Attempt questions to the best of your ability. This review package consists of 26 pages, including 1 cover page and 30 questions. The questions are meant to be the level of a real examination or slightly above, in order to prepare you for the real exam. Material from lectures and from the relevant textbook sections is examinable, and the problems for this package were chosen with that in mind, as well as considerations based on past examination question difficulty and style. Problems are ranked in difficulty as (*) for easy, (**) for medium, and (****) for difficult. Note that sometimes difficulty can be subjective, so do not be discouraged if you are stuck on a (*) problem.

The solutions to these problems will be posted at the following web address: [https://ubcengineers.ca/services/academic/tutoring/](https://ubcengineers.ca/services/academic/tutoring/) If you believe that there is an error in these solutions, or have any questions, comments, or suggestions regarding EUS Tutoring sessions, please e-mail us at: tutoring@ubcengineers.ca. If you are interested in helping with EUS tutoring sessions in the future or other academic events run by the EUS, please e-mail vpacademic@ubcengineers.ca.

Some of the problems in this package were not created by the EUS. Those problems originated from one of the following sources:

- Schuam’s Outline of Calculus 2 ed; Ayres Jr., Frank
- Calculus – Early Transcendentals 7 ed; Stewart, James
- Calculus – 3 ed; Spivak, Michael
- Calculus Volume 1 2 ed; Apostol, Tom

All solutions prepared by the EUS.

Good Luck!
1. Considering \( x \) and \( y \) as independent variables, find \( \frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y} \) when \( x = e^{2r} \cos \theta, y = e^{3r} \sin \theta \)

**Solution:**
With the identities \( x = e^{2r} \cos \theta \) and \( y = e^{3r} \sin \theta \), differentiate both sides with respect to \( x \), recalling that \( r = r(x, y) \) and \( \theta = \theta(x, y) \).

\[
1 = 2e^{2r} \frac{\partial r}{\partial x} \cos \theta - e^{2r} \sin \theta \frac{\partial \theta}{\partial x}
\]

\[
0 = 3e^{3r} \frac{\partial r}{\partial x} \sin \theta + e^{3r} \cos \theta \frac{\partial \theta}{\partial x}
\]

Inverting the linear system,
\[
\frac{\partial r}{\partial x} = \frac{\cos \theta}{e^{2r}(2 + \sin^2 \theta)}
\]

\[
\frac{\partial \theta}{\partial x} = \frac{-3 \sin \theta}{e^{2r}(2 + \sin^2 \theta)}
\]

With the identities \( x = e^{2r} \cos \theta \) and \( y = e^{3r} \sin \theta \), differentiate both sides with respect to \( y \), recalling that \( r = r(x, y) \) and \( \theta = \theta(x, y) \).

\[
0 = 2e^{2r} \frac{\partial r}{\partial y} \cos \theta - e^{2r} \sin \theta \frac{\partial \theta}{\partial y}
\]

\[
1 = 3e^{3r} \frac{\partial r}{\partial y} \sin \theta + e^{3r} \cos \theta \frac{\partial \theta}{\partial y}
\]

Inverting the linear system,
\[
\frac{\partial r}{\partial y} = \frac{\sin \theta}{e^{3r}(2 + \sin^2 \theta)}
\]

\[
\frac{\partial \theta}{\partial y} = \frac{2 \cos \theta}{e^{3r}(2 + \sin^2 \theta)}
\]

2. Evaluate \( \iint_{D} \frac{1}{\sqrt{2y - y^2}} \, dx \, dy \), where \( D \) is the region in the first quadrant bounded by \( x^2 = 4 - 2y \)

**Solution:** Graph the function: \( x^2 = 4 - 2y \)

\[
2y = 4 - x^2
\]

\[
y = -\frac{1}{2} x^2 + 2
\]

Since \( D \) is the region in the first quadrant, we want \( x \geq 0 \) and \( y \geq 0 \). We now need to solve for
the bounds of $x$ and $y$. From the graph, we can tell that $y \in [0, 2]$. Next, we solve for the bounds of $x$.

\[ x^2 = 4 - 2y \Rightarrow x = \pm \sqrt{4 - 2y} \]

$x \in [0, \sqrt{4 - 2y}]$

Note that $D$ is the region in the first quadrant, so we only want the value in which $x \geq 0$. Now that we have our bounds, we will evaluate the following integral.

\[
\int_{y=0}^{y=2} \int_{x=0}^{x=\sqrt{4-2y}} \frac{1}{\sqrt{2y-y^2}} \, dx \, dy = \int_{y=0}^{y=2} \frac{1}{\sqrt{2y-y^2}} \bigg|_{x=0}^{x=\sqrt{4-2y}} \, dy
\]

\[
= \int_{y=0}^{y=2} \frac{\sqrt{4-2y}}{\sqrt{2y-y^2}} \, dy
\]

\[
= \int_{y=0}^{y=2} \frac{\sqrt{2(2-y)}}{\sqrt{y(2-y)}} \, dy
\]

\[
= \int_{y=0}^{y=2} \sqrt{2} \, dy
\]

\[
= \sqrt{2} \int_{y=0}^{y=2} \frac{1}{\sqrt{y}} \, dy
\]

\[
= \sqrt{2} [2^{1/2}]_{y=0}^{y=2}
\]

\[
= 2\sqrt{2}\sqrt{2}
\]

\[
= 4
\]

3. Evaluate $\int_{0}^{\infty} \frac{e^{-x} - e^{-ax}}{x} \, dx$ as follows:

a) Compute $\int_{1}^{a} e^{-xy} \, dy$.

b) Use part (a) to rewrite the integral you wish to evaluate as a double integral. Evaluate the double integral by exchanging the order of integration.

**Solution:** Part a: Compute: $\int_{1}^{a} e^{-xy} \, dy$

\[
\int_{1}^{a} e^{-xy} \, dy = \frac{-1}{x} [e^{-xy}]_{y=1}^{y=a} = \frac{-1}{x} e^{-ax} + \frac{1}{x} e^{-x} = \frac{e^{-x} - e^{-ax}}{x}
\]
Part b: Evaluate $\int_a^\infty \int_1^a e^{-xy} \, dy \, dx$

$$\int_1^a \int_0^\infty e^{-xy} \, dx \, dy \Rightarrow \int_1^a -\frac{1}{y} e^{-xy} \bigg|_{x=0}^{x=\infty} \Rightarrow \int_1^a \frac{1}{y} \, dy \Rightarrow \ln |y| \bigg|_{x=0}^{x=\infty} \Rightarrow \ln |a| - \ln |1| \Rightarrow$$

4. Consider the function $f(x, y) = x^3 - xy^2 - 4x^2 + 3x + x^2y$

a) Find the maximum and minimum values of the directional derivative at the point $(1/2, 1)$ as the direction varies.

b) In which directions do the maximum and minimum values occur?

c) In what directions is the directional derivative zero?

Solution:

Part a:

Since the value of $\cos \theta$ will always be between $-1$ and $1$. When the angle between the gradient vector and the unit vector is $0$, the directional derivative will be at its maximum value. When $\theta = \pi$, the directional derivative will be at its minimum value.

Recall the following formula: $D_u f(x, y) = \nabla f \cdot u = |\nabla f| |u| \cos \theta = |\nabla f| \cos \theta$

Note: $\theta$ is the angle between $\nabla f$ and $u$.

$$\nabla f(1/2, 1) = \langle 3x^2 - y^2 - 8x + 3 + 2xy, -2xy + x^2 \rangle \Rightarrow \langle (3/2)^2 - (1)^2 - 8(1/2) + 3 + 2(1/2)(1), -2(1/2)(1) + (1/2)^2 \rangle \Rightarrow \langle -1/4, -3/4 \rangle$$

The maximum rate of change: $|\nabla f(1/2, 1)| \cos(0) \Rightarrow \sqrt{(-1/4)^2 + (-3/4)^2} \Rightarrow \sqrt{\frac{10}{16}} \Rightarrow \frac{\sqrt{10}}{4}$

The minimum rate of change: $|\nabla f(1/2, 1)| \cos(\pi) \Rightarrow -\sqrt{(-1/4)^2 + (-3/4)^2} \Rightarrow -\sqrt{\frac{10}{16}} \Rightarrow -\frac{\sqrt{10}}{4}$

Part b:

The maximum direction derivative occurs when the unit vector is in the same direction as the gradient vector. From Part a, the gradient vector is found to be $\langle -\frac{1}{4}, -\frac{3}{4} \rangle$.

To find the direction, we need to convert this vector into a unit vector.

$$u = \frac{v}{|v|} \Rightarrow \frac{\langle -1/4, -3/4 \rangle}{\sqrt{10/16}} \Rightarrow \frac{4}{\sqrt{10}} \left\langle -\frac{1}{4}, -\frac{3}{4} \right\rangle$$

To find the direction in which the directional derivative is at its minimum, the vector is opposite to the direction at which the maximum occurs.

Therefore, $\frac{1}{\sqrt{10}} \langle 1, 3 \rangle$. 

Part c:
\[ \nabla f \cdot \mathbf{u} = 0 \Rightarrow \left\langle -\frac{1}{4}, -\frac{3}{4} \right\rangle \cdot \mathbf{u} = 0 \Rightarrow -\frac{1}{4} x - \frac{3}{4} y = 0 \Rightarrow -\frac{3}{4} y = \frac{1}{4} x \Rightarrow -3y = x \]

Arbitrarily choose \( y = 1 \Rightarrow x = -3 \). We want the direction where the directional derivative is equal to 0, so we need to find the unit vector.

\[ \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} \Rightarrow \left\langle -\frac{3}{4}, \frac{1}{4} \right\rangle \frac{1}{\sqrt{\left(\frac{3}{2}^2 + \frac{1}{2}^2\right)}} \Rightarrow \left\langle -\frac{3}{4}, \frac{1}{4} \right\rangle \frac{1}{\sqrt{10}} \]

Note that \(-\mathbf{u}\) also works.

(* ) 5. Find the maximum and minimum distances from the origin to the curve \( 5x^2 + 6xy + 5y^2 = 8 \).

Solution:
Use Lagrange Multipliers as we are given a constraint in this problem. Recall the distance equation: \( d = \sqrt{x^2 + y^2} \), which can be rewritten as \( d^2 = x^2 + y^2 \). Rewrite the equation to \( 5x^2 + 6xy + 5y^2 - 8 = 0 \).

\[ \nabla f(x, y) = \lambda \nabla g(x, y) \Rightarrow \nabla (x^2 + y^2) = \lambda \nabla (5x^2 + 6xy + 5y^2 - 8) \]
\[ = 2x = \lambda(10x + 6y) \]
\[ = 2y = \lambda(6x + 10y) \]

Solve for lambda in both equations:
\[ 2x = \lambda(10x + 6y) \Rightarrow \lambda = \frac{2x}{10x + 6y} = \frac{x}{5x + 3y} \]

Part b:
\[ 2y = \lambda(6x + 10y) \Rightarrow \lambda = \frac{2y}{6x + 10y} = \frac{y}{3x + 5y} \]

Next we want to solve for the points that will give us the minimum and maximum distance from the origin.
\[ \frac{x}{5x + 3y} = \frac{y}{3x + 5y} \]
\[ x(3x + 5y) = y(5x + 3y) \]
\[ 3x^2 + 5xy = 3y^2 + 5xy \]
\[ 3x^2 = 3y^2 \]
\[ x^2 = y^2 \]
\[ y = \pm x \]

Case 1:
For \( y = x \)
\[ 5x^2 + 6xy + 5y^2 - 8 = 0 \]
\[ 5x^2 + 6x(x) + 5(x)^2 - 8 = 0 \]
\[ 5x^2 + 6x^2 + 5x^2 - 8 = 0 \]
\[ 16x^2 = 8 \]
\[ x^2 = \frac{1}{2} \]
\[ x = \pm \sqrt{\frac{1}{2}} \]

Since \( y = x \)
\[ (x, y) \Rightarrow \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \text{ and } \left( \frac{-1}{\sqrt{2}}, -1 \right) \]

Distance: \( d = \sqrt{x^2 + y^2} \)
\[ \sqrt{\left( \frac{1}{\sqrt{2}} \right)^2 + \left( \frac{1}{\sqrt{2}} \right)^2} \Rightarrow 1 \]

Case 2:
For \( y = -x \)
\[ 5x^2 + 6xy + 5y^2 - 8 = 0 \]
\[ 5x^2 + 6x(-x) + 5(-x)^2 - 8 = 0 \]
\[ 5x^2 - 6x^2 + 5x^2 - 8 = 0 \]
\[ 4x^2 = 8 \]
\[ x^2 = 2 \]
\[ x = \pm \sqrt{2} \]

Since \( y = -x \)
\[ (x, y) \Rightarrow \left( \sqrt{2}, -\sqrt{2} \right) \text{ and } \left( -\sqrt{2}, \sqrt{2} \right) \]

Distance: \( d = \sqrt{x^2 + y^2} \)
\[ \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} \Rightarrow 2 \]

Since \( 1 < 2 \), the minimum distance from the origin is 1. This occurs at points: \( \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \) and \( \left( \frac{-1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \).
The maximum distance from the origin is 2. This occurs at points: \( \left( \sqrt{2}, -\sqrt{2} \right) \) and \( \left( -\sqrt{2}, \sqrt{2} \right) \).

(**) 6. Assume \( a \) and \( b \) are fixed positive numbers.
   a) Find the extreme values of \( z = x/a + y/b \) subject to the condition \( x^2 + y^2 = 1 \)
   b) Find the extreme values of \( z = x^2 + y^2 \) subject to the condition \( x/a + y/b = 1 \)
Solution: a) 
\[ \nabla z = \lambda \nabla (x^2 + y^2) \Rightarrow \left\langle \frac{1}{a}, \frac{1}{b} \right\rangle = \lambda \left\langle 2x, 2y \right\rangle \]
Using the constraint \( x^2 + y^2 = 1 \), we find that \( x = \frac{b}{\sqrt{a^2 + b^2}} \) and \( y = \frac{a}{\sqrt{a^2 + b^2}} \).
Thus \( z \) at this extreme value is
\[ z_{\text{max}} = \frac{\sqrt{a^2 + b^2}}{ab} \]

b) 
\[ \nabla z = \lambda \nabla (\frac{x}{a} + \frac{y}{b}) \Rightarrow \left\langle 2x, 2y \right\rangle = \lambda \left\langle \frac{1}{a}, \frac{1}{b} \right\rangle \]
Using the constraint \( \frac{x}{a} + \frac{y}{b} = 1 \), we find \( x = \frac{a b^2}{a^2 + b^2} \) and \( y = \frac{a^2 b}{a^2 + b^2} \).
Thus \( z \) at the extreme value is
\[ z_{\text{max}} = \frac{a^2 b^2}{a^2 + b^2} \]

(∗) 7. Maximize \( x^2 + y^2 \) subject to \( x^2 + xy + y^2 = 4 \).

Solution: Use Lagrange Multipliers since we are given a constraint in this problem.
\[ \nabla f(x, y) = \lambda \nabla g(x, y) \]
\[ \nabla (x^2 + y^2) = \lambda \nabla (x^2 + xy + y^2 - 4) \]
\[ 2x = \lambda (2x + y) \]
\[ 2y = \lambda (x + 2y) \]

Multiply line 3 by \( x \) and multiply line 4 by \( y \). We want to cancel out \( \lambda \).
\[ 2xy = \lambda y (2x + y) \]
\[ 2xy = \lambda x (x + 2y) \]
\[ \lambda y (2x + y) = \lambda x (x + 2y) \]
\[ 2xy + y^2 = x^2 + 2xy \]
\[ y^2 = x^2 \]
\[ \pm \sqrt{y^2} = x \]
\[ \pm y = x \]

Case 1: \( x = y \)
\[ x^2 + xy + y^2 = 4 \]
\[ (y^2) + (y) y + y^2 = 4 \]
\[ y^2 + y^2 + y^2 = 4 \]
\[ 3y^2 = 4 \]
\[ y = \pm \sqrt{\frac{4}{3}} \]

Case 2: \( x = -y \)

\[
\begin{align*}
(-y)^2 + (-y)y + y^2 &= 4 \\
y^2 - y^2 + y^2 &= 4 \\
y &= \pm 2
\end{align*}
\]

Since we want to maximize \( x^2 + y^2 \), we need to determine which of the two cases will yield a bigger number.

Case 1: \( \left( \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}} \right) \) and \( \left( -\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}} \right) \)

\[
x^2 + y^2 \Rightarrow \left( \frac{2}{\sqrt{3}} \right)^2 + \left( \frac{2}{\sqrt{3}} \right)^2 \Rightarrow \frac{8}{3}
\]

Case 2: \( (2, -2) \) and \( (-2, 2) \)

\[
x^2 + y^2 \Rightarrow (2)^2 + (-2)^2 = 8
\]

Since \( 8 > \frac{8}{3} \), we get \( x^2 + y^2 = 8 \).

(∗) 8. Find the relative maxima and minima of \( f(x, y) = x^3 + y^3 - 3x - 12y + 20 \).

**Solution:**

We first locate the critical points by setting the partial derivatives equal to 0.

\[
f_x = 3x^2 - 3 \Rightarrow 0 = 3x^2 - 3 \\
\Rightarrow 3x^2 = 3 \\
\Rightarrow x^2 = 1 \\
\Rightarrow x = \pm \sqrt{1} \\
\Rightarrow x = \pm 1
\]

\[
f_y = 3y^2 - 12 \Rightarrow 3y^2 = 12 \\
\Rightarrow y^2 = 4 \\
\Rightarrow y = \pm \sqrt{4} \\
\Rightarrow y = \pm 2
\]

Therefore, the four critical points are: \((x, y) = (\pm 1, \pm 2)\). Next, we use the second derivative test to classify the critical points.
\[ D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2 \]
\[ D = \begin{vmatrix} 6x & 0 \\ 0 & 6y \end{vmatrix} = (6x)(6y) - 0 = 36xy \]

Point 1: \((1, 2)\)
\[ 36xy \Rightarrow 36(1)(2) = 72 \]
\[ f_{xx}(1, 2) = 6x \Rightarrow 6(1) \Rightarrow 6 \]
Since \(D > 0\) and \(f_{xx}(1, 2) > 0\), \((1, 2)\) is a local minimum.

Point 2: \((1, -2)\)
\[ 36xy \Rightarrow 36(1)(-2) = -72 \]
Since \(D < 0\), \((1, -2)\) is a saddle point.

Point 3: \((-1, 2)\)
\[ 36xy \Rightarrow 36(-1)(2) = -72 \]
Since \(D < 0\), \((-1, 2)\) is a saddle point.

Point 4: \((-1, -2)\)
\[ 36xy \Rightarrow 36(-1)(-2) = 72 \]
\[ f_{xx}(-1, -2) = 6x \Rightarrow 6(-1) \Rightarrow -6 \]
Since \(D > 0\) and \(f_{xx}(-1, -2) < 0\), \((-1, -2)\) is a local maximum.

\[(*)\] 9. Find the extreme values of \(z\) on the surface \(2x^2 + 3y^2 + z^2 - 12xy + 4xz = 35\)

**Solution:**
Set partial derivatives equal to 0 to find critical points.

\[ f_y = \frac{12x - 6y}{2z + 4x} \]
\[ 0 = \frac{6y - 12x}{6y - 12x} \]
\[ 12x = 6y \]
\[ 2x = y \]

We conclude that \(y = 2x\).
\[ f_x = -\frac{4x - 12y + 4z}{2z + x} \]
\[ 0 = 4x - 12y + 4z \]
\[ 0 = 4x - 24x + 4z \]
\[ 20x = 4z \]
\[ 5x = z \]

To find the minimum and maximum values of \( z \), we will substitute our critical points into our original equation.

\[ 2x^2 + 3y^2 + z^2 - 12xy + 4xz = 35 \]
\[ 2x^2 + 3(2x)^2 + (5x)^2 - 12x(2x) + 4x(5x) = 35 \]
\[ 2x^2 + 12x^2 + 25x^2 - 24x^2 + 20x^2 = 35 \]
\[ 35x^2 = 35 \]
\[ x^2 = \pm \sqrt{1} \]
\[ x = \pm 1 \]

Since \( z = 5x \) \( \Rightarrow \) \( z = \pm 5 \)

() 10. Find and classify the extreme values (if any) of the following function: \( f(x, y) = y^2 + x^2y + x^4 \)

**Solution:**

\[ f_x = 2xy + 4x^3 \]
\[ f_y = 2y + x^2 \]

The only solution to this system is \((x, y) = (0, 0)\).

So \((0, 0)\) is a critical point.

\[ D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2 \]
\[ D = \begin{vmatrix} 2y + 12x^2 & 2x \\ 2x & 2 \end{vmatrix} = 4y + 24x^2 - 4x^2 = 0 \]

The second derivative test is inconclusive and \((0,0)\) could either be a minimum, maximum or saddle point. Since \( f_{xx} = 0 \), we must be more clever to determine if it is a max or min.

\[ f(x, y) = y^2 + x^2y + x^4 = \left(y + \frac{x^2}{2}\right)^2 - \frac{x^4}{4} + x^4 = \left(y + \frac{x^2}{2}\right)^2 + \frac{3x^4}{4} \]

Since we have a sum of squares, the function is always positive, and \((0,0)\) is a global minimum.

(\*) 11. Find and classify the extreme values (if any) of the following function: \( f(x, y) = (x - 1)^4 + (x - y)^4 \)
Solution:

\[ f(x, y) = (x - 1)^4 + (x - y)^4 \]
\[ f_x = 4(x - 1)^3 + 4(x - y)^3 \]
\[ 0 = 4(x - 1)^3 + 4(x - y)^3 \]
\[ x = 1 \]

\[ f_y = 4(x - y)^3(-1) \]
\[ 0 = 4(x - y)^3(-1) \]
\[ 0 = 4(1 - y)^3(-1) \]
\[ 0 = ((y - 1)^3)^{1/3} \]
\[ y = 1 \]

\[ f_x = 4(x - 1)^3 \]
\[ f_y = 4(x - y)^3 \]
\[ 0 = 4(1 - y)^3 \]
\[ 0 = ((y - 1)^3)^{1/3} \]
\[ y = 1 \]

\[ D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2 \]
\[ D = \begin{vmatrix} 12(x - 1)^2 + 12(x - y)^2 & 12(x - y)^2(-1) \\ 12(x - y)^2(-1) & 12(x - y)^2 \end{vmatrix} = 144(x - 1)^2(x - y)^2 + 144(x - y)^3 - 144(x - y)^4 = 0 \]

Since the function is a sum of squares, we conclude that (1,1) must be a minimum point.

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(*) 12. Compute the gradient vector \( \nabla f(x, y) \) at those points \((x, y)\) in \( \mathbb{R} \) where it exists.

a) \( f(x, y) = x^2y^2 \log(x^2 + y^2) \)
b) \( f(x, y) = xy \sin \left( \frac{1}{x^2 + y^2} \right) \)

Solution: Part a:

\[ f(x, y) = x^2y^2 \log (x^2 + y^2) \]
\[ \nabla f(x, y) = (f_x, f_y) \]

\[ f_x: \]
\[ y^2 \left( \frac{\partial}{\partial x} (x^2 \log (x^2 + y^2)) \right) \text{ (Apply Product and Chain Rule)} \]
\[ \Rightarrow y^2 \left[ 2x \log (x^2 + y^2) + \frac{2x^3}{x^2 + y^2} \right] \]
\[ \Rightarrow y^2 \left[ 2x \log (x^2 + y^2) + \frac{2x^3}{x^2 + y^2} \right] \]

\[ f_y: \]
\[ x^2 \left( \frac{\partial}{\partial y} (y^2 \log (x^2 + y^2)) \right) \text{ (Apply Product and Chain Rule)} \]
\[ \Rightarrow x^2 \left[ 2y \log (x^2 + y^2) + \frac{y^2(2y)}{x^2 + y^2} \right] \]
\[ \Rightarrow x^2 \left[ 2y \log (x^2 + y^2) + \frac{2y^3}{x^2 + y^2} \right] \]
\[ \nabla f(x, y) = \left\langle y^2 \left[ 2x \log (x^2 + y^2) + \frac{2x^3}{x^2 + y^2} \right], x^2 \left[ 2y \log (x^2 + y^2) + \frac{2y^3}{x^2 + y^2} \right] \right\rangle \]

Part b:
\[ f(x, y) = xy \sin \left( \frac{1}{x^2 + y^2} \right) \]
\[ \nabla f(x, y) = \langle f_x, f_y \rangle \]
\[ f_x: \]
\[ y \left( \frac{\partial}{\partial x} (x \sin[(x^2 + y^2)^{-1}]) \right) \quad \text{(Apply Product and Chain Rule)} \]
\[ \Rightarrow y \sin \left( \frac{1}{x^2 + y^2} \right) + xy \cos \left( \frac{1}{x^2 + y^2} \right) \left( -(x^2 + y^2)^{-2}(2x) \right) \]
\[ f_y: \]
\[ x \left( \frac{\partial}{\partial y} (y \sin[(x^2 + y^2)^{-1}]) \right) \quad \text{(Apply Product and Chain Rule)} \]
\[ \Rightarrow x \sin \left( \frac{1}{x^2 + y^2} \right) + xy \cos \left( \frac{1}{x^2 + y^2} \right) \left( -(x^2 + y^2)^{-2}(2y) \right) \]
\[ \nabla f(x, y) = \left\langle y \sin \left( \frac{1}{x^2 + y^2} \right) + xy \cos \left( \frac{1}{x^2 + y^2} \right) \left( -(x^2 + y^2)^{-2}(2x) \right), x \sin \left( \frac{1}{x^2 + y^2} \right) + xy \cos \left( \frac{1}{x^2 + y^2} \right) \left( -(x^2 + y^2)^{-2}(2y) \right) \right\rangle \]

\[ \nabla f(x, y) = \langle \frac{-128}{49}, \frac{-256}{49} \rangle \]

(*) 13. The temperature \( T \) of a heated circular plate at any of its points \( (x, y) \) is given by \( T = \frac{64}{x^2 + y^2 + 2} \), the origin being the center of the plate. At the point \( (1, 2) \) find the rate of change of \( T \) in the direction \( \theta = \pi/3 \).

**Solution:** First, we need to determine the unit vector. We are given the angle in radians and we can determine the unit vector by using the formula: \( \mathbf{u} = (\cos \theta, \sin \theta) \).

\[ \mathbf{u} = \left\langle \cos \left( \frac{\pi}{3} \right), \sin \left( \frac{\pi}{3} \right) \right\rangle = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle \]

\[ T = \frac{64}{x^2 + y^2 + 2} \Rightarrow 64(x^2 + y^2 + 2)^{-1} \]

\[ D_{\mathbf{u}} f = \nabla f(x, y) \cdot \mathbf{u} = \nabla f(1, 2) = \langle -64(x^2 + y^2 + 2)^{-2}(2x), -64(x^2 + y^2 + 2)^{-2}(2y) \rangle \Rightarrow \]
\[ \langle -64((1)^2 + (2)^2 + 2)^{-2}(2(1)), -64((1)^2 + (2)^2 + 2)^{-2}(2(2)) \rangle = \langle \frac{-128}{49}, \frac{-256}{49} \rangle \]
14. Find the directional derivative of the function at the given point \( P \) in the direction of the vector \( \mathbf{v} \).

\[ g(x, y) = \arctan(xy), \quad P = (1, 2), \quad \mathbf{v} = \mathbf{i} + 3\mathbf{j}. \]

**Solution:**

\[ g_x = \frac{y}{1 + x^2y^2} \]

\[ g_y = \frac{x}{1 + x^2y^2} \]

Hence, \( \nabla g(x, y) = \left( \frac{y}{1 + x^2y^2}, \frac{x}{1 + x^2y^2} \right) \)

\[ \Rightarrow \nabla g(1, 2) = \left( \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right) \]

Recall: \( D_u g = \nabla g(x, y) \cdot \mathbf{u} \). Before we continue, we need to convert \( \mathbf{v} \) to a unit vector \( \mathbf{u} \).

\[ \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{10}} \mathbf{i} + \frac{3}{\sqrt{10}} \mathbf{j} \]

\[ D_u g(1, 2) = \nabla g(1, 2) \cdot \mathbf{u} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}} \]

15. **Challenge Problem**

Define

\[ f(x) = \left( \int_0^x e^{-t^2} \, dt \right)^2 \quad g(x) = \int_0^1 \frac{e^{-x(t^2+1)}}{t^2+1} \, dt. \]

a) Show that \( f'(x) + g'(x) = 0 \) for all \( x \).

b) What is the value of \( f(x) + g(x) \)? Recall that a function’s derivative is 0 for all \( x \) if and only if it is a constant function for all \( x \).

c) Prove that

\[ \int_0^\infty e^{-t^2} \, dt = \frac{\sqrt{\pi}}{2} \]

**Solution:**

a)

\[ f'(x) = 2e^{-x^2} \int_0^x e^{-t^2} \, dt \]

\[ g'(x) = \int_0^1 \frac{(t^2+1)(-2x)e^{-x(t^2+1)}}{t^2+1} \, dt \]
\[
f'(x) + g'(x) = 2e^{-x^2} \int_0^x e^{-t^2} dt - 2 \int_0^1 \frac{(t^2 + 1)(x) e^{-x^2(t^2 + 1)}}{t^2 + 1} dt \\
= 2 \left[ e^{-x^2} \int_0^x e^{-t^2} dt - x \int_0^1 e^{-x^2(t^2 + 1)} dt \right] \\
= 2 \left[ e^{-x^2} \int_0^x e^{-t^2} dt - x \int_0^1 e^{-x^2 t^2} dt \right] \\
= 2 \left[ e^{-x^2} \int_0^x e^{-t^2} dt - x e^{-x^2} \int_0^1 e^{-x^2 t^2} dt \right] \\
= 2e^{-x^2} \left[ \int_0^x e^{-t^2} dt - x \int_0^1 e^{-x^2 t^2} dt \right] \\
\text{Let } t = sx, dt = xds \text{ in the first integral} \\
= 2e^{-x^2} \left[ \int_0^1 s^2 e^{-s^2} ds - x \int_0^1 e^{-x^2 t^2} dt \right] \\
\text{Renaming } s \text{ back to } t, \\
= 2e^{-x^2} \left[ \int_0^1 e^{-x^2 t^2} dt - x \int_0^1 e^{-x^2 t^2} dt \right] \\
= 0
\]

b) If \( f'(x) + g'(x) = (f(x) + g(x))' = 0 \) for all \( x \), this implies that \( f(x) + g(x) = C \) for some constant \( C \). Thus we can plug in any variable for \( x \) to find \( C \). Let \( x = 0 \). Then \( f(0) = 0 \), and \( g(0) = \int_0^1 \frac{1}{t^2 + 1} dt \).

Thus \( g(0) = \int_0^1 \frac{1}{t^2 + 1} dt = \arctan(x)|_0^1 = \frac{\pi}{4} \).

So \( f(x) + g(x) \equiv \frac{\pi}{4} \).

c) We take the limit as \( x \to \infty \) of both sides.

\[
\lim_{x \to \infty} f(x) + \lim_{x \to \infty} g(x) = \frac{\pi}{4}
\]

\( \lim_{x \to \infty} g(x) \) is clearly 0 because the decaying exponential will cause the integrand to become 0. So we are left with \( \lim_{x \to \infty} f(x) = \left( \int_0^\infty e^{-t^2} dt \right)^2 = \frac{\pi}{4} \). Thus

\[
\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}
\]

(* *) 16. Find the volume cut from \( 9x^2 + 4y^2 + 36z = 36 \) by the plane \( z = 0 \).
**Solution:** This will be a downward opening elliptic paraboloid. Rearranging, \( z = 1 - \frac{x^2}{4} - \frac{y^2}{9} \).

The region in the \( x-y \) plane will be elliptic, with the major radius being 3, in the \( y \) direction, and the minor radius being 2, in the \( x \) direction. Thus the desired volume will be

\[
V = \int_{-2}^{2} \int_{-\sqrt{36-x^2}}^{\sqrt{36-x^2}} 1 - \frac{x^2}{4} - \frac{y^2}{9} \, dy \, dx
\]

To simplify the problem, we will make the change of variables \( x = 2x', y = 3y' \). Thus \( \, dy \, dx = 6 \, dx' \, dy' \).

Now the integral becomes

\[
6 \int_{-1}^{1} \int_{-\sqrt{1-x'^2}}^{\sqrt{1-x'^2}} 1 - x'^2 - y'^2 \, dx' \, dy'
\]

Now we can make the change of variables \( x' = r \cos \theta, y' = r \sin \theta \) to polar coordinates to obtain

\[
6 \int_{0}^{2\pi} \int_{0}^{1} (1 - r^2) r \, dr \, d\theta = 9 \pi
\]

(*** 17. Find the volume in the first octant inside \( y^2 + z^2 = 9 \) and outside \( y^2 = 3x \)

**Solution:** To find the bounds for the double integral, graph \( y^2 + z^2 = 9 \) and \( y^2 = 3x \) in the \( xy \) plane.

\[
\int_{y=0}^{y=3} \int_{x=0}^{x=y^2/3} \sqrt{9-y^2} \, dx \, dy = \int_{y=0}^{y=3} x \sqrt{9-y^2} \bigg|_{x=0}^{x=y^2/3} \, dy
\]

Use substitution:

\( y = 3 \sin(u) \)

\( dy = 3 \cos(u) \, du \)

\( u = \sin^{-1}(\frac{y}{3}) = \sin^{-1}(\frac{3}{3}) = \frac{\pi}{2} \)

\[
\int_{y=0}^{y=3} 1^{y^2} \sqrt{9-y^2} \, dy = \int_{0}^{\pi/2} \frac{1}{3} (9 \sin^2(u)) \sqrt{9-9 \sin^2(u)(3 \cos(u))} \, du
\]

Note: \((\sqrt{9-9 \sin^2(u)} = 3 \cos(u) \Rightarrow \text{Trigonometric Identity: } \cos^2 \theta + \sin^2 \theta = 1)\)

\[
\Rightarrow \int_{0}^{\pi/2} 27 \sin^2(u) \cos^2(u) \, du \Rightarrow 27 \int_{0}^{\pi/2} \sin^2(u) [1 - \sin^2(u)] \, du \Rightarrow 27 \int_{0}^{\pi/2} \sin^2(u) \, du - 27 \int_{0}^{\pi/2} \sin^4(u) \, du
\]

Useful Trigonometric Identities:

\( \cos^2(\theta) = \frac{1}{2} + \frac{1}{2} \cos(2\theta) \)
\[ \sin^2(\theta) = \frac{1}{2} - \frac{1}{2} \cos(2\theta) \]

Let’s integrate one integral at a time:

\[ 27 \int_0^\pi \sin^2(u) du \Rightarrow 27 \left[ u - \frac{1}{2} \sin(2u) \right]_0^\pi = 27 \frac{\pi}{2} \]

\[ -27 \int_0^\pi \sin^4(u) du \Rightarrow -27 \left[ \frac{1}{2} - \frac{1}{2} \cos(2u) \right]^2 du \Rightarrow -\frac{27}{4} \int_0^\pi [1 - 2 \cos(2u) + \cos^2(2u)] du \]

\[ \Rightarrow -\frac{27}{4} \left[ \frac{\pi}{2} - \frac{27}{8} \right] \]

Total Volume = \( 27 \left( \frac{\pi}{2} \right) - \frac{27}{4} \left( \frac{\pi}{2} \right) \Rightarrow 27 \frac{\pi}{16} \)

(**) 18. Find the volume inside \( x^2 + y^2 = 9 \), bounded below by \( x^2 + y^2 + 4z = 16 \) and above by \( z = 4 \).

**Solution:** Bounded Volume = (Volume below \( z = 4 \)) - (Volume below \( x^2 + y^2 + 4z = 16 \))

\[ \Rightarrow 27 \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=3} \left[ 4 - (r^2 - 9) \right] r dr d\theta - 27 \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=4} \left[ 4 + \frac{1}{4}(r^2 - 16) \right] r dr d\theta \]

\[ \Rightarrow 27 \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=3} \left[ 4 - (r^2 - 9) \right] r dr d\theta - 27 \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=4} \left[ 4 + \frac{1}{4}(r^2 - 4r) \right] r dr d\theta \]

\[ \Rightarrow (2\pi) \left[ \frac{4r^2}{2} - \frac{r^4}{4} + \frac{9r^2}{2} \right]_{r=0}^{r=3} - (2\pi) \left[ \frac{4r^2}{2} + \frac{r^4}{4(4)} - \frac{4r^2}{2} \right]_{r=0}^{r=4} \Rightarrow \frac{153}{2} \pi - 32 \pi \Rightarrow \frac{89}{2} \pi \]

(**) 19. A thin plate is bounded by an art of the parabola \( y = 2x - x^2 \) and the interval \( 0 \leq x \leq 2 \). Determine its mass if the density at each point \( (x, y) \) is \( \delta(x, y) = \frac{1 - y}{1 + x} \)

**Solution:**

\[ \int_{x=0}^{x=2} \int_{y=0}^{y=2x-x^2} \frac{1 - y}{1 + x} dy dx \Rightarrow \int_{x=0}^{x=2} \int_{y=0}^{y=2x-x^2} \frac{1}{1 + x} - \frac{y}{1 + x} dy dx = \int_{x=0}^{x=2} \left[ \frac{1}{1 + x} y - \frac{y^2}{2(1 + x)} \right]_{y=0}^{y=2x-x^2} dx \]
\[ \Rightarrow \int_{x=a}^{x=2} \frac{(2x-x^2)^2}{1+x} - \frac{(2x-x^2)^2}{2(1+x)} \, dx \Rightarrow \int_{x=0}^{x=2} -x^4 + 4x^3 - 6x^2 + 4x \, dx \]

Use long division to break up the integral.

Result: \[ \frac{p(x)}{q(x)} = m(x) + \frac{r(x)}{q(x)} \Rightarrow -\frac{1}{2} x^3 + \frac{5}{2} x^2 - \frac{11}{2} x + \frac{15}{2} - \frac{15}{2(x+1)} \]

\[ \int_{x=0}^{x=2} -\frac{1}{2} x^3 + \frac{5}{2} x^2 - \frac{11}{2} x + \frac{15}{2} - \frac{15}{2(x+1)} \, dx \Rightarrow -\frac{1}{8} x^4 + \frac{5}{6} x^3 - \frac{11}{4} x^2 + \frac{15}{2} x - \frac{15}{2} \ln |x+1| \bigg|_{x=0}^{x=2} \]

\[ \Rightarrow \frac{26}{3} - \frac{15 \ln(3)}{2} \]

(**) 20. Find the centre of mass of a thin plate in the shape of a rectangle ABCD if the density at any point is the product of the distances of the point from two adjacent sides AB and AD.

**Solution:** Let point A be at the origin, point B be the point (B,0), point C be the point (0,C), and point D be (B,C).

Density Function: \( f(x, y) = xy \)

Total mass: \( M = \int_{y=0}^{y=C} \int_{x=0}^{x=B} xy \, dxdy \Rightarrow \int_{y=0}^{y=C} 1 \cdot y^2 \bigg|_{x=0}^{x=B} \, dy \Rightarrow \int_{y=0}^{y=C} \frac{1}{2} y^2 \bigg|_{y=0}^{y=C} \)

\[ \Rightarrow \frac{1}{4}(B^2)(C^2) \]

\[ \hat{x} = \frac{\iint xyf(x,y)dA}{M} \Rightarrow \int_{y=0}^{y=C} \int_{x=0}^{x=B} x^2 y \, dxdy \Rightarrow \int_{y=0}^{y=C} \frac{1}{3} x^3 \bigg|_{x=0}^{x=B} \, dy \Rightarrow \int_{y=0}^{y=C} \frac{1}{3} y[B^3] \, dy \]

\[ \Rightarrow \frac{1}{3}[B^3 - 0^3] - \frac{1}{2} y^2 \bigg|_{y=0}^{y=C} \Rightarrow \frac{1}{6}[B^3][C^2] \]

\[ \hat{x} = \frac{\iint xyf(x,y)dA}{M} \Rightarrow \frac{1}{6}[B^3][C^2] \Rightarrow \frac{2[B^3]}{3[B^2]} = \frac{2B}{3} \]
\[ \hat{y} = \int \int \frac{yf(x,y)\,dA}{M} \Rightarrow \int_{y=0}^{y=C} \int_{x=B}^{x=0} xy^2 \, dx\, dy \Rightarrow \int_{y=0}^{y=C} \frac{1}{2} x^2 y^2 \bigg|_{x=0}^{x=B} \, dy \Rightarrow \int_{y=0}^{y=C} \frac{1}{2} y^2 [B^2] \, dy \]

\[ \Rightarrow \frac{1}{2} [B^2] \frac{1}{3} y^3 \bigg|_{y=0}^{y=C} = \frac{1}{6} [B^2] [C^3] \]

\[ \hat{y} = \int \int \frac{xf(x,y)\,dA}{M} \Rightarrow \int_{x=1}^{x=2} \int_{y=0}^{y=C} \frac{1}{6} [B^2] [C^3] \Rightarrow \frac{2}{3} [C^3] = \frac{2C}{3} \]

Thus, \((\hat{x}, \hat{y}) = \left(\frac{2B}{3}, \frac{2C}{3}\right)\)

\[ \hat{y} = \int \int \frac{xyf(x,y)\,dA}{M} \Rightarrow \int_{y=0}^{y=C} \int_{x=0}^{x=B} xy \, dx\, dy \Rightarrow \int_{y=0}^{y=C} \frac{1}{2} y^2 [B^2] \, dy \]

\[ \Rightarrow \frac{1}{2} [B^2] \frac{1}{3} y^3 \bigg|_{y=0}^{y=C} = \frac{1}{6} [B^2] [C^3] \]

\[ \Rightarrow \hat{x} = \int \int \frac{xyf(x,y)\,dA}{M} \Rightarrow \int_{x=1}^{x=2} \int_{y=0}^{y=C} xy \, dx\, dy \Rightarrow \int_{x=1}^{x=2} x(4-x^2) + \frac{1}{2} (4-x)^2 \, dx \]

\[ \Rightarrow \int_{x=1}^{x=2} 4x - x^3 + 8x - 4x^2 + \frac{1}{2} x^4 \, dx \Rightarrow \int_{x=1}^{x=2} x(4-x^2) + \frac{1}{2} (4-x)^2 \, dx \]

\[ \Rightarrow \int_{x=1}^{x=2} x^3 - 3x^2 + 8x - 4x^2 + \frac{1}{2} x^4 \, dx \Rightarrow \int_{x=1}^{x=2} 4x^2 - \frac{1}{4} x^4 + \frac{8}{2} x^2 - \frac{4}{3} x^3 + \frac{1}{2} \left(\frac{1}{5}\right) x^5 \bigg|_{x=1}^{x=2} = \frac{241}{60} \]

\((***)\) 21. An electrically charged plate has surface charge density \(\sigma(x, y) = x + y\). Compute the total charge of the plate in the plane region bounded by \(y = 4 - x^2\), \(y = 0\), \(x = 1\), \(x = 2\). Evaluate it first with respect to \(x\), then with respect to \(y\), and then interchange the order of integration and show that the two methods give the same solution.

**Solution:**

Total Charge:

\[ \int_{x=1}^{x=2} \int_{y=0}^{y=4-x^2} (x + y) \, dy\, dx \Rightarrow \int_{x=1}^{x=2} \int_{y=0}^{y=x} x + y \, dy \bigg|_{y=x}^{y=4-x^2} \, dx \Rightarrow \int_{x=1}^{x=2} x(4-x^2) + \frac{1}{2} (4-x)^2 \, dx \]

\[ \Rightarrow \int_{x=1}^{x=2} 4x - x^3 + 8x - 4x^2 + \frac{1}{2} x^4 \, dx \Rightarrow \int_{x=1}^{x=2} x(4-x^2) + \frac{1}{2} (4-x)^2 \, dx \]

As a result of switching the order of integration, we need to determine the new bounds.

\(y = 4 - x^2 \Rightarrow x = \pm \sqrt{4 - y}\) \Rightarrow Our region restricts us to only consider the positive value of \(x\), since \(x \geq 1\) as given in the question.

Total Charge:

\[ \int_{y=0}^{y=3} \int_{x=1}^{x=\sqrt{4-y}} (x + y) \, dx\, dy \Rightarrow \int_{y=0}^{y=3} \int_{x=1}^{x=\sqrt{4-y}} \left(\frac{1}{2} x^2 + xy\right) \bigg|_{x=1}^{x=\sqrt{4-y}} \, dy \]

\[ \Rightarrow \int_{y=0}^{y=3} \frac{1}{2} \left(\sqrt{4-y} - 1\right)^2 + \left(\sqrt{4-y} - 1\right) \, dy \Rightarrow \int_{y=0}^{y=3} \frac{3}{2} y \, dy + \int_{y=0}^{y=3} y \sqrt{4-y} \, dy \]

\[ \Rightarrow \frac{3}{2} y^2 - \frac{3}{4} y^2 \bigg|_{y=0}^{y=3} + \int_{y=0}^{y=3} y \sqrt{4-y} \, dy \Rightarrow \frac{9}{2} - \frac{27}{4} + \int_{y=0}^{y=3} y \sqrt{4-y} \, dy \]

Use u substitution to solve the final integral:

Let \(u = 4 - y \Rightarrow y = 4 - u\)

\[ du = -dy \]

\[ -du = dy \]
\[
\int_{y=0}^{y=3} y \sqrt{4 - y} \, dy = \int_{y=0}^{y=3} (u - 4) \sqrt{u} \, du = \int_{y=0}^{y=3} u^{3/2} - 4u^{1/2} \, du
\]
\[
\Rightarrow \frac{2}{5} u^{5/2} - 4\left(\frac{2}{3}\right) u^{3/2} \bigg|_{y=0}^{y=3} \Rightarrow -\frac{34}{15} - \left(-\frac{128}{15}\right) \Rightarrow \frac{94}{15}
\]

Final Answer:
\[
\frac{9}{2} - \frac{27}{4} + \int_{y=0}^{y=3} y \sqrt{4 - y} \, dy \Rightarrow \frac{9}{2} - \frac{27}{4} + \frac{94}{15} \Rightarrow \frac{241}{60}
\]

22. Set up but Do Not Evaluate the integrals that give the center of mass of the region bounded by \(x^2/3 + y^2/3 = 1, x = 0, y = 0\) in the first quadrant, with density \(\delta(x, y) = x^2/(y + 1)\).

**Solution:** General form of a circle: \(x^2 + y^2 = r^2\).

Rewrite \(\frac{x^2}{3} + \frac{y^2}{3} = 1 \Rightarrow x^2 + y^2 = 3 \Rightarrow \text{Circle with radius } \sqrt{3}\)

Change to polar coordinates:

Since we are restricting the region to the first quadrant, \(\theta = 0\) to \(\theta = \frac{\pi}{2}\).

\(x^2 + y^2 = 3\)

\(r^2 \cos^2 \theta + r^2 \sin^2 \theta = 3\)

\(r^2 (\cos^2 \theta + \sin^2 \theta) = 3\)

\(r^2 (1) = 3\)

\(r = \pm \sqrt{3}, \text{ but } r \geq 0 \Rightarrow r = \sqrt{3}\)

\[
\int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=\sqrt{3}} r^2 \cos^2 \theta \sin \theta + 1 \, r \, dr \, d\theta = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=\sqrt{3}} \frac{r^3 \cos^2 \theta}{r \sin \theta + 1} \, dr \, d\theta
\]

(∗∗) 23. Evaluate the iterated integral: \(\int_0^1 \int_x^{x^2} \frac{1}{\sqrt{x^2 + y^2}} \, dy \, dx\)

**Solution:** Our first step is to convert the bounds from Cartesian to Polar Coordinates. We do this by graphing \(y = x\) and \(y = x^2\). We can see that \(\theta\) varies from 0 to \(\frac{\pi}{4}\) and \(r\) varies from the curve \(y = x^2\) to the origin.

To determine the bounds for the curve \(y = x^2\), we will rewrite the equation as follows:
\[ y = x^2 \]
\[ r \sin \theta = r^2 \cos^2 \theta \]
\[ r = \tan \theta \sec \theta \]

\[
\int_{x=0}^{x=1} \int_{y=x}^{y=x^2} \frac{1}{\sqrt{x^2 + y^2}} \, dy \, dx = \int_{\theta=0}^{\theta=\pi/4} \int_{r=0}^{r=\tan \theta \sec \theta} \frac{1}{\sqrt{r^2}} \, r \, dr \, d\theta
\]
\[
= \int_{\theta=0}^{\theta=\pi/4} \int_{r=0}^{r=\tan \theta \sec \theta} 1 \, r \, dr \, d\theta
\]
\[
= \int_{\theta=0}^{\theta=\pi/4} \tan \theta \sec \theta \, d\theta
\]
\[
= \sec \theta \bigg|_{\theta=0}^{\theta=\pi/4}
\]
\[
= \sqrt{2} - 1
\]

(*) 24. Compute \( dz/dx \) and \( dz/dy \) given \( z = f(x, y) = x^2 + 2xy + 4y^2 \), \( y = e^{ax} \).

**Solution:**

Compute: \( \frac{\partial z}{\partial x} \)

\[ z = f(x, y) = x^2 + 2xy + 4y^2 \quad \text{where} \quad y = e^{ax} \]

\[ \Rightarrow x^2 + 2x(e^{ax}) + 4(e^{2ax}) \]

\[ \Rightarrow \frac{\partial z}{\partial x} = 2x + 2xe^{ax} + 2e^{ax} + 4(2a)e^{2ax} \Rightarrow 2x + 2xe^{ax} + 2e^{ax} + 8ae^{2ax} \]

Compute: \( \frac{\partial z}{\partial y} \)

\[ z = f(x, y) = x^2 + 2xy + 4y^2 \quad \text{where} \quad y = e^{ax} \]

\[ \Rightarrow \frac{\partial z}{\partial y} = 2x + 8y \]

\[ \Rightarrow 2x + 8(e^{ax}) \]
25. Compute all first and second partial derivatives of \( z = x^2 + xy + y^2 \) with respect to \( r \) and \( s \). \( x = 2r + s \) and \( y = r - 2s \).

Solution:
There are two solutions to this problem. We will show the one that is shorter in this case, but in a general case, is the wrong choice to use. In the general case, one must use the chain rule on partial derivatives.

Plug \( x \)'s and \( y \)'s dependence on \( r, s \) into \( z \).

\[
z = (2r + s)^2 + (2r + s)(r - 2s) + (r - 2s)^2 = 7r^2 + 3s^2 - 3rs.
\]

\[
z_r = 14r - 3s, \quad z_s = 6s - 3r, \quad z_{rs} = -3 = z_{sr}, \quad z_{rr} = 14, \quad z_{ss} = 6.
\]

26. Find expressions for partial derivatives of the following functions:
   i) \( F(x, y) = f(g(x)k(y), g(x) + h(y)) \)
   ii) \( F(x, y, z) = f(g(x + y), h(y + z)) \)
   iii) \( F(x, y, z) = f(x^y, y^z, z^x) \)
   iv) \( F(x, y) = f(x, g(x), h(x, y)) \)

Solution: i)

\[
\frac{\partial F}{\partial x} = (D_1 f) \cdot g'(x)k(y) + (D_2 f) \cdot g'(x)
\]

\[
\frac{\partial F}{\partial y} = (D_1 f) \cdot k'(y) + (D_2 f) \cdot h'(y)
\]

ii)

\[
\frac{\partial F}{\partial x} = (D_1 f) \cdot g'(x + y)
\]

\[
\frac{\partial F}{\partial y} = (D_1 f) \cdot g'(x + y) + (D_2 f) \cdot h'(y + z)
\]

\[
\frac{\partial F}{\partial z} = (D_2 f) \cdot h'(y + z)
\]

iii)

\[
\frac{\partial F}{\partial x} = (D_1 f) \cdot yx^{y-1} + (D_3 f) \cdot z^x \log z
\]

\[
\frac{\partial F}{\partial y} = (D_1 f) \cdot x^y \log x + (D_2 f) \cdot yz^{z-1}
\]

\[
\frac{\partial F}{\partial z} = (D_2 f) \cdot y^z \log y + (D_3 f) \cdot xz^{x-1}
\]

iv)

\[
\frac{\partial F}{\partial x} = (D_1 f) + (D_2 f) \cdot g'(x) + (D_3 f) \cdot \frac{\partial h}{\partial x}
\]
27. Find and classify any maxima and minima of the surface \( z = (5x + 7y - 25)e^{-(x^2 + xy + y^2)} \)

**Solution:** First set the first partial derivatives to zero to obtain the critical points.

\[
\begin{align*}
z & = (5x + 7y - 25)e^{-(x^2 + xy + y^2)} \quad \text{[Apply Product and Chain Rule]} \\
f_x & = 5e^{-(x^2 + xy + y^2)} + (5x + 7y - 25)e^{-(x^2 + xy + y^2)}(-2x - y) \\
0 & = 5e^{-(x^2 + xy + y^2)} + (5x + 7y - 25)e^{-(x^2 + xy + y^2)}(-2x - y) \\
(5x + 7y - 25)e^{-(x^2 + xy + y^2)}(2x + y) & = 5e^{-(x^2 + xy + y^2)} \\
(5x + 7y - 25)(2x + y) & = 5 \\
(5x + 7y - 25) & = \frac{5}{2x + y}
\end{align*}
\]

Leave the answer like this for now, as we will have another factor of \((5x + 7y - 25)\) when we compute \(f_y\). You don’t need to simplify it even more, as we can do a substitution into the \((5x + 7y - 25)\) term directly.

\[
\begin{align*}
z & = (5x + 7y - 25)e^{-(x^2 + xy + y^2)} \quad \text{[Apply Product and Chain Rule]} \\
f_y & = 7e^{-(x^2 + xy + y^2)} + (5x + 7y - 25)e^{-(x^2 + xy + y^2)}(-x - 2y) \\
0 & = 7e^{-(x^2 + xy + y^2)} + (5x + 7y - 25)e^{-(x^2 + xy + y^2)}(-x - 2y) \\
(5x + 7y - 25)e^{-(x^2 + xy + y^2)}(x + 2y) & = 7e^{-(x^2 + xy + y^2)} \\
(5x + 7y - 25)(x + 2y) & = 7
\end{align*}
\]

This is where we will perform our substitution \((5x + 7y - 25) = \frac{5}{2x + y}\) into the above equation.

\[
\begin{align*}
(x + 2y)(5x + 7y - 25) & = 7 \\
(x + 2y) \left( \frac{5}{2x + y} \right) & = 7 \quad \text{[Substitution was done here]} \\
\frac{5x + 10y}{2x + y} & = 7 \\
5x + 10y & = 7(2x + y) \\
3y & = 9x \\
y & = 3x
\end{align*}
\]
Substitute \( y = 3x \) into either \((5x + 7y - 25)(2x + y) = 5\) or \((5x + 7y - 25)(x + 2y) = 7\); both will give you the same results.

Substituting \( y = 3x \)

\[
(5x + 7y - 25)(2x + y) = 5
\]

\[
(5x + 7(3x) - 25)(x + 2(3x)) = 7
\]

\[
7x(26x - 25) = 7
\]

\[
26x^2 - 25x - 1 = 0
\]

Use the quadratic equation \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)

\[
x = \frac{-(-25) \pm \sqrt{(-25)^2 - 4(26)(-1)}}{2(26)} \Rightarrow \frac{25 \pm 27}{52} \Rightarrow x = 1 \text{ or } x = -\frac{1}{26}
\]

Since \( y = 3x \), our critical points are \((1, 3)\) and \((-\frac{1}{26}, -\frac{3}{25})\). Finally, we want to substitute our points into our original equation to determine the local minimum and maximum.

\[
z(1, 3) = (5x + 7y - 25)e^{-(x^2+xy+y^2)} \Rightarrow \frac{5x + 7y - 25}{e^{x^2+xy+y^2}} \Rightarrow \frac{1}{e^{13}} \approx 2.26 \cdot 10^{-6} \text{ [Local Maximum]}
\]

\[
z(-\frac{1}{26}, -\frac{3}{26}) = (5x + 7y - 25)e^{-(x^2+xy+y^2)} \Rightarrow \frac{5x + 7y - 25}{e^{x^2+xy+y^2}} \Rightarrow -\frac{26}{e^{12}} \Rightarrow \approx -25.505 \text{ [Local Minimum]}
\]

(\(\ast\ast\)) 28. For the following iterated integral, sketch the region of integration, then interchange the order of integration. Then, express the integral in polar coordinates.

\[
\int_{-6}^{2} \int_{(x^2-4)/4}^{2-x} f(x, y) dy dx
\]
Solution:

Integral with order of integration switched: We will need to express this as the sum of two double integrals.

\[
\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y) dy dx + \int_{0}^{2} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} f(x, y) dy dx
\]

Integral in polar coordinates:

\[
\int_{0}^{\pi} \int_{\frac{2}{\sin\theta + \cos\theta}}^{\frac{4}{\cos\theta - \sin\theta}} f(r \cos\theta, r \sin\theta) r dr d\theta + \int_{\pi - \arctan(4/3)}^{2\pi} \int_{0}^{\frac{4}{\cos\theta - \sin\theta}} f(r \cos\theta, r \sin\theta) r dr d\theta
\]

(*** ) 29. For the following iterated integration, sketch the region of integration, then interchange the order of integration. Then, express the integral in polar coordinates. Do not evaluate the integral.

\[
\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{1-x^2} \sec^2(x^2 + y^2) dy dx
\]
Interchange of order of integrals: We will have a sum of two other double integrals.

\[
\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sec^2(x^2 + y^2) dx dy + \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sec^2(x^2 + y^2) dx dy
\]

Integral in polar coordinates:

\[
\int_{\pi}^{0} \int_{0}^{1} \frac{1}{\sin \theta + \cos \theta} \sec^2(r^2) r dr d\theta + \int_{2\pi}^{\pi} \int_{0}^{1} \sec^2(r^2) r dr d\theta
\]

\((***)\) 30. When a double integral was set up for the volume \(V\) of the solid under the paraboloid \(z = x^2 + y^2\), and above a region \(S\) of the \(xy\) - plane, the following sum of iterated integrals was obtained:

\[
V = \int_{0}^{1} \int_{0}^{y} (x^2 + y^2) dx dy + \int_{1}^{2} \int_{0}^{2-y} (x^2 + y^2) dx dy.
\]

Sketch the region \(S\) and express \(V\) as an iterated integral in which the order of integration is reversed. Also, carry out the integration and compute \(V\).
Solution:

Switching the order of integration,

\[
\int_0^1 \int_x^{2-x} (x^2 + y^2) dy \, dx
\]

The integral is just polynomials, so we leave it to the reader to show that it evaluates to 4/3.