Attempt questions to the best of your ability. This review package consists of 19 pages, including 1 cover page and 43 questions. The questions are meant to be the level of a real examination or slightly above, in order to prepare you for the real exam. Material from lectures and from the relevant textbook sections is examinable, and the problems for this package were chosen with that in mind, as well as considerations based on past examination question difficulty and style. Problems are ranked in difficulty as (*) for easy, (**) for medium, and (***) for difficult. Note that sometimes difficulty can be subjective, so do not be discouraged if you are stuck on a (*) problem.

The solutions to these problems will be posted at the following web address: https://ubcengineers.ca/services/academic/tutoring. If you believe that there is an error in these solutions, or have any questions, comments, or suggestions regarding EUS Tutoring sessions, please e-mail us at: tutoring@ubcengineers.ca. If you are interested in helping with EUS tutoring sessions in the future or other academic events run by the EUS, please e-mail vpcademic@ubcengineers.ca.

Some of the problems in this package were not created by the EUS. Those problems originated from one of the following sources:

- Schuam’s Outline of Calculus 2 ed; Ayres Jr., Frank
- Calculus – Early Transcendentals 7 ed; Stewart, James
- Calculus – 3 ed; Spivak, Michael
- Calculus Volume 1 2 ed; Apostol, Tom

Note on notation: Whenever log(x) is used without a subscript to indicate the base, it is assumed to be base e in math courses. Thus in this review package, log(x) and ln(x) are used interchangeably. For inverse trigonometric functions, sin⁻¹(x) = arcsin(x), and the other inverse trigonometric functions are similarly denoted.

All solutions prepared by the EUS.

Good Luck!
1. Show that the vectors $A = 3i - 2j + k$, $B = i - 3j + 5k$, $C = 2i + j - 4k$ form a right triangle.

**Solution:**

\[ A \cdot B = (3, -2, 1) \cdot (1, -3, 5) = 14 \]
\[ A \cdot C = (3, -2, 1) \cdot (2, 1, -4) = 0 \]
\[ B \cdot C = (1, -3, 5) \cdot (2, 1, -4) = -21 \]

Dot product is zero means that the 2 vectors form a right angle. It is shown that the three vectors form 1 right angle and 2 non-right angles, thus forming a right triangle.

2. Find the constant $a$ such that the vectors $2i - j + k$, $i + 2j - 3k$, and $3i + aj + 5k$ are coplanar.

**Solution:** Cross product the first two vectors to find the vector perpendicular to both of them.

\[ (2, -1, 1) \times (1, 2, -3) = (1, 7, 5) \]

For the third vector to be coplanar to the first two, it must also be perpendicular to the vector $(1, 7, 5)$. The dot product of them must be 0. $\langle 1, 7, 5 \rangle \cdot (3, a, 5) = 0$, solving and get $a = -4$.

3. Find the area of the parallelogram having diagonals $A = 3i + J - 2k$ and $B = i - 3j + 4k$.

**Solution:**

First, find the sides of the parallelogram. Let the sides be $x$, and $y$.

\[ 2y = B - A, \text{ and } 2x = A + B. \]

Then $x \times y = -i - 7j - 3k$

\[ |x \times y| = 5\sqrt{3}. \]

4. Compute the volume of the parallelepiped whose edges are represented by $A = 2i - 3j + 4k$, $B = i + 2j - k$, $C = 3i - j + 2k$.

**Solution:**

Volume of parallelepiped is the scalar triple product.

\[
\begin{vmatrix}
2 & -3 & 4 \\
1 & 2 & -1 \\
3 & -1 & 2 \\
\end{vmatrix} = 2 \begin{vmatrix} 2 & -1 \\ 2 & 3 \end{vmatrix} + 3 \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} + 4 \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 2(3) + 3(5) + 4(-7) = -7
\]

Volume is 7.

5. Show that the sum of the intercepts of the tangent plane of the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ at any of its points is $a$.
Solution:

\[ f_x = \frac{1}{2\sqrt{x}} \]
\[ f_y = \frac{1}{2\sqrt{y}} \]
\[ f_z = \frac{1}{2\sqrt{z}} \]

Tangent plane at any point \((x_o, y_o, z_o)\) is

\[
\frac{1}{2\sqrt{x_o}}(x - x_o) + \frac{1}{2\sqrt{y_o}}(y - y_o) + \frac{1}{2\sqrt{z_o}}(z - z_o) = 0
\]

x-intercept of this plane: \(\frac{y_o}{2\sqrt{y_o}} - \frac{z_o}{2\sqrt{z_o}} = 0\)

\[ x = \left(\frac{y_o}{2\sqrt{y_o}} + \frac{z_o}{2\sqrt{z_o}}\right) \cdot 2\sqrt{x_o} + x_o \]

y-intercept of this plane: \(\frac{x_o}{2\sqrt{x_o}} - \frac{z_o}{2\sqrt{z_o}} = 0\)

\[ y = \left(\frac{x_o}{2\sqrt{x_o}} + \frac{z_o}{2\sqrt{z_o}}\right) \cdot 2\sqrt{y_o} + y_o \]

x-intercept of this plane: \(\frac{y_o}{2\sqrt{y_o}} - \frac{x_o}{2\sqrt{x_o}} = 0\)

\[ z = \left(\frac{y_o}{2\sqrt{y_o}} + \frac{x_o}{2\sqrt{x_o}}\right) \cdot 2\sqrt{z_o} + z_o \]

Adding them yields:

\[
2\sqrt{x_o}\sqrt{y_o} + 2\sqrt{y_o}\sqrt{z_o} + 2\sqrt{x_o}\sqrt{z_o} + x_o + y_o + z_o = (\sqrt{x_o} + \sqrt{y_o} + \sqrt{z_o})^2 = a
\]

(*) 6. Examine the following function for relative maximum and minimum values. \(z = 3x - 3y - 2x^3 - xy^2 + 2x^2y + y^3\).

Solution:

\[ f_x = 3 - 6x^2 - y^2 + 4xy = 0, \]
\[ 3 + 4xy = 6x^2 + y^2 \]

\[ f_y = -3 - 2xy + 2x^2 + 3y^2 = 0, \]
\[ 3 + 2xy = 2x^2 + 3y^2 \]

Solving the system yields \(y = \pm 1, x = \pm 4/3\).

(*) 7. Find the area of the portion of the sphere \(x^2 + y^2 + z^2 = 25\) within the elliptic cylinder \(2x^2 + y^2 = 25\).
Solution:

\[ SA = \int \int_A \sqrt{f_x^2 + f_y^2 + 1} \, dA \]

\[ z = \sqrt{25 - x^2 - y^2} \]

\[ f_x = \frac{-x}{\sqrt{25 - x^2 - y^2}} \]

\[ f_y = \frac{-y}{\sqrt{25 - x^2 - y^2}} \]

The bound of integration on the xy plane is \( 2x^2 + y^2 = 25 \). So the integral for the surface area is

\[ \int_{x=-5}^{5} \int_{y=-\sqrt{25-2x^2}}^{\sqrt{25-2x^2}} \sqrt{1 + \frac{x^2}{25 - x^2 - y^2} + \frac{y^2}{25 - x^2 - y^2}} \, dy \, dx = 5 \int_{x=-5}^{5} \int_{y=-\sqrt{25-2x^2}}^{\sqrt{25-2x^2}} \sqrt{\frac{1}{25 - x^2 - y^2}} \, dy \, dx \]

Changing to polar coordinates we obtain

\[ 5 \int_{\theta=0}^{2\pi} \int_{\rho=0}^{\sqrt{\frac{5}{\cos^2 \theta + 1}}} \rho \sqrt{\frac{1}{25 - \rho^2}} \, d\rho \, d\theta \]

\[ 5 \int_{\theta=0}^{2\pi} \left[ \sqrt{\frac{5}{\cos^2 \theta + 1}} \right]_0^5 \, d\theta = 50 \pi \]

However, that is the surface area in the top portion, so we multiply by 2 to find the total surface area of \( 100 \pi \).

(*) 8. Compute the surface area of a hemisphere of radius \( a \) cut off by a cylinder having this radius as its diameter.

Solution: Use the formula for surface area

\[ f(x, y) = \sqrt{a^2 - (x^2 + y^2)}, \quad f_x = \frac{-x}{\sqrt{a^2 - (x^2 + y^2)}}, \quad f_y = \frac{-y}{\sqrt{a^2 - (x^2 + y^2)}}, \]

\[ f_x^2 = \frac{x^2}{a^2 - (x^2 + y^2)} \quad f_y^2 = \frac{y^2}{a^2 - (x^2 + y^2)} \]

Integral is

\[ \int \int_A \sqrt{\frac{x^2 + y^2 + a^2 - (x^2 + y^2)}{a^2 - (x^2 + y^2)}} \, dA \]

Change to polar coordinate

\[ \int_{\theta=0}^{2\pi} \int_{\rho=0}^{a/\sqrt{a^2 - \rho^2}} \frac{a \rho}{\sqrt{a^2 - \rho^2}} (d\rho)(d\theta) \]
\[ = (2\pi) \left( -\frac{a^2\sqrt{3}}{2} + a^2 \right) = a^2\pi(-\sqrt{3} + 2) \]

9. Compute the triple integral of \( f(\rho, \theta, z) \) over the region \( R \) bounded by the paraboloid \( \rho^2 = 9 - z \) and the plane \( z = 0 \).

**Solution:** Convert the equation to Cartesian Coordinate to calculate the bound of integration.

\[ z = 9 - (x^2 + y^2) \]

When \( z \) is zero, it is \( x^2 + y^2 = 9 \), this is the bound of integration.

\[
\int_{\theta=0}^{2\pi} \int_{\rho=0}^{3} \int_{z=0}^{9-\rho^2} \rho (9 - \rho^2) (d\rho)(d\theta) = (2\pi) \left( \frac{9}{2} \rho^2 - \frac{1}{4} \rho^4 \right) \bigg|_{0}^{3} = \frac{81\pi}{2}
\]

10) Evaluate \( \iiint_{R} \frac{dxdydz}{(x^2 + y^2 + z^2)^{3/2}} \), where \( R \) is the region bounded by the spheres \( x^2 + y^2 + z^2 = a^2 \) and \( x^2 + y^2 + z^2 = b^2 \), where \( a > b > 0 \).

**Solution**

Use Spherical Coordinate.

\[
\int_{\phi=0}^{\pi} \int_{\theta=0}^{\pi} \int_{r=b}^{a} r^2 \sin \phi (dr)(d\theta)(d\phi) = (2\pi) \int_{\phi=0}^{\pi} \sin \phi (d\phi)(\ln \frac{a}{b}) = 4\pi(\ln \frac{a}{b})
\]

(*) 10) Evaluate \( \iiint_{R} \frac{dxdydz}{(x^2 + y^2 + z^2)^{3/2}} \), where \( R \) is the region bounded by the spheres \( x^2 + y^2 + z^2 = a^2 \) and \( x^2 + y^2 + z^2 = b^2 \), where \( a > b > 0 \).

**Solution:** Use Spherical Coordinate.

\[
\int_{\phi=0}^{\pi} \int_{\theta=0}^{\pi} \int_{r=b}^{a} r^2 \sin \phi (dr)(d\theta)(d\phi) = (2\pi) \int_{\phi=0}^{\pi} \sin \phi (d\phi)(\ln \frac{a}{b}) = 4\pi(\ln \frac{a}{b})
\]

(*) 11. Compute the volume of the region bounded above by the sphere \( \rho = 2a \cos \phi \) and below by the cone \( \phi = \alpha \) where \( 0 < \alpha < \pi/2 \). Leave your answer in terms of \( \alpha \).

**Solution:**

\[
\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\alpha} \int_{\rho=0}^{2a \cos \phi} \rho^2 \sin \phi (d\rho)(d\phi)(d\theta)
\]
\[
\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\alpha} 8 \sin^3 \phi \cos \phi \, d\phi \, d\theta = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\alpha} 3 \sin \phi \cos \phi \, d\phi \, d\theta
\]

\[
\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\alpha} -2 \cos^4 \phi \, d\phi \, d\theta = \frac{4\pi}{3} \left(1 - \cos^4 \alpha\right)
\]

(∗) 12. Compute the triple integral of \(F(x, y, z) = z\) over the region \(R\) in the first octant bounded below by the planes \(y = 0, z = 0, x + y = 2, 2y + x = 6,\) and the cylinder \(y^2 + z^2 = 4.\)

**Solution:**

\[
\int_{y=0}^{2} \int_{x=2-y}^{6-2y} \int_{z=0}^{\sqrt{4-y^2}} z \, dz \, dx \, dy = \int_{y=0}^{2} \int_{x=2-y}^{6-2y} \frac{1}{2} z^2 \left|_{0}^{\sqrt{4-y^2}} \right. \, dx \, dy
\]

\[
= \int_{y=0}^{2} \int_{x=2-y}^{6-2y} \frac{1}{2} \left(4 - y^2\right) \, dx \, dy
\]

\[
= \int_{y=0}^{2} \frac{4 - y^2}{2} \left|_{2-2y}^{6-2y} \right. \, dy
\]

\[
= \frac{1}{2} \int_{y=0}^{2} 4 - y^2 \left(4 - y\right) \, dy
\]

\[
= \frac{1}{2} \left[16y - 2y^3 - 4/3y^3 + 1/4y^4\right]_0^2
\]

\[
= \frac{1}{2} \left(32 - 8 - 32/3 + 4\right)
\]

\[
= \frac{26}{3}
\]

(∗) 13. For the following iterated integral, sketch the region of integration, then interchange the order of integration. \(\int_{0}^{4} \int_{-(y-4)/2}^{(y-4)/2} f(x, y) \, dx \, dy\)

**Solution:** \(x = -\sqrt{4-y}, x^2 = 4 - y, y = 4 - x^2\) \(x = \frac{y-4}{2}, 2x = y - 4, 2x + 4 = y\)

\[
\int_{x=-2}^{0} \int_{y=2x+4}^{4-x^2} f(x, y) \, dy \, dx
\]
14. Find an equation of the tangent plane to the surface \( f(x, y) = \sin(x - y) + \sqrt{x^2 + 3y^2} \) at the point \((2, 2)\).

**Solution:**

\[
\begin{align*}
    f_x &= \cos(x - y) + \frac{1}{2}(x^2 + 3y^2)^{-1/2} - \frac{1}{2}(2x) \\
    	ext{at } (2, 2), \text{ it is } 1 + 1/2 &= \frac{3}{2} \\
    f_y &= -\cos(x - y) + \frac{1}{2}(x^2 + 3y^2)^{-1/2}(6y) \\
    	ext{at } (2, 2), \text{ it is } -1 + 3/2 &= \frac{1}{2} \\
    f(2, 2) &= 4 \\
    z &= 4 + \frac{3}{2}(x - 2) + \frac{1}{2}(y - 2)
\end{align*}
\]

15. When a double integral was set up for the volume \( V \) of the solid under the surface \( z = f(x, y) \) and above a region \( S \) of the \( xy \)-plane, the following sum of iterated integrals was obtained:

\[
V = \int_{\alpha \sin c}^{\beta \sin c} \int_{\sqrt{a^2 - y^2}}^{\sqrt{b^2 - y^2}} f(x, y) \, dx \, dy + \int_{\alpha \sin c}^{\beta \sin c} \int_{\sqrt{y \cot c}}^{\sqrt{b^2 - y^2}} f(x, y) \, dx \, dy
\]

Given that \( 0 < a < b \) and \( 0 < c < \pi/2 \), sketch the region \( S \), giving the equations of all curves which form its boundary.
16. \( \int_0^1 e^{-t^2} \, dt \), and \( B = \int_0^{1/2} e^{-t^2} \, dt \). Evaluate the iterated integral

\[
I = 2 \int_{-1/2}^1 \int_0^x e^{-y^2} \, dy \, dx
\]
in terms of \( A \) and \( B \).

---

**Solution:** Interchange the order of integration, so

\[
I = 2 \int_{-1/2}^1 \int_0^x e^{-y^2} \, dy \, dx = 2 \left( - \int_{-1/2}^0 \int_{-1/2}^y e^{-y^2} \, dx \, dy + \int_0^1 \int_y^1 e^{-y^2} \, dx \, dy \right)
\]

\[
= -2 \int_{-1/2}^0 (y + 1/2)e^{-y^2} \, dy + 2 \int_0^1 (1 - y)e^{-y^2} \, dy
\]

\[
= 2A - B + e^{-1} - e^{-1/4}
\]

17. Given \( a > 0 \), transform the integral to polar coordinates and compute its value.

\[
\int_0^a \int_0^{\sqrt{a^2 - y^2}} x^2 + y^2 \, dx \, dy
\]

---

**Solution:**

\[
\int_{\theta=0}^{\pi/2} \int_{\rho=0}^{a} (\rho)^3 (d\rho) (d\theta) = \left( \frac{1}{4} \rho^4 \right) |_{\rho=0}^{\rho=a} \frac{\pi}{2} = \frac{a^4 \pi}{8}
\]
18. For the three iterated integrals, describe the region of integration $S$ by means of a sketch, showing its projection onto the $xy$-plane. Then express the triple integral as one or more iterated integrals in which the first integration is with respect to $y$.

Solution: i)

$$
\int_0^1 \int_0^{2-2x} \int_{z-x}^{1-x} f(x,y,z) \, dy \, dz \, dx
$$

ii)

$$
\int_0^2 \int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y,z) \, dy \, dx \, dz - \int_0^2 \int_0^z \int_0^{z-x^2} f(x,y,z) \, dy \, dx \, dz
$$

19. Find and classify all critical points of $f(x,y) = xy - x^4 - y^2 + 2$

Solution: Find the partial derivatives and equate them to 0.

$$
f_x = y - 4x^3, \, f_y = x - 2y, \, 0 = f_x = x - 2y, \, 0 = y - 4x^3
$$

Need to solve for $x$ and $y$. $2y = x$ and $2y = 8x^3$, so $8x^3 = x$, $x = 0$ and $x = \frac{1}{2\sqrt{2}}$. Substitute back to
get y values. Found 3 pairs:

- \((0, 0)\)
- \(\left(\frac{1}{2\sqrt{2}}, \frac{1}{2^{5/2}}\right)\)
- \(\left(-\frac{1}{2\sqrt{2}}, -\frac{1}{2^{5/2}}\right)\)

Calculate the discriminant to see the maximum and minimum. \(f_{xx} = -12x^2, f_{yy} = -2, f_{xy} = 1, f_{yx} = 1\)

For point \((0, 0)\), \(D = \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -1\) since discriminant is less than 0, \((0, 0)\) is a saddle point.

For point \(\left(\frac{1}{2\sqrt{2}}, \frac{1}{2^{5/2}}\right)\), \(D = \begin{vmatrix} -3/2 & 1 \\ 1 & -2 \end{vmatrix} = 2\) since discriminant is bigger than 0 and \(f_{xx}\) is less than 0, \(\left(\frac{1}{2\sqrt{2}}, \frac{1}{2^{5/2}}\right)\) is a local maximum.

For point \(\left(-\frac{1}{2\sqrt{2}}, -\frac{1}{2^{5/2}}\right)\), \(D = \begin{vmatrix} -3/2 & 1 \\ 1 & -2 \end{vmatrix} = 2\) since discriminant is bigger than 0 and \(f_{xx}\) is less than 0, \(\left(-\frac{1}{2\sqrt{2}}, -\frac{1}{2^{5/2}}\right)\) is a local maximum.

(***) 20. Compute the mass \(M\) of the upper half of the annulus \(1 < x^2 + y^2 < 9\) with density \(\delta = y/(x^2 + y^2)\).

**Solution:** Mass is the double integral of density function over the region. Use polar coordinate.

\[
\int_{\theta=0}^{\pi} \int_{\rho=1}^{3} \frac{\rho^2 \sin \theta}{\rho^2} (d\rho)(d\theta) = (2)(-\cos \theta \bigg|_0^\pi) = 4
\]

() 21. Consider the surface \(S\) given by the equation \(z = (x^2 + y^2 + z^2)^2\).

a) Show that \(S\) lies in the upper half space \((z \geq 0)\). b) Write out the equation for the surface in spherical coordinates. c) Using the equation obtained in part b, give an iterated integral, with explicit integrand and limits of integration, which gives the volume of the region inside this surface. Do not evaluate the integral.

**Solution:** a) \(z = (x^2 + y^2 + z^2)^2\), square anything always gives positive, so \(z\) is always larger than 0.

b) \(z = r^4, r \cos \phi = r^4, \cos \phi = r^3\)

c)

\[
\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{r=0}^{(\cos \phi)^{1/3}} r^2 \sin \phi (dr)(d\phi)(d\theta)
\]

(**) 22. Let \(f(x, y) = xy^2 - x^3\). a) Find the gradient of \(f\) at \(P : (1, 1)\). b) Detemine an approximate formula showing how small changes \(\Delta x\) and \(\Delta y\) produce a small change \(\Delta u\) in the value of \(u = f(x, y)\) at the point \((x, y) = (2, 1)\).
Solution: a) \( f_x = y^2 - 3x^2 = -2, \ f_y = 2xy = 2 \)
Gradient is \((-2,2)\)

b) at \((2,1), f_x = -11, f_y = 4, \Delta u = -11\Delta x + 4\Delta y.\)

\((*)\) 23. Find the equation of the tangent plane to the surface \( x^4y + z - z^2x + 1 = 0 \) at the point \((-1,1,2)\).

Solution: \( f_x = 4x^3y - z^2 = -4 - 4 = -8, \ f_y = x^4 = 1, \ f_z = 1 - 2zx = 5. \)

\[ 0 = -8(x + 1) + (y - 1) + 5(z - 2) \]
\[ 19 = -8x + y + 5z \]

\((*)\) 24. Using Lagrange multipliers, maximize \( f(x, y) = xe^{x^2-y} \) subject to \( x^2 + 2y^2 = 4/3. \)

Solution:

\[
\nabla f = \lambda \nabla g \\
f_x = e^{x^2-y} + xe^{x^2-y}2x \\
f_y = -xe^{x^2-y} \\
g_x = 2x, \ g_y = 4y
\]

Thus \( e^{x^2-y} + xe^{x^2-y}2x = \lambda 2x, \) and \(-xe^{x^2-y} = \lambda 4y. \) Multiplying the second equation by 2x and adding gives \( e^{x^2-y} = 2x\lambda(1 + 4y), \) which means that \( 4y\lambda = -2x^2(1 + 4y)\lambda. \) Thus \( 4y = -2(4/3 - 2y^2)(1 + 4y), \) so we have \(-2y = 4/3 - 8y^3 - 2y^2 + 16y/3. \) Multiplying both sides by 3 gives \( 24y^3 + 6y^2 - 22y - 4 = 0 \Rightarrow 12y^3 + 3y^2 - 11y - 2 = 0. \) Since no one is evil enough to give this cubic and make it hard to factor, so by inspection we see that \( y = -1 \) solves it so \( y + 1 \) is a factor. Now we can rewrite \( 12y^3 + 3y^2 - 11y - 2 = (y + 1)(12y^2 - 9y - 2) = 0. \) The second factor gives \( y = -9 \pm \sqrt{177} \). To find which one is the max we plug back into the original expression.

Since we have the constraint we can rewrite \( f(x, y) = xe^{x^2-y} = \sqrt{4/3 - 2y^2}e^{4/3 - 2y^2-y}. \) It helps to estimate \( \sqrt{177} \approx 13 \) to determine which gives the maximum value. Doing the estimation, we find that \( y = \frac{9 - \sqrt{177}}{24} \) gives the maximum value, where \( x = \pm \sqrt{\frac{4}{3} - 2 \left( \frac{9 - \sqrt{177}}{24} \right)^2}. \) Thus there are two solutions.

\[
(x, y) = \left( \sqrt{\frac{4}{3} - 2 \left( \frac{9 - \sqrt{177}}{24} \right)^2}, \frac{9 - \sqrt{177}}{24} \right)
\]
\[
(x, y) = \left( -\sqrt{\frac{4}{3} - 2 \left( \frac{9 - \sqrt{177}}{24} \right)^2}, \frac{9 - \sqrt{177}}{24} \right)
\]

\((*)\) 25. Let \( x, y, z \in \mathbb{R} \) such that \( x^2 + y^2 + z^2 + xyz = 4. \) Find the minimal value of the expression \( x + y + z. \)
**Solution:** First we identify $g(x, y, z) = x^2 + y^2 + z^2 + xyz = 4$, and $f(x, y, z) = x + y + z$.

Applying Lagrange multipliers, $\nabla f = \lambda \nabla g \Rightarrow (1, 1, 1) = \lambda (2x + yz, 2y + xz, 2z + xy)$.

Starting with $1 = \lambda (2x + yz)$, multiply both sides by $x$ to obtain $x = \lambda (2x^2 + yz)$

Also with $1 = \lambda (2y + xz)$, multiply both sides by $y$ to obtain $y = \lambda (2y^2 + xz)$.

Subtracting these equations, $x - y = 2\lambda (x^2 - y^2) - 2\lambda (x - y)(x + y)$.

There are two options here; $x = y$, or $2\lambda(x + y) = 1$.

Also it is important to note that $\lambda$ cannot be $0$ because that would contradict the above equation.

**Case 1:** $(x = y)$

Then $(2z + x^2)\lambda = 1$, and $(2x + xz\lambda) = 1$. Multiplying the equations by $x$ and $z$ respectively, we obtain $(2z + x^3)\lambda = 1$, and $(2xz + x^2z\lambda) = 1$. Subtracting equations, we obtain $\lambda x(z^2 - x^2) = z - x$.

This gives us two sub-cases

**Sub-case 1-1: $(z = x)$**

Plugging this into the constraint, we obtain $x^2 + x^2 + x^2 + x^3 = 3x^2 + x^3 = 4 \Rightarrow x^3 + 3x^2 - 4 = 0$. It is clear that $x = 1$ is a solution to this equation. Then we can factor $x^3 + 3x^2 - 4 = (x - 1)(x^2 + 4x + 4)$.

So then we get $x = -2$.

Thus our first two solutions to the problem is $(x, y, z) = (1, 1, 1)$, $(x, y, z) = (-2, -2, -2)$

**Sub-case 1-2 $(z + x)\lambda x = 1$**

Since $1 = \lambda (2x + xz) \Rightarrow 1/\lambda = 2x + xz$, we can multiply the equations to obtain $x^2 + xz = 2x + xz \Rightarrow x^2 = 2x$, so $x = 2$ or $x = 0$, so then $y = 2$ or $y = 0$ as well. Plugging this $(x = 2)$ into the constraint gives $2^2 + 2^2 + z^2 + (2)(2)z = 4 \Rightarrow z^2 + 4z + 4 = 0$. So $z = -2$. If $x = 0$, then $z = \pm 2$. Thus we have three solutions from here, $(x, y, z) = (2, 2, -2)$, $(x, y, z) = (0, 0, -2)$, and $(x, y, z) = (0, 0, 2)$.

**HOWEVER,** all of these solutions are invalid because when you plug them into the original equations, you obtain 0 = 1. So then the only three valid solutions are from the first sub-case.

**Case 2:** $(2\lambda(x + y) = 1)$

Then $2x + 2y = 1/\lambda$, so we can equate this with $1/\lambda = 2x + yz$ to obtain $2x + yz = 2x + 2y$, and thus $y(2 - x) = 0$.

Then we have two sub-cases: $y = 0$, and $z = 2$.

**Sub-case 2-1 $(y = 0)$**

Plugging $y = 0$ into the gradient equation produces $x = 0$ or $z = 2$. This means that we do not have to check the second sub-case, and also this does not add to the solution count because this was already a known solution. **HOWEVER,** these known solutions are also invalid for the reasons described above.

Thus the extrema are at $(x, y, z) = (1, 1, 1)$ and $(x, y, z) = (-2, -2, -2)$. Plugging these into the original function, the maximum value is $f(1, 1, 1) = 1 + 1 + 1 = 3$, and the minimum value is $f(-2, -2, -2) = -2 - 2 - 2 = -8$.

26) Locate and classify extremal points (if any) of the surface having Cartesian equation given. $z = x - 2y + \log(\sqrt{x^2 + y^2}) + 3\arctan(y/x)$

(* * *) 26. Locate and classify extremal points (if any) of the surface having Cartesian equation given. $z = x - 2y + \log(\sqrt{x^2 + y^2}) + 3\arctan(y/x)$
Solution:

\[
\begin{align*}
    f_x &= 1 + \frac{2x}{\sqrt{x^2 + y^2} \sqrt{x^2 + y^2}} + \frac{3}{1 + \frac{y^2}{x^2}} \frac{-y}{x^2} = 1 + \frac{x}{x^2 + y^2} - \frac{3y}{x^2 + y^2} = \frac{x^2 + y^2 + x - 3y}{x^2 + y^2} = 0 \\
    f_y &= -2 + \frac{3}{2(x^2 + y^2)} \frac{2y}{x(1 + \frac{y^2}{x^2})} = -2 + \frac{y}{x^2 + y^2} + \frac{3x}{x^2 + y^2} = \frac{-2 + y + 3x}{x^2 + y^2} \\
    -2 + y + 3x &= 0, y = -3x + 2 \\
    x^2 + y^2 + x - 3y &= 0
\end{align*}
\]

plug \(y = -3x + 2\) into \(x^2 + y^2 + x - 3y = 0\)

\[
\begin{align*}
    0 &= x^2 + (-3x + 2)^2 + x - 3(-3x + 2) \\
    &= x^2 + 9x^2 - 12x + 4 + x + 9x - 6 \\
    &= 10x^2 - 2x - 2
\end{align*}
\]

There is no real roots to this equation, so there is no extremal point.

(\(*\)\) 27. Locate and classify the extremal points (if any) of the surface having Cartesian equation given. \(z = x^3 + y^3 - 3xy\)

Solution:

\[
\begin{align*}
    f_x &= 3x^2 - 3y = 0, x^2 = y \\
    f_y &= 3y^2 - 3x = 0, x = y^2
\end{align*}
\]

Solve for \(x\) and \(y\) and obtain \((0,0)\) and \((1,1)\).

\[
\begin{align*}
    f_{xx} &= 6x, f_{yx} = -3, f_{xy} = -3, f_{yy} = 6y
\end{align*}
\]

at \((0,0)\), \(f_{xx} = 0\) and discriminant \(D\) is \(-9\), so it is a saddle point.

at \((1,1)\), \(f_{xx} = 6\) and discriminant \(D\) is \(27\), so it is a minimum.

(\(\ast\ \ast\ \ast\)) 28. Locate and classify the extremal points (if any) of the surface having Cartesian equation given. \(z = e^{2x+3y}(8x^2 - 6xy + 3y^2)\)

Solution:

\[
\begin{align*}
    f_x &= 2e^{2x+3y}(16x - 6y) = 0 \\
    16x &= 6y \\
    \frac{8}{3}x &= y
\end{align*}
\]

\[
\begin{align*}
    f_x &= 3e^{2x+3y}(-6x + 6y) = 0 \\
    6x &= 6y \\
    x &= y
\end{align*}
\]
29. At Oktoberfest, someone is standing on the surface \( f(x, y) = x^2 - y^2 + 2 \) at the point \((0, 1)\). The individual spills a glass of beer such that, when the stain is projected to the \( xy \) plane it forms a circle of radius 1 (part of the stain travelled upward). How much surface area must be cleaned?

Solution:

\[
\int_{\theta=0}^{2\pi} \int_{\rho=0}^{1} \rho \sqrt{4\rho^2 + 1} (d\rho)(d\theta) \\
= (2\pi)(\frac{1}{12}(4\rho^2 + 1)^{3/2} |^{1}_{0}) \\
= (2\pi)(\frac{5^{3/2}}{12} - \frac{1}{12}) = \frac{\pi(5^{3/2}) - 1}{6}
\]

30. Match each contour map to its function.

- \( f(x, y) = \sin(x^2 - y) + y = E \)
- \( f(x, y) = x^2 + 2y^2 - x = F \)
- \( f(x, y) = \sqrt{y} \cos x = C \)
- \( f(x, y) = \frac{x^2+y^2+6x}{x^2+4} = B \)
- \( f(x, y) = |x + \log(y^2)| = D \)
- \( f(x, y) = x^{2/5} + y^{4/7} = G \)
- \( f(x, y) = (3x - y/2)(x + 4y) = A \)
- \( f(x, y) = x^3 - 6xy = H \)
31. Compute the first partial derivatives of the function \( f(x, y, z) = x^{y^z} \)

\[
\begin{align*}
  f_x &= y^z x^{y^z-1} \\
  f_y &= x^{y^z} \ln x (zy^{z-1}) \\
  f_z &= x^{y^z} \ln x (y^{z} \ln y)
\end{align*}
\]

32. Compute the first partial derivatives of the function \( f(x, y) = \int_a^y g(t) dt \)

\[ f_y = g \left( \int_b^y g(t) dt \right) \frac{d}{dy} \left( \int_b^y g(t) dt \right) \]

\[ f_y = g \left( \int_b^y g(t) dt \right) g(y) \]

\[ f_x = 0 \]

since the function doesn’t depend on \( x \).
33. \( \star \star \star \) Compute the first partial derivatives of the function \( f(x, y) = \int_0^y g(t) \, dt \).

**Solution:** Apply the Fundamental Theorem of Calculus.

\[
  f_x = g(x, y) \cdot y \\
  f_y = g(x, y) \cdot x
\]

34. \( \star \star \star \) Given \( u = x^2 + 2y^2 + 2z^2 \), and \( x = \rho \sin \beta \cos \theta \), \( y = \rho \sin \beta \sin \theta \), \( z = \rho \cos \beta \), compute \( \frac{\partial u}{\partial \rho}, \frac{\partial u}{\partial \beta}, \frac{\partial u}{\partial \theta} \).

**Solution:**

\[
  \frac{\partial u}{\partial \rho} = 2x(\sin \beta)(\cos \theta) + 4y(\sin \beta)(\sin \theta) + 4z(\cos \beta)
\]

\[
  \frac{\partial u}{\partial \beta} = 2x(\rho(\cos \beta)(\cos \theta)) + 4y(\rho(\sin \theta)(\cos \beta)) + 4z(-\rho(\sin \beta))
\]

\[
  \frac{\partial u}{\partial \theta} = -2x(\rho(\sin \beta)(\sin \theta)) + 4y(\rho(\sin \beta)(\cos \theta))
\]

35. \( \star \star \star \) Compute \( \frac{du}{dx} \) given \( u = f(x, y, z) = xy + yz + xz \), \( y = 1/x \), \( z = x^2 \).

**Solution:**

\[
  u = 1 + x + x^3 \\
  \frac{du}{dx} = 1 + 3x^2
\]

36. \( \star \star \star \) Use differentials to compute the diagonal of a rectangular box of dimensions 3.03 by 5.98 by 6.01 ft.

**Solution:** Define a function that calculates the diagonal.

\[
  d = \sqrt{l^2 + w^2 + h^2}
\]

Choose \( l_0 = 3, w_0 = 6, h_0 = 6 \)

\[
  \Delta d = d(l, w, h)\Delta l + d(w, l, h)\Delta w + d(h, l, w)\Delta h
\]
\[ d_l = \frac{l}{\sqrt{l^2 + w^2 + h^2}} \]
\[ d_w = \frac{w}{\sqrt{l^2 + w^2 + h^2}} \]
\[ d_h = \frac{h}{\sqrt{l^2 + w^2 + h^2}} \]

\[ \Delta d = \frac{3}{9}(3.03 - 3) + \frac{6}{9}(-0.02) + \frac{6}{9}(0.01) \approx 0.0033 \]

\[ d_o \text{ is 9 and } d_o + \Delta d = 9 + 0.0033 = 9.0033 \]

\((***)\) 37. Compute all first and second partial derivatives of \( z \), given \( x^2 + 2yz + 2xz = 1 \)

**Solution:**

**\( f_x \):**

\[ 2x + 2y \frac{\partial z}{\partial x} + 2(z + x \frac{\partial z}{\partial x}) = 0 \]

\[ \frac{\partial z}{\partial x} = \frac{x + z}{-y - x} \]

**\( f_y \):**

\[ 2(z + y \frac{\partial z}{\partial y}) + 2x \frac{\partial z}{\partial y} = 0 \]

\[ \frac{\partial z}{\partial y} = \frac{z}{-x - y} \]

**\( f_{xx} \):**

\[ f_{xx} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{x + z}{-y - x} \right) = \frac{(1 + \frac{\partial z}{\partial x})(-y - x) + (x + z)}{(-y - x)^2} \]

**\( f_{yy} \):**

\[ f_{yy} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{z}{-x - y} \right) = \frac{\partial z}{\partial y} \left( \frac{-x - y}{(-x - y)^2} \right) \]

\((***)\) 38. On a hill represented by \( z = 8 - 4x^2 - 2y^2 \), find

a) the direction of the steepest grade at (1,1,2)

b) the direction of the contour line

**Solution:**

a) \( f_x = -8x = -8 \), \( f_y = -4y = -4 \) Direction with steepest grade is \((-8, -4)\).

b) \((2, -4)\) because the direction of contour line is perpendicular to the gradient.

\((***)\) 39. Find the equation of the plane through \((2,-1,-1)\) and \((1,2,3)\) and perpendicular to \(2x + 3y - 5z - 6 = 0\).
**Solution:** Need two directions of this plane. One direction is obtained by subtracting the two vectors. The other direction is from the perpendicular plane (the perpendicular direction of the perpendicular plane).

\[
(2, -1, -1) - (1, 2, 3) = (1, -3, -4)
\]

\[
(2, 3, -5)
\]

Cross product these two vectors and obtain:

\[
\langle -27, 3, -9 \rangle
\]

Now we have the perpendicular direction of this plane and we choose a point. Plug in the equation of plane.

\[-27(x - 2) + 3(y + 1) - 9(z + 1) = 0\]

\[-27x + 3y - 9z = -48\]

(***) 40. Find the shortest distance between the line through A(2, -1, -1) and B(6, -8, 0) and the line through C(2, 1, 2) and D(0, 2, -1).

**Solution:** Direction of line AB is \(\langle 6, -8, 0 \rangle \) - \(\langle 2, -1, -1 \rangle = \langle 4, -7, 1 \rangle\).

Direction of line CD is \(\langle 0, 2, -1 \rangle - \langle 2, 1, 2 \rangle = \langle -2, 1, -3 \rangle\).

Write them in parametric form:

Line AB:

\[
x = 2 + 4t, y = -1 - 7t, z = -1 + t
\]

Line CD:

\[
x = -2s, y = 2 + s, z = -1 - 3s
\]

The vector of shortest distance between these two lines must be perpendicular to both lines. This vector has direction \((-2s - 4t - 2, 3 + s + 7t, -3s - t)\). In order for it to be perpendicular to line AB:

\[
\langle -2s - 4t - 2, 3 + s + 7t, -3s - t \rangle \cdot \langle 4, -7, 1 \rangle = 0
\]

\[-18s - 66t = 29\]

In order for it to be perpendicular to line CD:

\[
\langle -2s - 4t - 2, 3 + s + 7t, -3s - t \rangle \cdot \langle -2, 1, -3 \rangle = 0
\]

\[14s + 18t = -7\]

Solving these two equations and get \(t = -0.42\) and \(s = -1.04\).

This vector of shortest distance is \(\langle 1, 76, -0.98, 3.54 \rangle\) and the shortest distance is \(\approx 4.07\).

(***) 41. Find a pair of linear Cartesian equations for the line which is tangent to both the surfaces \(x^2 + y^2 + 2z^2 = 4\) and \(z = e^{x-y}\) at the point (1,1,1).
Solution: A line is the intersection between two planes. Since this line is tangent to both surfaces at the point (1,1,1), then the tangent planes of the surfaces will be the two planes that define this line.

\[ f(x, y, z) = x^2 + y^2 + 2z^2 - 4 \]

\[ f_x = 2x = 2, f_y = 2y = 2, f_z = 4z = 4 \] Tangent plane is \( 2x + 2y + 4z = 8 \).

\[ g(x, y, z) = z - e^{x-y} \]

\[ g_x = -e^{x-y}, g_y = e^{x-y} = 1, g_z = 1 \] Tangent plane is \( -x + y + z = 1 \). These two planes define this line.

42. \( \star \star \star \) The temperature at any point \((x, y)\) in the \(xy\)-plane is given by \( T = \frac{100xy}{x^2 + y^2} \).

a) Find the directional derivative at the point (2,1) in a direction making an angle of 60° with the positive \(x\)-axis.

b) In what direction from (2,1) would the derivative be a maximum?

c) What is the value of this maximum?

Solution:

\[ f_x = \frac{100y(x^2 + y^2) - 100xy(2x)}{(x^2 + y^2)^2} = \frac{-300}{25} = -12 \]

\[ f_y = \frac{100x(x^2 + y^2) - 100xy(2y)}{(x^2 + y^2)^2} = 24 \]

\[ \vec{u} = \left( \cos \frac{\pi}{3}, \sin \frac{\pi}{3} \right) = \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \]

\[ D_{\vec{u}} f(x, y) = (-12)(1/2) + (24)(\sqrt{3}/2) = -6 + 12\sqrt{3} \]

b) Maximum derivative is in the direction of gradient vector \( \langle f_x, f_y \rangle \), so it is \( \langle -12, 24 \rangle \).

c) Maximum value is the length of gradient vector

\[ \sqrt{(-12)^2 + 24^2} = 12\sqrt{5} \]

43. \( \star \star \star \) Find the value of \( a \) such that \( f(x, y, t) = e^{at} \sin x \sin y \) solves the two dimensional heat equation \( f_t = f_{xx} + f_{yy} \).

Solution:

\[ f_x = \cos(x) \sin(y)e^{at}, \quad f_{xx} = -\sin(x) \sin(y)e^{at} \]

\[ f_y = \cos(y) \sin(x)e^{at}, \quad f_{yy} = -\sin(y) \sin(x)e^{at} \]

\[ f_t = ae^{at} \sin(x) \sin(y) \]

\[ f_t = f_{xx} + f_{yy} \]

equating both sides:

\[ ae^{at} \sin(x) \sin(y) = -2\sin(y) \sin(x)e^{at} \]

Therefore \( a = -2 \).