Mathematics 101 Final Exam Review Package

UBC Engineering Undergraduate Society

Problems are ranked in difficulty as (*) for easy, (**) for medium, and (***) for difficult. Note that sometimes difficulty can be subjective, so do not be discouraged if you are stuck on a (*) problem.

Solutions posted at: [http://ubcengineers.ca/tutoring/](http://ubcengineers.ca/tutoring/)

If you believe that there is an error in these solutions, or have any questions, comments, or suggestions regarding EUS Tutoring sessions, please e-mail us at: tutoring@ubcengineers.ca. If you are interested in helping with EUS tutoring sessions in the future or other academic events run by the EUS, please e-mail vpacademic@ubcengineers.ca.

The first 7 problems are review of high school material and are highly optional. They cover the basics of the different functions covered in high school.

Some of the problems in this package were not created by the EUS. Those problems originated from one of the following sources:

- Schuam’s Outline of Calculus 2 ed; Ayres Jr., Frank
- Calculus – Early Transcendentals 7 ed; Stewart, James
- Calculus – 3 ed; Spivak, Michael
- Calculus Volume 1 2 ed; Apostol, Tom

Want a warm up? These are the easier problems
2, 5, 6, 7, 9

Short on study time? These cover most of the material
4, 10, 11, 14, 16, 18, 19, 22, 29

Want a challenge? These are some tougher questions
37, 38, 41, 42

EUS Health and Wellness Study Tips

- **Eat Healthy**—Your body needs fuel to get through all of your long hours studying. You should eat a variety of food (not just a variety of ramen) and get all of your food groups in.
- **Take Breaks**—Your brain needs a chance to rest: take a fifteen minute study break every couple of hours. Staring at the same physics problem until your eyes go numb wont help you understand the material.
- **Sleep**—We have all been told we need 8 hours of sleep a night, university should not change this. Get to know how much sleep you need and set up a regular sleep schedule.
1. Determine whether the following series converges or diverges.

\[
\sum_{n=1}^{\infty} \sqrt{2}^{n}
\]

**Solution:** The general term of the sum, \(a_n\), is given by \(a_n = \sqrt{2} = 2^{1/n}\).
Thus
\[
\lim_{n \to \infty} 2^{1/n} = 1 \neq 0
\]
By the divergence test, the series diverges.

2. Determine whether the following series converges or diverges.

\[
\sum_{n=1}^{\infty} \frac{n^3}{(\log 3)^n}
\]

**Solution:** Apply the ratio test.

\[
\lim_{n \to \infty} \left| \frac{(n+1)^3}{(\log 3)^{n+1}} \cdot \frac{n^3}{(\log 3)^n} \right| = \lim_{n \to \infty} \left( \frac{1 + \frac{1}{n}}{\log 3} \right)^3
\]
\[
= \frac{1}{\log 3} < 1
\]
Therefore the series converges (absolutely) by the ratio test.

3. Find the area bounded by the parabolas \(y = 6x - x^2\) and \(y = x^2 - 2x\).

**Solution:** First we need to find the points of intersection of the parabolas. Solving the equation
\[6x - x^2 = x^2 - 2x\]
gives the solutions \(x = 0, 4\). The area is then given by
\[
A = \int_{0}^{4} 6x - x^2 - (x^2 - 2x) \, dx
\]
\[
= \int_{0}^{4} 8x - 2x^2 \, dx
\]
\[
= \left[ 4x^2 - \frac{2}{3} x^3 \right]_{0}^{4}
\]
\[
= 4(4)^2 - \frac{2}{3} (4)^3
\]
\[
= 64 - \frac{2}{3} \cdot 64
\]
\[
= \frac{64}{3}
\]
4. Find the volume of the solid generated by revolving the given plane area about the $y$ axis: $y = e^{-x^2}, y = 0, x = 1, x = 0$

**Solution:** First note that at $x = 1$, we have $y(1) = e^{-1}$. Thus since the area is being rotated about the $y$ axis, it can be split up into a cylindrical section, as well as an irregularly shaped section. The cylindrical section has height $e^{-1}$ and radius 1. So we have

\[
V = \int_{e^{-1}}^{1} \pi x^2 dy + \pi (1)^2 e^{-1}
\]

\[
V = \int_{e^{-1}}^{1} \pi x^2 dy + \pi e^{-1}
\]

\[
= \int_{e^{-1}}^{1} -\pi \log y dy + \pi e^{-1}
\]

\[
= -\pi (y \log y - y)|_{e^{-1}}^{1} + \pi e^{-1}
\]

\[
= -\pi (1 \log 1 - 1 - (e^{-1} \log e^{-1} - e^{-1})) + \pi e^{-1}
\]

\[
= -\pi (0 - 1 + e^{-1} + e^{-1}) + \pi e^{-1}
\]

\[
= \pi - 2\pi e^{-1} + \pi e^{-1}
\]

\[
= \pi - \pi/e
\]

5. Compute the derivative of the function $f(x)$.

\[
f(x) = \int_{\cos x}^{\sin x} \log(e - t^3) dt
\]

**Solution:**

\[
f'(x) = \cos x \log(e - \sin^3 x) + \sin x \log(e - \cos^3 x)
\]

6. Compute the derivative of $g(t)$, where $y$ is some differentiable function.

\[
g(t) = \int_{t^2 - 3}^{1/(t-1)} (y(x))^n + (y'(x))^n dx
\]

**Solution:**

\[
g'(t) = \frac{1}{(t-1)^2} \left[ \left( y \left( \frac{1}{t-1} \right) \right)^n + \left( y' \left( \frac{1}{t-1} \right) \right)^2 \right] - 2t \left[ (y(t^2 - 3))^n + (y'(t^2 - 3))^2 \right]
\]
7. Evaluate the integral.
\[ \int_{-4}^{4} (x + \sin x) \log(x^2 + \cos x) \, dx \]

**Solution:** Let
\[ f(x) = (x + \sin x) \log(x^2 + \cos x) \]

Notice that \( f(x) \) is odd, that is
\[
\begin{align*}
  f(-x) &= ((-x + \sin(-x)) \log((-x)^2 + \cos(-x))) \\
        &= ((-x - \sin(x)) \log(x^2 + \cos x)) \\
        &= -(x + \sin x) \log(x^2 + \cos x) \\
        &= -f(x)
\end{align*}
\]

Therefore, since the interval of integration is symmetric and the integrand is odd,
\[ \int_{-4}^{4} (x + \sin x) \log(x^2 + \cos x) \, dx = 0 \]

8. Find the average value of \( f(x) = 2^x \) on the interval \([1, 3]\).

**Solution:**
\[
\begin{align*}
  \mathcal{J} &= \frac{1}{2} \int_{1}^{3} 2^x \, dx \\
       &= \frac{1}{2} \left( \frac{2^x}{\log 2} \right) \bigg|_{1}^{3} \\
       &= \frac{1}{2} \log 2 (8 - 2) \\
       &= \frac{3}{\log 2}
\end{align*}
\]

9. Determine whether the following series converges or diverges.
\[ \sum_{n=1}^{\infty} \frac{n + 1}{n\sqrt{3n - 2}} \]

**Solution:** Apply the limit comparison test. Compare with
\[ \sum_{n=1}^{\infty} n^{-1/2} \]
\[
\lim_{n \to \infty} \frac{n + 1}{n\sqrt{3n - 2}} \frac{1}{n^{1/2}} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) \sqrt{\frac{n}{3n - 2}} \\
= \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) \sqrt{\frac{1}{3 - 2/n}} \\
= \sqrt{\frac{1}{3}}
\]

Since \(0 < 1/\sqrt{3} < \infty\),

\[
\sum_{n=1}^{\infty} n^{-1/2}
\]
diverges by the \(p\) - test for series, the given series

\[
\sum_{n=1}^{\infty} \frac{n + 1}{n\sqrt{3n - 2}}
\]
also diverges

(**) 10. Rewrite the following Riemann sum as a definite integral.

\[
\lim_{n \to \infty} \frac{4}{n} \sum_{i=1}^{n} \log \left(\left(2 + \frac{4i}{n}\right)^2 + 5\right)
\]

**Solution:** The increment in \(x\) can be read off as

\[
\Delta x = \frac{4}{n}
\]

and since we have \(x_i = 2 + \frac{4i}{n}\), we conclude that \(a = x_0 = 2\). Since

\[
\Delta x = \frac{b - a}{n}
\]

we can calculate that \(b = 6\). Matching with the formula for the Riemann sum,

\[
f(x_i) = \log \left(\left(2 + \frac{4i}{n}\right)^2 + 5\right) = \log \left((x_i)^2 + 5\right)
\]

From this we conclude that

\[
f(x) = \log \left(x^2 + 5\right)
\]

Now equipped with the bounds of integration and the function, we can write it as the integral:

\[
\lim_{n \to \infty} \frac{4}{n} \sum_{i=1}^{n} \log \left(\left(2 + \frac{4i}{n}\right)^2 + 5\right) = \int_{2}^{6} \log \left(x^2 + 5\right) dx
\]

Note that an alternative correct answer is

\[
\int_{0}^{4} \log((x + 2)^2 + 5)dx
\]
11. Evaluate the integral by using a Riemann sum.

$$\int_{-2}^{3} (x^2 - 3x) \, dx$$

**Hint.** The following formulas may be useful.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

**Solution:**

$$a = -2, \quad b = 3, \quad \Delta x = \frac{5}{n}, \quad x_i = -2 + \frac{5i}{n}$$

$$\sum_{i=1}^{n} f(x_i) \Delta x = \sum_{i=1}^{n} f \left( -2 + \frac{5i}{n} \right) \frac{5}{n}$$

$$= \frac{5}{n} \sum_{i=1}^{n} \left[ \left( -2 + \frac{5i}{n} \right)^2 - 3 \left( -2 + \frac{5i}{n} \right) \right]$$

$$= \frac{5}{n} \sum_{i=1}^{n} \left[ -20i/n + 25i^2/n^2 + 6 - 15i/n \right]$$

$$= \frac{5}{n} \sum_{i=1}^{n} \left[ 10 - \frac{35i}{n} + \frac{25i^2}{n^2} \right]$$

$$= \frac{5}{n} \left[ 10n - \frac{35n(n+1)}{2} + \frac{25n^2(n+1)(2n+1)}{6} \right]$$

Taking the limit $$n \to \infty$$ yields the true value of the integral.

$$\int_{-2}^{3} x^2 - 3x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x$$

$$= \lim_{n \to \infty} \frac{5}{n} \left[ 10n - \frac{35n(n+1)}{2} + \frac{25n^2(n+1)(2n+1)}{6} \right]$$

$$= 50 - \frac{175}{2} + \frac{125}{3} = \frac{25}{6}$$

Thus

$$\int_{-2}^{3} (x^2 - 3x) \, dx = \frac{25}{6}$$

12. Evaluate the integral by using a limit of Riemann sums.

$$\int_{0}^{2} (x^3 - 4x^2 + 5) \, dx$$

**Hint.** The following formulas may be useful.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$
Solution: With the formulas
\[ \Delta x = \frac{b - a}{n} = \frac{2}{n} \]
and
\[ x_i = a + i\Delta x = 0 + \frac{2i}{n} \]
we can use the definition of the integral to write it as a Riemann sum.

\[
\int_0^2 (x^3 - 4x^2 + 5) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x
\]

\[
= \lim_{n \to \infty} \Delta x \sum_{i=1}^{n} f(x_i)
\]

\[
= \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} \left( x_i^3 - 4x_i^2 + 5 \right)
\]

\[
= \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} \left( \left( \frac{2i}{n} \right)^3 - 4 \left( \frac{2i}{n} \right)^2 + 5 \right)
\]

\[
= \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} \left( \left( \frac{8}{n^3} \right)i^3 - \left( \frac{16}{n^2} \right)i^2 + 5 \right)
\]

\[
= \lim_{n \to \infty} \frac{2}{n} \left( \frac{8}{n^3} \sum_{i=1}^{n} i^3 - \frac{16}{n^2} \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} 5 \right)
\]

\[
= \lim_{n \to \infty} \frac{2}{n} \left( \frac{8}{n^3} \frac{n^2(n+1)^2}{4} - \frac{16}{n^2} \frac{n(2n+1)(n+1)}{6} + 5n \right)
\]

\[
= \lim_{n \to \infty} \frac{2}{n} \left( \frac{2(n+1)^2}{1} - \frac{8}{n} \left( \frac{2n+1}{3} \right) + 5n \right)
\]

\[
= \lim_{n \to \infty} \left( \frac{4}{1} \left( 1 + \frac{1}{n} \right)^2 - 16 \frac{1}{3} \left( \frac{2 + \frac{1}{n}}{3} \right) + 10 \right)
\]

\[
= 10 - \frac{32}{3} + 10
\]

\[
= \frac{10}{3}
\]

(**) 13. Evaluate the integral.

\[
\int \csc(2x) \, dx
\]
Solution:

\[
\int \csc(2x) \, dx = \int \frac{1}{\sin(2x)} \, dx
\]

\[= \int \frac{1}{2 \sin(x) \cos(x)} \, dx\]

\[= \int \frac{\cos(x)}{2 \sin(x) \cos^2(x)} \, dx\]

\[= \frac{1}{2} \int \frac{\sec^2(x)}{\tan(x)} \, dx\]

\[= \frac{1}{2} \log |\tan x| + C\]

The final answer is then

\[
\int \csc(2x) \, dx = \frac{1}{2} \log |\tan x| + C
\]

14. Evaluate the integral.

\[
\int \frac{1}{x + x^{1/3}} \, dx
\]

Solution: Let \( x = z^3 \), and \( dx = 3z^2 \, dz \). Making these substitutions yields

\[
\int \frac{1}{z^3 + z} \, 3z^2 \, dz = 3 \int \frac{z}{z^2 + 1} \, dz
\]

\[= \frac{3}{2} \log(z^2 + 1) + C\]

\[= \frac{3}{2} \log(x^{2/3} + 1) + C\]

Thus the final answer is

\[
\int \frac{1}{x + x^{1/3}} \, dx = \frac{3}{2} \log(x^{2/3} + 1) + C
\]

15. Find the value of the integral.

\[
\int_{1}^{e} \log x \, dx
\]
Solution:

\[
\int_{1}^{e} \log x \, dx = (x \log x - x)|_{1}^{e} \\
= e \log e - e - (1 \log 1 - 1) \\
= e - e + 1 \\
= 1
\]

\[
\int_{1}^{e} \log x \, dx = 1
\]

(∗∗) 16. A solid has a circular base of radius 4 metres. Find the volume of the solid if every cross section perpendicular to a fixed diameter (therefore also perpendicular to the plane of the base) is an equilateral triangle.

Solution:

Equation of base:

\[x^2 + y^2 = 16\]

- Base of a triangle: \( b = 2x \)
- Height of a triangle: \( h = \tan(\pi/3)x \)
- Area of a triangle: \( bh/2 = \sqrt{3}x^2 \)

The area of cross section will be

\[A(y) = x^2\sqrt{3} = \sqrt{3}(16 - y^2)\]
which means the volume can be calculated as
\[ V = \int_{-4}^{4} A(y)dy \]
\[ = \int_{-4}^{4} x^2 \sqrt{3} dy \]
\[ = \sqrt{3} \int_{-4}^{4} 16 - y^2 dy \]
\[ = \sqrt{3} \left( 16y - \frac{y^3}{3} \right) \bigg|_{-4}^{4} \]
\[ = \frac{256 \sqrt{3}}{3} \]

(**) 17. Find the area enclosed between the x axis and the first arch of \[ y = e^{-ax} \sin(ax) \]

**Solution:** The function is 0 at \( x = 0 \), and \( x = \pi/a \), and thus the first arch is contained in that interval. The area of the first arch is then given by
\[ \int_{0}^{\pi/a} e^{-ax} \sin(ax) dx \]
First, we find the antiderivative of the integrand. Let \( u = ax \) and \( du = adx \).
\[ \int e^{-ax} \sin(ax) dx = \frac{1}{a} \int e^{-u} \sin u du \]
\[ = \frac{1}{a} \left( -e^{-u} \sin u + \int e^{-u} \cos u du \right) \]
\[ = \frac{1}{a} \left( -e^{-u} \sin u - e^{-u} \cos u - \int e^{-u} \sin u du \right) \]
Rearranging,
\[ \frac{1}{a} \int e^{-u} \sin u du = \frac{1}{a} \left( \frac{-1}{2} e^{-u} (\sin u + \cos u) \right) \]
\[ = \frac{-e^{-ax}}{2a} (\sin(ax) + \cos(ax)) \]
Evaluating,
\[ \int_{0}^{\pi/a} e^{-ax} \sin(ax) dx = \left[ \frac{-e^{-ax}}{2a} (\sin(ax) + \cos(ax)) \right]_{0}^{\pi/a} \]
\[ = \frac{-1}{2a} [e^{-\pi} \cos(\pi) - \cos 0] \]
\[ = \frac{e^{-\pi} + 1}{2a} \]
18. A conical vessel is 12 ft across the top and 15 ft deep. If it contains a liquid weighing \( w \) lb/ft\(^3\) to a depth of 10 ft, find the work done in pumping the liquid to a height 3 ft above the top of the vessel. You may express your answer in terms of \( w \).

**Solution:** The weight of a small disk of thickness \( \Delta y \) at height where the radius of the disk is \( x \) of the liquid will be \( W = \pi x^2 w \Delta y \). The work required to lift this disk to a height of \( y = 18 \) will be

\[
\Delta W = \pi x^2 w (18 - y) \Delta y
\]

By similar triangles, we have

\[
\frac{x}{y} = \frac{6}{15}
\]

This gives us

\[
W = \int_0^{10} \frac{4}{25} \pi wy^2 (18 - y) dy
\]

\[
= \frac{4}{25} \pi w \int_0^{10} y^2 (18 - y) dy
\]

\[
= 560 \pi w \text{ ft} \cdot \text{lb}
\]

Thus it takes 560 ft \cdot lb to lift all of the liquid to a height of 18 m.

19. Evaluate the integral.

\[
\int x^3 \sqrt{a^2 - x^2} dx
\]

**Solution:** Let \( u = a^2 - x^2 \) and \( du = -2xdx \).

\[
\int x^3 \sqrt{a^2 - x^2} \, dx = -\frac{1}{2} \int x^2 \sqrt{u} \, du = \frac{1}{2} \int (u - a^2) \sqrt{u} \, du = \frac{1}{2} \int u^{3/2} - a^2 \sqrt{u} \, du = \frac{1}{2} \left( \frac{2}{5} u^{5/2} - \frac{2}{3} a^2 u^{3/2} \right) + C = \frac{1}{5} (a^2 - x^2)^{5/2} - \frac{a^2}{3} (a^2 - x^2)^{3/2} + C
\]

20. Evaluate the integral.

\[
\int \sin^4(3x) \cos^5(3x) \, dx
\]
**Solution:** Let $3x = u$, and $3dx = du$.

\[
\int \sin^3(3x) \cos^5(3x) dx = \frac{1}{3} \int \sin^3(u) \cos^5(u) du = \frac{1}{3} \int (1 - \cos^2 u) \sin u \cos^5(u) du
\]

Let $\cos u = z$, and $dz = -\sin u du$.

\[
\frac{1}{3} \int (1 - \cos^2 u) \sin u \cos^5(u) du = \frac{1}{3} \int (z^2 - 1)z^5 dz = \frac{1}{3} \int z^7 - z^5 dz = \frac{1}{3} \left( \frac{z^8}{8} - \frac{z^6}{6} \right) = \frac{1}{3} \left( \frac{\cos^8(3x)}{8} - \frac{\cos^6(3x)}{6} \right) + C
\]

Thus the final answer is

\[
\int \sin^3(3x) \cos^5(3x) dx = \frac{1}{3} \left( \frac{\cos^8(3x)}{8} - \frac{\cos^6(3x)}{6} \right) + C
\]

(**) 21. Evaluate the integral.

\[
\int \tan^5 x dx
\]

**Solution:**

\[
\int \tan^5 x dx = \int (\sec^2 x - 1) \tan^3 x dx = \int \sec^2 x \tan^3 x - \tan^3 x dx = \frac{\tan^4 x}{4} - \int (\sec^2 x - 1) \tan x dx = \frac{\tan^4 x}{4} - \int \sec^2 x \tan x - \tan x dx = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log |\sec x| + C
\]

(**) 22. Evaluate the integral.

\[
\int \tan^3(2x) \sec^3(2x) dx
\]
Solution: First, let $2x = u$ and $2dx = du$. 

$$\int \tan^3(2x) \sec^3(2x) dx = \frac{1}{2} \int \tan^3 u \sec^3 u \, du$$

$$= \frac{1}{2} \int (\sec^2 u - 1) \sec^3 u \tan u \, du$$

Now let $\sec u = z$ and $dz = \sec u \tan u \, du$.

$$\frac{1}{2} \int (\sec^2 u - 1) \sec^3 u \tan u \, du = \frac{1}{2} \int (z^2 - 1)z^2 \, dz$$

$$= \frac{1}{2} \int z^4 - z^2 \, dz$$

$$= \frac{1}{2} \left( \frac{z^5}{5} - \frac{z^3}{3} \right) + C$$

$$= \frac{1}{2} \left( \frac{\sec^5(2x)}{5} - \frac{\sec^3(2x)}{3} \right) + C$$

(**) 23. Evaluate the integral.

$$\int \frac{x^4 - x^3 - x + 1}{x^3 - x^2} \, dx$$

Solution:

$$\int \frac{x^4 - x^3 - x + 1}{x^3 - x^2} \, dx = \int x - \frac{x - 1}{x^3 - x^2} \, dx$$

$$= \frac{1}{2} x^2 - \int \frac{x - 1}{x^2(x - 1)} \, dx$$

$$= \frac{1}{2} x^2 - \int \frac{1}{x^2} \, dx$$

$$= \frac{1}{2} x^2 + \frac{1}{x} + C$$

(**) 24. Evaluate the integral

$$\int \frac{2x + 3}{9x^2 - 12x + 8} \, dx$$

Solution:

$$\int \frac{2x + 3}{9x^2 - 12x + 8} \, dx = \frac{1}{9} \int \frac{18x + 27}{9x^2 - 12x + 8} \, dx$$

$$= \frac{1}{9} \int \left( \frac{18x - 12}{9x^2 - 12x + 8} + \frac{39}{(3x - 2)^2 + 4} \right) \, dx$$
Let $3x - 2 = 2z$, and $3dx = 2dz$.

\[
\frac{1}{9} \int \left( \frac{18x - 12}{9x^2 - 12x + 8} + \frac{39}{(3x - 2)^2 + 4} \right) dx = \frac{1}{9} \log |9x^2 - 12x + 8| + \frac{1}{18} \int \frac{13}{z^2 + 1} dz
\]
\[
= \frac{1}{9} \log |9x^2 - 12x + 8| + \frac{13}{18} \arctan(z) + C
\]
\[
= \frac{1}{9} \log |9x^2 - 12x + 8| + \frac{13}{18} \arctan \left( \frac{3x - 2}{2} \right) + C
\]

(**) 25. Evaluate the integral

\[
\int \frac{\sqrt{9 - 4x^2}}{x} dx
\]

**Solution:** Let $x = \frac{3}{2} \sin z$ and $dx = \frac{3}{2} \cos z dz$

\[
\int \frac{\sqrt{9 - 4x^2}}{x} dx = \int \frac{\cos z \sqrt{9 - 9 \sin^2 z}}{\sin z} dz
\]
\[
= 3 \int \frac{\cos z \sqrt{1 - \sin^2 z}}{\sin z} dz
\]
\[
= 3 \int \cos^2 z dz
\]
\[
= 3 \int \csc z - \sin z dz
\]
\[
= -3 \log |\csc z + \cot z| + 3 \cos z + C
\]
\[
= -3 \log \left| \frac{3}{2x} + \frac{\sqrt{9 - 4x^2}}{2x} \right| + \sqrt{9 - 4x^2} + C
\]
\[
= \sqrt{9 - 4x^2} - 3 \log \left| \frac{3}{x} + \frac{\sqrt{9 - 4x^2}}{x} \right| + C
\]

(**) 26. (a) Find the approximations for $n = 8$ to the integral $\int_0^1 \cos(x^2) dx$ with

(i) The trapezoidal rule
(ii) The midpoint rule

(b) Estimate the errors in the approximations of part (a).

(c) How large do we have to choose $n$ so that

(i) The $n$th approximation to the integral with the trapezoidal rule is accurate to 0.0001?
(ii) The $n$th approximation to the integral with the midpoint rule is accurate to 0.0001?

**Solution:**
27. Solve the differential equation:

\[ y'(\cos y)(x - 1) = 2x \sin y \]

**Solution:** Expressing in the Leibniz notation:

\[ \frac{dy}{dx} (x - 1) \cos y = 2x \sin y \]

Rearranging:

\[ \cot yy' = \frac{2x}{x - 1} dx \]

Integrating both sides:

\[ \int \cot yy' = \int \frac{2x}{x - 1} dx \]

\[ \log |\sin y| = 2 \int \frac{x}{x - 1} dx \]

\[ = 2 \int \frac{x - 1 + 1}{x - 1} dx \]

\[ = 2x + 2 \log |x - 1| + C \]

Solving for \( y = y(x) \):

\[ |\sin y| = e^C(x - 1)^2 e^{2x} \]

\[ \sin y = C(x - 1)^2 e^{2x} \]

\[ y(x) = \arcsin \left( C(x - 1)^2 e^{2x} \right) \]

Where \( C \in \mathbb{R} \) is an arbitrary constant.

28. Solve the differential equation.

\[ y' = y \log y \cot x \]

**Solution:** Expressing in the Leibniz notation:

\[ y \log y \cot x = \frac{dy}{dx} \]

Rearranging:

\[ \frac{dy}{y \log y} = \cot x dx \]

Integrating both sides:

\[ \int \frac{dy}{y \log y} = \int \cot x dx \]
\[
\begin{align*}
\log |y| &= \log |\sin x| + C \\
\log |y| &= C \sin x \\
|y| &= Ce^{\sin x}, \quad C > 0 \\
y &= Ce^{\sin x}
\end{align*}
\]

Where \( C \in \mathbb{R} \) is an arbitrary constant.

(**) 29. Solve the differential equation.

\[ xdy + (1 + y^2) \arctan ydx = 0 \]

**Solution:** Rearranging:

\[
x dy = -(1 + y^2) \arctan ydx
\]

\[
\frac{dy}{(1 + y^2) \arctan y} = -\frac{dx}{x}
\]

Integrating:

\[
\int \frac{dy}{(1 + y^2) \arctan y} = -\int \frac{dx}{x}
\]

\[
\log |\arctan y| = -\log |x| + C
\]

Solving for \( y = y(x) \):

\[
|\arctan y| = \frac{C}{x}
\]

\[
y = \tan \left( \frac{C}{x} \right)
\]

Where \( C \in \mathbb{R} \) is an arbitrary constant.

(**) 30. Determine whether the following series converges or diverges.

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{3} n^{1/3}}
\]

**Solution:** Apply the divergence test.

\[
\lim_{n \to \infty} \frac{(-1)^{n-1}}{\sqrt{3} n^{1/3}} = DNE \neq 0
\]

The numerator oscillates, and the denominator approaches 1. Thus the series diverges.

(**) 31. Find the sum of the series.

\[
\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n+1)}
\]
Solution: Consider the $n$th partial sum $S_n$

$$S_n = \sum_{i=1}^{n} \frac{1}{(4i - 3)(4i + 1)}$$

$$= \frac{1}{4} \sum_{i=1}^{n} \left[ \frac{1}{4i - 3} - \frac{1}{4i + 1} \right]$$

$$= \frac{1}{4} \left[ \sum_{i=1}^{n} \frac{1}{4i - 3} - \sum_{i=1}^{n} \frac{1}{4i + 1} \right]$$

$$= \frac{1}{4} \left[ \sum_{i=0}^{n-1} \frac{1}{4i + 1} - \sum_{i=1}^{n-1} \frac{1}{4i + 1} - \frac{1}{4n + 1} \right]$$

$$= \frac{1}{4} \left[ 1 + \sum_{i=1}^{n-1} \frac{1}{4i + 1} - \sum_{i=1}^{n-1} \frac{1}{4i + 1} - \frac{1}{4n + 1} \right]$$

$$= \frac{1}{4} \left[ 1 - \frac{1}{4n + 1} \right] = S_n$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1}{4} \left[ 1 - \frac{1}{4n + 1} \right] = \frac{1}{4}$$

Thus,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \log n}{3n + 2} = \frac{1}{4}$$

(**) 32. Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \log n}{3n + 2}$$

Solution: Apply alternating series test

Let

$$a_n = \frac{\log n}{3n + 2}$$

In order to show that $a_{n+1} < a_n$, let $a_n = f(n)$, and show that $f(x)$ is decreasing.

$$f(x) = \frac{\log x}{3x + 2} \Rightarrow f'(x) = \frac{\frac{1}{x}(3x + 2) - 3 \log x}{(3x + 2)^2}$$

Because the denominator is always positive, we must show that

$$\frac{1}{x}(3x + 2) - 3 \log x < 0$$

$$3x + 2 - 3x \log x < 0 \Rightarrow 3x(\log x - 1) > 2$$
Since \( \log x - 1 > 1 \) for \( x > e^2 \), and \( 3x > 3 \) for \( x > 1 \), we have that \( 3x(\log x - 1) > 3 > 2 \) for \( x > e^2 \). Thus the general term is eventually decreasing in absolute value.

Now, we show that the limit of the general term goes to 0.

\[
\lim_{x \to \infty} \frac{\log x}{3x + 2} = \lim_{x \to \infty} \frac{\frac{1}{x}}{3} = \lim_{x \to \infty} \frac{1}{3x} = 0
\]

Thus the series

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \log n}{3n + 2}
\]

converges.

(\(*\)) **33.** Determine whether the following series converges, converges absolutely, or diverges.

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2 + 1}
\]

**Solution:** First test for absolute convergence. Apply the integral test. Consider

\[
\int_1^{\infty} \frac{x}{x^2 + 1} \, dx = \lim_{a \to \infty} \left. \frac{1}{2} \log(x^2 + 1) \right|_1^a = \text{diverges to } +\infty
\]

Thus the series does not converge absolutely. Now we test for conditional convergence. Consider

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2 + 1}
\]

and apply the alternating series test. Let \( a_n = f(n) \), and show that \( f(x) \) is decreasing. Let

\[
a_n = \frac{n}{n^2 + 1} = f(n)
\]

We want to show that \( f(x) \) is decreasing.

\[
f(x) = \frac{x}{x^2 + 1}
\]

\[
f'(x) = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2}
\]

\[
= \frac{1 - x^2}{(x^2 + 1)^2}
\]

\[
\leq 0
\]

For all \( x > 1 \). Now we compute

\[
\lim_{n \to \infty} \frac{n}{n^2 + 1} = 0
\]

Since \( f(x) \) decreasing, and the general term goes to 0, the alternating series test applies, and the series converges conditionally.
34. Determine the interval of convergence for the series.

\[ \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} \]

**Solution:**

\[ \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)(n+2)} \cdot \frac{n(n+1)}{x^n} \right| = \lim_{n \to \infty} \left| \frac{n}{n+2} \cdot \frac{x}{1} \right| = |x| \left( \lim_{n \to \infty} \frac{n}{n+2} \right) = |x| \left( \lim_{n \to \infty} \frac{1}{1 + 2/n} \right) = |x| < 1 \]

⇒ radius = 1

Test Endpoints.

(i) \( x = 1 \)

\[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \]

Which converges by the comparison test, because

\[ \frac{1}{n(n+1)} < \frac{1}{n^2} \]

(ii) \( x = -1 \)

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} \]

Which converges absolutely by the above.

Thus the interval of convergence is given by:

\[ x \in [-1, 1] \]

35. Determine the interval of convergence for the series.

\[ \sum_{n=1}^{\infty} \frac{n}{2^n x^n} \]
**Solution:**

\[
\lim_{n \to \infty} \left| \frac{n+1}{2^n x^n} \right| = \lim_{n \to \infty} \frac{n+1}{2^n |x|^n} = \lim_{n \to \infty} \left( \frac{1 + 1/n}{2|x|} \right) = \frac{1}{2|x|} < 1
\]

Thus we have \(|2x| > 1 \Rightarrow x > \frac{1}{2}, x < -\frac{1}{2}\)

**Test Endpoints**

(i) \(x = \frac{1}{2}\)

\[
\sum_{n=1}^{\infty} \frac{n}{2^n \left( \frac{1}{2} \right)^n} = \sum_{n=1}^{\infty} n
\]

Thus the series diverges at this endpoint

(ii) \(x = -\frac{1}{2}\)

\[
\sum_{n=1}^{\infty} \frac{n}{2^n \left( -\frac{1}{2} \right)^n} = \sum_{n=1}^{\infty} n(-1)^n =
\]

Thus the series diverges at this endpoint

The interval of convergence is then

\(x \in (-\infty, -\frac{1}{2}) \cup (\frac{1}{2}, \infty)\)

(**) 36. Determine the interval of convergence for the series.

\[
\sum_{n=0}^{\infty} \frac{(x+3)^n}{(n+1)^2}
\]

**Solution:**

\[
\lim_{n \to \infty} \left| \frac{(x+3)^{n+1}}{(n+2)^2} \right| = \lim_{n \to \infty} \frac{|x+3| (n+1)^2}{(n+2)^2} = \lim_{n \to \infty} \frac{|x+3| (1+1/n)^2}{(1+2/n)^2} = |x+3| < 1
\]

\(\Rightarrow \) radius = 1
Test Endpoints:
- At $x = -2$, we have the convergent sum (by $p$ test)
  \[ \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \]
- At $x = -4$, we have the convergent sum (by absolute convergence and $p$ test)
  \[ \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} \]

Thus the series converges in the interval
\[ x \in [-4, -2] \]

(*** 37. **Determine whether the following series converges or diverges.**

\[ \sum_{n=1}^{\infty} \frac{1}{n^{n-1}} \]

**Solution:** Apply the ratio test.

\[
\lim_{n \to \infty} \left| \frac{\frac{1}{(n+1)^{n-1}}}{\frac{1}{n^{n-1}}} \right| = \lim_{n \to \infty} \left| \frac{\frac{n^{n-1}}{n(n+1)^{n-1}}}{\frac{1}{n^{n-1}}} \right| = \lim_{n \to \infty} \left| \frac{\frac{n}{n+1}}{1} \right| = \lim_{n \to \infty} \frac{1}{n} \cdot \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^{n} \]

We must use L’hopital’s rule on the second limit, and in order to do calculus, we change the discrete variable $n$ into the continuous variable $x$.

Let
\[ \lim_{x \to \infty} \left( \frac{x}{x+1} \right)^{x} = L \]
Thus
\[
\log(L) = \lim_{x \to \infty} x \log \left( \frac{x}{x+1} \right)
\]
\[
= \lim_{x \to \infty} x (\log x - \log(x+1))
\]
\[
= \lim_{x \to \infty} \frac{\log x - \log(x+1)}{1/x}
\]
\[
= \lim_{x \to \infty} \frac{1/x - 1/(x+1)}{-1/x^2}
\]
\[
= \lim_{x \to \infty} \frac{x+1-x}{(x+1)x} 
\]
\[
= \lim_{x \to \infty} \frac{-x^2}{x(x+1)}
\]
\[
= \lim_{x \to \infty} \frac{-1}{(1+1/x)}
\]
\[
= -1
\]

Thus we have \(\log(L) = -1\) \(\Rightarrow L = 1/e\).

Plugging this back into the above equation yields
\[
\lim_{n \to \infty} \frac{1}{n} \cdot \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n = \lim_{n \to \infty} \frac{1}{e \cdot n} = 0 < 1
\]

Therefore the series converges by the ratio test.

(∗ ∗ ∗) 38. Determine whether the following series converges or diverges.

\[
\sum_{n=1}^{\infty} \left( \sqrt{n^2 + 1} - n \right)
\]

**Solution:** Use comparison test

\[
\frac{3n^2}{9n^2} > 1
\]
\[
\frac{9n^4 + 9n^2}{1 + 6n^2 + 9n^4} > 1
\]
\[
\frac{9n^2(n^2 + 1)}{(1 + 3n^2)^2} > 1
\]
\[
\frac{3n\sqrt{n^2 + 1}}{1 + 3n^2} > 1
\]
\[
\frac{3n\sqrt{n^2 + 1} - 3n^2}{1} > 1
\]
\[
\frac{\sqrt{n^2 + 1} - n}{\frac{1}{3n}}
\]

Since
\[
\sum_{n=1}^{\infty} \frac{1}{3n}
\]
diverges by the $p$-test (this is the harmonic series), therefore, by the comparison test,

$$\sum_{n=1}^{\infty} \left( \sqrt{n^2 + 1} - n \right)$$

also diverges.

(***) 39. Obtain the Maclaurin expansion for $f(x) = \sqrt{1 + \sin x}$.

**Hint.** Use a double angle formula.

**Solution:** First, we need to simplify the expression. We will apply the double angle formula

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

and then note that we need a sine instead of cosine on the right hand side. Thus we will shift by $\pi/4$.

$$\cos^2 \left( \frac{\pi}{4} - \frac{x}{2} \right) = \frac{1 + \cos \left( 2 \left( \frac{\pi}{4} - \frac{x}{2} \right) \right)}{2} = \frac{1 + \cos \left( \frac{\pi}{2} - x \right)}{2} = \frac{1 + \sin x}{2}$$

Rearranging,

$$2 \cos^2 \left( \frac{\pi}{4} - \frac{x}{2} \right) = 1 + \sin x$$

$$\Rightarrow \sqrt{1 + \sin x} = \sqrt{2} \cos \left( \frac{\pi}{4} - \frac{x}{2} \right) = \sqrt{2} \left( \cos \left( \frac{\pi}{4} \right) \cos \left( \frac{x}{2} \right) + \sin \left( \frac{\pi}{4} \right) \sin \left( \frac{x}{2} \right) \right) = \cos \left( \frac{x}{2} \right) + \sin \left( \frac{x}{2} \right) = f(x)$$

The expansions for the individual sine and cosine are

$$\cos \left( \frac{x}{2} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(2n)!}$$

$$\sin \left( \frac{x}{2} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}(2n + 1)!}$$
\[ f(x) = \cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n (2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{n+1} (2n+1)!} \]

\[ f(x) = \sum_{n=0}^{\infty} \left[ \frac{(-1)^n x^{2n}}{2^n (2n)!} + \frac{(-1)^n x^{2n+1}}{2^{n+1} (2n+1)!} \right] \]

\(*\ast\) 40. Obtain the Maclaurin expansion for \( f(x) = \sin^2 x \)

**Solution:**

\[ f(x) = \sin^2 x = \frac{1 - \cos 2x}{2} \]

\[ \cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} \]

With the power series expansion for \( \cos(2x) \), we can calculate \( 1 - \cos(2x) \)

\[ 1 - \cos 2x = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n}}{(2n)!} \]

\[ \frac{1 - \cos 2x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n}}{(2n)!} = \sin^2 x \]

\(*\ast\) 41. Let \( a > 0 \). Find the area enclosed by the loop.

\[ 9ay^2 = x(3a - x)^2 \]

**Hint.** Draw a picture, noting that the graph is symmetric with respect to the \( x \) axis, and has zeros at \( x = 0, 3a \).

**Solution:** First, to get an idea of what the graph looks like, observe that it is symmetric with respect to the \( x \)-axis, because replacing \( y \) by \( -y \) leaves the equation unchanged. Solving for \( y \),

\[ y = \pm \frac{\sqrt{x}[3a - x]}{3\sqrt{a}} \]

Note that the function has zeroes at \( x = 0 \) and \( x = 3a \). Thus it will be a closed loop cross the \( x \)
axis at those two points. The area is then given by

\[ A = 2 \int_0^{3a} \frac{\sqrt{x(3a-x)}}{3\sqrt{a}} \, dx \]

\[ = \frac{2}{3\sqrt{a}} \left( \left. \frac{2}{3} x^{3/2} - \frac{2}{5} x^{5/2} \right|_0^{3a} \right) \]

\[ = \frac{2}{3\sqrt{a}} \left( \frac{2}{3} a^{3/2} - \frac{2}{5} (3a)^{5/2} \right) \]

\[ = \frac{2}{3} \left( \frac{2}{3} a^{3/2} - \frac{2}{5} a^{5/2} \right) \]

\[ = \frac{4a^2}{3} \left( \sqrt{3} - \frac{3^{5/2}}{5} \right) \]

(*** 42. Evaluate the integral.

\[ \int x \arcsin x \, dx \]

**Solution:** We will need to integrate by parts. Let \( u = x, \) and \( v' = \arcsin x. \) First we need to find \( v(x). \) For this integral, integrate by parts, with the two functions being 1 and \( \arcsin x. \)

\[ v(x) = \int \arcsin x \, dx = x \arcsin x + \sqrt{1-x^2} \]

\[ \int x \arcsin x \, dx = x \left( x \arcsin x + \sqrt{1-x^2} \right) - \int \left( x \arcsin x + \sqrt{1-x^2} \right) \, dx \]

\[ = x^2 \arcsin x + x \sqrt{1-x^2} - \int x \arcsin x \, dx - \int \sqrt{1-x^2} \, dx \]

Rearranging,

\[ 2 \int x \arcsin x \, dx = x^2 \arcsin x + x \sqrt{1-x^2} - \int \sqrt{1-x^2} \, dx \]

Now we need to find the antiderivative of \( \sqrt{1-x^2}. \) To do this, let \( x = \sin z, \) and \( dx = \cos z \, dz. \)

\[ \int \sqrt{1-x^2} \, dx = \int \cos z^2 \, dz \]

\[ = \frac{1}{2} \int 1 + \cos 2z \, dz \]

\[ = \frac{z}{2} + \frac{\sin 2z}{4} \]

\[ = \frac{z}{2} + \frac{\sin z \cos z}{2} \]

\[ = \frac{\arcsin x}{2} + \frac{x \sqrt{1-x^2}}{2} \]
\[
2 \int x \arcsin x \, dx = x^2 \arcsin x + x \sqrt{1 - x^2} - \left( \frac{\arcsin x}{2} + \frac{x \sqrt{1 - x^2}}{2} \right)
\]
\[
2 \int x \arcsin x \, dx = x^2 \arcsin x + \frac{x \sqrt{1 - x^2}}{2} - \frac{\arcsin x}{2} + C
\]

Thus the final answer is
\[
\int x \arcsin x \, dx = \frac{x^2 \arcsin x}{2} + \frac{x \sqrt{1 - x^2}}{4} - \frac{\arcsin x}{4} + C
\]

(*** ) 43. Evaluate the integral.
\[
\int \frac{dx}{(4x^2 - 24x + 27)^{3/2}}
\]

Solution:
\[
\int \frac{dx}{(4x^2 - 24x + 27)^{3/2}} = \int \frac{dx}{(4(x - 3)^2 - 9)^{3/2}}
\]
Let \(2(x - 3) = 3 \sec z\) and \(2 \, dx = 3 \sec z \tan z \, dz\).
\[
\int \frac{dx}{(4x^2 - 24x + 27)^{3/2}} = \frac{3}{2} \int \frac{\sec z \tan z \, dz}{(9 \sec^2 z - 9)^{3/2}}
\]
\[
= \frac{1}{18} \int \frac{\sec z \tan z}{\tan^3 z} \, dz
\]
\[
= \frac{1}{18} \int \frac{\sec z}{\tan^2 z} \, dz
\]
\[
= \frac{1}{18} \int \cot z \csc z \, dz = \frac{1}{18} \left( -\csc z \right) + C
\]
\[
= \frac{-1}{18 \sin z} + C
\]
\[
= \frac{2(3 - x)}{18 \sqrt{4x^2 - 24x + 27}} + C
\]

The final answer is then
\[
\int \frac{dx}{(4x^2 - 24x + 27)^{3/2}} = \frac{3 - x}{9 \sqrt{4x^2 - 24x + 27}} + C
\]