USING COMMUTATIVE ALGEBRA TO EXAMINE A ROGERS-RAMANUJAN IDENTITY

NATE MANKOVICH

ABSTRACT. The study of a subspace of the level 1 standard $\hat{sl}_2$ module by Capparelli-Lepowsky-Milas has given rise to a modern reinterpretation the celebrated Rogers-Ramanujan identities. This subspace can be realized using commutative algebra of graded infinite dimensional complex polynomial ring modules. This project studies a certain finitization of this space from two points of view: a short exact sequence of quotients of polynomial rings and a free resolution of ideals. Then it investigates the connection between a Rogers-Ramanujan identity and the graded dimension of these quotients.

1. INTRODUCTION

This paper investigates a finite commutative algebra realization of the level 1 standard $\hat{sl}_2$ module studied by Capparelli-Lepowsky-Milas. We the space studied by Capparelli-Lepowsky-Milas as

$$W = \mathbb{C}[x_1, x_2, ...]/I$$

W is a quotient, that uses

$$R(t) = \sum_{\substack{0<m_1, 0<m_2 \\ m_1+m_2=t}} x_{m_1} x_{m_2}$$

Then we consider the finite subspaces of W defined as

$$W(n) = \mathbb{C}[x_1, x_2, ...]/(I + \langle x_{n+1} \rangle_\partial)$$

This commutative algebra approach simplifies dimension calculations and module structure to make them accessible to the undergraduate mathematician. The graded dimension of $W(n)$ provides an interesting approach to proving a Rogers-Ramanujan identity. The paper begins with a review of modules, morphism diagrams and free resolutions. Then we use these tools to prove the graded dimension of $W(n)$ via short exact sequences. This approach encounters the Fibonacci Sequence and involves some simple abstract algebra methods of proof. Next, we look at an outline for a proof of the graded dimension of $W(n)$ that uses free resolutions. This approach has not been used in previous studies of $W(n)$. We will look at some examples for small values of n then provide a short explanation of how the proof would follow.

2. BACKGROUND

This section covers the basic machinery necessary to understand the meat of the paper. As usual, each important definition will be enriched with an example.

2.1. Modules over a Ring.

We will begin the paper by introducing an algebraic structure that functions as a generalized vector space. We call these structures modules.

Definition 2.1. Let R be a commutative ring with unity where 1_R is the multiplicative identity in R. An abelian group under addition M is called a *module* over R (R-Module) if it has an action $R \times M \to M$ such that for any $a, b \in R$ and $r, s \in M$:

- $a(r+s) = ar + as$
- $(a+b)s = as + bs$
- $a(sb) = (ab)s$
- $1_R s = s$ for all $s \in M$

Example 2.1. The following are three examples of modules:

- $R = k$ is any field and $M = V$ is a vector space over k.
- $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
- R is a commutative ring with unity and $I \subset R$ is an ideal of R. I and R/I are both R-modules.

Like vector spaces, we are interested in the building blocks of modules.

Definition 2.2. An ordered set $\{m_1, m_2, ...\} \in M$ is a *generating set* (aka spanning set) of M if any $m \in M$ can be written as $r_1m_1 + r_2m_2 + ... = m$ where each $r_i \in R$. This set is a *basis* of M if $r_1m_1 + r_2m_2 + ... = 0$ only when each $r_i = 0 \in R$. We say M is a *free module* if it has a basis.

Definition 2.3. The *dimension* of a module is the size of its basis.

Example 2.2. Let R be a ring and M be a free R-module with the basis $B = \{m_1, m_2, ..., m_n\}$. We can write

$$M = \bigoplus_{i=1}^{n} Rm_i$$

where Rm_i is the free module generated by m_i. Take $v \in Rm_i \bigcap Rm_j$ for $m_i, m_j \in B$. So for some $r_1, r_2 \in R$, $v = r_1m_i = r_2m_j$ so $r_1m_i - r_2m_j = 0$. This means $r_1, r_2 = 0$ so $v = 0$. Thus

$$\bigcap_{i=1}^{n} Rm_i = \{0\}$$

.

Example 2.3. A module without a basis is the group of integers modulo two, $\mathbb{Z}/2\mathbb{Z}$ (aka $\mathbb{Z}_2$) is a $\mathbb{Z}$ module. It is not a free module because for $1 \in \mathbb{Z}/2\mathbb{Z}$ and $2 \in \mathbb{Z}$ we have $2\dot{1} = 0$ which is a non-trivial linear combination of elements in $\mathbb{Z}/2\mathbb{Z}$ that is zero.

Example 2.4. Consider the free module $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$. Its basis is $\{(1,0),(0,1)\}$. The *dimension* of a module as the number of basis elements. So we say $\dim(\mathbb{R}^2) = 2$.

Definition 2.4. Let A be an abelian semigroup under addition and R be a ring. The ring R is *A-graded* if it decomposes as

$$R = \bigoplus_{a \in A} R_a$$

where are all R_a are subgroups of the group $(R, +)$. For $r \in R_a$ and $s \in R_b$, $rs \in R_{a+b}$. If $r \in R_a$ then we say $\mathrm{gr}(r) = a$.

Example 2.5. Let $R = \mathbb{C}[x, y]$. Let $A = \mathbb{Z}_{\geqslant 0}$. Each

$$R_n = \{p \text{ is a polynomial of homogeneous degree } n\} = \{p \in R : \deg(p) = n\}$$

We can write $R = \bigoplus_{0 \leqslant i} R_i$.

If we choose to define $\deg(p)$ to be the highest degree of each monomial in any given polynomial $p \in \mathbb{C}[x, y]$. The problem $\deg(x^2 + 2y) = \deg(x^2 - 5x)$ so $x^2 + 2y, x^2 - 5x \in R_2$. But $x^2 + 2y - x^2 - 5x = 2y - 5x \in R_1$ so R_2 is not closed under addition and therefore cannot be a subgroup of $(R, +)$. Therefore it is important to grade by homogenious degree.

Definition 2.5. Let A be an abelian semigroup under addition. Let R be a graded ring and M be an R-module. We call M a *graded* module if $M = \oplus_{a \in A} M_a$ and for $a, b \in A$ we have $R_a M_b \subseteq M_{a+b}$

Example 2.6. Take $R = \mathbb{C}[x, y]$ and $A = \mathbb{Z}_{\geqslant 0}$. Take the $M = R/\langle x^2, y^3\rangle$. The basis of M is $\{1, x, xy, xy^2, y, y^2\}$. Define M_n to be the set of polynomials in $R/\langle x^2, y^3\rangle$ of homogeneous degree n. Notice

$$\mathbb{C}[x, y]/\langle x^2, y^3\rangle = \bigoplus_{i=0}^{3} M_i$$

We can write out each basis:

- $\text{basis}(M_0) = \{1\}$
- $\text{basis}(M_1) = \{x, y\}$
- $\text{basis}(M_2) = \{xy, y^2\}$
- $\text{basis}(M_3) = \{xy^2\}$

Also for any $p \in M_a$ and $q \in M_b$ we have $pq \in M_{a+b}$. For example, take $y^2 + xy \in M_3$ and take $2x \in M_1$. Notice $(y^2 + xy)2x = 2xy^2 + 2x^2y = 2xy^2 \in M_3$.

Definition 2.6. Let R be a ring. A *derivation*, ∂, is a linear operator that distributes across addition and follows Leibniz's law:

$$\partial(rs) = \partial(r)s + r\partial(s)$$

Example 2.7. For our example we let R be the ring of $(n \times n)$ matrices with real coefficients. We represent R as $R = \text{Mat}_{n\times n}\mathbb{R}$. Fix $A \in R$ and define $\partial_A(B) = AB - BA$ for any $B \in R$. We will show ∂_A is a derivation. Take any $B, C \in R$

$$\partial_A(BC) = A(BC) - (BC)A = (AB - BA)C + B(AC - CA) = \partial_A(B)C + B\partial_A(C)$$

also notice ∂_A is additive

$$\partial_A(B + C) = A(B + C) - (B + C)A = AB - BA + AC - CA = \partial_A(B) + \partial_A(C)$$

Definition 2.7. Let R be a ring and N, M be two R-modules. $\phi : N \to M$ is a *module homomorphism* if $\phi(n + n') = \phi(n) + \phi(n')$ and $\phi(rn) = r\phi(n)$ for all $n, n' \in N$ and $r \in R$. The kernel of ϕ is a subset of N and is a module. We define $\ker(\phi) = \{p \in N : \phi(p) = 0\}$. The image of ϕ is a submodule of M. We define $\text{im}(\phi) = \{\phi(p) : p \in N\}$

Example 2.8. In vector spaces, module homomorphisms are linear transformations. Let $R = \mathbb{R}$, $N = \mathbb{R}^3$ and $M = \mathbb{R}^2$. We will define ϕ as multiplication by the matrix

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

ϕ is a module homomorphism because matrix multiplication distributes over addition and multiplication by a real scalar. It is fun to notice that this matrix is a projection from three dimensions to two dimensions including a reflection across the second dimension axis. We find the kernel is

$$\ker(\phi) = \{ \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} : r \in \mathbb{R} \}$$

Take

$$n = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

in N. So

$$\phi(n) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

We can string homomorphisms together to create sequences of maps between R-modules for some ring R.

2.2. **Diagrams of Morphisms in Algebraic Structures.** We will discuss exact sequences and commutative diagrams in general for any algebraic structure and its associated morphisms. This allows us to provide examples of the following definitions with homomorphisms between groups, rings and modules.

Definition 2.8. A sequence of morphisms between the algebraic structures $M_0, M_1, ...$

$$M_0 \xrightarrow{\phi_1} M_1 \xrightarrow{\phi_2} M_2 \xrightarrow{\phi_3} ...$$

is *exact* if for every $i \in \mathbb{N}$ $\phi_i \circ \phi_{i-1}(m) = 0$ for any $m \in M_i$ and $\ker(\phi_i) = \text{im}(\phi_{i-1})$.

We are particularly interested in short exact sequences.

Definition 2.9. Let A, B, C be algebraic structures. Consider the sequence of homomorphisms

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

This is a *short exact sequence* if $\text{im}(f) = \ker(g)$, f is an injection and g is a surjection. If a sequence is exact then $\dim(B) = \dim(A) + \dim(C)$.

Example 2.9. Again, we will use some linear algebra for an example. Consider this sequence of maps between $\mathbb{R}$ modules

$$0 \to \mathbb{R}^3 \xrightarrow{f} \mathbb{R}^5 \xrightarrow{g} \mathbb{R}^2 \to 0$$

f is defined by matrix multiplication by

$$\begin{bmatrix} 55/3 & -18 & -18 \\ -54 & 54 & 54 \\ -55/2 & 55/2 & 27 \\ -18 & 18 & 18 \\ 18 & -18 & -18 \end{bmatrix}$$

We define g by matrix multiplication by

$$\begin{bmatrix} 0 & 0 & 0 & 3 & 3 \\ 0 & -1 & 2 & 0 & 1/18 \end{bmatrix}$$

which is a projection of $\mathbb{R}^5$ onto $\mathbb{R}^2$. If we take any three dimensional real vector $\mathbf{x}$, we have $g \circ f(\mathbf{x}) = \mathbf{0}$. Also $\ker(g) = \mathbb{R}^3 = \text{im}(f)$. Since all these are $\mathbb{R}$-modules we have $\dim(\mathbb{R}^5) = \dim(\mathbb{R}^2) + \dim(\mathbb{R}^3) = 2 + 3 = 5$.

Example 2.10. The rank nullity theorem states that if $T : V \to W$ is a linear transformation between two vector spaces then

$$\text{rank}(T) + \text{null}(T) = \dim(V)$$

We define the short exact sequence

$$0 \to \ker(T) \xrightarrow{\iota} V \xrightarrow{T} \text{im}(T) \to 0$$

where ι is an inclusion map. So $\dim(V) = \dim(\ker(T)) + \dim(\text{im}(T))$. Notice $\dim(\ker(T)) = \text{null}(T)$ and $\dim(\text{im}(T)) = \text{rank}(T)$.

Example 2.11. Exact sequences can also be used with groups. However, the concept of dimensions does not apply. Suppose there is an exact sequence

$$0 \to N \xrightarrow{f} G \xrightarrow{g} H \to 0$$

So f must be an injection and g must be a surjection and $\text{im}(f) = \ker(g) = N$. Now recall the first isomorphism theorem for groups which says $H \cong G/\ker(g) = G/N$.

Example 2.12. For example consider the sequence of maps

$$0 \to S_3 \xrightarrow{\psi} D_6 \xrightarrow{\phi} \mathbb{Z}_2 \to 0$$

For some $n \in \mathbb{N}$, D_n is generated by rotations (r) and reflections (s). S_n is generated by $(1,2)$ and $(1,2,...,n)$. To define a homomorphism we just need to know what it does to the generators. So we define $\psi((1,2)) = s$, $\psi((1,2,3)) = r^2$ and $\phi(r) = 1$, $\phi(s) = 0$. Notice $\ker(\iota) = \{e, r^2, r^4, s, sr^2, sr^4\} = \text{im}(\pi) = D_3$. ψ is an injection and ϕ is a surjection. Using the first isomorphism theorem for groups $D_6/S_3 \cong \mathbb{Z}_2$.

Now we will introduce commutative diagrams and eventually link these diagrams with exact sequences via a group theory example.

Definition 2.10. Let A, B, C, D be algebraic structures and f, g, ϕ, ψ be morphisms between them.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \phi\downarrow & & \downarrow\psi \\ C & \xrightarrow{g} & D \end{array}$$

We say the diagram above commutes if for all $a \in A$, $\psi \circ f(a) = g \circ \phi(a)$.

Example 2.13. Consider this diagram:

$$\begin{array}{ccc} D_4 & \xrightarrow{f} & D_8 \\ \phi\downarrow & & \downarrow\psi \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \xrightarrow{g} & \mathbb{Z}_4 \end{array}$$

Below are the definitions of each map where r, s are the generators for D_n, $n \in \mathbb{N}$

- $f(r) = r^2$
- $f(s) = s$
- $\phi(r) = (1, 0)$

- $\phi(s) = (0, 1)$
- $g((1, 0)) = g((0, 1)) = 2$
- $\psi(r) = 1$
- $\psi(s) = 2$

This is a commutative diagram. For example $\psi \circ f(sr) = \psi(sr^2) = 2 + 2 = 4 = 0$ and $g \circ \phi(sr) = g(1, 1) = g(0, 1) + g(1, 0) = 2 + 2 = 4 = 0$.

Now let's look at the kernel of each of these maps.

- $\ker(f) = \{e\}$
- $\ker(\phi) = \{e, r^2\}$
- $\ker(g) = \{(1, 1), (0, 0)\}$
- $\ker(\psi) = \{e, r^4, sr^2, sr^6\}$

Using a generic injective identity maps $\iota_f, \iota_\phi, \iota_g, \iota_\psi$ we can make a larger diagram with 4 exact sequences and a commutative center.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \ker(\phi) & \longrightarrow & \ker(\psi) & & \\
 & & & & \downarrow{\scriptstyle \iota_\phi} & & \downarrow{\scriptstyle \iota_\psi} & & \\
0 & \longrightarrow & \ker(f) & \xrightarrow{\iota_f} & D_4 & \xrightarrow{f} & D_8 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \\
0 & \longrightarrow & \ker(g) & \xrightarrow{\iota_g} & \mathbb{Z}_2 \times \mathbb{Z}_2 & \xrightarrow{g} & \mathbb{Z}_4 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Note: In order to simplify the diagram the symbol 0 has been chosen to represent the trivial group of the identity of multiplicative and additive groups. For example: the 0 to the right of D_8 represents $\{1\}$.

2.3. **Free Resolutions of R-Modules.**

Now we will consider a sequence of module homomorphisms which is another way to calculate the graded dimension of ideals and quotients of modules. First we need to understand some basic notation:

Let A be an abelian monoid under addition and R be a graded module with $\text{gr}(r) = a \in A$ for $r \in R$. We use the notation $R(-b)$ ($b \in \mathbb{A}$) to define a graded module that has all the same elements as R but the grading has changed to $\text{gr}(r') = a + b$ for any $r' \in R(-b)$. For the rest of the paper all our modules will be $\mathbb{N}$ graded.

Example 2.14. Consider the module $R_n = \mathbb{C}[x_1, x_2, ..., x_n]$ with the grading called weight which is the sum of the subscripts of a monomial in a given homogeneously weighted polynomial. For example, $\text{wt}(x_2^3 + x_5 x_1) = 6$. If we view $x_2^3 + x_5 x_1 \in R_n(-3)$ then $\text{wt}(x_2^3 + x_5 x_1) = 9$.

Definition 2.11. If we have an ideal I of a module R we define a free resolution of I as a sequence of maps between free modules R^{n_i}

$$0 \to ... \xrightarrow{f_3} R^{n_2} \xrightarrow{f_2} R^{n_1} \xrightarrow{f_1} I \to 0$$

We call each f_i is defined by a $n_{i-1} \times n_i$ syzygy matrix with entries $s_{k,l}$ for the kth row and lth column. Each $R^{n_i} = \oplus_{j=1}^{n_i} R(-a_{i,j})$ where each $a_{i,j} \in \mathbb{Z}$. We can calculate each $a_{i,j} = a_{i-1,j} + s_{j,j}$.

It is well known that $\mathbb{Z}$-modules are finitely generated abelian groups.

Example 2.15. Consider the $\mathbb{Z}$-module $\mathbb{Z}/8\mathbb{Z}$ We will find a free resolution of this module.

$$0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}/8\mathbb{Z} \to 0$$

where f is defined by multiplication and g is a projection map. Specifically, $f(a) = 8a$ and $g(b) = b + 8\mathbb{Z}$ for $a, b \in \mathbb{Z}$. $\text{im}(f) = \ker(g) = 8\mathbb{Z}$ and f is defined on all $\mathbb{Z}$ and all of $\mathbb{Z}/8\mathbb{Z}$ is mapped to by g.

Definition 2.12. A *Hilbert Series* for a singly graded R-Module called M is a function $F(M, q)$ of a single variable where the coefficient of the term q^k in $F(M, q)$ ($k \in \mathbb{Z}_{\geqslant 0}$) is the dimension of the submodule of M with only elements of grading k.

Note: $W(n)$ we will use $\chi_n(q, x)$ to represent the Hilbert series for the doubly graded $W(n)$.

Example 2.16. Let $R = \mathbb{C}[x_1, x_2]$ with the weight grading from earlier. Take the ideal $I = \langle x_1^2 \rangle$. We use the res command from Macauly 2 to calculate the free resolution

$$0 \to R^1 \xrightarrow{f} I \to 0$$

where f is the one by one matrix x_1^2. So $R^1 = R(-(2+0) = R(-2))$.

We can use these resolutions to compute Hilbert series for I and for $\mathbb{C}[x_1, x_2]/I$. To do this we must know the Hilbert series

$$F(\mathbb{C}[x_1, x_2, ..., x_n], q) = \frac{1}{\prod_{i=1}^{n}(1 - q^i)}$$

The Hilbert series for an arbitrary free $\mathbb{C}[x_1, x_2, ..., x_n]$ - module

$$M = \oplus_{i \geqslant 1} \mathbb{C}[x_1, x_2, ..., x_n](-m_i)$$

for $m_i \in \mathbb{Z}_{\geqslant 0}$ can be written as

$$F(M, q) = \frac{\sum_{i \geqslant 1} q^{m_i}}{\prod_{i=1}^{n}(1 - q^i)}$$

Definition 2.13. Consider the following free resolution where I is an ideal of the module R

$$0 \to ... \to R_2^{i_2} \to R_1^{i_1} \to I \to 0$$

where $k, i_k \in \mathbb{N}$. The Hilbert series

$$F(I, q) = \sum_{k \geqslant 1} (-1)^{k+1} F(R_k^{i_k}, q)$$

and

$$F(R/I, q) = F(R, q) - F(I, q)$$

Also if some

$$I = \bigoplus_{i \in \mathbb{N}} R_i(-m_i)$$

for $m_i \in \mathbb{N}$ then

$$F(I, q) = \sum_{i \in \mathbb{N}} F(R_i(-m_i), q)$$

Example 2.17. Continuing the example above

$$F(I,q) = \frac{q^2}{(1-q)(1-q^2)}$$

$$F(\mathbb{C}[x_1,x_2]/I,q) = \frac{1-q^2}{(1-q)(1-q^2)} = \frac{1}{1-q}$$

3. Constructing W(n)

Now we will provide more holistic definitions of W and $W(n)$ from the introduction. Given $x_i \in \mathbb{C}[x_1,x_2,...]$ define the derivation, ∂ as $\partial(x_i) = ix_{i+1}$ for $i \in \mathbb{N}$. Define $n \in \mathbb{N}$ compositions of ∂ as $\partial^n(p) = \partial(\partial(...(\partial(p)...))$ where $\partial^0(p) = p$. For a complex polynomial ring ideal generated by the polynomials $p_1, p_2, ..., p_n$ we define

$$\langle p_1, p_2, ..., p_n\rangle_\partial = \langle \partial^i(p_j) : i \in \mathbb{Z}_{\geqslant 0},\ 1 \leqslant j \leqslant n\rangle$$

To begin our first construction of $W(n)$ we will define the t^th relation as $\partial^{t-2}(x_1^2)$ for $t \in \mathbb{N}$ and $t \geqslant 2$.

Now we can produce the first definition of W and $W(n)$ using the relation definition above as

$$W = \mathbb{C}[x_1,x_2,...]/\langle x_1^2\rangle_\partial$$

A finitization of W is $W(n)$ which we define as

$$W(n) = \mathbb{C}[x_1,x_2,...]/\langle x_1^2, x_{n+1}\rangle_\partial$$

Another construction of $W(n)$ uses the relation definition

$$R(t) = \sum_{\substack{m_1,m_2>0\\ m_1+m_2=t}} x_{m_1}x_{m_2}$$

for $m_1, m_2 \in \mathbb{N}$. Then we can use

$$I = \langle R(t) : t \geqslant 2\rangle$$

to define

$$\mathbb{C}[x_1,x_2,...]/I$$

and

$$\mathbb{C}[x_1,x_2,...]/(I + \langle x_{n+1}\rangle_\partial)$$

Proposition 3.1. *The two ideals I and $\langle x_1^2\rangle_\partial$ of $\mathbb{C}[x_1,x_2,...]$ are equal. Thus*

$$W = \mathbb{C}[x_1,x_2,...]/I$$

$$W(n) = \mathbb{C}[x_1,x_2,...]/(I + \langle x_{n+1}\rangle_\partial)$$

Proof. To do this we will show each $\langle x_1^2\rangle_\partial$ is a scalar multiple of $R(t)$ which implies that $\langle \partial^0(x_1^2), \partial^1(x_1^2), \partial^2(x_1^2)...\rangle = \langle R(2), R(3), ...\rangle = I$.

We claim $R(t) = \frac{1}{(t-2)!}\partial^{t-2}(x_1^2)$.

We will proceed by induction on t for the usual assumptions $t \geqslant 2$ and $t \in \mathbb{N}$. For the base case $t = 2$ so $\frac{1}{(2-2)!}\partial^{2-2}(x_1^2) = x_1x_1$. Suppose for some $t > 2 \in \mathbb{N}$ we have $R(t) = \frac{1}{(t-2)!}\partial^{t-2}(x_1^2)$. Now we will show $R(t+1) = \frac{1}{(t-1)!}\partial^{t-1}(x_1^2)$.

$$
\begin{aligned}
\frac{1}{t-1}\partial R(t) &= \frac{1}{t-1}\partial\left(\frac{1}{(t-2)!}\partial^{t-2}(x_1^2)\right)\\
&= \frac{1}{t-1}\partial\left(\sum_{\substack{m_1,m_2>0\\ m_1+m_2=t}} x_{m_1}x_{m_2}\right)\\
&= \frac{1}{t-1}\sum_{\substack{m_1,m_2>0\\ m_1+m_2=t}} \partial(x_{m_1}x_{m_2})\\
&= \frac{1}{t-1}\sum_{\substack{m_1,m_2>0\\ m_1+m_2=t}} (m_1x_{m_1+1}x_{m_2}+m_2x_{m_1}x_{m_2+1})\\
&= \frac{1}{t-1}\Bigg((t-1)x_tx_1+(t-1)x_1x_t\\
&+ \sum_{\substack{m_1>0,m_2>1\\ m_1+m_2=t}} m_1x_{m_1+1}x_{m_2} + \sum_{\substack{m_1>1,m_2>0\\ m_1+m_2=t}} m_2x_{m_1}x_{m_2+1}\Bigg)\\
&= \frac{1}{t-1}\Bigg((t-1)x_tx_1+(t-1)x_1x_t\\
&+ \sum_{\substack{m_1,m_2>1\\ m_1+m_2=t+1}} ((m_1-1)x_{m_1}x_{m_2}+(m_2-1)x_{m_1}x_{m_2+1})\Bigg)\\
&= \frac{1}{t-1}\Bigg((t-1)x_tx_1+(t-1)x_1x_t\\
&+ (t-1)\sum_{\substack{m_1,m_2>1\\ m_1+m_2=t+1}} x_{m_1}x_{m_2}\Bigg)\\
&= \sum_{\substack{m_1,m_2>0\\ m_1+m_2=t+1}} x_{m_1}x_{m_2}\\
&= R(t+1)
\end{aligned}
$$

By way of mathematical induction, the two ideals I and $\langle x_1^2\rangle_\partial$ of $\mathbb{C}[x_1, x_2, ...]$ are equal. □

Define

$$R_n(t) = \sum_{\substack{0<m_1,m_2\leqslant n\\ m_1+m_2=t}} x_{m_1}x_{m_2}$$

for $m_1, m_2 \in \mathbb{N}$. We use this to build

$$I_n = \langle R_n(t) : t \geqslant 2\rangle$$

Using this we can realize $W(n)$ as a quotient of $\mathbb{C}[x_1, x_2, ..., x_n]$ without any derivation structure using the following result:

Theorem 3.1.
$$\mathbb{C}[x_1, x_2, ..., x_n]/I_n \cong W(n)$$

Proof. Define
$$\phi : \mathbb{C}[x_1, x_2, ..., x_n] \to W(n)$$
$$\phi(p(x_1.x_2, ..., x_n)) = p(x_1.x_2, ..., x_n) + I + \langle x_{n+1}\rangle_\partial$$
for $p(x_1.x_2, ..., x_n) \in \mathbb{C}[x_1, x_2, ..., x_n]$.

ϕ is a well defined homomorphism because is is the composition of an inclusion map
$$\mathbb{C}[x_1, x_2, ..., x_n] \to \mathbb{C}[x_1, x_2, ...]$$
and a projection map
$$\mathbb{C}[x_1, x_2, ...] \to \mathbb{C}[x_1, x_2, ...]/(I + \langle x_{n+1}\rangle_\partial)$$

We want to show ϕ is a surjection with $\ker(\phi) = I_n$. Then using the first isomorphism theorem $\mathbb{C}[x_1, x_2, ..., x_n]/I_n \cong W(n)$.

$$\begin{aligned}\ker(\phi) &= \{p(x_1, x_2, ..., x_n) \in \mathbb{C}[x_1, x_2, ..., x_n] : \phi(p(x_1, x_2, ..., x_n)) = 0\} \\ &= \{p(x_n, x_2, ..., x_n) \in \mathbb{C}[x_n, x_2, ..., x_n] : \phi(p(x_1, x_2, ..., x_n)) \in I + \langle x_{n+1}\rangle_\partial\}\end{aligned}$$
because $0 \in W(n)$ is a $q(x_1, x_2, ..., x_n) \in I + \langle x_{n+1}\rangle_\partial$.

First we will show ϕ is a surjection. If we take any $p(x_1, x_2, ...) + I + \langle x_{n+1}\rangle_\partial \in W(n)$ then we can separate
$$\begin{aligned}p(x_1, x_2, ...) + I + \langle x_{n+1}\rangle_\partial &= p'(x_1, x_2, ..., x_n) + q(x_{n+1}, x_{n+2}, ...) + I + \langle x_{n+1}\rangle_\partial \\ &= p'(x_1, x_2, ..., x_n) + I + \langle x_{n+1}\rangle_\partial\end{aligned}$$
Then $\phi(p'(x_1, x_2, ..., x_n)) = p(x_1, x_2, ...) + I + \langle x_{n+1}\rangle_\partial \in W(n)$ therefore ϕ is a surjection.

Now we will show $\ker(\phi) \supseteq I_n$. Take an arbitrary $p(x_1, x_2, ..., x_n) \in I_n$. Then we can write $p(x_1, x_2, ..., x_n) = p_2 R_n(2) + p_3 R_n(3) + ... + p_N R_n(N)$ for some $N \in \mathbb{N}$. We say
$$R_n(t) = R(t) - \sum_{\substack{m_1, m_2 \geqslant n+1 \\ m_1 + m_2 = t}} x_{m_1} x_{m_2} \in I + \langle x_{n+1}\rangle_\partial$$
Since each term in $p(x_1, x_2, ..., x_n)$ has some $R_n(t)$ as a factor, all of $p(x_1, x_2, ..., x_n) \in I + \langle x_{n+1}\rangle_\partial$. Then $p(x_1, x_2, ..., x_n) \in \ker(\phi)$.

Now to prove $\ker(\phi) \subseteq I_n$. Take some $p(x_1, x_2, ...) \in \ker(\phi)$. Then we can write
$$p(x_1, x_2, ...) = p_2 R(2) + p_3 R(3) + ... + p_M R(M) + x_{n+1} q_1 + x_{n+2} q_2 + ... + x_{n+T} q_T$$
where $M, T \in \mathbb{N}$ and p_i, q_j are polynomials. Now separate every $p_i = p_i^0 + p_i^1$ where $p_i^0 \in \mathbb{C}[x_1, x_2, ..., x_n]$ and $p_i^1 \in \langle x_{n+1}\rangle_\partial$. So we say
$$\begin{aligned}p(x_1, x_2, ...) = &(p_2^0 + p_2^1)R(2) + (p_3^0 + p_3^1)R(3) + ... + (p_M^0 + p_M^1)R(M) \\ &+ x_{n+1} q_1 + x_{n+2} q_2 + ... + x_{n+T} q_T\end{aligned}$$
Recall we can take
$$R(t) = R_n(t) - \sum_{\substack{m_1, m_2 \geqslant n+1 \\ m_1 + m_2 = t}} x_{m_1} x_{m_2}$$
Now all terms that are multiples of x_k where $k \geqslant n + 1$ are 0 so
$$p(x_1, x_2, ...) = p_2^0 R_n(2) + p_3^0 R_n(3) + ... + p_M^0 R_n(M) \in I_n$$

Thus $\ker(\phi) \subseteq I_n$ so $\ker(\phi) = I_n$. Since ϕ is a surjection and $\ker(\phi) = I_n$ the first isomorhpism theorem tells us $\mathbb{C}[x_1, x_2, ..., x_n]/I_n \cong W(n)$. □

4. The Graded Dimension of W(n)

4.1. **Using Short Exact Sequences.** We can use short a exact sequences to find the dimension of $W(n)$ as a recursion.

First we will need to define six maps.

Definition 4.1.

$$\begin{array}{ccccc} \mathbb{C}[x_1, ..., x_{n-2}] & \xrightarrow{f} & \mathbb{C}[x_1, ..., x_n] & \xrightarrow{g} & \mathbb{C}[x_1, ..., x_{n-1}] \\ \downarrow{\scriptstyle \pi_{n-2}} & & \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_{n-1}} \\ W(n-2) & \xrightarrow{\bar{f}} & W(n) & \xrightarrow{\bar{g}} & W(n-1) \end{array}$$

Take and $p(x_1, ..., x_{n-2}) \in \mathbb{C}[x_1, ..., x_{n-2}]$, $s(x_1, ..., x_{n-1}) \in \mathbb{C}[x_1, ..., x_{n-1}]$ and $q(x_1, ..., x_n) \in \mathbb{C}[x_1, ..., x_n]$. We define

$$f(p(x_1, ...x_{n-2})) = x_1 p(x_3, ...x_n)$$

$$g(q(x_1, ...x_n)) = q(0, x_1, ...x_{n-1})$$

$$\pi_{n-2}(p(x_1, ...x_{n-2})) = p(x_1, ...x_{n-2}) + I_{n-2}$$

$$\pi_{n-1}(q(x_1, ...x_{n-1})) = q(x_1, ...x_{n-1}) + I_{n-1}$$

$$\pi_n(q(x_1, ...x_n)) = q(x_1, ...x_n) + I_n$$

Take $p(x_1, x_2, ...x_{n-2}) \in W(n-2)$ and $q(x_1, x_2, ...x_n) \in W(n)$. We define

$$\bar{f}(p(x_1, x_2, ...x_{n-2})) = x_1 p(x_3, x_4, ..., x_n)$$

$$\bar{g}(q(x_1, x_2, ...x_n)) = p(0, x_1, x_2, ..., x_{n-1})$$

The following lemma is the key to proving the dimension of $W(n)$.

Lemma 4.1.

$$g(I_n + x_1\mathbb{C}[x_1, ..., x_n]) = I_{n-1}$$

Proof. First take any $p(x_1, ..., x_{n-1}) \in I_{n-1}$ and $p(0, x_2, ..., x_n) \in I_n \subseteq I_n + x_1\mathbb{C}[x_1, ..., x_n]$. Notice that

$$g(p(0, x_2, ..., x_n)) = p(x_1, ..., x_{n-1}) \in g(I_n) \subseteq g(I_n + x_1\mathbb{C}[x_1, ..., x_n])$$

Thus $I_{n-1} \subseteq g(I_n + x_1\mathbb{C}[x_1, ..., x_n])$. Now take any

$$p(x_1, ..., x_{n-1}) \in g(I_n + x_1\mathbb{C}[x_1, ..., x_n])$$

Then $p \in g(x_1\mathbb{C}[x_1, ..., x_n])$ or $p \in g(I_n)$. So if $p \in x_1\mathbb{C}[x_1, ..., x_n]$ then $g(p) = 0$ because $g(x_1) = 0$. If $p \in I_n$ then $g(p) \in I_{n-1}$ because

$$g : \mathbb{C}[x_1, ..., x_n] \to \mathbb{C}[x_1, ..., x_{n-1}]$$

Thus $g(I_n + x_1\mathbb{C}[x_1, ..., x_n]) \subseteq I_{n-1}$. □

Theorem 4.1.

$$0 \to W(n-2) \xrightarrow{\bar{f}} W(n) \xrightarrow{\bar{g}} W(n-1) \to 0$$

is a short exact sequence.

Proof. Recall that an exact map has $\mathrm{im}(\bar{f}) = \ker(\bar{g})$, $\bar{f}$ is an injection and $\bar{g}$ is a surjection.

If we can show $\ker(\bar{f}) = \{0\}$ then $\bar{f}$ is an injection. Suppose by way of contradiction $0 \neq p(x_1, x_2, ..., x_{n-2}) \in \ker(\bar{f})$. So $x_1 p(x_3, x_4, ..., x_n) \in I_n$. This means there exists at least one $R_n(t)$ and a polynomials $q_i \in \mathbb{C}[x_1, x_2, .., x_n]$ where

$$x_1 p(x_3, x_4, ..., x_n) = \sum_{i=1}^{2n} q_i R_n(i)$$

Notice that at least one $R_n(t)$ must have an x_2 and $x_1 p(x_3, x_4, ..., x_n)$ does not have an x_2. Thus we have reached a contradiction so $\bar{f}$ is an injection.

Take any $q(x_1, x_2, ..., x_{n-1}) \in W(n-1)$. If we take $q'(x_2, x_3, ..., x_n) \in W(n)$ notice $\bar{g}(q'(x_2, x_3, ..., x_n)) = q(x_1, x_2, ..., x_{n-1})$, So $\bar{g}$ is indeed a surjection.

Now we will show for $p(x_1, ..., x_n) \in \mathbb{C}[x_1, ..., x_n]$ that $\pi_{n-1} \circ g(p) = \bar{g} \circ \pi_n(p)$.

$$\pi_{n-1} \circ g(p) = \pi_{n-1}(p(0, x_1, ..., x_{n-1})) = p(0, x_1, ..., x_{n-1}) + I_{n-1} \in W(n-1)$$

$$\bar{g} \circ \pi_n(p) = \bar{g}(p(x_1, ..., x_n) + I_n) = p(0, x_1, ..., x_{n-1}) + I_{n-1} \in W(n-1)$$

Therefore the diagram bellow commutes.

$$\begin{array}{ccccc}
\mathbb{C}[x_1, ..., x_{n-2}] & \xrightarrow{f} & \mathbb{C}[x_1, ..., x_n] & \xrightarrow{g} & \mathbb{C}[x_1, ..., x_{n-1}] \\
\downarrow{\scriptstyle \pi_{n-2}} & & \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_{n-1}} \\
W(n-2) & \xrightarrow{\bar{f}} & W(n) & \xrightarrow{\bar{g}} & W(n-1)
\end{array}$$

Now we will use the lemma to show $\ker(\bar{g}) = I_n + x_1\mathbb{C}[x_1, ..., x_n] = \mathrm{im}(\bar{f})$.

The first half of the equality is proved by showing $p \in \ker(\bar{g})$ if and only if $p \in I_n + x_1\mathbb{C}[x_1, ..., x_n]$. Take and arbitrary $p(x_1, ..., x_n) \in \ker(\bar{g})$. Notice

$$\bar{g}(p(x_1, ..., x_n)) = g(p(x_1, ..., x_n)) = p(0, x_1, ..., x_{n-1}) = 0$$

This only happens when $p(x_1, ..., x_n) = x_1 p'(x_2, ..., x_n)$. So $p(x_1, ..., x_n) \in x_1\mathbb{C}[x_1, ..., x_n]$ which implies $p(x_1, ..., x_n) \in I_n + x_1\mathbb{C}[x_1, ..., x_n]$. If $p \in \ker(\bar{g})$ then $p \in I_n + x_1\mathbb{C}[x_1, ..., x_n]$. Now take some $p(x_1, ..., x_n) \in I_n + x_1\mathbb{C}[x_1, .., x_n]$. It is true that

$$\pi_{n-1} \circ g(p(x_1, ..., x_n)) = \bar{g} \circ \pi_n(p(x_1, ..., x_n))$$

Also notice

$$I_n + x_1\mathbb{C}[x_1, ..., x_n] \subseteq I_n + \mathbb{C}[x_1, ..., x_n] = W(n)$$

Then

$$\bar{g} \circ \pi_n(p(x_1, ..., x_n)) = \bar{g}(p(x_1, ..., x_n) + I_n)$$

By the fact that

$$g(I_n + x_1\mathbb{C}[x_1, ..., x_n]) = I_{n-1}$$

for $g(p(x_1, ..., x_n)) \in I_{n-1}$. So

$$\bar{g}(p(x_1, ..., x_n)) = \bar{g} \circ \pi_n(p(x_1, ..., x_n)) = \pi_{n-1} \circ g(p(x_1, ..., x_n)) = 0$$

which means $p(x_1, ..., x_n) \in \ker(\bar{g})$.

The second half of the equality is proved by showing $p \in \mathrm{im}(\bar{f})$ if and only if $p \in I_n + x_1\mathbb{C}[x_1, ..., x_n]$. Now suppose $p(x_1, ..., x_n) \in \mathrm{im}(\bar{f})$. This means $p(x_1, ..., x_n) = \bar{f}(p'(x_1, ..., x_{n-2})) = x_1 p'(x_3, ..., x_n) \in x_1\mathbb{C}[x_1, ..., x_n]$. So $p(x_1, ..., x_n) \in I_n + x_1\mathbb{C}[x_1, ..., x_n]$. Now take some $p(x_1, ..., x_n) \in I_n + x_1\mathbb{C}[x_1, ..., x_n]$. Notice $x_1^2 = x_1 x_2 = 0$ in $I_n + x_1\mathbb{C}[x_1, ..., x_n]$ for all $n \geqslant 3$. Then we can write $p(x_1, ..., x_n) =$

$x_1p'(x_1,...,x_n) = x_1p'(0,0,x_3,...,x_n) = x_1p'(x_3,...,x_n) \in \text{im}(\bar{f})$.
Thus this is an exact sequence of maps. □

Theorem 4.2. *The dimension of* $W(n)$ *is*

$$\dim(W(n)) = \dim(W(n-1)) + \dim(W(n-2))$$

Where $\dim(W(0)) = 1$ *and* $\dim(W(1)) = 2$

Proof. We will begin by finding then dimensions of $W(1)$ and $W(2)$. $W(0) = \mathbb{C}/I_0 = \mathbb{C}/\langle 0\rangle = \mathbb{C}$.

So $\dim(W(0)) = \dim(\mathbb{C}) = 1$. $W(1) = \mathbb{C}[x_1]/I_1 = \mathbb{C}[x_1]/\langle R_1(t) : 2 \leqslant t\rangle = \mathbb{C}[x_1]/\langle x_1^2\rangle = \mathbb{C} \oplus \mathbb{C}[x_1]$. So we can say $\dim(W(1)) = \dim(\mathbb{C} \oplus \mathbb{C}[x_1]) = 2$.

Then we have our seeds $\dim(W(0)) = 1$ and $\dim(W(1)) = 2$

$$0 \to W(n-2) \xrightarrow{f} W(n) \xrightarrow{g} W(n-1) \to 0$$

is an exact map so $\dim(W(n)) = \dim(W(n-1)) + \dim(W(n-2))$. □

4.2. **Grading W(n).** $W(n)$ is doubly graded by weight and charge

- Charge is the degree
- Weight is the sum of the indices

$W(n)$ with the above grading is represented as

$$W(n)_{(m,k)} = \{p \in W(n) : \text{ch}(p) = k, \text{wt}(p) = m\}$$

Example 4.1. Let us look at some examples of monomials to give us insight as to how $\bar{f}$ and $\bar{g}$ effect grading. In order to simplify things we will use f and g in place of $\bar{f}$ and $\bar{g}$ so we don't have to worry about certain factors of a monomial making the monomial zero.

$$f(x_1x_4^2) = x_1x_3x_6^2$$
$$g(x_1x_4^2) = 0$$
$$f(x_2x_3x_5) = x_1x_4x_6$$
$$g(x_2x_3x_5) = x_1x_2x_4$$

Using the definition of $\bar{f}$ we can see that the charge and weight are both increased by one from the multiplication by x_1. The addition of two to all the indexes by f is responsible for the $+2k$ for the weight.

$$\bar{f} : W(n-2)_{(m-2k-1,k-1)} \to W(n)_{(m,k)}$$

Since the indexes are all decreased by 1 the weight is decreased by one while the charge remains the same.

$$\bar{g} : W(n)_{(m,k)} \to W(n-1)_{(m-k,k)}$$

This induces a short exact sequence of graded components.

$$0 \to W(n-2)_{(m-2k-1,k-1)} \to W(n)_{(m,k)} \to W(n-1)_{(m-k,k)} \to 0$$

Since this is exact

$$\dim W(n)_{(m,k)} = \dim W(n-1)_{(m-2k-1,k-1)} + \dim W(n-2)_{(m-k,k)}$$

Definition 4.2.

$$\chi_n(q,x) = \sum_{0\leqslant m,k} \dim(W(n)_{(m,k)})q^m x^k$$

Where x is the charge and q is the weight.

Once we have a better understanding of $\chi_n(q,x)$, we can use it to determine the graded dimension of a $W(n)_{(m,k)}$ as the coefficient of $q^m x^k$.

Using the exact sequence result

$$\chi_n(q,x) = \sum_{0\leqslant m,k} \text{dim}W(n-1)_{(m-k,k)}q^m x^k + \sum_{0\leqslant m,k} \text{dim}W(n-2)_{(m-2k-1,k-1)}q^m x^k$$

$$= \sum_{0\leqslant m,k} \text{dim}W(n-1)_{(m,k)}q^{m+k}x^k + \sum_{0\leqslant m,k} \text{dim}W(n-2)_{(m,k)}q^{m+2k+1}x^{k+1}$$

$$= \sum_{0\leqslant m,k} \text{dim}W(n-1)_{(m,k)}q^m(qx)^k + px\sum_{0\leqslant m,k} \text{dim}W(n-2)_{(m,k)}q^m(q^2x)^k$$

$$\chi_n(q,x) = \chi_{n-1}(q,qx) + xq\chi_{n-2}(q,q^2x)$$

For notational convenience recall the q-analauge to the classic binomial coefficient and the q-Pocchamer symbol. The q binomial coefficient

$$\binom{n}{r}_q = \frac{(1-q^n)(1-q^{n-1})...(1-q^{n-r+1})}{(1-q)(1-q^2)...(1-q^r)}$$

When $r \geqslant n$, $\binom{n}{r}_q = 0$

Another useful piece of notation is.

$$(q)_m = (1-q)...(1-q)^m$$

If we solve the recursion $\chi_n(q,x) = \chi_{n-1}(q,qx) + xq\chi_{n-2}(q,q^2x)$ we find the following result.

Theorem 4.3. *We have*

$$\chi_n(q,x) = \sum_{0\leqslant m} x^m q^{m^2}\binom{n+1-m}{m}_q = \sum_{m=0}^{\lfloor\frac{n+1}{2}\rfloor} x^m q^{m^2}\binom{n+1-m}{m}_q$$

Proof. Suppose

$$\chi_n(q,x) = \sum_{m=0}^{\lfloor\frac{n+1}{2}\rfloor} x^m q^{m^2}\binom{n+1-m}{m}_q$$

Then this would have to satisfy

$$\chi_n(q,x) = \chi_{n-1}(q,qx) + qx\chi_{n-2}(q,q^2x)$$

$$= \sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}(qx)^m q^{m^2}\binom{n-m}{m}_q + qx\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}(q^2x)^m q^{m^2}\binom{n-m}{m}_q$$

$$= 1 + \sum_{m=1}^{\lfloor\frac{n}{2}\rfloor}(qx)^m q^{m^2}\binom{n-m}{m}_q + qx\sum_{m=1}^{\lfloor\frac{n-1}{2}\rfloor+1}(q^2x)^{m-1}q^{(m-1)^2}\binom{n-m}{m-1}_q$$

Now define

$$\delta_{r,s} = \begin{cases}1 & r=s\\ 0 & r\neq s\end{cases}$$

Let

$$L = \delta_{n\text{mod}(2),1}(q^2x)^{\lfloor\frac{n-1}{2}\rfloor}q^{\lfloor\frac{n-1}{2}\rfloor}\binom{n+1-\lfloor\frac{n-1}{2}\rfloor}{\lfloor\frac{n-1}{2}\rfloor}$$

Then

$$\begin{aligned}
\chi_n(q,x) &= 1 + L + \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} (qx)^m q^{m^2} \binom{n-m}{m}_q + qx \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} (q^2x)^{m-1} q^{(m-1)^2} \binom{n-m}{m-1}_q \\
&= 1 + L + \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \left[(qx)^m q^{m^2} \binom{n-m}{m}_q + qx(q^2x)^{m-1} q^{(m-1)^2} \binom{n-m}{m-1}_q \right] \\
&= 1 + L + \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} x^m q^{m^2} \left[\binom{n-m}{m}_q q^m + \binom{n-m}{m-1}_q \right] \\
&= 1 + L + \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} x^m q^{m^2} \binom{n+1-m}{m}_q \\
&= L + \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} x^m q^{m^2} \binom{n+1-m}{m}_q \\
&= \sum_{m=0}^{\lfloor \frac{n+1}{2} \rfloor} x^m q^{m^2} \binom{n+1-m}{m}_q
\end{aligned}$$

□

Example 4.2. Now we have an easy way to find the dimension of $W(4)_{(3,1)}$. Using the theorem above we find

$$\chi_4(6,3) = \sum_{m=0}^{2} x^m q^{m^2} \binom{5-m}{m}_q \qquad = 1 + xq(1 + 2q^2 + q^3) + x^2q^4(1 + q + q^2)$$

Since the coefficient of xq^3 is 2 then

$$\dim(W(4)_{(3,1)}) = 2$$

By setting $x = 1$ we create a singly graded structure.

$$\chi_n(q,1) = \sum_{m=0}^{\lfloor \frac{n+1}{2} \rfloor} q^{m^2} \binom{n+1-m}{m}_q$$

To determine the normal dimension of $W(n)$ take the combinatorial limit as $q \to 1$.

$$\lim_{q \to 1} \chi_n(q,1) = \sum_{m=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-m}{m}_q = F(n+2)$$

Consider $\binom{n+1-m}{m}_q$ as $n \to \infty$ which is

$$\frac{1}{(1-q)(1-q^2)...(1-q^m)} = \frac{1}{(q)_m}$$

because $0 \leqslant q \leqslant 1$.

So we have

$$\lim_{n \to \infty} \chi_n(q,1) = \chi_\infty(q,1) = \sum_{0 \leqslant m} \frac{q^{m^2}}{(q)_m}$$

5. Free Resolutions of W(n)

Now we explore $W(n)$ using free resolutions as a tool to determine graded dimension. We will show a couple of basic examples for low values of n then make a skeleton of a second proof that would show the dimension of $W(n)$.

Let $n = 2$

$$W(2) = \mathbb{C}[x_1, x_2]/\langle x_1^2, x_1x_2, x_2^2\rangle$$

The basis of this $\mathbb{C}[x_1, x_2]$ -modules is $\{1, x_1, x_2\}$. Using this basis we know the Hilbert polynomial for $W(2)$ should be $F(W(2), q) = 1 + q + q^2$. In Macaulay 2 when we define $C_2 = \mathbb{C}[x_1, x_2]$ we have

$$0 \to C_2^2 \to C_2^3 \to I_2 \to 0$$

We can use the free resolution definition from the background section we can calculate the change in weight and charge between the modules using the syzygy matrices provided by Macauly 2.

$$\begin{bmatrix} x_1^2 & x_1x_2 & x_2^2 \end{bmatrix}$$

and

$$\begin{bmatrix} -x_2 & 0 \\ x_1 & -x_2 \\ 0 & x_1 \end{bmatrix}$$

Using these matrices we can see the weight and change changes as follows

$$0 \to C_2(-4, -3) \oplus C_2(-5, -3) \to C_2(-2, -2) \oplus C_2(-3, -2) \oplus C_2(-4, -2) \to I_2 \to 0$$

The change in charge is not that interesting since each syzygy matrix is of uniform charge. So we will only use the weight to create the Hilbert series.

$$F(W(2), q) = \frac{1 - q^2 - q^3 + q^5}{(1 - q)(1 - q^2)} = 1 + q + q^2$$

Now let $n = 3$

$$W(3) = \mathbb{C}[x_1, x_2, x_3]/\langle x_1^2, x_1x_2, 2x_1x_3 + x_2^2, x_2x_3, x_3^3\rangle$$

The basis of this $\mathbb{C}[x_1, x_2, x_3]$ -modules is $\{1, x_1, x_2, x_3, x_1x_3\}$. So we know the Hilbert polynomial for $W(3)$ should be $F(W(3), q) = 1 + q + q^2 + q^3 + q^4$. In Macaulay 2 we define $C_3 = \mathbb{C}[x_1, x_2, x_3]$ and use the res command to find the free resolution to be:

$$0 \to C_3^1 \to C_3^5 \to C_3^5 \to I_3 \to 0$$

The syzygy matrices from right to left are:

$$\begin{bmatrix} x_1^2 & x_1x_2 & 2x_1x_3 + x_2^2 & x_2x_3 & x_3^2 \end{bmatrix}$$

$$\begin{bmatrix} -x_2 & -2x_3 & 0 & 0 & 0 \\ x_1 & -x_2 & -x_3 & 0 & 0 \\ 0 & x_1 & 0 & -x_3 & 0 \\ 0 & 0 & x_1 & x_2 & -x_3 \\ 0 & 0 & 0 & 2x_1 & x_2 \end{bmatrix}$$

$$\begin{bmatrix} -x_3^2 \\ \frac{1}{2}x_2x_3 \\ -\frac{1}{2}x_2^2 - x_1x_3 \\ \frac{1}{2}x_1x_2 \\ -x_1^2 \end{bmatrix}$$

Again we use these matrices from right to left to find the details of this resolution.

$$0 \to C_3(-10,-5) \to C_3(-4,-3) \oplus C_3(-5,-3) \oplus C_3(-6,-3) \oplus C_3(-7,-3) \oplus C_3(-8,-3)$$
$$\to C_3(-2,-2) \oplus C_3(-3,-2) \oplus C_3(-4,-2) \oplus C_3(-5,-2) \oplus C_3(-6,-2) \to I_3 \to 0$$

So we find

$$\begin{aligned} F(W(3),q) &= \frac{1-q^2-q^3+q^7+q^8-q^{10}}{(1-q)(1-q^2)(1-q^3)} \\ &= \frac{1+q^2(1-q^3)}{1-q} \\ &= 1+q+q^2+q^3+q^4 \end{aligned}$$

The syzygy matrices for $W(4)$ are in the appendix. The use the Macaulay 2 output to calculate the free resolution:

$$0 \to C_4(-14,-6) \oplus C_4(-15,-6) \oplus C_4(-16,-6) \to$$
$$C_4(-10,-5) \oplus C_4(-11,-5) \oplus C_4(-12,-5) \oplus C_4(-12,-5) \oplus C_4(-13,-5) \oplus$$
$$C_4(-13,-5) \oplus C_4(-14,-5) \oplus C_4(-15,-5) \to$$
$$C_4(-4,-3) \oplus C_4(-5,-3) \oplus C_4(-6,-3) \oplus C_4(-7,-3) \oplus C_4(-8,-3) \oplus C_4(-9,-3) \oplus$$
$$C_4(-10,-3) \oplus C(-11,-3) \oplus C_4(-9,-4) \oplus C_4(-10,-4) \oplus C_4(-11,-4) \to$$
$$C_4(-2,-2) \oplus C_4(-3,-2) \oplus C_4(-4,-2) \oplus C_4(-5,-2) \oplus C_4(-6,-2) \oplus C_4(-7,-2) \oplus C_4(-8,-2) \to$$
$$I_4 \to 0$$

So the Hilbert series is

$$\begin{aligned} F(W(4),q) &= \frac{1-q^2-q^3+2q^9+q^{10}+q^{11}-2q^{12}-2q^{13}+q^{16}}{(1-q)(1-q^2)(1-q^3)(1-q^4)} \\ &= \frac{1+q^5(q^6+q^4-q^4-q^3-q)}{(1-q)(1-q^4)} \end{aligned}$$

Now we can consider $n=5$. Using Macaulay 2 we can see the free resolution for I_5 is

$$0 \to R_5^2 \to R_5^{15} \to R_5^{26} \to R_5^{21} \to R_5^9 \to I_5 \to 0$$

We will not calculate the changes in grading for the free resolution. We will just use Macaulay 2 to compute the Hilbert series.

$$\begin{aligned} F(W(5),q) &= \frac{1-q^2-q^3+q^9+2q^{11}+q^{12}-q^{14}-2q^{15}-2q^{16}-q^{17}+q^{18}+q^{19}+q^{20}+q^{22}-q^{24}}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)} \\ &= \frac{1+q^7(q^7+q^4-q^3-q-1)}{(1-q)(1-q^4)} \end{aligned}$$

In the previous section we proved that the χ function as $n \to \infty$ evaluated at $x=1$ is

$$\prod_{0 \leqslant k} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}$$

This graded free resolution method of calculating Hilbert functions should converge to the same product. We still must prove this. So far we have noticed the pattern that we can factor the Hilbert series for a given n into

$$F(n,q) = \frac{1-q^k p(q)}{\prod(1-q^i)}$$

Where $i \equiv 1, 4 \mod (5)$, $0 < i \leqslant n$ and as $n \to \infty$, $q^k \to 0$. k is increasing so that $k \to \infty$ as $n \to \infty$. To prove this factorization for every n would most likely involve a looking carefully at the syzygy matrices used to compute the graded free resolutions. Sadly, these matrices significantly increase in size and complexity as n increases and do not follow an obvious pattern.

Assuming we can tie up the loose ends of this proof we are left with two representaions for the graded dimension of $W(n)$ as n goes to ∞ which gives us the Rogers-Ramanujan Identity:

$$\sum_{0 \leqslant m} \frac{q^{m^2}}{(q)_m} = \prod_{0 \leqslant k} \frac{1}{(1 - q^{5k+1})(1 - q^{5k+4})}$$

6. Appendix

```
     7                                                                   11
1 : R  <---------------------------------------------------------------- R   : 2
          {2} | -x2 -2x3 -x4 0    0    0   0   0   -x3x4 -x4^2 0     |
          {2} | x1  -x2  -x3 -5x4 0    0   0   0   0     0     -x4^2 |
          {2} | 0   x1   0   -x3  -2x4 0   0   0   0     0     0     |
          {2} | 0   0    x1  x2   -x3  -x4 0   0   0     0     0     |
          {2} | 0   0    0   2x1  x2   0   -x4 0   0     0     0     |
          {2} | 0   0    0   0    5x1  x2  x3  -x4 x1^2  0     0     |
          {2} | 0   0    0   0    0    x1  2x2 x3  0     x1^2  x1x2  |

     11                                                                            8
2 : R  <------------------------------------------------------------------------- R  : 3
           {3} | -x3^2+.5x2x4 -x3x4 -2.5x4^2 x4^2 0       0     0     0            |
           {3} | .5x2x3-2x1x4 0     -.5x3x4  0    x4^2    0     0     0            |
           {3} | -.5x2^2-x1x3 -x1x4 .5x2x4   0    0       0     -x4^2 0            |
           {3} | .5x1x2       0     -.5x1x4  0    0       0     0     -x4^2        |
           {3} | -x1^2        0     0        0    .5x1x4  0     0     .5x3x4       |
           {3} | 0            -x1^2 0        0    -.5x1x3 0     -x1x4 -.5x3^2-x2x4 |
           {3} | 0            0     -x1^2    0    .5x1x2  0     0     .5x2x3-2x1x4 |
           {3} | 0            0     0        0    0       -x1^2 -x1x2 -x2^2+.5x1x3 |
           {4} | 5x1          x2    x3       0    -2.5x4  -x4   0     0            |
           {4} | 0            x1    2x2      -x2  .5x3    x3    x4    0            |
           {4} | 0            0     0        x1   -x2     0     x3    5x4          |

     8                             3
3 : R  <------------------------ R  : 4
          {5} | x4  0   0    |
          {5} | -x3 -x4 0    |
          {5} | x2  0   -2x4 |
          {5} | 2x2 -x3 -5x4 |
          {5} | 2x1 0   -x3  |
          {5} | 0   -x2 .5x3 |
          {5} | 0   x1  -x2  |
          {5} | 0   0   x1   |
```

```
        3
4 : R  <----- 0 : 5
```

References

[1] George Andrews, Number Theory, *W.B. Saunders Company* (1971).

[2] Michael Artin, Algebra, *Prentice Hall, Inc.* (1991).

[3] C. Calinescu, J. Lepowski and A. Milas, Vertex-Algebraic Structure of The Principal Subspaces of Certain $A_i^{(1)}$ -Modules, I: Level One Case.

[4] David Cox, John Little, Donal O-Shea, Ideals Varieties, and Algorithms *Addison Wesely Publishing Company*, University of Oxford, (1969).

[5] Elena Grigorescu, Hilbert Series- Senior Project *Bard College* (2003).

[6] Mike Stillman, Dave Bayer, Computations of Hilbert functions, *J. Symbolic Computations* (1992) 14, 31-50.

Department of Mathematics and Computer Science, Colorado College, Colorado Springs, CO 80903

E–mail address: nathan.mankovich@coloradocollege.edu