Recall

Last lecture we discussed risk measures.

- Value-at-risk;
- Coherent risk measures;
- Risk-based performance measures;
- Estimating risk measures;
- Extreme value theory; and,
- Types of convergence.

Today we will talk about diversification.
Diversification

Chapter 9, *A Quantitative Primer on Investments with R*
This part discusses diversification, specifically:

- Risk aversion;
- Utility and indifference;
- Allocation between risk-free and risky assets;
- Finding the optimal risky portfolio; and,
- Finding the optimal complete portfolio.
Risk Aversion and Utility

- Can think of risk-averse investor’s *utility* function.
  - Values expected return less some multiple of risk.
  - What captures risk? If returns log-normal, variance $\sigma^2$.

\[
U_\lambda(E(r_P), \sigma_P) = E(r_P) - \lambda \text{Risk}^\omega
\]  \hspace{1cm} (1)

(usually proxied as:) \quad = E(r_P) - \frac{\lambda}{2} \sigma_P^2. \hspace{1cm} (2)

- To invest: Maximize utility when choosing a portfolio.

- $\lambda$ expresses the investor’s risk aversion.

- How to know your risk aversion $\lambda$? HARD.
  - How to simulate the fear of losing your life’s savings?
  - For same variance, do you prefer more/less skewness, kurtosis?

- Ignoring these concerns: common $\lambda$’s in 2–4 range.

$^2$Log-returns are not normal, but this can (should) be done with ES.
Risk Aversion and Utility: Example

- Consider 3 portfolios:

<table>
<thead>
<tr>
<th>Risk</th>
<th>Risk Prem.</th>
<th>$E(r_p)$</th>
<th>$\sigma_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>None</td>
<td>0%</td>
<td>4%</td>
<td>0%</td>
</tr>
<tr>
<td>Low</td>
<td>2%</td>
<td>6%</td>
<td>8%</td>
</tr>
<tr>
<td>High</td>
<td>8%</td>
<td>12%</td>
<td>20%</td>
</tr>
</tbody>
</table>

- The utilities of these for various risk aversions $\lambda$:

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$U(\text{no-risk})$</th>
<th>$U(\text{low-risk})$</th>
<th>$U(\text{high-risk})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.04 = 4%</td>
<td>6.00%</td>
<td>12.00%</td>
</tr>
<tr>
<td>2</td>
<td>4%</td>
<td>5.36%</td>
<td>8.00%</td>
</tr>
<tr>
<td>4</td>
<td>4%</td>
<td>4.72%</td>
<td>4.00%</td>
</tr>
<tr>
<td>6</td>
<td>4%</td>
<td>4.08%</td>
<td>0.00%</td>
</tr>
<tr>
<td>7</td>
<td>4%</td>
<td>3.76%</td>
<td>-2.00%</td>
</tr>
</tbody>
</table>
We are *indifferent* between outcomes of equal utility. In other words, we would be equally happy with any of them.

Rearrange (2) to get *indifference curve*.

\[ E(r_P) = U + \frac{\lambda}{2} \sigma_P^2. \] (3)

*N.B.* Different \( \lambda \)'s yield different indifference curves.

If \( \lambda = 3 \), we’re equally happy with:

- \( E(r_P) = 0.04, \sigma_P^2 = 0; \)
- \( E(r_P) = 0.10, \sigma_P^2 = 0.04, \sigma_P = 0.20; \)
- \( E(r_P) = 0.20, \sigma_P^2 = 0.106, \sigma_P = 0.327. \)
How do we create different portfolios on indifference curve?

- Idea: Put some money in risk-free investment $F$.
- Complete portfolio includes risk-free, risky portfolio $(F, P)$.

Say $P$ is 40% corp bonds, 60% equities; investing $300k:

- 1/3 in $F$, 2/3 in $P$: $100k$ T-bills, $80k$ corp bonds, $120k$ equities.
- 1/2 in $F$, 1/2 in $P$: $150k$ T-bills, $60k$ corp bonds, $90k$ equities.

As we change allocation between $F$ and $P$:

- We do not change what is in $F$ and $P$. 
Allocate fraction $y$ of investment to $P$, $1 - y$ to $F$.

Let $r_f$, $r_P$ and $0$, $\sigma_P$ be rates of return, volatilities of $F$, $P$.

Then the rate of return for complete portfolio $C$ is:

$$E(r_C) = yE(r_P) + (1 - y)r_f = r_f + y\left(E(r_P) - r_f\right)$$  \hspace{1cm} (4)

The volatility of the complete portfolio $C$ is:

$$\sigma_C = y\sigma_P \Rightarrow y = \frac{\sigma_C}{\sigma_P}.$$  \hspace{1cm} (5)

Thus we can restate the expected return on $C$ as:

$$E(r_C) = r_f + \sigma_C \frac{E(r_P) - r_f}{\sigma_P}.$$  \hspace{1cm} (6)

\textsuperscript{3}Why 0 for $F$ volatility?
Look more closely at what we have:

\[ E(r_C) = r_f + yE(r_p - r_f) = r_f + \frac{E(r_p - r_f)}{\sigma_P} \sigma_C = r_f + S_P \sigma_C. \]  

(7)

Defines a line relating \( E(r_C) \) and \( \sigma_C \).

We call this line the *capital allocation line* (CAL).

- Slope: Sharpe ratio of portfolio \( P \).
- Gives transition from all \( F \) to all \( P \): \( F \) at \( y = 0 \), \( P \) at \( y = 1 \).
Where to be along the CAL? What \( y \) makes us happiest?

To find out, maximize utility along the CAL.

\[
U^* = \max_y U_\lambda(y) = \max_y E(r_C) - \frac{\lambda}{2} \sigma_C^2
\]  

(8)

\[
= \max_y r_f + y(E(r_P) - r_f) - \frac{\lambda}{2} y^2 \sigma_P^2.
\]  

(9)

We could also constrain \( 0 \leq y \leq 1 \) (i.e. no SS).

Since \( U \) concave, set \( U' = 0 \) and solve:

\[
U' = E(r_P) - r_f - y^* \lambda \sigma_P^2 = 0 \quad \implies y^* = \frac{E(r_P) - r_f}{\lambda \sigma_P^2}
\]  

(10)

N.B. Must be very risk averse to invest only risk-free.
Recall the indifference curve definition (for portfolio $C$):
- Equally-pleasing $(E(r_C), \sigma_C)$ pairs given our $\lambda$.
- Thus any pair of $(\lambda, \text{utility})$ defines a curve.

Utility gives indifference curves $E(r_C) = U + \lambda/2\sigma_C^2$: 

Ceteris paribus... 

...prefer a higher curve for given $\lambda$.

But higher curves may not be possible.
Optimal Risky Portfolios
Diversification and Risk

- **Diversification** is splitting capital among many instruments.
- Idea: Reduces instrument-specific risk.

Some risk eliminated (or greatly reduced) by diversification.
- Call this *idiosyncratic risk* or *specific risk*.

However, some risk may be common to all instruments.
- Call this *systematic risk*. (“Market risk” is inaccurate.)
- This risk cannot be made to vanish under diversification.

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4 This is merely an expression of the Gauss-Markov Theorem.
Explore diversification further; consider two-asset portfolio:

- Invest in risky bond portfolio $B$ or equity portfolio $E$.
- Shares invested: $w_B$ and $w_E$; $w_B + w_E = 1$.

Portfolio return: \[ r_P = w_B r_B + w_E r_E. \]

Portfolio variance: \[ \sigma_P^2 = w_B^2 \sigma_B^2 + w_E^2 \sigma_E^2 + 2 w_B w_E \text{Cov}(r_B, r_E). \]

Don’t forget: \[ \text{Cov}(r_B, r_E) = \rho_{BE} \sigma_B \sigma_E. \]
Two Risky Assets: Varying Weights

- Can vary $w_B$ ($w_E = 1 - w_B$).
- Frontiers: $B - E$ curves.
- $|\rho_{BE}| = 1$? Can make $\sigma_P = 0$.

Minimum variance portfolio may have $\sigma_P < \sigma_i$ for all stocks $i$.
- N.B. green and blue curves have points with lower volatility.
- Points right of curve define opportunity set of risky assets (OSRA).
Two Risky and One Risk-Free Asset

- What if we add money market to “bond and equity” universe?
  - CAL gives possible allocations to risky, risk-free portfolios.
  - OSRA gives possible risky portfolios.
  - Thus we must invest on the CAL and OSRA.
- Also means the all-risk endpoint of CAL must be on OSRA.
  - This gives many possible CALs; which to use?
  - Want CAL with greatest slope (i.e. Sharpe ratio).
Find the CAL

* Want to maximize Sharpe ratio.

* CAL for $P$ has slope $\frac{E(r_P) - r_f}{\sigma_P}$.

\[
\max_{P \in OSRA} S_P = \max_{P \in OSRA} \frac{E(r_P) - r_f}{\sigma_P}. 
\]  

(11)

That gives the line from $F$ tangent to the top of the OSRA.

* Point of tangency yields the *optimal risky portfolio*.

  * One Risky Portfolio to Rule Them All! (…Sort of.)

* Why not pick a point on red dashed frontier?
Maximizing utility on CAL gives *optimal complete portfolio*.

Thus utility guides allocation between $F$ and optimal $P$.

Find tangency indifference curve to get tangency point $C$.

$C$ is optimal complete portfolio, mix of $F$ and $P$. 

![Graph showing the relationship between expected return ($E(r_P)$) and standard deviation ($\sigma_P$) for different portfolios, including the optimal complete portfolio (C), mean-variance (MV), and F. The graph is labeled OSRA.](image)
We can now outline the process to find $C$:

- Get $rf \rightarrow F$;
- Get expected returns $E(r_i)$, variances $\sigma_i^2$, correlations $\rho_{ij} \rightarrow$ OSRA;
- Find optimal risky portfolio $P$ and resulting CAL;
- Use risk aversion $\lambda$ to get indifference curves; and,
- Find optimal complete portfolio $C$.

This is the idea of Markowitz (1952) and Roy (1952).

Usually, we skip finding the OSRA and just solve for $P$.

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superscript 5 Harry Markowitz received the Nobel Prize for this. He has said that Arthur Roy should have also won.
Markowitz-Roy Terminology

- Markowitz-Roy approach has yielded useful terminology.
- OSRA includes minimum variance portfolio $MV$;
- Minimal variance for any target return: *minimum variance frontier*.
- Top-half of OSRA is the *efficient frontier* (of risky assets).
- Bottom-half of OSRA is the *inefficient frontier* (of risky assets).

Diagram:

- $E(r_P)$ vs $\sigma_P$
- $F$, $MV$, and $P$ points
- OSRA shaded area

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Quantitative Investments
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Why doesn’t everyone use the same optimal risky portfolio?

- People estimate inputs with different data;
- Taxes differ based on holding period, etc.
- Self-imposed constraints (risk, ethical/religious issues).
- Life exposures: Do not forget exposure to your employer!
- Home bias: You don’t own Canadian, Mexican stocks??
- Different levels of financial education.
Other Problems with Markowitz-Roy

- Even with same inputs, taxes, Markowitz-Roy is not perfect.
- Must estimate $> n^2/2$ inputs; quickly exhausts info in recent data.
- Consider randomness! $\max E(\text{utility}) \gg \max$ based on averages.
  - However, stochastic optimization is not easy.
- Why not use higher moments, better risk measures?

$$U = E(r) - \frac{\lambda}{2} \sigma^2 + \frac{\lambda^2}{6} \gamma - \frac{\lambda^3}{24} \kappa + \ldots.$$ (12)

- Should consider transactions costs: lower return, add risk.
Diversification in the Limit

- How well does diversification work? Consider an example:
  - Suppose we invest equal amounts in \( n \) risky assets\(^6\).
  - All risky assets: same \( E(r) \), \( \sigma \), pairwise correlation \( \rho \).

- Portfolio variance is then:

\[
\sigma^2_P = \sum_{i=1}^{n} \frac{\sigma^2}{n^2} + \sum_{i=1}^{n} \sum_{j \neq i} \frac{\rho \sigma^2}{n^2} = \frac{\sigma^2}{n} + \frac{n-1}{n} \rho \sigma^2. \quad (13)
\]

- As \( n \to \infty \), \( \frac{\sigma^2}{n} \downarrow 0 \), \( \frac{n-1}{n} \rho \sigma^2 \to \rho \sigma^2 \).
- Thus \( \rho \sigma^2 \) is the common (systematic) variance.
- Portfolio variance mostly due to correlation among assets.
- Why we like assets that are uncorrelated with other assets.

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\(^6\)The well-known “1/\( n \) portfolio” which sometimes also beats MPT.
We have covered diversification; on to statistical modeling next time!

- Measuring: Statistical Modeling;
- Valuation: Fixed Income, Yield Curves, Equity Valuation;
- Valuation II: Factor Models, Microfoundations, Global Investing, FX;
- Risk Alleviation: Futures, Options, Credit, Structured Products; and,
- All Together Now: Active Portfolios, Investment Firms, Crises.