosen with that in mind, as well as considerations based on past examination question difficulty and style. Problems are ranked in difficulty as (*) for easy, (**) for medium, and (***) for difficult. Note that sometimes difficulty can be subjective, so do not be discouraged if you are stuck on a (*) problem.

The solutions to these problems will be posted at the following web address: https://ubcengineers.ca/services/academic/tutoring/. If you believe that there is an error in these solutions, or have any questions, comments, or suggestions regarding EUS Tutoring sessions, please e-mail us at: tutoring@ubcengineers.ca. If you are interested in helping with EUS tutoring sessions in the future or other academic events run by the EUS, please e-mail vpacademic@ubcengineers.ca.

Some of the problems in this package were not created by the EUS. Those problems originated from one of the following sources:

- Schuam’s Outline of Calculus 2 ed; Ayres Jr., Frank
- Calculus – Early Transcendentals 7 ed; Stewart, James
- Calculus – 3 ed; Spivak, Michael
- Calculus Volume 1 2 ed; Apostol, Tom

All solutions prepared by the EUS.

Good Luck!
1. Find the partial derivatives $f_x, f_y$ of the function $f$.

$$f(x, y) = \left[\sin(xy)\right]^{\cos^3}$$

**Solution:**

$$f_x = (\cos 3)\left[\sin(xy)\right]^{\cos^3 - 1} \cdot \cos(xy)y$$

$$f_y = (\cos 3)\left[\sin(xy)\right]^{\cos^3 - 1} \cdot \cos(xy)x$$

2. Find all second partial derivatives of $z$.

$$z = x \cos - y \cos x$$

**Solution:**

First find the first partial derivatives.

$$\frac{\partial z}{\partial x} = \cos y + y \sin x$$

$$\frac{\partial z}{\partial y} = -x \sin y - \cos x$$

Second derivatives are:

$$\frac{\partial^2 z}{\partial x^2} = y \cos x$$

$$\frac{\partial^2 z}{\partial y \partial x} = -\sin y + \sin x$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\sin y + \sin x$$

$$\frac{\partial^2 z}{\partial y^2} = -x \cos y$$

3. Find the first partial derivatives of the function. $g(x, y) = \sin(x \sin y)$

**Solution:**

$$g_x = \cos(x \sin y) \cdot \sin y$$

$$g_y = \cos(x \sin y) \cdot x \cos y$$

4. Find the partial derivatives $f_x, f_y$ of the function $f$.

$$f(x, y) = x^2 + y^2 \sin(xy)$$
Solution:

\[ f_x = 2x + y^3 \cos(xy) \]
\[ f_y = 2y \sin(xy) + y^2 x \cos(xy) \]

(\*) 5. Find the partial derivatives \( f_x, f_y \) of the function \( f \).

\[ f(x, y) = \tan(\frac{x^2}{y}) \]

Solution:

\[ f_x = \frac{2x}{y} \sec^2(\frac{x^2}{y}) \]
\[ f_y = -\frac{x^2}{y^2} \sec^2(\frac{x^2}{y}) \]

(**) 6. Let \( v(r, t) = t^n e^{-r^2/(4t)} \). Find the value of the constant \( n \) such that \( v \) satisfies the following equation:

\[ \frac{\partial v}{\partial t} = 1 \frac{\partial}{\partial r} \left( r^2 \frac{\partial v}{\partial r} \right) \]

Solution:

\[ nt^{n-1}e^{-r^2/(4t)} + t^n e^{-r^2/(4t)} \cdot \frac{r^2}{4t^2} = \frac{1}{r^2} \cdot \frac{\partial}{\partial r} \left( -\frac{2r}{4t} r^2 t^n e^{-r^2/(4t)} \right) \]
\[ = -\frac{t^{n-1}}{2r^2} \left( \frac{\partial}{\partial r} \left[ r^3 e^{-r^2/(4t)} \right] \right) \]
\[ = -\frac{t^{n-1}}{2r^2} \left( 3r^2 e^{-r^2/(4t)} - \frac{2r^4}{4t} e^{-r^2/(4t)} \right) \]

Cancelling,
\[ nt^{n-1} + \frac{r^2 t^{n-2}}{4} = -\frac{t^{n-1}}{2r^2} \left( 3r^2 - \frac{r^4}{2t} \right) \]
\[ = -\frac{3}{2} t^{n-1} + \frac{r^2 t^{n-2}}{4} \]

Cancelling again,
\[ nt^{n-1} = -\frac{3}{2} t^{n-1} \]
\[ n = -\frac{3}{2} \]

(**) 7. If \( z = e^{x/y} \sin \frac{x}{y} + e^{y/x} \cos \frac{y}{x} \), show that \( z \) satisfies the following partial differential equation:

\[ x \frac{\partial z}{\partial y} + y \frac{\partial z}{\partial x} = 0 \]
\[ \frac{\partial z}{\partial x} = \frac{1}{y} e^{x/y} \sin(x/y) + \frac{1}{y} e^{x/y} \cos(x/y) - \frac{y}{x^2} e^{y/x} \cos(y/x) + \frac{y}{x^2} e^{y/x} \sin(y/x) \]

\[ \frac{\partial z}{\partial y} = -\frac{x}{y^2} e^{x/y} \sin(x/y) - \frac{x}{y^2} e^{x/y} \cos(x/y) + \frac{1}{x} e^{y/x} \cos(y/x) - \frac{1}{x} e^{y/x} \sin(y/x) \]

\[ x \frac{\partial z}{\partial y} + y \frac{\partial z}{\partial x} = x \left( \frac{x}{y^2} e^{x/y} \sin(x/y) - \frac{x}{y^2} e^{x/y} \cos(x/y) + \frac{1}{x} e^{y/x} \cos(y/x) - \frac{1}{x} e^{y/x} \sin(y/x) \right) + \\
\quad y \left( \frac{1}{y} e^{x/y} \sin(x/y) + \frac{1}{y} e^{x/y} \cos(x/y) - \frac{y}{x^2} e^{y/x} \cos(y/x) + \frac{y}{x^2} e^{y/x} \sin(y/x) \right) = 0 \]

(\*) 8. (a) Prove that the vectors \( \mathbf{A} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}, \mathbf{B} = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}, \) and \( \mathbf{C} = 4\mathbf{i} - 2\mathbf{j} - 6\mathbf{k} \) can form the sides of a triangle.

(b) Find the lengths of the sides of the triangle.

(c) Find the equation of the plane that contains the triangle.

(d) Find a normal vector to the triangle.

Solution:

(a) It is sufficient to show that \( \mathbf{A} = \mathbf{B} + \mathbf{C} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k} \)

(b) \( |\mathbf{A}| = \sqrt{14}, |\mathbf{B}| = \sqrt{26}, |\mathbf{C}| = \sqrt{56} \)

(c) \( \mathbf{A} \times \mathbf{B} = 10\mathbf{i} - 10\mathbf{j} + 10\mathbf{k} \rightarrow x - y + z = 0 \)

(d) \( \mathbf{N} = \mathbf{i} - \mathbf{j} + \mathbf{k} \)

\( \ast \) 9. Find the angles which the vector \( \mathbf{A} = 3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} \) makes with each of the coordinate axes.

Solution:
Let the angles with the \( x, y, z \) axes be \( \alpha, \beta, \gamma \) respectively.

\( \mathbf{A} \cdot \mathbf{i} = 3 = |\mathbf{A}| \cos \alpha \rightarrow \cos \alpha = 3/7 \rightarrow \alpha = \arccos(3/7) \)

\( \mathbf{A} \cdot \mathbf{j} = -6 = |\mathbf{A}| \cos \beta \rightarrow \cos \beta = -6/7 \rightarrow \beta = \arccos(-6/7) \)
\[ \mathbf{A} \cdot \mathbf{k} = 2 = |\mathbf{A}| \cos \gamma \rightarrow \cos \gamma = 2/7 \rightarrow \gamma = \arccos(2/7) \]

(*) 10. Find the projection of the vector \( \mathbf{A} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \) in the direction of \( \mathbf{B} = 4\mathbf{i} - 4\mathbf{j} + 7\mathbf{k} \)

Solution:
\[
\text{proj}_\mathbf{B} \mathbf{A} = \frac{(\mathbf{A} \cdot \mathbf{B}) \mathbf{B}}{|\mathbf{B}|^2} = \frac{19}{81} \mathbf{B}
\]

(*) 11. (a) Find the work done in moving an object along a vector \( \mathbf{r} = 3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k} \) if the applied force is \( \mathbf{F} = 2\mathbf{i} - \mathbf{j} - \mathbf{k} \)
(b) Find the angle between the applied force and the displacement.

Solution:
1. \( W = \mathbf{r} \cdot \mathbf{F} = 6 - 2 + 5 = 9 \text{J} \)
2. \( \mathbf{r} \cdot \mathbf{F} = 9 = |\mathbf{r}| \cdot |\mathbf{F}| \cos \theta \rightarrow \theta = \arccos \left( \frac{9}{\sqrt{6} \cdot \sqrt{38}} \right) \)

(**) 12. (a) Find an equation for the plane perpendicular to the vector \( \mathbf{A} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k} \) and passing through the terminal point of the vector \( \mathbf{B} = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k} \)
(b) Find the distance from the origin to the plane.

Solution:
(a) \( 2(x - 1) + 3(y - 5) + 6(z - 3) = 0 \)
(b) Let \( \mathbf{a} \) be the vector from the origin to the plane. \( \mathbf{a} = c \mathbf{A} \) because the minimizing distance will occur when the vector from the origin is perpendicular to the plane. Plug the components of the vector \( \mathbf{a} \) into the equation for the plane. \( 2(2c) + 3(3c) + 6(6c) = 35 \rightarrow c = 35/49 = 5/7 \rightarrow |\mathbf{a}| = c \cdot |\mathbf{A}| = 5 \)

(**) 13. Find the minimum distance between the point \( (3,2,6) \) and the line \( \mathbf{r}(t) = (3t - 2)\mathbf{i} + t\mathbf{j} - (2t + 5)\mathbf{k} \)

Solution:
Form the general displacement vector \( \mathbf{a} \) between the point in question and the line in question.
**14.** Find the minimum distance between the point \((9,0,-2)\) and the plane \(z = 3x - 2y + 6\)

*Solution:*

Find the parametric form of the plane. First find 3 points inside the plane by inspection. Three points could be \((0, 0, 6)\), \((0, 1, 4)\), \((1, 0, 9)\).

By subtracting points, we can then find 2 vectors which are in the plane. \(\mathbf{u} = \mathbf{j} - 2\mathbf{k}\) and \(\mathbf{v} = \mathbf{i} + 3\mathbf{k}\). If you are not convinced by this, take \(\mathbf{u} \times \mathbf{v}\) and see that it is indeed the normal vector of the original plane.

The parametric form of the plane is then \(\mathbf{r}(s,t) = s\mathbf{i} + t\mathbf{j} + (6 - 2t + 3s)\mathbf{k}\) Then, the general vector between the point of interest \((9,0,-2)\) and the plane is:

\[\mathbf{d}(s,t) = (s - 9)\mathbf{i} + t\mathbf{j} + (8 - 2t + 3s)\mathbf{k}\]

Since the minimum distance between the point and plane occurs if the point lies along the normal vector to the plane, we must ensure that the displacement vector between the point and the plane is orthogonal to the plane itself. Thus, \(\mathbf{d} \cdot \mathbf{u} = 0\), and \(\mathbf{d} \cdot \mathbf{v} = 0\)

We then obtain two equations in two unknowns.

\[5t - 6s = 16, \text{ and } 6t - 10s = 15\]

The solution is \(s = 3/2\), and \(t = 5\)

The magnitude of the displacement vector evaluated at \(s = 3/2\), and \(t = 5\) will be the distance between the point and the plane.

\[\mathbf{d}(3/2,5) = -(15/2)\mathbf{i} + 5\mathbf{j} + (5/2)\mathbf{k}\]
\[|\mathbf{d}(3/2,5)| = \sqrt{175}/2\]
Solution:
With the line \( r(t) = (3t - 2)i + tj - (2t + 5)k \) and plane \( z = 3x - 2y + 6 \), we simply plug in the components of the line into the plane.

Thus \(-2t - 5 = 3(3t - 2) - 2t + 6\). Solving for \( t \) gives \( t = -5/9 \).

We plug this value of \( t \) into the equation of the line \( r(-5/9) = -11/3i - 5/9j - 35/9k \)

So the point of intersection is \((-11/3, -5/9, -35/9)\).

(*) 16. Find the equation of the tangent plane to the surface \( f(x, y) = x^4y - 3y^2x + 4y \) at the point \((-1, 2)\).

Solution:
\[
z = 22 + f_x(-1, 2)(x + 1) + f_y(-1, 2)(y - 2) = 22 - 20(x + 1) + 17(y - 2)
\]

(*) 17. For \( z = x^2 + 3xy + y^2 \), Find all second partial derivatives of \( z \).

Solution:
\[
f_{xx} = 2
\]
\[
f_{yy} = 2
\]
\[
f_{xy} = 3 = f_{yx}
\]

(*) 18. Find all first partial derivatives of the function \( f(x, y) = y^4 \log(x^3 + \sqrt{y}) \)

Solution:
\[
f_x = \frac{3x^2y^4}{x^3 + \sqrt{y}}
\]
\[
f_y = 4y^3 \log(x^3 + \sqrt{y}) + \frac{y^4}{2\sqrt{y}(x^3 + \sqrt{y})}
\]

(**) 19. Find all first partial derivatives of \( g. g(x, y) = \log(y - x) \cdot \int_{x^2}^{\cos y} e^{t^2 - t} dt \)
20. Find all first partial derivatives of \( f(x, y, z) = \sqrt{\sin(x^2y) + z^4} \)

\[
\begin{align*}
\frac{\partial f}{\partial x} &= \frac{xy \cos(x^2y)}{\sqrt{\sin(x^2y) + z^4}} \\
\frac{\partial f}{\partial y} &= \frac{x^2 \cos(x^2y)}{2\sqrt{\sin(x^2y) + z^4}} \\
\frac{\partial f}{\partial z} &= \frac{2z^3}{\sqrt{\sin(x^2y) + z^4}}
\end{align*}
\]

21. Find the equation of the plane tangent to the surface \( h(x, y) = \sqrt{x^4y^3 - 7x - y} \) at the point \((2, 1, 1)\).

\[
\begin{align*}
h_x &= \frac{4x^3y^3 - 7}{2\sqrt{x^4y^3 - 7x - y}} \quad \Rightarrow \quad h_x(2, 1) = \frac{25}{2} \\
h_y &= \frac{3x^4y^2 - 1}{2\sqrt{x^4y^3 - 7x - y}} \quad \Rightarrow \quad h_y(2, 1) = \frac{47}{2} \\
L(x, y) &= 1 + \frac{25}{2}(x - 2) + \frac{47}{2}(y - 1)
\end{align*}
\]

22. Find an approximate value for \( g(x, y, z) = \sqrt{xyz} \) at the point \((2.05, 1.97, 4.03)\).

\[
\begin{align*}
g_x &= \frac{1}{2} \sqrt{\frac{yz}{x}} \quad \Rightarrow \quad g_x(2, 2, 4) = 1 \\
g_y &= \frac{1}{2} \sqrt{\frac{zx}{y}} \quad \Rightarrow \quad g_y(2, 2, 4) = 1
\end{align*}
\]
\[ g_z = \frac{1}{2} \sqrt{\frac{xy}{z}} \Rightarrow g_z(2, 2, 4) = \frac{1}{2} \]

\[ L(x, y, z) = g(2, 2, 4) + g_x(2, 2, 4)(x - 2) + g_y(2, 2, 4)(y - 2) + g_z(2, 2, 4)(z - 4) = x + y + \frac{1}{2} z - 4 \]

\[ L(2.05, 1.97, 4.03) = 2.05 + 1.97 + \frac{1}{2} \cdot 4.03 - 4 = 4.035 \]

(**) 23. Find an approximate value for \( M(2.94, 8.03) \) if \( M(x, y) = \log |x^2 - y| \)

**Solution:**

\[ M_x = \frac{2x}{x^2 - y} \]

\[ M_y = \frac{1}{y - x^2} \]

\[ L(x, y) = M(3, 8) + M_x(3, 8)(x - 3) + M_y(3, 8)(y - 8) = 6x - y - 10 \]

\[ L(2.94, 8.03) = -0.39 \]

(***) 24. Match the following functions with their contour maps.

(i) \( f(x, y) = \log |x \sin y| \)

(ii) \( f(x, y) = x^2 + 5y \)

(iii) \( f(x, y) = \sqrt{y + |x|} \)

(iv) \( f(x, y) = x^2 + xy - y^2 \)

(v) \( f(x, y) = \cos x \sin(y/2) \)

(vi) \( f(x, y) = (2y + 3)(x - 2) \)

(vii) \( f(x, y) = x^2 - xy + y^2 \)

(viii) \( f(x, y) = \sin x + \sin y \)
Solution:

(i) D
(ii) H
(iii) B
(iv) G
(v) C
(vi) A
(vii) F
(viii) E