20 min

6+ years

Exponential Growth

Explore how quickly numbers can grow

Learning Goals

- Explain how exponential growth occurs.
- Understand how exponential growth is important to quantum computing.

Importance in Quantum Computing

The number of operations a quantum computer can perform grows exponentially because each qubit can occupy two states simultaneously.

Materials

One or more of the following:

- ❏ Checkerboard and at least 31 identical small objects
- ❏ 2 or more sheets of paper
- ❏ 32 or more small tiles of the same size (e.g., dominoes, Magna-Tiles)

Preparation

- If you plan on using the *Colored Tiles* activity, identify a space to cover with the tiles (e.g., table top).

Background Knowledge

If it's been awhile since you've thought about exponents, remember the basis for this mathematical operation is multiplication. If we square (or raise to the power of 2) a number, we multiply that number by itself rather than multiplying by 2. For example, 8 times 2 is 16 ($8 \times 2 = 16$), but $8^2 = 8 \times 8 = 64$. On an exponential scale, the rate of change is ever increasing. Think about raising the number 2 to consecutive exponents. 2 is a small number to start with but as we raise it to higher exponents, the numbers increase rapidly: $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$, $2^6 = 64$, $2^7 = 128$, $2^8 = 256$… Exponents are a powerful mathematical tool with many applications.

In quantum computing, qubits function on an exponential scale. In classical computing, each bit occupies a single state at a given time. In quantum, however, qubits have the ability to be in superposition, which means they can occupy two states simultaneously. Therefore, when you add a qubit to a quantum computer, the number of possible states of the system doubles.

Facilitating the Activities

ENGAGE

Literacy Connection

1. Consider reading or having available one or more of the following:
    a. *On Beyond a Million: An Amazing Math Journey* by David M. Schwartz
    b. *One Grain of Rice: A Mathematical Folktale* by Demi

   In these books, the characters experience numbers that grow exponentially. Ask participants questions to get them thinking about this type of growth: What do you notice about the numbers in the story? Have you ever seen patterns like this before? What do you think the next number might be?

ACTIVITIES

Facilitation Note: Choose one or more of the following activities, depending on the available materials and time. The more ways that participants can engage with the concept of exponential growth, the better they will understand it.

<u>Growth on a Checkerboard</u>

1. Start with a blank grid. If you have a standard sized chess/checkers board, you will have 64 squares to work with. If your grid has a different number of squares, adjust accordingly - the process will be the same.
2. Tell participants that you are going to fill the square with objects by putting twice the number of objects in a square as there are in the previous square. If one square has one object, the next square will have two objects. Ask: How many objects will be in the next (third) square? [*There will be twice as many as the 2 in the previous square, or 2 x 2 = 4 objects.*]
3. Begin by putting one object in the first square. The next square will have 2, the next 4, then 8, and so on. If you start with 31 items, you have enough items to fill 5 squares.
4. Ask participants questions like the following:
    a. How many objects do you think we will need in the last square? [*If you are using a standard chess/checkers board with 64 squares, you would need 2^{63} objects, which is* ***9,223,372,036,854,775,808*** *or 9 quintillion 223 quadrillion 372 trillion 36 billion 854 million 775 thousand 808.*]
    b. Do you know a name for this kind of growth? [*This is called exponential growth.*]
    c. Do you know any examples of exponential growth? [*On a global scale, the human population is growing exponentially. It doubles in size in about 50 years. Bacteria often show exponential growth.*]

Paper Folding

1. Start with a piece of paper. Ask participants questions like the following:
   a. How many times in a row do you think you can fold a piece of paper?
   b. Have you ever tried to fold a piece of paper many times in a row? What happened?
   c. How many sections do you get when you fold a piece of paper in half and then unfold it?
   d. How do you think the number of sections relates to the number of folds?
   e. Each time you fold the paper, how many sections are added?
2. Have a participant fold the paper once, then unfold it. Ask what they observe. [*There are two sections, separated by the fold.*]
3. Ask participants to predict what will happen if we refold the paper and then add a second fold to fold it in half again.
4. Have another participants refold the paper and then add a second fold. How many sections are there when you unfold it? [*There are twice as many, or 4*.]
5. Ask participants to predict what will happen after a 3rd fold [*there will be 8 sections*] and a 4th fold [*there will be 16 sections*]. Then have participants complete these and check their predictions. Ask: Do you notice a pattern? [*The number of sections doubles each time you add a fold. The number of sections grows exponentially.*]
6. Direct participants' attention to the thickness of the paper by asking if anyone has noticed any changes in it. [*The paper gets thicker each time you fold it.*] Ask: How many times do you think we can fold the paper before we can't fold it any more? Have a participant see how many times they can fold it. [*In theory, you can fold a standard piece of paper no more than 7 times, which would give you 128 sections.*]
7. Tell participants that just like the number of sections is growing exponentially with each fold, so is the thickness of the paper as the layers stack up. Have participants observe an unfolded piece of paper and piece of paper participants folded as many times as they could to reinforce the difference in thickness between the two.
8. Ask: How tall do you think the paper would be if we were able to fold it 17 times? Wait for participants to make predictions, even if they seem like wild guesses. Then tell them that we can figure it out using exponents!
   a. The number of sections would be 2^{17} = 131,072.
   b. Assuming the unfolded paper is 0.001 cm thick, the folded paper would be 0.001 cm x 131,072 = 131.072 cm or 51.6 inches tall.
9. Ask participants to make similar predictions for other numbers of folds.
   a. 25 folds: 2^{25} = 33,554,432 sections. 33,554,432 × 0.001 cm = 33,554.432 cm or 13,210.4 inches or 1,100.9 feet. This is about the same height as

the Chrysler Building in New York City and is a little less than a quarter of a mile.

b. 30 folds: 2^{30} = 1,073,741,824 sections. 1,073,741,824 x 0.001 cm = 1,073,741.824 cm or 422,2733 inches or 35,227.8 feet or 6.7 miles. Commercial airplanes fly at an altitude of about 35,000 feet!

c. 40 folds: 2^{40} = 1,099,511,627,776 sections. 1,099,511,627,780 x 0.001 cm = 1,099,511,627.78 cm or 432,878,594 inches or 36,073,216.1 feet or 6,832 miles. That's about the distance from the United States to China!

d. 45 folds would make a paper stack that could reach the Moon!

<u>Colored Tiles</u>

1. Start with one colored tile and ask participants questions such as:
    a. How many tiles do you think could fit in this space?
    b. If we put down one tile, then put down 2, and then 4, so we are doubling the number of tiles each time… how quickly do you think we would fill the space?
2. Work with participants to place tiles in your space. Place 1, then place 2 more adjacent to the first. Ask: Is this how big you thought it would be, adding twice as many as the time before? Can you imagine what 4 more tiles would look like?
3. Work with participants to place 4 additional tiles, so you have 7 total. Then continue the pattern.
4. Participants should soon realize and get a sense of how quickly the tiles are covering the space. Tell participants that when you double the number each time, that's an exponential pattern.
5. When you run out of either tiles or space, count how many times you added tiles. For example, if you end up with 15 tiles, you must have done 4 iterations, starting with 1 tile, then adding 2, then adding 4, then adding 8 because 15 = 1 + 2 + 4 + 8.)

Facilitation Note: This activity is similar to Growth on a Checkerboard, but instead of adding double the amount of small objects to each new square on the board, we are laying tiles side by side all in one space.

DISCUSSION

Quantum Connection

1. Tell participants that researchers are developing a new kind of computer called a quantum computer. Quantum computers us the power of exponents! The power of quantum computers is not that they're faster than a classical computer, but that they can do very different operations than a classical computer can. This is because of the exponential growth aspects of qubits and their superposition abilities. For certain types of problems, quantum computers can pick out patterns and come to a solution much more quickly because they can look at all possible combinations at once.
2. Tell participants that for bigger problems that require looking at many options at once, quantum computers are the way to go. Ask: Can you think of any really big problems that quantum computers might be useful for?
3. Ask participants to think back to one of the activities they engaged in:
    a. As we kept going, we ended up with immensely huge numbers! Now apply this exponential growth to building a quantum computer. Each time you add one qubit to the computer, the number of operations the computer can perform doubles. A quantum computer with 64 qubits could perform 2^{64} = 18,446,744,073,709,551,616 operations simultaneously in order to answer a question! A classical computer would take about 400 years to complete the same problem because it would have to run through each of the 18,446,744,073,709,551,616 possible answers one by one.
4. Here are some examples of problems that can only be solved once we have more powerful quantum computers.
    a. **Designing new medicines**: Scientists are interested in finding new medicines and many chemical reactions are still not completely understood. Quantum computers could replicate chemical systems to give us new insights into molecules and reactions by simulating how the electrons in the atoms that make up molecules interact with each other.
    b. **Cracking codes and better encryption**: Our daily lives rely on modern cryptography, in everything from banking to secure web searches. Computer cryptography is based on the fact that it is much easier to multiply two numbers to find their product than it is to find specific factors of a given number — and this gets harder as the numbers get larger. Current encryption algorithms, called RSA protocols, can be broken if one can figure out the two prime factors of a number with hundreds of digits. This is a problem that classical computers would need an enormous amount of time to solve. However, an algorithm on a quantum computer could quickly calculate the prime numbers used in current encryption schemes. Currently, quantum computers are too small and error prone to accomplish this. However, once quantum computers get large enough,

new encryption schemes will need to be developed that are unsolvable by quantum computers.

c. **Designing new fertilizers**: The Haber Process is the basis for creating fertilizer, which is key in food production. Scientists hope quantum computers will give them a better understanding of this process in the near future. This would help scientists find more energy-efficient ways to make fertilizer which in turn increases food production. Solving this problem requires a quantum computer with between 1,000 and 10,000 qubits, but the biggest quantum computer right now is 100 qubits.
d. **Finding meaning in Big Data**: Many compelling questions in science boil down to finding patterns in large datasets. Finding these patterns is harder as the datasets get larger - and they are getting huge in many scientific fields. Quantum computers offer a fundamentally different and faster way to explore these large datasets and could help solve this important type of problem. On a practical level, quantum computers could make web searches faster and better.
e. **Designing better solar energy materials**: Quantum computer could help scientists and engineers design better photovoltaics, which are solar energy cells that absorb sunlight and convert it to electricity. Solar energy is an important type of renewable energy and quantum computers could help us find better, more efficient ways to make them.

Connections to Standards

Next Generation Science Standards*

Crosscutting Concepts: Scale, Proportion, and Quantity
Science and Engineering Practices: Using Mathematical and Computational Thinking

Common Core State Standards

Standards for Mathematical Practice: Reason Abstractly and Quantitatively, Model with Mathematics, Use Appropriate Tools Strategically

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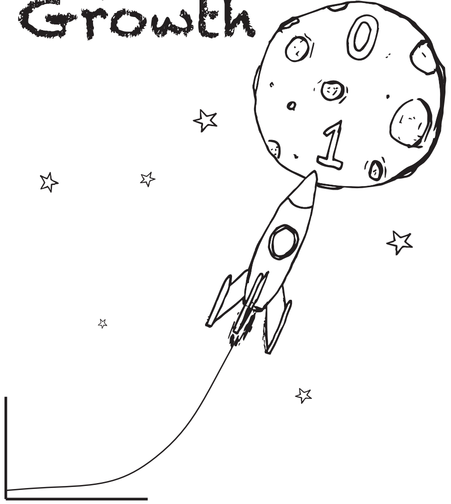
Share what you've learned about exponential growth!

- Explain to someone else what exponential growth is.
- Find a big space, like a playground or field. Estimate how many steps it would take you to get from one end to the other. How many times would you have to double your steps to get to there?
- Put a small amount (1 or 2 tablespoons) of uncooked rice in a cup. Use a grid, such as a chess/checkers board, and count out the rice in an exponential pattern (1 grain in the first square, 2 grains in the second, 4 grains in the third, etc.). Make a prediction: How many squares will you use before you run out of rice grains?

Exponential Growth

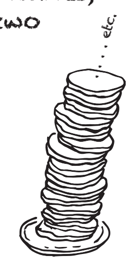


In Quantum Computing

Tough Choice?

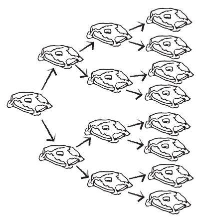
Imagine this: one day you're walking along, when suddenly you meet a pancake fairy!

You ask her for pancakes, and she gives you two options:



A million (1,000,000) pancakes right now!

OR



1 pancake today,
2 tomorrow,
4 the next,
then 8, ...
for 30 days

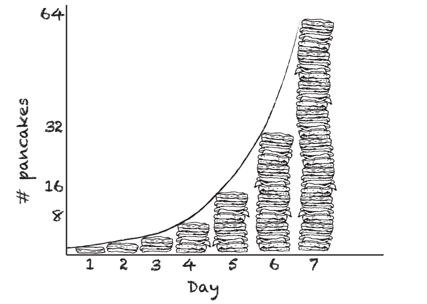
How do you choose?

(in order to maximize pancakes)

Adding up the pancakes from the second option. . .

$$1+2+4+8+\cdots+2^{29} = 2^{30}-1$$
$$= 1,073,741,823 \text{ pancakes}$$

Wow! That's a lot of pancakes!



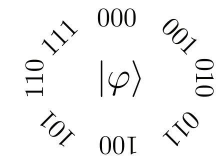
Doubling the amount of pancakes each day is a form of **exponential growth**

Superposition is Powerful!

Given n bits, there are 2^n permutations of these bits

000 001 010 011
100 101 110 111

n classical bits can only represent 1 of these permutations at a time



With superposition, n qubits can be a combination of all permutations at once

Example

8 classical bits:

1	0	0	1	0	0	0	0

8 pieces of information

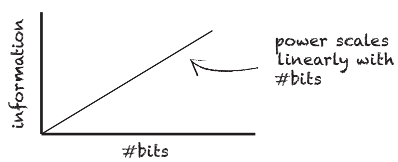
8 qubits in superposition:

0	0	1	1	0	0	1	0

$2^8 = 256$ possible states in superposition, giving 256 pieces of information

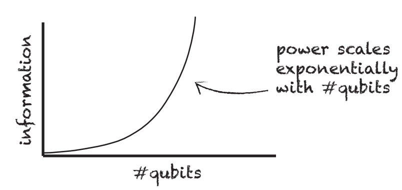
Adding a qubit

Add a classical bit:



8 -> 9 pieces of info

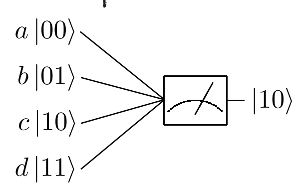
Add a qubit:



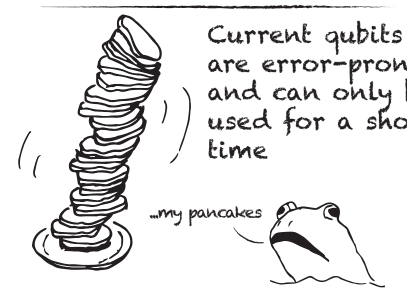
256 -> 512 pieces of info

Caveats

Superposition allows a qubit to hold multiple values at once



But we can only read out one value, and doing so destroys all the others!



Current qubits are error-prone and can only be used for a short time

Find more Quantum Computing zines here:

https://www.epiqc.cs.uchicago.edu/resources/

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AN NSF EXPEDITION IN COMPUTING
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