Dantzig already said it: if we do not know how to make decisions under uncertainty, we are not planning the problem right👇

"Those of us who were doing the planning right from the very beginning understood that the real problem was to be able to do planning under uncertainty. This, I feel, is the real field we should all be working on."

— G. B. Dantzig, E-Optimization (2001)
Robust Linear Complementarity Problems

Martin Schmidt
October 26, 2020, University of Maryland/Zoom

Trier University
Overview

A Primer on Robust Optimization

Linear Complementarity Problems

(Why) Robust Linear Complementarity Problems

Γ-Robust LCPs

Adjustable Robust LCPs

Conclusion
This is joint work with

- Vanessa Krebs
- Christian Biefel
- Emre Çelebi
- Anja Kramer
- Frauke Liers
- Michael Müller
- Jan Rolfes
A Primer on Robust Optimization

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Convex Quadratic Programming

\[
\min_{x \in \mathbb{R}^n} \quad \frac{1}{2} x^\top Q x + c^\top x \quad \text{s.t.} \quad Ax \leq b, \quad Cx = d
\]

- \( Q \in \mathbb{R}^{n \times n} \) is a symmetric and positive semi-definite matrix
- \( A \in \mathbb{R}^{m \times n}, \ C \in \mathbb{R}^{k \times n} \)
- \( b \in \mathbb{R}^m, \ d \in \mathbb{R}^k, \ c \in \mathbb{R}^n \)
Towards the Robust Counterpart

Uncertain QP data

- QP data \((Q, A, C, c, b, d)\) are uncertain
- In particular: contained in a given uncertainty set \(U\)
Uncertain QP data

- QP data \((Q, A, C, c, b, d)\) are uncertain
- In particular: contained in a given uncertainty set \(U\)

Uncertain convex QP

\[
\left\{ \begin{array}{l}
\min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} x^T Q x + c^T x : A x \leq b, \; C x = d \right\} \\
\end{array} \right\}_{(Q,A,C,c,b,d) \in U}
\]

- Family of optimization problems of the nominal type
- Abbreviation \(u := (Q, A, C, c, b, d)\)
Towards the Robust Counterpart

Uncertain QP data

- QP data \((Q, A, C, c, b, d)\) are uncertain
- In particular: contained in a given uncertainty set \(\mathcal{U}\)

Uncertain convex QP

\[
\left\{ \min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} x^\top Q x + c^\top x : A x \leq b, \ C x = d \right\} \right\}_{(Q, A, C, c, b, d) \in \mathcal{U}}
\]

- Family of optimization problems of the nominal type
- Abbreviation \(u := (Q, A, C, c, b, d)\)

Robust counterpart

\[
\min_{x \in \mathbb{R}^n} \left\{ \sup_{u \in \mathcal{U}} \left\{ \frac{1}{2} x^\top Q x + c^\top x : A x \leq b, \ C x = d \text{ for all } u \in \mathcal{U} \right\} \right\}
\]
The first paper and the standard textbook

- Soyster (OR, 1973): First paper on robust (linear) optimization
Some Literature

The first paper and the standard textbook

- Soyster (OR, 1973): First paper on robust (linear) optimization

Many extensions

- Bertsimas, Sim (2003, 2004), Sim (2004): $\Gamma$-robustness
- Fischetti, Monaci (2009): light robustness
- Ben-Tal, Goryashko, Guslitzer, Nemirovski (2004): adjustable robustness
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The Linear Complementarity Problem (LCP)

Given $q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n \times n}$, find a vector $z$ that satisfies

$$z \geq 0, \quad Mz + q \geq 0, \quad z^T(Mz + q) = 0$$

or show that no such vector exists.
The Linear Complementarity Problem (LCP)

Given \( q \in \mathbb{R}^n, M \in \mathbb{R}^{n \times n} \), find a vector \( z \) that satisfies

\[
z \geq 0, \quad Mz + q \geq 0, \quad z^T(Mz + q) = 0
\]

or show that no such vector exists.

**Alternative notation for the LCP(\( q, M \))**

\[
0 \leq z \perp Mz + q \geq 0
\]
Why Would Anyone Care?

The applications are extremely manifold!
Why Would Anyone Care?

The applications are extremely manifold!

- Matrix theory
- Optimality conditions of QPs
- The bimatrix game is an LCP
- Market equilibrium modeling
- Optimal stopping
- Contact mechanics
- Special case of variational inequalities
- ...
The QP

$$\min_x c^\top x + \frac{1}{2} x^\top Q x \quad \text{s.t.} \quad x \geq 0$$

with positive semi-definite $Q$ is equivalent to the LCP($q, M$).
Example #1

The QP

$$\min_x c^T x + \frac{1}{2} x^T Q x \quad \text{s.t.} \quad x \geq 0$$

with positive semi-definite $Q$ is equivalent to the LCP($q, M$).

- Simply write down its KKT conditions
- Can be generalized to QPs with arbitrary inequality constraints
Example #2: Market Equilibrium Modeling

Standard micro-economic setting

Production/Generation
+ Demand (depending on market price)
+ Market clearing conditions
= Market equilibrium problem
Example #2: Market Equilibrium Modeling

Standard micro-economic setting

Production/Generation
+ Demand (depending on market price)
+ Market clearing conditions
= Market equilibrium problem
= LCP (under suitable assumptions)
Production (modeled as an LP)

$$\min_{x \in \mathbb{R}^n} \quad c^T x$$

s.t. $Ax \geq b$ \quad [\lambda]

$Bx \geq r$ \quad [\pi]

$x \geq 0$

Demand

$$r = Dp + d$$

Equilibrating condition

$$p = \pi$$
Take the production KKTs and massage the terms . . .

\[ 0 \leq x \perp c - A^\top \lambda - B^\top p \geq 0 \]
\[ 0 \leq \lambda \perp -b + Ax \geq 0 \]
\[ 0 \leq p \perp -Dp - d + Bx \geq 0 \]

This is the LCP\((q, M)\) with

\[
\begin{pmatrix}
  x \\
  \lambda \\
  p
\end{pmatrix}, \quad
M = \begin{bmatrix}
  0 & -A^\top & -B^\top \\
  A & 0 & 0 \\
  B & 0 & -D
\end{bmatrix}, \quad
q = \begin{pmatrix}
  c \\
  -b \\
  -d
\end{pmatrix}
\]
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Why Robust LCPs?

Production

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad c^T x \\
\text{s.t.} & \quad Ax \geq b \\
& \quad Bx \geq r \\
& \quad x \geq 0
\end{align*}
\]

Demand

\[
r = Dp + d
\]
Why Robust LCPs?

Production

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad c^T x \\
\text{s.t.} & \quad Ax \geq b \\
& \quad Bx \geq r \\
& \quad x \geq 0
\end{align*}
\]

Demand

\[ r = Dp + d \]

Uncertainties are everywhere!

- Price sensitivity \( D, d \)
- Production data \( B \) (e.g., renewable power production)
- Cost data \( c \) (e.g., feed-in tariffs for renewables)
Consider the LCP’s gap function QP

\[
\min_{x \in \mathbb{R}^n} g(x) := x^\top (Mx + q)
\]

s.t. \( x \in X := \{x \in \mathbb{R}^n : x \geq 0, \ Mx + q \geq 0\} \)
Consider the LCP’s gap function QP

\[
\min_{x \in \mathbb{R}^n} g(x) := x^\top (Mx + q)
\]
\[
s.t. \quad x \in \mathcal{X} := \{x \in \mathbb{R}^n: x \geq 0, Mx + q \geq 0\}
\]

**No-Brainer**: A point \(x \in \mathbb{R}^n\) is a solution of the LCP if and only if it is global minimizer of the gap function with objective function value 0.
Consider the LCP’s gap function QP

\[
\min_{x \in \mathbb{R}^n} g(x) := x^\top (Mx + q)
\]

s.t. \( x \in \mathcal{X} := \{ x \in \mathbb{R}^n : x \geq 0, Mx + q \geq 0 \} \)

**No-Brainer**: A point \( x \in \mathbb{R}^n \) is a solution of the LCP if and only if it is global minimizer of the gap function with objective function value 0.

**Expected Gap Minimization Problem**

\[
\min_{\mathcal{X}} \mathbb{E}_{(u_M, u_q)} [g(x; u_M, u_q)]
\]

with

\[
g(x; u_M, u_q) := x^\top (M(u_M)x + q(u_q)).
\]

**Some(!) articles**: Chen, Fukushima (MOR 2005),
Lin, Fukushima (OMS 2006),
Chen, Zhang, Fukushima (Math. Prog. 2009),
Chen, Wets, Zhang (SIOPT 2012)
The Robust LCP

- Consider LCP data $M$ and $q$ to be uncertain
- No assumptions on probability distributions
- $M(u_M)$ and $q(u_q)$ with $u_M \in \mathcal{U}_M$ and $u_q \in \mathcal{U}_q$
- $\mathcal{U}_M$, $\mathcal{U}_q$ are given (deterministic) uncertainty sets
- Example: $q(u_q) := \bar{q} + u_q$ with nominal value $\bar{q}$ and $u_q \in \mathcal{U}_q$
Consider LCP data $M$ and $q$ to be uncertain

- No assumptions on probability distributions
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- Example: $q(u_q) := \bar{q} + u_q$ with nominal value $\bar{q}$ and $u_q \in \mathcal{U}_q$

The robust LCP (= family of LCPs)

$$\{0 \leq x \perp M(u_M)x + q(u_q) \geq 0\}_{(u_M, u_q) \in \mathcal{U}_M \times \mathcal{U}_q}$$
We call a point \( x \) strictly robust feasible if
\[
x \geq 0, \quad M(u_M)x + q(u_q) \geq 0
\]
holds for all \( (u_M, u_q) \in \mathcal{U}_M \times \mathcal{U}_q \).
We call a point $x$ strictly robust feasible if

$$x \geq 0, \quad M(u_M)x + q(u_q) \geq 0$$

holds for all $(u_M, u_q) \in \mathcal{U}_M \times \mathcal{U}_q$.

The point is called a strictly robust LCP solution if it additionally satisfies

$$x^T (M(u_M)x + q(u_q)) = 0 \quad \text{for all} \ (u_M, u_q) \in \mathcal{U}_M \times \mathcal{U}_q.$$
Robustifying the Gap Function QP

\[
\min_{x \in \mathcal{X}(u_M, u_q)} \sup_{(u_M, u_q) \in U_M \times U_q} g(x; u_M, u_q)
\]

with robust feasible set

\[
\mathcal{X}(u_M, u_q) := \{ x \in \mathbb{R}^n : x \geq 0, M(u_M)x + q(u_q) \geq 0 \}
\]
The Robust Gap Function Formulation

Robustifying the Gap Function QP

$$\min_{x \in \mathcal{X}(u_M, u_q)} \sup_{(u_M, u_q) \in U_M \times U_q} g(x; u_M, u_q)$$

with robust feasible set

$$\mathcal{X}(u_M, u_q) := \{x \in \mathbb{R}^n : x \geq 0, M(u_M)x + q(u_q) \geq 0\}$$

This is surprisingly new stuff . . .

- First paper by Wu, Han, Zhu in (2011)
- Latest paper (before our articles): Xie, Shanbhag (2014, 2016)
- Nothing in between!
Our starting point was Xie, Shanbhag (SIOPT 2016).

**Proposition**

A vector $x$ solves

$$0 \leq x \perp M(u_M)x + q(u_q) \geq 0 \quad \text{for all } (u_M, u_q) \in U_M \times U_q$$

if and only if $x$ is a solution of the robust gap function formulation with optimal objective function value of zero.
Our starting point was Xie, Shanbhag (SIOPT 2016).

**Proposition**

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$$0 \leq x \perp M(u_M)x + q(u_q) \geq 0 \quad \text{for all } (u_M, u_q) \in U_M \times U_q$$

if and only if $x$ is a solution of the robust gap function formulation with optimal objective function value of zero.

**Bad news**

- This is almost never the case!
- “almost never” = only in trivial cases
The Economist’s Problem

Our starting point was Xie, Shanbhag (SIOPT 2016).

Proposition

A vector \( x \) solves

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0 \leq x \perp M(u_M)x + q(u_q) \geq 0 \quad \text{for all } (u_M, u_q) \in U_M \times U_q
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if and only if \( x \) is a solution of the robust gap function formulation with optimal objective function value of zero.

Bad news

- This is almost never the case!
- “almost never” = only in trivial cases

This means (for instance):

If the LCP models a market equilibrium,
there is no “robust market equilibrium”.

• Let’s mimic the robust optimization literature starting from 2003 on
• Thus: Consider less conservative notions of robustness
• Xie, Shanbhag (SIOPT 2016) “only” considered the strictly robust case, which delivers the most conservative solutions of all robustness concepts
Remedies?

- Let’s mimic the robust optimization literature starting from 2003 on
- Thus: Consider less conservative notions of robustness
- Xie, Shanbhag (SIOPT 2016) “only” considered the strictly robust case, which delivers the most conservative solutions of all robustness concepts

What we did:

- $\Gamma$-robust LCPs (à la Bertsimas, Sim (2003, 2004) and Sim (2004))
  - Krebs, S. (OMS, 2020): $\ell_1$ and $\ell_\infty$ norm uncertainties
  - Krebs, Müller, S. (Preprint, 2019): ellipsoidal uncertainty sets
- Adjustable Robustness (à la Ben-Tal et al. (2004))
  - Biefel, Rolfes, Liers, S. (Preprint, 2020)
- Applications in power market equilibrium models
Another Economist’s Problem

If existence of robust equilibria cannot be established . . .

What about approximate equilibria?
Another Economist’s Problem

If existence of robust equilibria cannot be established . . .

What about approximate equilibria?

That is, we consider solutions of

$$\min_{x \in \mathcal{X}(u_M, u_q)} \sup_{(u_M, u_q) \in \mathcal{U}_M \times \mathcal{U}_q} g(x; u_M, u_q)$$

with strictly positive optimal objective function values.
If existence of robust equilibria cannot be established . . .

What about approximate equilibria?

That is, we consider solutions of

$$\min_{x \in \mathcal{X}(u_M, u_q)} \sup_{(u_M, u_q) \in \mathcal{U}_M \times \mathcal{U}_q} g(x; u_M, u_q)$$

with strictly positive optimal objective function values.

Questions?

- Do these approximate equilibria exist?
Another Economist’s Problem

If existence of robust equilibria cannot be established . . .

What about approximate equilibria?

That is, we consider solutions of

$$\min_{x \in \mathcal{X}(u_M, u_q)} \sup_{(u_M, u_q) \in U_M \times U_q} g(x; u_M, u_q)$$

with strictly positive optimal objective function values.

Questions?

- Do these approximate equilibria exist?
- What about uniqueness?

The Optimizer’s Problem:
Another Economist’s Problem

If existence of robust equilibria cannot be established . . .

What about approximate equilibria?

That is, we consider solutions of

$$
\min_{x \in \mathcal{X}(u_M, u_q)} \sup_{(u_M, u_q) \in \mathcal{U}_M \times \mathcal{U}_q} g(x; u_M, u_q)
$$

with strictly positive optimal objective function values.

Questions?

- Do these approximate equilibria exist?
- What about uniqueness?

The Optimizer’s Problem:

What about tractability?
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Conclusion
• Consider

\[ \{0 \leq x \perp Mx + q(u) \geq 0\}_{u \in \mathcal{U}} \]

• Given uncertainty set \( \mathcal{U} \subset \mathbb{R}^n \)

• \( \Gamma \)-version of the uncertainty set

\[ \mathcal{U}_\Gamma := \{ u \in \mathcal{U} : |\{ i \in [n] : u_i \neq 0 \}| \leq \Gamma \} \]

• Robust gap function problem

\[
\min_x \sup_{u \in \mathcal{U}_\Gamma} \left\{ x^\top Mx + x^\top q(u) : x \geq 0, \ Mx \geq -q(u) \text{ for all } u \in \mathcal{U}_\Gamma \right\}
\]
Robust gap function problem

\[
\min_x \sup_{u \in \mathcal{U}_\Gamma} \left\{ x^\top Mx + x^\top q(u) : x \geq 0, \ Mx \geq -q(u) \ \text{for all } u \in \mathcal{U}_\Gamma \right\}
\]

Proposition (Krebs, S. (OMS, 2020))

A vector \( x \) solves

\[
0 \leq x \perp Mx + q(u) \geq 0 \quad \text{for all } u \in \mathcal{U}_\Gamma
\]

if and only if \( x \) is a solution of the problem above with optimal objective function value of zero.
Robust gap function problem

$$\min_x \sup_{u \in \mathcal{U}_\Gamma} \left\{ x^T M x + x^T q(u) : x \geq 0, \ M x \geq -q(u) \text{ for all } u \in \mathcal{U}_\Gamma \right\}$$

Proposition (Krebs, S. (OMS, 2020))

A vector $x$ solves

$$0 \leq x \perp M x + q(u) \geq 0 \text{ for all } u \in \mathcal{U}_\Gamma$$

if and only if $x$ is a solution of the problem above with optimal objective function value of zero.

Roadmap

- Tractability of the robust gap function problem
- Existence and uniqueness of approximate robust equilibria
Uncertain $q$: Box Uncertainties

- $\mathcal{U}_\Gamma$: box uncertainty set
  
  $$\mathcal{U}_{\Gamma, \bar{u}}^{\text{box}} := \{ u \in \mathbb{R}^n : -\bar{u}_i \leq u_i \leq \bar{u}_i, \ i \in [n], |\{ i \in [n] : u_i \neq 0 \}| \leq \Gamma \}$$

- $\bar{u}_i \geq 0$ for all $i \in [n]$

- $q(u) := \bar{q} + u$ with $u \in \mathcal{U}_{\Gamma, \bar{u}}^{\text{box}}$
Uncertain $q$: Box Uncertainties

- $\mathcal{U}_\Gamma$: box uncertainty set
  
  \[ \mathcal{U}_{\Gamma, \bar{u}}^{\text{box}} := \{ u \in \mathbb{R}^n : -\bar{u}_i \leq u_i \leq \bar{u}_i, \ i \in [n], |\{i \in [n] : u_i \neq 0\}| \leq \Gamma \} \]

- $\bar{u}_i \geq 0$ for all $i \in [n]$

- $q(u) := \bar{q} + u$ with $u \in \mathcal{U}_{\Gamma, \bar{u}}^{\text{box}}$

The robust counterpart of the gap function problem in this case reads

\[
\min_{x \geq 0} x^\top M x + x^\top \bar{q} + \max_{\{I \subseteq [n] : |I| \leq \Gamma\}} \sum_{i \in I} \bar{u}_i x_i
\]

s.t. \[ M x \geq -\bar{q} + \sum_{i \in I} \bar{u}_i e_i \] for all $I \subseteq [n], |I| \leq \Gamma$
Theorem

The robust counterpart (of the last slide) is equivalent to

\[
\min_{x, \alpha, \beta} \quad x^T M x + x^T \tilde{q} + \alpha \Gamma + \sum_{i=1}^{n} \beta_i \\
\text{s.t.} \quad M_i . x \geq -\tilde{q}_i + \bar{u}_i, \quad i \in [n] \\
\alpha + \beta_i \geq \bar{u}_i x_i, \quad i \in [n] \\
\alpha \geq 0 \\
x_i \geq 0, \quad \beta_i \geq 0, \quad i \in [n]
\]
Theorem

The robust counterpart (of the last slide) is equivalent to

$$\min_{x, \alpha, \beta} x^\top M x + x^\top \bar{q} + \alpha \Gamma + \sum_{i=1}^{n} \beta_i$$

s.t. $M_i.x \geq -\bar{q}_i + \bar{u}_i, \quad i \in [n]$

$\alpha + \beta_i \geq \bar{u}_i x_i, \quad i \in [n]$

$\alpha \geq 0$

$x_i \geq 0, \beta_i \geq 0, \quad i \in [n]$

Proof.

Read the PhD thesis of Melvyn Sim and apply the same techniques. \qed
Results

- The robust counterpart is convex if $M$ is positive semi-definite.
- In this case, existence of approximate robust equilibria can be shown.
- If $M$ is positive definite, the approximate robust equilibrium is unique in $x$.
- Uniqueness of the other “primal” variables cannot be achieved.
Uncertain $q$: Box Uncertainties

Results

- The robust counterpart is convex if $M$ is positive semi-definite
- In this case, existence of approximate robust equilibria can be shown
- If $M$ is positive definite, the approximate robust equilibrium is unique in $x$
- Uniqueness of the other “primal” variables cannot be achieved

We have similar results for the case of $\ell_1$ norm uncertainties
• We now consider the problem

$$\{ 0 \leq x \perp M(u)x + q \geq 0 \}_{u \in U}$$
Uncertain $M$: Box Uncertainties

- We now consider the problem

$$\{0 \leq x \perp M(u)x + q \geq 0\}_{u \in \mathcal{U}}$$

- Let’s start with a definition of $M(u)$ in analogy to $q(u)$:

$$\tilde{M} := [\tilde{m}_{ij}]_{i,j \in [n]}$$

with

$$M(u) := [\tilde{m}_{ij} + u_{ij}]_{i,j \in [n]}$$

and

$$[u_{ij}]_{i,j \in [n]} \in \mathcal{U}.$$
Uncertain $M$: Box Uncertainties

- We now consider the problem

$$\{0 \leq x \perp M(u)x + q \geq 0\}_{u \in \mathcal{U}}$$

- Let’s start with a definition of $M(u)$ in analogy to $q(u)$:

$$\tilde{M} := [\tilde{m}_{ij}]_{i,j \in [n]}$$

with

$$M(u) := [\tilde{m}_{ij} + u_{ij}]_{i,j \in [n]}$$

and

$$[u_{ij}]_{i,j \in [n]} \in \mathcal{U}.$$ 

- Box uncertainties for entries of $M$ (row-wise)

$$\mathcal{U}^{\text{box}}_{\Gamma, \bar{u}, i} := \{ u_i \in \mathbb{R}^n : -\bar{u}_{ij} \leq u_{ij} \leq \bar{u}_{ij}, j \in [n], |\{j \in [n] : u_{ij} \neq 0\}| \leq \Gamma_i \}$$
Uncertain $M$: Box Uncertainties

- We now consider the problem
  \[ \{0 \leq x \perp M(u)x + q \geq 0\}_{u\in\mathcal{U}} \]
- Let's start with a definition of $M(u)$ in analogy to $q(u)$:
  \[ \bar{M} := [\bar{m}_{ij}]_{i,j\in[n]} \]
  with
  \[ M(u) := [\bar{m}_{ij} + u_{ij}]_{i,j\in[n]} \]
  and
  \[ [u_{ij}]_{i,j\in[n]} \in \mathcal{U}. \]
- Box uncertainties for entries of $M$ (row-wise)
  \[ \mathcal{U}_{\Gamma,\bar{u},i} := \{ u_i \in \mathbb{R}^n : -\bar{u}_{ij} \leq u_{ij} \leq \bar{u}_{ij}, j \in [n], |\{ j \in [n] : u_{ij} \neq 0 \}| \leq \Gamma_i \} \]
- Robust counterpart
  \[
  \begin{aligned}
  \min_{x \geq 0} & \quad x^T \bar{M}x + x^T q + \sum_{i \in [n]} \max_{\{ l_i \subseteq [n] : |l_i| \leq \Gamma_i \}} \sum_{j \in l_i} \bar{u}_{ij}x_i x_j \\
  \text{s.t.} & \quad \sum_{j \in [n]} \bar{m}_{ij}x_j - \max_{\{ l_i \subseteq [n] : |l_i| \leq \Gamma_i \}} \sum_{j \in l_i} \bar{u}_{ij}x_j \geq -q_i, \quad i \in [n]
  \end{aligned}
  \]
Theorem

Let $\mathcal{U}_{\Gamma, \bar{u}, i}^{\text{box}}$ be the uncertainty set of row $i \in [n]$ in $M(u)x + q \geq 0$. Then, the robust counterpart (of the last slide) is equivalent to

$$\min_{x, \alpha, \beta, \gamma, \delta, \varepsilon, \xi} \quad x^T Mx + x^T q + \sum_{i \in [n]} \left( \gamma_i \Gamma_i + \sum_{j \in [n]} \delta_{ij} \right)$$

subject to

$$\sum_{j \in [n]} \bar{m}_{ij} x_j - \varepsilon_i \Gamma_i - \sum_{j \in [n]} \xi_{ij} \geq -q_i, \quad i \in [n]$$

$$\varepsilon_i + \xi_{ij} \geq \bar{u}_{ij} x_j, \quad j \in [n]$$

$$\varepsilon_i \geq 0, \quad i \in [n]$$

$$\xi_{ij} \geq 0, \quad i, j \in [n]$$

$$\gamma_i + \delta_{ij} \geq \bar{u}_{ij} x_i x_j, \quad i, j \in [n]$$

$$\gamma_i \geq 0, \quad i \in [n]$$

$$\delta_{ij} \geq 0, \quad i, j \in [n]$$
The “correct” uncertainty modeling:

\[ M(u) := \bar{M} + \sum_{\ell \in [L]} u_\ell M^\ell \]

with \( L \in \mathbb{N} \) and \( M^\ell := [m^\ell_{ij}]_{i,j \in [n]} \in \mathbb{R}^{n \times n} \)
The “correct” uncertainty modeling:

\[ M(u) := \bar{M} + \sum_{\ell \in [L]} u_\ell M^\ell \]

with \( L \in \mathbb{N} \) and \( M^\ell := [m^\ell_{ij}]_{i,j \in [n]} \in \mathbb{R}^{n \times n} \)

Uncertainty set:

\[ \mathcal{U}_{\Gamma, \bar{u}}^{box} := \{ u \in \mathbb{R}^L : 0 \leq u_\ell \leq \bar{u}_\ell, \; \ell \in [L], \; |\{ \ell \in [L] : u_\ell \neq 0 \}| \leq \Gamma \} \]
Good-News-Theorem

Theorem

Consider the “correct” uncertainty set with \( L > \Gamma \). Furthermore, suppose that \( \tilde{M} \) and \( M^\ell, \ell \in [L] \), are positive semidefinite. Then, the robust counterpart is equivalent to the convex, and thus tractable, problem

\[
\min_{x, \alpha, \beta, \gamma, \delta} \quad x^T \tilde{M} x + x^T q + \Gamma \alpha + \sum_{\ell \in [L]} \beta_\ell \\
\text{s.t.} \quad \alpha + \beta_\ell \geq \bar{u}_\ell x^T M^\ell x, \quad \ell \in [L] \\
\alpha \geq 0 \\
\beta_\ell \geq 0, \quad \ell \in [L] \\
\gamma_i \geq 0, \quad i \in [n] \\
\delta_{i\ell} \geq 0, \quad i \in [n], \ \ell \in [L] \\
x \geq 0 \\
\tilde{M}_{i,.} x + q_i - \gamma_i \Gamma - \sum_{\ell \in [L]} \delta_{i\ell} \geq 0, \quad i \in [n] \\
\gamma_i + \delta_{i\ell} \geq -\bar{u}_\ell M_{i,.} x, \quad i \in [n], \ \ell \in [L]
\]
Further Results

**Theorem**
Assume that the Problem of the last slide is feasible and that $\bar{M}$ and $M^\ell$, $\ell \in [L]$, are positive semidefinite. Then, there exists a solution.
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**Proposition**
Suppose that the matrix $\bar{M}$ is positive definite. Then, the solution of Problem is unique in $x$. 
Theorem
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Proposition
Suppose that the matrix $\tilde{M}$ is positive definite. Then, the solution of Problem is unique in $x$.

Comparable results for $\ell_1$ norm uncertainty sets
Uncertain $q$ and $M$

**No-Brainer**

- If the uncertainties for $q$ and $M$ are independent, we can simply combine the separate robustifications.

**Open problem**

- Correlation between uncertainty in $q$ and $M$
Ellipsoidal Uncertainties (Krebs, Müller, S. (2019))

- We obtain qualitatively comparable results
- The required techniques are a bit different
- Some of the results change as expected
  - Tractable counterparts under the assumption positive semidefinite LCP matrix . . .
  - . . . counterpart is an SOCP
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Existence & Uniqueness
• Much harder to achieve
• Before: mainly Frank–Wolfe theorem
• Now: quasi-Frank-and-Wolfe sets
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Existence & Uniqueness

- Much harder to achieve.
- Before: mainly Frank–Wolfe theorem.

A convex set $C \subseteq \mathbb{R}^n$ is called a quasi-Frank-and-Wolfe set, if every quadratic function $f$, which is quasi-convex and bounded from below on $C$, attains its infimum on $C$. 
Overview

A Primer on Robust Optimization

Linear Complementarity Problems

(Why) Robust Linear Complementarity Problems

Γ-Robust LCPs

Adjustable Robust LCPs

Conclusion
Reminder: The Economist’s Problem!

Is there an established robustness concept that allows to prove the existence of robust equilibria?
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**Bad news so far**

- **Strict robustness**
  - No for all relevant geometries of the uncertainty sets
- **Γ-robustness**
  - No for all relevant geometries of the uncertainty sets
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**Bad news so far**

- Strict robustness
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**Good news**

- Adjustable robustness works!
Find a vector $r \in \mathbb{R}^n$, which can be adjusted for all uncertainties $(\zeta, u) \in \mathcal{U}_M \times \mathcal{U}_q$ by a vector $y(\zeta, u)$ such that $z(\zeta, u) := r + y(\zeta, u)$ satisfies

$$0 \leq z(\zeta, u) \perp M(\zeta)z(\zeta, u) + q(u) \geq 0 \quad \text{for all} \quad (\zeta, u) \in \mathcal{U}_M \times \mathcal{U}_q.$$
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**Biefel, Liers, Rolfes, S. (2020)**

- Box uncertainties
- Affine decision rules
- **Existence, characterization, and uniqueness of solutions**
Overview

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Conclusion

Done

- Robust LCPs is a very young field of research
  - First two papers: 2011 and 2014/2016
  - Nothing more up to now except the papers I talked about
- Existence of robust equilibria is hard to establish . . .
- . . . but adjustable robustness does the job!
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Done

- Robust LCPs is a very young field of research
  - First two papers: 2011 and 2014/2016
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To-Do

- Real-World Applications
  - Challenge: Calibration of uncertainty sets
- Other robustness concepts
  - Light robustness, distributional robustness, . . .
- Correlated uncertainties between $q$ and $M$
• Γ-Robust Linear Complementarity Problems
  Jointly with Vanessa Krebs
  In: Optimization Methods and Software. 2020.

• Γ-Robust Linear Complementarity Problems with Ellipsoidal Uncertainty Sets
  Jointly with Vanessa Krebs and Michael Müller

• Affinely Adjustable Robust Linear Complementarity Problems
  Jointly with Christian Biefel, Frauke Liers, and Jan Rolfes

• Γ-Robust Electricity Market Equilibrium Models with Transmission and Generation Investments
  Jointly with Emre Çelebi and Vanessa Krebs
  Accepted for publication (10/2020) in Energy Systems

• Strictly and Γ-Robust Counterparts of Electricity Market Models: Perfect Competition and Nash-Cournot Equilibria
  Jointly with Anja Kramer and Vanessa Krebs
Thanks!