Quantitative Investments

Dale W.R. Rosenthal

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1 info@q36llc.com
Recall

Last lecture we discussed returns.

- Time value of money;
- Compounding: annual to continuous;
- Real and nominal interest rates; dangers of inflation;
- Comparing rates of returns; and,
- Log-returns.

Today we will talk about risk & returns.
Risk versus Returns

Chapter 7, *A Quantitative Primer on Investments with R*
This part discusses risky returns:

- Randomness;
- Statistical metrics for returns;
- How actual returns differ from normality; and,
- Risk premia.
Randomness

- When returns are uncertain, our perspective changes.
- Compounding is uncertain, may be bad: what if losses compound?
- Ideas on efficient markets guide how we model, assess risk.
- Recall Regnault and Bachelier modeling Paris stock prices.
- They, A.N.K., used a random walk (sum of disturbances):

\[ X_0 = 0, \quad X_t = X_{t-1} + \epsilon_t \quad \epsilon_t \sim iid \begin{cases} +1 & \text{w.p. } p \\ -1 & \text{w.p. } 1 - p \end{cases} \quad \forall t = 1, 2, \ldots \]

- If \( p = 1/2 \), \( X_t \) is a martingale \((E(X_{t+1}|X_t) = X_t)\).
Honey, I Shrunk the Step Size

- This is not a bad model for prices, but it is a bit too “chunky.”
- Shrink time and space steps: process looks more like market prices.
  - Shrink time by factor of $k$, space by factor of $\sqrt{k}$.
  - Why $\sqrt{k}$ for space? Compounding, else growth differs.
- Get arithmetic or geometric Brownian motion (aka Wiener process):
  \[
  dS_t = \mu dt + \sigma dW_t; \quad (\text{arithmetic}) \quad \text{or,} \quad \frac{dS_t}{S_t} = \mu dt + \sigma dW_t \quad (\text{geometric}).
  \]
  \[dS_t = \mu dt + \sigma dW_t; \quad (\text{arithmetic}) \quad \text{or,} \quad \frac{dS_t}{S_t} = \mu dt + \sigma dW_t \quad (\text{geometric}).
  \]
- $E(W_t|W_0) = W_0$, $\text{Var}(W_t) = t$.
- Usually use geometric BM (stays $>0$); why we use log-returns.
Log-Returns Redux

- Derivative $d \log(S_t) = \frac{dS_t}{S_t} =$ continuous % change.
- To estimate $\mu$ and $\sigma$, sample market prices regularly.
- Then look at log-returns, differences of log(prices):

$$r_{t_i} = \log(S_{t_i}) - \log(S_{t_{i-1}}) = \mu \Delta t_i + \sigma (W_{t_i} - W_{t_{i-1}}).$$

(3)

- Breaking News! Beautiful mathematical model slain by ugly facts!
- Geometric Brownian motion $\implies$ log-returns normally-distributed.
- Problem: They are certainly not normally-distributed.
- Thus we often look at moments: expected powers, e.g. $E(r_{t_i}^2)$
- Especially central moments: remove mean, scale to unit variance.
Statistics Review: Mean, Variance, Volatility

- In general: we look at these metrics scaled to be annual.
- **Expected return**: probability-weighted average possible return.

\[
\mu_r = E(r) = \int_{-\infty}^{\infty} r f(r) \, dr \text{ or } \sum_{i=1}^{N} r(i) p(i) \tag{4}
\]

- **Variance**: central second moment (aka \(\text{var}(r)\)).

\[
\sigma_r^2 = \int_{-\infty}^{\infty} (r - \mu_r)^2 f(r) \, dr = \sum_{i=1}^{N} (r(i) - \mu_r)^2 p(i). \tag{5}
\]

- **Volatility**: is square root of variance: \(\sigma_r = \sqrt{\text{Var}(r)}\).

\[\text{Volatility}^2 \text{ is square root of variance: } \sigma_r^2 = \sqrt{\text{Var}(r)}.\]
Higher-order moments are of concern for non-normal data.

**Skewness**: standardized third moment.

\[
\gamma_r = \int_{-\infty}^{\infty} \left( \frac{r - \mu_r}{\sigma_r} \right)^3 f(r) dr = \sum_{i=1}^{N} \left( \frac{r(i) - \mu_r}{\sigma_r} \right)^3 p(i). \quad (6)
\]

**Kurtosis**: standardized fourth moment.

\[
\kappa_r = \int_{-\infty}^{\infty} \left( \frac{r - \mu_r}{\sigma_r} \right)^4 f(r) dr = \sum_{i=1}^{N} \left( \frac{r(i) - \mu_r}{\sigma_r} \right)^4 p(i). \quad (7)
\]

Normal distribution: \( \gamma = 0, \kappa = 3. \)
Sample Distribution Metrics

- For regularly-spaced returns, can easily estimate metrics.\(^3\)
- Assume each observation is equally likely: \(p(i) = \frac{1}{N}\).
- Put “hats” on quantities we estimate.

\[
\text{mean} = \hat{\mu}_r = \frac{1}{N} \sum_{i=1}^{N} r_i
\]  
(8)

\[
\text{variance} = \hat{\sigma}_r^2 = \frac{1}{N-1} \sum_{i=1}^{N} (r_i - \hat{\mu}_r)^2
\]  
(9)

\[
\text{skewness} = \hat{\gamma}_r = \frac{1}{N-2} \sum_{i=1}^{N} \left( \frac{r_i - \hat{\mu}_r}{\hat{\sigma}_r} \right)^3
\]  
(10)

\[
\text{kurtosis} = \hat{\kappa}_r = \frac{1}{N-2} \sum_{i=1}^{N} \left( \frac{r_i - \hat{\mu}_r}{\hat{\sigma}_r} \right)^4
\]  
(11)

\(^3\)This estimation gets much more complicated with intraday returns.
Need dependence measures to find portfolio variance.

Covariance of $X$, $Y$: $\Sigma_{XY} = E(XY) - E(X)E(Y)$.

Correlation is covariance over $\sigma_X\sigma_Y$: $\rho_{XY} = \frac{\Sigma_{XY}}{\sigma_X\sigma_Y}$.

Correlation is bounded: $-1 \leq \rho_{XY} \leq +1$.

Thus covariance is bounded: $-\sigma_X\sigma_Y \leq \Sigma_{XY} \leq \sigma_X\sigma_Y$.

Often write variance, covariances in covariance matrix $\Sigma$:

$$
\Sigma = \begin{pmatrix}
\sigma_X^2 & \Sigma_{XY} & \cdots & \Sigma_{XZ} \\
\Sigma_{XY} & \sigma_Y^2 & \cdots & \Sigma_{YZ} \\
\vdots & \vdots & \ddots & \vdots \\
\Sigma_{ZX} & \Sigma_{ZY} & \cdots & \sigma_Z^2
\end{pmatrix}.
$$

(12)
Portfolio Variance

Example portfolio variance, $\sigma_P^2 = w^T \Sigma w$, for three-asset portfolio:

1. Write the weights along the top and left sides:

$$
\begin{array}{ccc}
   w_X & w_Y & w_Z \\
   w_X & \sigma_X^2 & \Sigma_{XY} & \Sigma_{XZ} \\
   w_Y & \Sigma_{XY} & \sigma_Y^2 & \Sigma_{YZ} \\
   w_Z & \Sigma_{XZ} & \Sigma_{YZ} & \sigma_Z^2 \\
\end{array}
$$

(13)

2. Then cross-multiply headings times cells:

$$
\begin{array}{ccc}
   w_X & w_Y & w_Z \\
   w_X & w_X^2 \sigma_X^2 & w_Y w_X \Sigma_{XY} & w_Z w_X \Sigma_{XZ} \\
   w_Y & w_X w_Y \Sigma_{XY} & w_Y^2 \sigma_Y^2 & w_Z w_Y \Sigma_{YZ} \\
   w_Z & w_X w_Z \Sigma_{XZ} & w_Y w_Z \Sigma_{YZ} & w_Z^2 \sigma_Z^2 \\
\end{array}
$$

(14)

3. Adding gives $\sigma_P^2 = w_X^2 \sigma_X^2 + w_Y^2 \sigma_Y^2 + w_Z^2 \sigma_Z^2$

$$
+ 2w_X w_Y \Sigma_{XY} + 2w_X w_Z \Sigma_{XZ} + 2w_Y w_Z \Sigma_{YZ}.
$$
For comparison, we generally scale metrics to be annualized.

Mean and variance \((\mu_r, \sigma_r^2)\) scale linearly with time.
- For US stocks, \(\mu_{\text{annual}} = 250 \mu_{\text{daily}}, \sigma^2_{\text{annual}} = 250 \sigma^2_{\text{daily}}\).
- A 0.04 daily volatility annualizes to \(0.04 \times \sqrt{250} = 0.63\) ("63").

Skewness, kurtosis \((\gamma_r, \kappa_r)\): No scaling needed.

A symmetric distribution has zero skewness.

Skewness \(<0\): more negative than positive surprises.

The normal distribution has a kurtosis of \(\kappa = 3\).
- If \(\kappa > 3\), we say data have fat tails.
- Fat tails: More data near mean and at extremes.
The problem with high kurtosis distributions is not just fat tails.

It’s the pointy heads! Returns show deceptively little variation.

This creates a false sense of security... so you take more risk.

Then a surprise hits — a big surprise. You lose extra big.
Let’s consider a risky game/investment:
- Coin flip: Heads wins $1\ BB + $2, tails loses $1\ BB.
- We expect to earn $1; how much does it cost to play?

If playing costs $1, we say this is a *fair game*.

What if playing costs less than $1?
- For some *risk-averse* investors, game will be worthwhile.
- Risk premium ($1–cost) must compensate for variability.
- This is *speculation*: taking a worthwhile risk.

If risk premium $\leq 0$, only *risk-loving (gamblers)* will play.

Speculator or gambler? Depends on investor’s expectations.
Government bonds determine a currency’s risk-free rate \( r_f \).

Governments need not default; can print money if needed.

\[ \text{And yet they do default... WHY?} \]

**Excess return**: actual risky return − risk-free return.

Investing in something riskier should earn more, on average.

\[ \text{Average compensation for that risk is the risk premium.} \]

\[ \text{Risk premium} = \text{average excess return} = E(r_i - r_f). \]

**N.B.** Some managers use a target rate instead of risk-free rate.

**Risk aversion**: how unwilling people are to take risk (\( \lambda \)).

We can then measure our outcomes in terms of utility:

\[ U = E(r) - \lambda \text{Risk}^\omega \quad \omega > 1. \]
Historical Risk Premia

- Mean excess returns suggest US stock risk premium of 6–8%.
- US risk premia similar to other industrialized nations.
- Do risk premia change over time?
  - Academic research, common sense both suggest they do.
- US stock risk premium ranges over 3–15%.
  - Varies across time, subset ("universe") of US stocks.
  - Probably persistent/exhibits serial correlations.
- Should consider various risk premia in a thorough analysis.
### Historical Record of Returns

<table>
<thead>
<tr>
<th>Return Distribution</th>
<th>Volatility $\sigma$</th>
<th>Skewness $\gamma$</th>
<th>Kurtosis $\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>N/A</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>Stocks</td>
<td>0.2–0.8</td>
<td>-0.5</td>
<td>5–7</td>
</tr>
<tr>
<td>Bonds</td>
<td>0.1–0.2</td>
<td>0.5</td>
<td>5–7</td>
</tr>
<tr>
<td>Currencies</td>
<td>0.1</td>
<td>-0.1</td>
<td>5–10</td>
</tr>
<tr>
<td>WTI Crude</td>
<td>0.4</td>
<td>0.1</td>
<td>6</td>
</tr>
<tr>
<td>Copper</td>
<td>0.2</td>
<td>-0.4</td>
<td>7.5</td>
</tr>
<tr>
<td>Electricity</td>
<td>0.5–1.5</td>
<td>0–1</td>
<td>7–25</td>
</tr>
<tr>
<td>Natural Gas</td>
<td>0.7</td>
<td>-1.0</td>
<td>31</td>
</tr>
<tr>
<td>Wheat</td>
<td>0.3</td>
<td>-0.8</td>
<td>60</td>
</tr>
</tbody>
</table>

- Skewness is negative, not “effectively” 0.
- $\gamma < 0$ is bad for longs.
- Kurtosis of large indices is less than for single stocks.
Whence Normality?

- Averaging real-life random numbers gives normal dist’n.\(^4\)
- But data show mild to severe departures from normality. Why?
- Not all random: competition; firm’s troubles may help competitors.
- Skewness < 0 for stocks, copper, natgas, wheat.
  - Stocks: surprises are often bad; firms rarely hide good news.\(^5\)
- Skewness > 0 for govt bonds and maybe crude, power.
  - Govt bonds: Occasional jolts of fear cause “flight to quality.”
- Kurtoses > 3 for everything.
  - Sporadic info releases mean sometimes there are big shocks.
  - In between info releases: price adjustments are small.
  - Also, market has different states (\(\uparrow\) , \(\downarrow\)), volatile periods.

\(^4\)Assuming that by “real-life” we mean having a finite volatility.
\(^5\)Stocks are also final claimants on a firm in bankruptcy.
The Road Ahead

We have covered risk & returns; on to risk measures next time!

- Measuring: Risk Measures, Diversification, and Modeling;
- Valuation: Fixed Income, Yield Curves, Equity Valuation;
- Valuation II: Factor Models, Microfoundations, Global Investing, FX;
- Risk Alleviation: Futures, Options, Credit, Structured Products; and,
- All Together Now: Active Portfolios, Investment Firms, Crises.