Mathematics 253 Midterm 2 Review Package

UBC Engineering Undergraduate Society

Attempt questions to the best of your ability. This review package consists of 5 pages, including 1 cover page and 43 questions. The questions are meant to be the level of a real examination or slightly above, in order to prepare you for the real exam. Material from lectures and from the relevant textbook sections is examinable, and the problems for this package were chosen with that in mind, as well as considerations based on past examination question difficulty and style. Problems are ranked in difficulty as (•) for easy, (••) for medium, and (•••) for difficult. Note that sometimes difficulty can be subjective, so do not be discouraged if you are stuck on a (•) problem.

The solutions to these problems will be posted at the following web address: https://ubcengineers.ca/services/academic/tutoring/. If you believe that there is an error in these solutions, or have any questions, comments, or suggestions regarding EUS Tutoring sessions, please e-mail us at: mailto:tutoring@ubcengineers.ca. If you are interested in helping with EUS tutoring sessions in the future or other academic events run by the EUS, please e-mail mailto:vpcademic@ubcengineers.ca.

Some of the problems in this package were not created by the EUS. Those problems originated from one of the following sources:

- Schuam’s Outline of Calculus 2 ed; Ayres Jr., Frank
- Calculus – Early Transcendentals 7 ed; Stewart, James
- Calculus – 3 ed; Spivak, Michael
- Calculus Volume 1 2 ed; Apostol, Tom

Note on notation: Whenever log(x) is used without a subscript to indicate the base, it is assumed to be base e in math courses. Thus in this review package, log(x) and ln(x) are used interchangeably. For inverse trigonometric functions, sin^(-1)(x) = arcsin(x), and the other inverse trigonometric functions are similarly denoted.

All solutions prepared by the EUS.

Good Luck!
1. Show that the vectors $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{B} = \mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$, $\mathbf{C} = 2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ form a right triangle.

2. Find the constant $a$ such that the vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, and $3\mathbf{i} + a\mathbf{j} + 5\mathbf{k}$ are coplanar.

3. Find the area of the parallelogram having diagonals $\mathbf{A} = 3\mathbf{i} + \mathbf{J} - 2\mathbf{k}$ and $\mathbf{B} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$.

4. Compute the volume of the parallelepiped whose edges are represented by $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{B} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{C} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

5. Show that the sum of the intercepts of the tangent plane of the surface $x^2 + y^2 + z^2 = 25$ cut off by a cylinder having this radius as its diameter.

6. Examine the following function for relative maximum and minimum values. $z = 3x - 3y - 2x^3 - xy^2 + 2x^2y + y^3$.

7. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 25$ within the elliptic cylinder $2x^2 + y^2 = 25$.

8. Compute the surface area of a hemisphere of radius $a$ cut off by a cylinder having this radius as its diameter.

9. Compute the triple integral of $f(\rho, \theta, z)$ over the region $R$ bounded by the paraboloid $\rho^2 = 9 - z$ and the plane $z = 0$.

10. Evaluate $\iiint_R \frac{dxdydz}{(x^2 + y^2 + z^2)^{3/2}}$, where $R$ is the region bounded by the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, where $a > b > 0$.

11. Compute the volume of the region bounded above by the sphere $\rho = 2a\cos \phi$ and below by the cone $\phi = \alpha$ where $0 < \alpha < \pi/2$. Leave your answer in terms of $\alpha$.

12. Compute the triple integral of $F(x, y, z) = z$ over the region $R$ in the first octant bounded below by the planes $y = 0$, $z = 0$, $x + y = 2$, $2y + x = 6$, and the cylinder $y^2 + z^2 = 4$.

13. For the following iterated integral, sketch the region of integration, then interchange the order of integration. $\int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} f(x, y)dx dy$

14. Find an equation of the tangent plane to the surface $f(x, y) = \sin(x - y) + x^2 + 3y^2$ at the point $(2, 2)$.

15. When a double integral was set up for the volume $V$ of the solid under the surface $z = f(x, y)$ and above a region $S$ of the $xy$-plane, the following sum of iterated integrals was obtained:

$$V = \int_0^a \int_{\sqrt{a^2 - y^2}}^{\sqrt{b^2 - y^2}} f(x, y)dx dy + \int_0^b \int_{\sqrt{b^2 - y^2}}^{\sqrt{a^2 - y^2}} f(x, y)dx dy + \int_{\sqrt{a^2 - y^2}}^{\sqrt{b^2 - y^2}} f(x, y)dx$$

Given that $0 < a < b$ and $0 < c < \pi/2$, sketch the region $S$, giving the equations of all curves which form its boundary.

16. Let $A = \int_0^1 e^{-t^2} dt$, and $B = \int_0^{1/2} e^{-t^2} dt$. Evaluate the iterated integral

$$I = 2 \int_{-1/2}^1 \int_0^x e^{-y^2} dy dx$$

in terms of $A$ and $B$. 

17. Given $a > 0$, transform the integral to polar coordinates and compute its value.

\[
\int_{0}^{a} \int_{0}^{\sqrt{a^2-y^2}} x^2 + y^2 \, dxdy
\]

18. For the three iterated integrals, describe the region of integration $S$ by means of a sketch, showing its projection onto the $xy$-plane. Then express the triple integral as one or more iterated integrals in which the first integration is with respect to $y$.

19. Find and classify all critical points of $f(x, y) = xy - x^4 - y^2 + 2$.

20. Compute the mass $M$ of the upper half of the annulus $1 < x^2 + y^2 < 9$ with density $\delta = y/(x^2 + y^2)$.

21. Consider the surface $S$ given by the equation $z = (x^2 + y^2 + z^2)^2$.
   a) Show that $S$ lies in the upper half space ($z \geq 0$).
   b) Write out the equation for the surface in spherical coordinates.
   c) Using the equation obtained in part b, give an iterated integral, with explicit integrand and limits of integration, which gives the volume of the region inside this surface. Do not evaluate the integral.

22. Let $f(x, y) = xy^2 - x^3$. a) Find the gradient of $f$ at $P : (1, 1)$. b) Determine an approximate formula showing how small changes $\Delta x$ and $\Delta y$ produce a small change $\Delta u$ in the value of $u = f(x, y)$ at the point $(x, y) = (2, 1)$.

23. Find the equation of the tangent plane to the surface $x^4y + z - z^2x + 1 = 0$ at the point $(-1, 1, 2)$.

24. Using Lagrange multipliers, maximize $f(x, y) = xe^{x^2-y}$ subject to $x^2 + 2y^2 = 4/3$.

25. Let $x, y, z \in \mathbb{R}$ such that $x^2 + y^2 + z^2 + xyz = 4$. Find the minimal value of the expression $x + y + z$.

26. Locate and classify extremal points (if any) of the surface having Cartesian equation given. $z = x - 2y + \log(\sqrt{x^2 + y^2}) + 3 \arctan(y/x)$

27. Locate and classify the extremal points (if any) of the surface having Cartesian equation given. $z = x^3 + y^3 - 3xy$

28. Locate and classify the extremal points (if any) of the surface having Cartesian equation given.

\[
z = e^{2x+3y}(8x^2 - 6xy + 3y^2)
\]

29. At Oktoberfest, someone is standing on the surface $f(x, y) = x^2 - y^2 + 2$ at the point $(0, 1)$. The individual spills a glass of beer such that, when the stain is projected to the $xy$ plane it forms a circle of radius 1 (part of the stain travelled upward). How much surface area must be cleaned?

30. Match each contour map to its function.
   i) $f(x, y) = \sin(x^2 - y) + y = E$
   ii) $f(x, y) = x^2 + 2y^2 - x = F$
   iii) $f(x, y) = \sqrt{y \cos x} = C$
   iv) $f(x, y) = \frac{x^2+y+6x}{x+1} = B$
   v) $f(x, y) = |x + \log(y^2)| = D$
   vi) $f(x, y) = x^{2/5} + y^{4/7} = G$
   vii) $f(x, y) = (3x - y/2)(x + 4y) = A$
   viii) $f(x, y) = x^3 - 6xy = H$
(*** 31.) Compute the first partial derivatives of the function \( f(x, y, z) = x^y z \)

(*** 32.) Compute the first partial derivatives of the function \( f(x, y) = \int_a^y g(t) dt \)
33. Compute the first partial derivatives of the function \( f(x,y) = \int_a^y g(t) dt \).

34. Given \( u = x^2 + 2y^2 + 2z^2 \), and \( x = \rho \sin \beta \cos \theta \), \( y = \rho \sin \beta \sin \theta \), \( z = \rho \cos \beta \), compute \( \frac{\partial u}{\partial \rho}, \frac{\partial u}{\partial \beta}, \frac{\partial u}{\partial \theta} \).

35. Compute \( \frac{du}{dx} \) given \( u = f(x,y,z) = xy + yz + xz \), \( y = \frac{1}{x} \), \( z = x^2 \).

36. Use differentials to compute the diagonal of a rectangular box of dimensions 3.03 by 5.98 by 6.01 ft.

37. Compute all first and second partial derivatives of \( z \), given \( x^2 + 2yz + 2xz = 1 \).

38. On a hill represented by \( z = 8 - 4x^2 - 2y^2 \), find
   a) the direction of the steepest grade at (1,1,2)
   b) the direction of the contour line

39. Find the equation of the plane thorough (2,-1,-1) and (1,2,3) and perpendicular to \( 2x + 3y - 5z - 6 = 0 \).

40. Find the shortest distance between the line through \( A(2,-1,-1) \) and \( B(6,-8,0) \) and the line through \( C(2,1,2) \) and \( D(0,2,-1) \).

41. Find a pair of linear Cartesian equations for the line which is tangent to both the surfaces \( x^2+y^2+2z^2 = 4 \) and \( z = e^{x-y} \) at the point (1,1,1).

42. The temperature at any point \((x, y)\) in the \( xy \) plane is given by \( T = \frac{100xy}{x^2 + y^2} \).
   a) Find the directional derivative at the point (2,1) in a direction making an angle of 60° with the positive \( x \)-axis.
   b) In what direction from (2,1) would the derivative be a maximum?
   c) What is the value of this maximum?

43. Find the value of \( a \) such that \( f(x,y,t) = e^{at} \sin x \sin y \) solves the two dimensional heat equation \( f_t = f_{xx} + f_{yy} \).