Self-fulfilling Asset Prices

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January 15, 2019

Abstract

I develop an asset pricing model and show that collateral constraints that are common to the literature—ones that guarantee no loan loss—can generate multiple equilibria in a dynamic rational expectations economy. In the model, expectations of the future value of collateral affect leverage and thus investor demand for assets. High expected collateral value implies high leverage and hence high asset prices, which confirms the initial beliefs. And conversely for low collateral value. As a consequence, asset prices can be self-fulfilling. Price crashes and booms, excess volatility, long price recoveries, price overshooting and misfiring, as well as leverage cycles transpire purely from shifts in investor expectations without corresponding shifts in fundamentals (i.e., from sentiments).

JEL classification: D84, G11, G12

Keywords: asset pricing, collateral constraint, self-fulfilling expectations, multiple equilibria, sentiments

∗I am grateful to Doug Diamond, Stavros Panageas, John Cochrane, and Pietro Veronesi for many fruitful conversations about this topic. I also thank Will Cong, Kent Daniel, Eduardo Davila (discussant), Gary Gorton, Veronica Guerrieri, Zhiguo He, Yunzhi Hu, Paymon Khorrami, Hanno Lustig, Aaron Pancost, Jung Sakong, Fabrice Tourre, and seminar participants at the Chicago Booth Finance Brownbag and AFA 2019 for their very helpful feedback and comments.
1 Introduction

The requirement to pledge collateral in exchange for leverage is ubiquitous in financial markets. Often the asset submitted as collateral is the one financed by the debt. Repurchase agreements on asset-backed securities is one example. Margin on a stock portfolio is another. The constraint is designed to restrict a borrower’s leverage in order to limit the lender's losses from possible default and in some cases guarantee repayment entirely.

A large body of work has extensively studied collateral and margin constraints, in particular their effects on asset prices. Kiyotaki and Moore (1997), Krishnamurthy (2003), Fostel and Geanakoplos (2008), Geanakoplos (2009), and Gârleanu and Pedersen (2011) are just a few among many. Generally, a central point of those papers has been the amplification of such leverage restrictions on asset price movements and the spiraling relation between tightening haircuts and falling prices.

In this paper, I show that collateral constraints that are regularly used in the literature—ones that ensure no loan loss—can also generate fluctuations in asset prices that are entirely due to changes in investor expectations. The presence of the constraint lets investor beliefs about future prices influence demand in a way that affirms the initial expectations in a market equilibrium. In other words, beliefs about asset prices can be self-fulfilling. Prices can display booms and busts, excess volatility, long recoveries after crashes, overshooting, misfiring, and the financial market can feature leverage cycles. These phenomena arise without any corresponding changes in fundamentals like cash flows or discount rates.

For this outcome to occur, the restriction on leverage via the collateral constraint crucially must depend on the expected future value of the collateral rather than its current value. The amount of debt a lender is willing to extend must depend not on the value he or she can recover at the time the loan is issued in the present, but rather on the value he or she believes is recoverable when payment is due in the future. This kind of constraint creates a mutually reinforcing relation between leverage and beliefs about future prices. That feedback leads to a multiplicity of equilibrium asset prices that all obey rational expectations.

The mechanism is the following: expectations about the future price of the collateral influences the amount of leverage available to borrowers, which affects aggregate demand for the collateralized asset. If investors expect the asset to command a high price in the future, they can borrow more from posting the asset as collateral. Leverage and demand for the asset will be high, which bids up the price and confirms the initial belief that the price would be high. Conversely, if investors expect the collateral value to be depressed in the future, leverage and demand will be low, and so too will the asset price.

In a dynamic economy, the constraint leads to an equilibrium asset price path that is indeterminate. The price path chosen depends on how investors coordinate their expectations over time. This feature of the financial market leaves room for an extrinsic source of variation to influence asset prices. These kinds of extrinsic “shocks” can be interpreted as investor sentiments (“animal spirits,” waves of optimism, or sudden pessimism) or changes in individual
investor beliefs about aggregate demand that are based on imperfect information. So long as these beliefs are correlated, they impact the market equilibrium.

Because the extrinsic variation is orthogonal to shocks that affect cash flows and discount rates, they do not influence investor beliefs about those fundamental objects of the economy. And yet the extrinsic process affects equilibrium expectations and alter beliefs about prices. These shocks to beliefs about an endogenous outcome generate the great number of interesting asset price dynamics mentioned earlier that are empirically observable in actual financial markets.

In the model, markets are perfectly competitive, investors are rational, they are infinitely-lived, and there is a continuum of them. Two risky assets and a risk-free, one-period asset are available for trade. The risky assets are in separate financial markets and their cash flows are perfectly negatively correlated. An investment in both will generate a riskless claim and constant consumption stream. However, making this trade is costly, since it requires expert knowledge of both markets.

No investor would invest in both markets and create the synthetic riskless claim when investment in the risk-free asset can be done at no charge. For markets to clear, investors must be compensated with a return on the synthetic riskless claim that is higher than the interest rate. This difference in risk-free returns generates an arbitrage opportunity. Any investor can feasibly short the risk-free asset, use the proceeds to invest in both markets, and earn positive profits with zero investment.

However, leverage of all investors is restricted by a collateral constraint that is based on expectations about the future value of the collateralized asset. This constraint limits leverage to ensure no loan loss: an investor can only borrow as much as the worst possible one-period return of the collateral. This collateral constraint is identical to a margin requirement that guarantees no default.

With bounded capacity to borrow, investors are unable to fully exploit the arbitrage. A persistent price difference emerges between two assets with identical cash flows. This basis arises because of the collateral constraint, which sidelines investors from closing the price gap.

Any investor is free to enter any financial market, as there is no exogenous market segmentation. In equilibrium, investors will endogenously segment into two types: a fraction will invest in only one market and the other fraction will pay the investment cost to hold both assets and exploit the arbitrage up to the limit. For this reason, the investment cost has an alternative interpretation as the cost of managing the arbitrage position. Free entry will require all agents to be indifferent from either investing in one asset only or becoming an arbitrageur.

The equilibrium asset price path is represented by a nonlinear dynamical system. A distinguishing feature of the system is that the relation between prices in one period and the next is characterized by a multi-valued function: a single asset price today can be followed by multiple different prices. The structure of the system allows for an arbitrary number of

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equilibrium price paths. Investors select equilibria based upon their expectations of future price paths. In this sense do asset prices become self-fulfilling and display a variety of interesting asset price dynamics that are driven solely by shifts in investor beliefs.

This paper shares a clear connection to the literature in economics on self-fulfilling beliefs; e.g. Azariadis (1981), Cass and Shell (1983), and Benhabib and Farmer (1999). See also Krugman (1979), Obstfeld (1986), and Hellwig et al. (2006) on self-fulfilling currency crises and Cole and Kehoe (2000) on self-fulfilling debt crises. Closely related are economies with sentiment-driven aggregate fluctuations such as those in Angeletos and La’O (2013) and Benhabib et al. (2015).

A distinctive aspect of this paper is the focus on both asset prices and collateral constraints in self-fulfilling equilibria. A number of papers have made interesting and important contributions examining either self-fulfilling prices or collateral constraints instead of the two combined:

Benhabib and Wang (2013) analyze how collateral constraints can lead to indeterminate equilibria in the real economy, but attention is on macroeconomic quantities and wages instead of risky asset prices.

Benhabib and Wang (2015) study sequential asset markets with private information and show the existence of multiple equilibria.

Benhabib et al. (2016) analyze asset pricing dynamics that are driven by sentiments and their feedback effects with the real sector.

Benhabib et al. (2018) describe how strategic complementarity in producing information emerges between financial markets and the macroeconomy which can lead to self-fulfilling equilibria.

These last three economies, however, do not feature collateral constraints.

The presence of an arbitrage opportunity that cannot be fully exploited by arbitrageurs links the model to the vast literature on the limits to arbitrage. See Shleifer and Vishny (1997) as well as Gromb and Vayanos (2010) for a survey. This paper connects to Gromb and Vayanos (2002), who also model arbitrageurs as investors required to collateralize their positions. There, the arbitrage opportunity emerges from a price discrepancy between two identical assets traded in segmented markets. Arbitrageurs have the unique ability to trade in both markets and close the price gap. Here, there is no exogenous market segmentation; instead investors will endogenously choose between trading in one or both markets.

This is not the first paper to recognize that collateral constraints have the capacity to generate multiple equilibria: Benhabib and Wang (2013), mentioned earlier, has discussed this feature in real economies.

Kiyotaki and Moore (1997) acknowledge that the nonlinear equilibrium system of their economy can feature multiplicity in collateral (land) prices, but their linearized solution makes the equilibrium unique.

economy and shows that with incomplete hedging contracts, a binding collateral constraint can lead to two equilibria: one in which collateral prices are high and another in which they are low.

Geanakoplos and Zame (2014) include collateral requirements within an Arrow-Debreu economy and discuss the possibility of multiple equilibria.

An important distinction of this paper is that the economy is dynamic, which allows a rich set of empirically relevant asset pricing behaviors to occur over time.

Finally, this paper shares similarities to the literature on limited financial market participation via the presence of an investment cost. See Merton (1987) for an early contribution and Gârleanu et al. (2015) for a more recent one. The cost in the model generates an arbitrage opportunity and splits the population into those who exploit the arbitrage (with limitations) and those who do not.

2 The Model

I describe the preferences of investors, the structure of uncertainty in the economy, the makeup of the financial markets, the cost of investing in both financial markets, and I end the section with the investors’ problem.

2.1 Preferences

A continuum of investors on the unit interval \( i \in [0, 1] \) populate the economy, having preferences

\[
U^i \equiv E_0 \left( \sum_{t=0}^{\infty} \delta^t \log (c^i_t) \right),
\]

over consumption \( c^i_t \), with \( \delta \in (0, 1) \) the common subjective discount factor.

2.2 Uncertainty

There are two states of the world: \( \omega \in \{H, T\} \). Let \( X_t \) denote the state in period \( t \). The random variable \( X_t \) obeys a two-state Markov chain with persistence. Specifically, the conditional probability of remaining in state \( H \) when starting in state \( H \), or \( \Pr (X_{t+1} = H \mid X_t = H) \), equals a constant \( q \in (\frac{1}{2}, 1) \). The same goes for the conditional probability of remaining in state \( T \) when starting in state \( T \), or \( \Pr (X_{t+1} = T \mid X_t = T) = q \). The probability of switching between states \( H \) and \( T \) is \( 1 - q \). The measure \( \Pr (\cdot) \) is the objective, rational expectations probability. A presentation of the stochastic transition matrix is given in Figure 1.
2.3 Financial markets

Competitive financial markets exist for a risk-free, one-period maturity asset and two risky, long-lived assets. The risky assets are denoted $A$ and $B$ and trade in separate markets.

Risky assets

The number of shares of each risky asset is normalized to one. The two risky assets have perfectly negatively correlated cash flows $D_{j,t}$ for $j \in \{A, B\}$. In any period, one risky asset yields a high cash flow, denoted by $\bar{D}$, whereas the other asset yields a low cash flow, denoted by $D$. The high and low cash flows are

$$\begin{pmatrix} \bar{D} \\ D \end{pmatrix} = \begin{pmatrix} \lambda D \\ (1 - \lambda) D \end{pmatrix},$$

for $\lambda \in (\frac{1}{2}, 1]$ and $D > 0$. Let $P_{j,t}$ for $j \in \{A, B\}$ denote the prices of the risky assets.

Examining the sum of the risky cash flows $\bar{D} + D$, one can see that the aggregate cash flow $D$ is constant, meaning there is no uncertainty in the payoff to the risky assets as a whole. Nevertheless, assets $A$ and $B$ individually are risky, as the share that each receives of the aggregate cash flow $D$ fluctuates randomly over time according to the process $\{X_t\}_{t \geq 0}$.

Risk-free asset

The risk-free asset has price $P_{f,t} = \frac{1}{r_{f,t}}$, where $r_{f,t}$ is the gross risk-free interest rate. The risk-free asset is in negative net supply in its financial market. There is an exogenous, constant supply of outside savings $Y > 0$ in the economy that is available to financial market investors to borrow from. The negative net supply of the risk-free asset permits all investors to be levered at once. How the presence of outside savings affects the number of equilibria is discussed in Appendix B.1.

Outside households supply savings perfectly elastically at a fixed rate of return (lending rate) $r_L$. Investors who save and lend to other investors via the risk-free asset, on the other hand, receive the interest rate $r_{f,t}$. The distinction in rates of return between “inside” savings (from investors) and “outside” savings (from households) is analogous to the distinction between short-term, risk-free repo rates and retail bank deposit (or money market mutual fund) rates.
Any difference in $r_L$ and $r_{f,t}$ is a profit or loss to a financial intermediation sector outside the model. For simplicity, the return to outside savings $r_L = 0$. The risk-free interest rate $R_{f,t}$ is endogenous.

Timing of decisions and information

Investors make their investment and consumption decisions at the beginning of each period $t$ while knowing the state $X_t$ and taking as given the prices $P_{j,t}$ of each risky asset and price $P_{f,t}$ of the risk-free asset. At the end of the period (start of period $t + 1$), the next period’s state $X_{t+1}$ is publicly revealed, which dictates which risky asset realizes the high or low cash flow via $D_{j,t+1}$.

2.4 Investment cost

Because investors are aware of the current state $X_t$ when making consumption and investment decisions, they are aware whether risky asset A or asset B received the low cash flow $D$ in the previous period (at the end of period $t - 1$). To invest in period $t$ in the asset that just received bad news requires paying a cost $\kappa$. The cost takes the form of a direct reduction in utility and must be paid each period an investment in the low cash-flow asset is made.

Because investors have log preferences, this utility cost has an analogous interpretation as a fraction of wealth that investors must sacrifice in order to invest in that market. For example, $\kappa = 0.05$ implies that an investor must pay 5% of wealth $W_t^i$ in period $t$ to invest in the low cash-flow asset. Investing in the risk-free asset or the risky asset that just received the high cash-flow is costless.

The investment cost’s perfect correlation with the low cash flow makes it a reduced-form representation of a cost of gathering information on a risky security after it received bad news. For example, in the case of public equity, Kothari et al. (2009) document that managers on average delay the release of bad news to investors, but quickly reveal good news to investors. Li (2008) documents that annual reports of firms with lower earnings are linguistically more complex and longer relative to those of firms with higher earnings. Hence, extracting information from injured firms can be more difficult and costly.

A larger expense of gathering information is not the unique interpretation of this investment cost. This cost will turn out to create an arbitrage in the market, and so another interpretation of it will be a cost of managing an arbitrage position.

2.5 Collateral constraint

If an investor wants to borrow, he or she cannot commit to repay the loan. This friction is akin to the commitment problems regarding repayment in Kehoe and Levine (1993); Kocherlakota (1996); Alvarez and Jermann (2000); Geanakoplos and Zame (2014). To borrow at all, an investor must post collateral. Only one of the two risky assets is acceptable as collateral, however. One justification for this kind of contract is that the lenders are only familiar with one of the risky assets—the one they invest in themselves. Or, because the two risky assets trade in different markets (exchanges), lenders may trade on one exchange but not the other.
And so, lenders are willing to accept as collateral only the asset traded on the exchange where they participate themselves in the period.

Let $\phi_i^{f,t}$ denote the fraction of wealth that a typical investor $i$ invests in the risk-free asset in period $t$. The value $\left(1 - \phi_i^{f,t}\right)$ is the fraction of wealth invested in risky assets and is the investor’s leverage. If $\left(1 - \phi_i^{f,t}\right) < 1$, the investor lends; if $\left(1 - \phi_i^{f,t}\right) > 1$, the investor borrows.

So that repayment is guaranteed, lenders will only extend an amount up to the minimum possible return of the collateral over one period. The collateral constraint is

$$
(1 - \phi_i^{f,t}) \phi_{coll,t}^i R_{coll,t+1}^{\min} + \phi_i^{f,t} R_{f,t} \geq 0,
$$

where $\phi_{coll,t}^i$ is the fraction of investor $i$’s risky portfolio allocated to the asset acceptable as collateral in period $t$, and $R_{coll,t+1}^{\min}$ is the minimum return on the collateral over the next period. The first term in (2) is the minimum return on the total fraction of wealth invested in the asset accepted as collateral, whereas the second term is the principal plus interest on the loan.\(^1\)

As an example, suppose the following: Asset $A$ is the asset acceptable as collateral in period $t$. Investor $i$ has $60$ of personal wealth (equity) in asset $B$ and $20$ of equity in asset $A$. The investor then borrows $20$ and invests that amount in asset $A$, which raises the total investment in that asset to $40$. The constraint in (2) involves posting the $40$ investment as collateral and requires that the minimum return on that investment meets or exceeds the principal and interest payments on the $20$ loan. In this example, because the total value of investor $i$’s risky portfolio is $100$, the risky share $\phi_{coll,t}^i = \frac{40}{100} = 0.4$, and the investor’s leverage $1 - \phi_i^{f,t} = \frac{100}{80} = 1.25$. Note that posting the asset not acceptable as collateral would automatically violate (2).\(^2\)

The leveraged investors post their entire investment in the acceptable asset as collateral in order to satisfy the constraint. Critically, the constraint depends on the minimum possible future return of the collateralized position over the next period, instead of the current value of the position, as in Gârleanu and Pedersen (2011). That way, beliefs about the distribution of future prices—in particular the value of the minimum possible price on the collateral $P_{coll,t+1}^{\min}$—will influence the amount of leverage and aggregate demand for assets. The minimum return of the collateral asset, denoted $R_{coll,t+1}^{\min}$, consists of the minimum price $P_{coll,t+1}^{\min}$ and minimum cash flow over one period $(1 - \lambda) D$.

\(^1\)The distribution of possible period $t + 1$ returns will be known to investors at time $t$. This distribution includes the minimum possible return on the collateral $R_{coll,t+1}^{\min}$. Hence, every component of (2) is measurable as of period $t$.

\(^2\)Although (2) applies to risky-asset collateral, technically an investor could purchase the risk-free security and then post it as collateral, which would be acceptable. However, repayment must also be guaranteed in that case as well, which means the most a borrower would receive is the amount he or she posted, making that strategy wash out. Mathematically, the constraint in that case would be $\phi_i^{f,t} R_{f,t} - \phi_i^{f,t} R_{f,t} \geq 0$, which holds trivially. Alternatively, investors could post the acceptable risky asset as collateral in order to take a levered position in the risk-free security, but it will turn out that no investor would do so in equilibrium.
2.6 Budget constraint

Let $\phi_{A,t}^i$ denote the fraction of the risky portfolio invested in asset $A$; and $\phi_{B,t}^i$, the fraction invested in $B$. Thus, $\phi_{A,t}^i + \phi_{B,t}^i = 1$. The fractions $\phi_{A,t}^i, \phi_{B,t}^i \in [0,1]$, which forbids risky asset shorting. The investor’s portfolio abides by the adding up constraint:

$$\phi_{f,t}^i + (1 - \phi_{f,t}^i) (\phi_{A,t}^i + \phi_{B,t}^i) = 1.$$  

The total fraction of wealth invested in risky asset $j$ is then $(1 - \phi_{f,t}^i) \phi_{j,t}^i$. Letting $W^i$ denote investor wealth, the dynamic budget constraint is

$$W_{t+1}^i = (W_t^i - c_t^i) \left[ \phi_{f,t}^i R_{f,t} + (1 - \phi_{f,t}^i) \sum_{j \in \{A,B\}} \phi_{j,t}^i R_{j,t+1} \right],$$

where

$$R_{f,t+1} = 1 + r_{f,t+1},$$
$$R_{j,t+1} = \frac{P_{j,t+1} + D_{j,t+1}}{P_{j,t}}, \quad j \in \{A,B\}$$

are gross returns.

2.7 Investor problem

For investor $i$, let $\phi^i = (\phi_{f}^i, \phi_{A}^i, \phi_{B}^i)$ denote the lifetime portfolio investment policy, where $\phi_{j}^i \equiv \{ \phi_{j,t}^i \}_{t \geq 0}$ for $j \in \{f, A, B\}$. And let $c^i$ be the consumption policy. The investor’s optimization problem is

$$V^i \equiv \max_{c^i,\phi^i} E_0 \sum_{t=0}^{\infty} \delta^t \left( \log (c_t^i) - \sum_{j \in \{A,B\}} \kappa_{j,t} 1_{\{ \phi_{j,t}^i \neq 0 \}} \right), \quad (3)$$

subject to the budget constraint

$$W_0^i = W_0,$$
$$W_{t+1}^i = (W_t^i - c_t^i) \left[ \phi_{f,t}^i R_{f,t} + (1 - \phi_{f,t}^i) \sum_{j \in \{A,B\}} \phi_{j,t}^i R_{j,t+1} \right], \quad (4)$$

the no-shorting condition

$$\phi_{A,t}^i, \phi_{B,t}^i \in [0,1], \quad (5)$$

and the collateral constraint

$$\left(1 - \phi_{f,t}^i\right) \phi_{coll,t}^i R_{coll,t+1}^{\min} + \phi_{f,t}^i R_{f,t} \geq 0. \quad (6)$$
Regarding the investment cost process, \( \kappa_{j,t} = \kappa \) if asset \( j \) bears the cost in period \( t \), whereas \( \kappa_{j,t} = 0 \) otherwise.

### 3 Equilibrium

What follows is the definition of an equilibrium in the consumption and financial markets that involves the usual sequence of prices and policies entailing optimality and market clearing.

**Definition.** An equilibrium is a set of price processes \( \{P_{j,t}\}_{t \in \mathbb{N}} \) for \( j \in \{f, A, B\} \) and consumption and portfolio decisions \( \{c^i_t, \phi^i_t\}_{t \in \mathbb{N}} \) for all \( i \in [0,1] \), such that (i) \( \{c^i_t, \phi^i_t\} \) solve the optimization problem in (3)-(6) given prices; and in every period \( t \), (ii) each risky asset market clears: \( P_{j,t} = \int_{i \in [0,1]} \left(1 - \phi^j_{f,t}\right) \phi^j_{j,t} W^i_t \, di \) for \( j \in \{A, B\} \), (iii) the risk-free asset market clears: \( Y + \int_{i \in [0,1]} \phi^f_{f,t} W^i_t \, di = 0 \), and (iv) the consumption market clears: \( \int_{i \in [0,1]} c^i_t \, di = \sum_{j \in \{A,B\}} D_{j,t} \).

#### 3.1 Market clearing implications

The structure of the financial market and preferences of investors allow a convenient representation of the risky asset prices. First, the optimal policy of log utility investors is to consume a constant fraction of personal wealth each period:

\[
c^i_t = (1 - \delta) W^i_t,
\]

This policy implies in the aggregate that

\[
\int_{i \in [0,1]} c^i_t \, di = (1 - \delta) \int_{i \in [0,1]} W^i_t \, di,
\]

where \( C_t \) and \( W_t \) are aggregate consumption and investor wealth, respectively.

Second, market clearing in the consumption market ensures that total cash flows match aggregate consumption:

\[
\sum_{j \in \{A,B\}} D_{j,t} = C_t.
\]

Furthermore, no matter the value of the state \( X_t \), aggregate cash flows are constant:

\[
\sum_{j \in \{A,B\}} D_{j,t} = D. \tag{8}
\]

Combining (8) with consumption market clearing expresses aggregate consumption as \( C_t = D \). Using the relation for aggregate consumption from (7) expresses aggregate investor wealth as

\[
W_t = \frac{D}{1 - \delta},
\]

meaning total wealth is constant in the financial market.

Finally, clearing in the risk-free market implies that the sum of risky asset prices matches
total wealth in the economy, which includes the wealth of investors and outside savings:

$$\sum_{j \in \{A,B\}} P_{j,t} = W + Y.$$  \hfill (9)

Because the sum of the risky asset prices are equal to a constant each period, once one of the risky asset prices is pinned down in equilibrium, so is the other. Finally, with no growth in the economy, the aggregate cash flow-to-price ratio $\frac{D}{W+Y}$ is constant.

### 3.2 Asset return symmetry

Because the high and low cash flows $\overline{D}$ and $D$ simply alternate between the two risky assets at random each period, a symmetry in risky asset returns emerges. To see why, suppose, without loss of generality, that $X_t = H$ implies asset $A$ receives $\overline{D}$ and asset $B$ receives $D$. Conversely, when $X_t = T$, the arrangement flips and asset $B$ instead receives the high cash flow $\overline{D}$ and asset $A$ receives the low cash flow $D$.

Suppose next that the current state $X_t = H$. If in the following period the value of the state persisted, meaning $X_{t+1} = H$, asset $A$ again would earn $\overline{D}$, whereas asset $B$ again would earn $D$. On the other hand, if the state switched, meaning $X_{t+1} = T$, asset $A$ now would earn the low cash flow $D$, whereas asset $B$ would earn the high cash flow $\overline{D}$.

Interestingly, if instead the current state were $X_t = T$, the distribution of cash flows would be identical to what was just described except asset $A$ and asset $B$ would exchange positions. Now asset $B$ would receive the high cash flow $\overline{D}$ if the state persisted, but asset $A$ would do so if the state switched. And conversely, asset $A$ now would receive the low cash flow $D$ if the state persisted, but asset $B$ would do so if the state switched.

This symmetry reveals that in every period, no matter the value of the state $X_t$, investors always face a constant investment opportunity set. Always available is one asset that earned $\overline{D}$ in period $t$ which can then receive either cash flow the following period $t + 1$. Also available is a second asset that earned $D$ in period $t$ which can then receive either cash flow the following period $t + 1$. The identities of those assets will change over time according to the process $\{X_t\}_{t \geq 0}$, but the distribution of payoffs on those assets will not. Under the stochastic transition matrix in Figure 1, each asset would receive in the following period the cash flow it received in the current period with probability $q$ and switch cash flows with probability $1 - q$.

The asset earning $\overline{D}$ will command a higher price than the one earning $D$. With this in mind, let $\overline{P}_t$ denote the price of the asset that received $\overline{D}$ in period $t$. Conversely, let $P_t$ denote the price of the asset that received $D$. By design, each period $\overline{P}_t \geq P_t$. Because the investment cost $\kappa$ is perfectly correlated with the low cash flow, it will always be costly to invest in the asset with price $P_t$.

These two prices will alternate between risky assets $A$ and $B$ at random. With this construction, however, it is convenient to refer to the risky assets by their prices rather than their names. In period $t$, the “$\overline{P}$” asset will have price $\overline{P}_t$ and the “$P$” asset will have price $P_t$. In the
next period, the $\bar{P}$ asset will have price $\bar{P}_{t+1}$ with probability $q$ (i.e., remain the $\bar{P}$ asset), or it will switch to have price $P_{t+1}$ (i.e., become the $P$ asset) with probability $1 - q$. Likewise, the $P$ asset will have price $P_{t+1}$ with probability $q$ or switch to have price $\bar{P}_{t+1}$ with probability $1 - q$.

The constant investment opportunity set implies that any decisions of investors, their expectations $E_t(\cdot)$, and all equilibrium objects need not be conditioned on the value of the current state $X_t$. Each asset will also have only two possible one-period gross returns. Those returns are expressed in Figure 2.

Figure 2: Risky Asset Returns

<table>
<thead>
<tr>
<th>The value of the state persists: $X_t = H \implies X_{t+1} = H$ or $X_t = T \implies X_{t+1} = T$</th>
<th>[ R_{\bar{P},t+1} = \frac{\bar{P}_{t+1} + D}{\bar{P}_t} ]</th>
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<tbody>
<tr>
<td>$\bar{P}$ asset</td>
<td>$P$ asset</td>
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<tr>
<td>$\bar{P}$ asset</td>
<td>$P_{t+1} = \frac{\bar{P}_{t+1} + D}{\bar{P}_t}$</td>
</tr>
<tr>
<td>The value of the state switches: $X_t = H \implies X_{t+1} = T$ or $X_t = T \implies X_{t+1} = H$</td>
<td>$R_{P,t+1} = \frac{P_{t+1} + D}{P_t}$</td>
</tr>
<tr>
<td>$\bar{P}$ asset</td>
<td>$P$ asset</td>
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<tr>
<td>$\bar{P}<em>{t+1} = \frac{P</em>{t+1} + D}{P_t}$</td>
<td></td>
</tr>
</tbody>
</table>

Notes: The first section of the figure presents the returns on the risky assets if the value of the state $X_t$ persisted, meaning either $X_t = H$ and $X_{t+1} = H$ the following period, or $X_t = T$ and $X_{t+1} = T$. The second section presents the returns on the risky assets if the value of the state $X_t$ switches, meaning either $X_t = H$ and $X_{t+1} = T$ the following period, or $X_t = T$ and $X_{t+1} = H$ the following period.

As an example from the figure, the first return, $R_{\bar{P},t+1}$, is the return on the $\bar{P}$ asset if the state persisted. In this case, the gross return on the asset is the capital price change $\bar{P}_{t+1}/\bar{P}_t$ plus the high cash-flow yield $D/\bar{P}_t$. Conversely, if in the next period the state switched, the return on the $\bar{P}$ asset, $R_{P,t+1}$, is the capital price change $P_{t+1}/P_t$ and the low cash-flow yield $D/P_t$.

### 3.3 Aggregate wealth shares

With the two asset prices expressible as $\bar{P}_t$ and $P_t$, the risky-asset market-clearing condition of (9) can be rewritten as

$$\bar{P}_t + P_t = W + Y.$$  

As explained before, once one risky asset price is determined in equilibrium, so too is the other because aggregate wealth is constant. For this reason, another way to interpret market clearing in the risky asset markets is to envision the *wealth share* as the single object pinned down in equilibrium. Let $\sigma_t \in (\frac{1}{2}, 1)$ denote the wealth share of the high cash-flow asset $\bar{P}$. The
risky-asset market-clearing conditions are then expressible as

\[ P_t = \sigma_t (W + Y), \]
\[ P_t = (1 - \sigma_t) (W + Y). \]

The high cash-flow asset \( \overline{P} \) will always earn a fixed share \( \lambda \) of aggregate cash flows. However, its share \( \sigma_t \) of aggregate wealth will be time-varying.

### 3.4 Collateral asset

I assign the sole collateralizable asset \( \text{coll}_{t} \) each period to be the one that has high price \( \overline{P}_t \). Investors will post the \( \overline{P} \) asset as collateral and use the issued leverage for risky investment. With this assignment, the minimum possible return on the collateral asset would transpire when the asset switches to have price \( P_{t+1} \) and receives the low cash flow. That minimum possible return on the collateral is \( R_{\overline{P},t+1} \). With this collateral assignment, the collateral constraint from (2) becomes

\[ (1 - \phi^i_{f,t}) \phi^i_{P,t} R_{\overline{P},t+1} + \phi^i_{f,t} R_{f,t} \geq 0. \]  

Because the constraint relies on the minimum return of the collateral, it creates a tight link between investor expectations about future prices and investor demand for the risky assets. For example, if investors expect the price \( \overline{P}_{t+1} \) to be very high—and hence expect the price \( P_{t+1} \) to be very low—the minimum possible return on the collateralized asset \( R_{\overline{P},t+1} \) would be very low, which restricts the obtainable leverage. With leverage constrained, demand for the \( \overline{P} \) asset would be lower, which confirms the initial expectations. Alternatively, if investors believe the price \( P_{t+1} \) will be relatively high (but still lower than price \( \overline{P}_{t+1} \) by construction), investors can then take on more leverage from the posted collateral, which increases the demand for the \( \overline{P} \) asset and raises its price in a way that coincides with the initial beliefs.

It is worth noting that the greater leverage will increase demand for both risky assets \( \overline{P} \) and \( P \). For a fixed amount of leverage, higher demand for one asset by construction implies lower demand for the other asset. The reinforcement of beliefs and the emergence of multiple equilibria will depend on which asset garners the larger change in demand from the change in leverage.

When the acceptable collateral is the \( \overline{P} \) asset, as in (10), demand must increase relatively more for the \( \overline{P} \) asset than the \( P \) asset if leverage were to increase from a loosened constraint. If instead the acceptable collateral were the \( P \) asset, the minimum return on the collateral would be \( R_{P,t+1} \), which would coincide with the \( P \) asset earning the low cash flow again and remaining the lower priced risky asset. For multiplicity, the demand must change relatively more for the \( P \) asset. That way, expectations about a higher \( P_{t+1} \) price, for example, would loosen the collateral constraint, raise leverage, boost demand for the \( P \) asset, and reinforce the initial beliefs.
4 Solving for the equilibrium

To build intuition for the full model, I first solve for the equilibrium of a frictionless economy where the investment cost $\kappa$ is absent. I then discuss how introducing a positive investment cost can create an arbitrage in the financial markets. Such an arbitrage can be closed if borrowing were not constrained by a collateral requirement. Once the margin constraint is imposed, however, the arbitrage persists and the law of one price is violated: two claims with identical cash flows trade at different prices, thus creating an endogenous basis (a price gap). I end by introducing an asymmetric equilibrium in which ex ante identical investors choose heterogeneous portfolios ex post.

4.1 Frictionless benchmark

To begin analyzing the equilibrium of the economy, I start with a frictionless benchmark in which the investment cost $\kappa = 0$. For the time being, suppose also that the collateral constraint in (10) is non-binding.

Investors are identical, so the continuum can be represented by a single investor. With constant aggregate cash flows, the representative investor faces a certain consumption stream. Hence the $\mathcal{P}$ and the $\mathcal{P}_1$ assets do not bear any risk premia. The expected returns on the risky assets match the return on the risk-free asset $R_{f,t}$, which itself equals the subjective rate of time preference $\frac{1}{\delta}$. The three equilibrium asset pricing equations are thus

$$R_{f,t} = \frac{1}{\delta},$$

$$E_t(R_{\mathcal{P},t+1}) = R_{f,t},$$

$$E_t(R_{\mathcal{P}_1,t+1}) = R_{f,t}.$$

These three moment conditions combined with the risky-asset market-clearing condition $P_t = \sigma_t(W + Y)$ pin down the equilibrium risky asset prices. Finally, the representative investor’s optimal portfolio policies $(\phi_{f,t}, \phi_{\mathcal{P},t}, \phi_{\mathcal{P}_1,t})$ are determined by market clearing in both the risk-free asset market and the individual risky asset markets. Following this overview of the frictionless equilibrium, Proposition 1 presents the full solution.

**Proposition 1.** (Equilibrium solution, frictionless economy) The equilibrium leverage of the representative investor in the frictionless economy without a collateral constraint is $1 - \phi_{f,t} = \frac{W + Y}{W}$, whereas the portfolio of risky assets is $\phi_{\mathcal{P},t} = \sigma_t$ and $\phi_{\mathcal{P}_1,t} = 1 - \sigma_t$. The equilibrium wealth share $\sigma_t$ of the high cash flow asset obeys a linear dynamical system represented by

$$\sigma_{t+1} = \frac{1}{\delta} \sigma_t - \left( \frac{1 - \delta}{\delta} \right) \left( \frac{1}{2} + \left( q - \frac{1}{2} \right) \left( \lambda - \frac{1}{2} \right) \right)$$

and initial value $\sigma_{t=0} = \sigma_0$. The system has a unique but unstable steady state $\sigma = \frac{1}{2} + \left( q - \frac{1}{2} \right) \left( \lambda - \frac{1}{2} \right)$. The wealth share process $\{\sigma_t\}$ sets the expected return of both the $\mathcal{P}$ and $\mathcal{P}_1$. 

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assets equal to the interest rate $R_{f,t} = \frac{1}{\delta}$ in every period $t$.

The proposition reveals how equilibrium risky asset prices in the frictionless economy are characterized by a linear dynamical system. In fact, that equilibrium system is deterministic. The reason is that the investment opportunity set was transformed from two assets with random cash flows into two assets with time-varying but non-random wealth shares. In every period, investors know with certainty the future prices of the two assets that will earn the high and low cash flows in the next period.

Of course, the underlying uncertainty in the economy has not disappeared. Investors are still unsure which asset (A or B) will earn which cash flow. The unique steady state $\sigma$ of the system should therefore be considered a stochastic steady state. It describes the long run wealth share of the asset earning the high cash flow, but that wealth share will alternate stochastically according to $\{X_t\}$.

A depiction of the linear system is presented in Figure 3. Where the system crosses the solid red $45^\circ$ line is the location of the unique stochastic steady state. The blue dot in the figure pinpoints that steady state wealth share at $\sigma = 0.58$.

**Figure 3: Frictionless Equilibrium Dynamical System**

\[ \sigma_{t+1} = a\sigma_t - b \quad \text{and} \quad \sigma_{t+1} = \sigma_t \]

*Notes: Parameters used are $\delta = 0.4$, $\lambda = 0.9$, $q = 0.7$, and $\kappa = 0$. The solid black line is the linear dynamical system of the high cash-flow asset’s wealth share in the frictionless economy: $\sigma_{t+1} = a\sigma_t - b$, where $a = \frac{1}{\delta}$ and $b = \left(\frac{1-\lambda}{\lambda}\right)\left(\frac{1}{2} + (q - \frac{1}{2})(\lambda - \frac{1}{2})\right)$. The red line indicates a stochastic steady state $\sigma_t = \sigma_{t+1} = \sigma$. The steady state value in the figure is the blue dot at $\sigma = 0.58$, which is the only sustainable wealth share along any equilibrium path.*
The function $\sigma_{t+1} = a\sigma_t - b$ in (11) is seen to cut the $45^\circ$ from below, which implies that the steady state wealth share is unstable. The instability of the steady state implies that the unique equilibrium path of the economy begins and stays at the steady state; i.e., $\sigma_t = \sigma$ for all $t$. Every other starting wealth share $\sigma_0 \neq \sigma$ initiates a divergent process that investors would recognize as unsustainable and hence off the equilibrium path. Interestingly, because of this, risky asset prices that began at any values other than those which set $\sigma_0 = \sigma$ would immediately jump in the market so that the initial wealth share matches the steady state value.

The steady state price $\bar{P}$ of the high cash flow asset is simply $\sigma (W + Y)$. The solution for the steady state in the proposition shows that the equilibrium price of the high cash flow asset increases if that asset earned a larger fraction $\lambda$ of the aggregate cash flow. The price would also rise if the probability $q$ of remaining the high cash flow asset increased (i.e., if there were greater persistence). Finally, if either the fraction or the probability are $\frac{1}{2}$, the aggregate wealth is split equally between the two assets, and their prices are identical.

**Non-binding collateral constraint**

For the risk-free asset market to clear in the frictionless economy, the representative investor must borrow the aggregate savings of outside households $Y$. Enough collateral must be posted to borrow this amount and satisfy the constraint in (10). (In this economy, the representative investor gives the collateral to the background intermediary that supplies the outside savings as leverage.)

The equilibrium prices presented in Proposition 1 were derived under a non-binding collateral constraint. The amount of outside savings $Y$ the representative investor borrows in equilibrium must be restricted to ensure that (10) is indeed non-binding. The next lemma provides that upper bound on the outside savings.

**Lemma 1. (Non-binding collateral constraint, frictionless economy)** The collateral constraint in (10) is non-binding at the equilibrium prices presented in Proposition 1 if and only if the ratio of outside savings to investor wealth satisfies

$$\frac{Y}{W} < \delta \left( \frac{1 - \sigma + (1 - \lambda) (1 - \delta)}{1 - \delta (1 - \sigma)} \right).$$

An increase in outside savings has two effects on the collateral constraint. Requiring the investor to take on more leverage tightens the constraint. But greater leverage raises aggregate investment in risky assets and boosts the price of the collateral, which loosens the constraint. The lemma provides the upper bound on the share of outside wealth to investor wealth at which the second effect on the collateral price dominates.

**4.2 Synthetic riskless claim**

Before introducing a positive investment cost $\kappa > 0$, I first discuss an important feature of the financial markets that prevails regardless of the presence of a cost. Recall that the risk-free
asset has price $P_{f,t} = \frac{1}{R_{f,t}}$ and delivers one unit of consumption (one dollar) at the end of a period no matter the realization of the state $X_{t+1}$. Now consider a $\beta$-dollar portfolio that invests $\beta \equiv \frac{W+Y}{W+Y+D} \in (0, 1)$ dollars in a portfolio of the $\overline{P}$ and $P$ assets. The portfolio allocates a weight $\sigma$ to the $\overline{P}$ asset and a weight $1 - \sigma$ to to the $P$ asset.

A simple calculation shows that the $\beta$-dollar portfolio also earns one dollar at the end of the period regardless of the value of the state.\(^3\) Hence the portfolio generates a synthetic riskless claim with the same payoff as the risk-free asset. Because the $\beta$-dollar portfolio and the risk-free asset have identical future cash flows, they must trade at the same price by the law of one price; otherwise, there is an arbitrage.

In the frictionless benchmark, the $\beta$-dollar portfolio and the risk-free asset did indeed match in price. From Proposition 1, the price of the risk-free asset $P_f = \delta$, which equaled the price $\beta$ of the portfolio. An equivalent relation between the two investments is that their returns match. The gross return on the synthetic riskless claim was $\frac{1}{\beta}$, which equaled the risk-free rate $R_{f} = \frac{1}{\delta}$ in the frictionless economy.\(^4\) With prices and returns of the risk-free asset and synthetic riskless claim matching, there was no arbitrage in the frictionless economy.

4.3 Positive $\kappa$

From now on, if an investor wishes to purchase the low cash-flow asset $P$ in a period, he or she must pay a fixed utility cost $\kappa > 0$ that can be interpreted as a fraction of wealth that investors must pay when investing in that asset.

4.4 Symmetric equilibria

In symmetric equilibria, all investors choose identical portfolio and consumption policies in the presence of the cost of investing in the $P$ asset. Because the investment cost is fixed, marginal decisions of a typical investor will not change from the optimal policy of the representative investor in the frictionless benchmark. Hence consumption will be fixed through time and across states, and market prices in a symmetric equilibrium will be the same as in Proposition 1.

The presence of the positive cost $\kappa$, however, requires one to verify whether a symmetric equilibrium is viable. A viable symmetric equilibrium is one in which no individual investor has an incentive to deviate from the symmetric policies given the symmetric equilibrium prices. It may be worthwhile, for instance, for an investor to bear some consumption risk by investing only in the $\overline{P}$ asset in order to avoid paying the investment cost.

One can verify whether a symmetric equilibrium is viable by examining a one-shot deviation. If a typical investor has greater utility following an optimal policy in one period that invests in the high cash flow asset only instead of allocating to both markets and paying $\kappa$ given the

\(^3\)Recall that $\sigma = \frac{\overline{P}_{\tau+1}}{\overline{P}_{\tau} + P_{\tau}}$ and $\overline{P}_{\tau} + P_{\tau} = W + Y$ in every period $\tau$. When the state persists, the return on this portfolio is $\beta \sigma \left( \frac{\overline{P}_{\tau+1}}{\overline{P}_{\tau}} + P_{\tau} \right) + \beta (1 - \sigma) \left( \frac{\overline{P}_{\tau+1} + D_{\tau}}{\overline{P}_{\tau}} \right) = 1$. A similar calculation shows the portfolio earns one dollar when the state switches as well.

\(^4\)The equality of prices and returns is condition (24) in the proof of the proposition that is presented in Appendix A.1.
market prices in a symmetric equilibrium, that equilibrium is not viable. Ruling out deviations of this kind is similar to confirming subgame perfection in extensive form games.

With investors having log preferences and facing a constant investment opportunity set, the solution to their dynamic portfolio problem is the same as that for a single period problem. A one-shot deviation can therefore be analyzed using an investor's problem over one period. If there is no profitable deviation in an arbitrary period \( t \), there is no profitable deviation in any period.

Let \( W^i_t \) be the wealth in period \( t \) of a typical investor \( i \), and let \( V_{\text{sym}}(W^i_t) \) be the value function of that investor in a symmetric equilibrium. Suppose also that the conditions of Lemma 1 are met so that the collateral constraint is non-binding in all periods.

A symmetric equilibrium implies that every investor holds the representative investor's portfolio from the frictionless benchmark in Proposition 1. With no aggregate portfolio risk, the wealth process of the typical investor satisfies \( W^i_{t+1} = W^i_t \), no matter the value of the state \( X_{t+1} \). Hence the value function \( V_{\text{sym}} \) is

\[
V_{\text{sym}}(W^i_t) = \log W^i_t - \kappa. 
\] (12)

If investor \( i \) considers deviating from the symmetric equilibrium policy, he or she would choose an optimal allocation between the risk-free asset and the \( P \) asset to maximize next period's wealth. In exchange for bearing some consumption risk, the investor avoids paying the investment cost \( \kappa \). The next lemma presents the value function \( V_{\text{dev}}(W^i_t) \) associated with a one-period deviation from the symmetric equilibrium policy.

**Proposition 2.** (Value function, one-period deviation) The value function \( V_{\text{dev}} \) of a typical investor \( i \) who deviates from the symmetric equilibrium policy by investing only in the risk-free asset and the \( P \) asset in one period \( t \) but chooses the symmetric equilibrium portfolio thereafter is

\[
V_{\text{dev}}(W^i_t) = \log W^i_t + \eta - \delta \kappa, 
\] (13)

where the constant

\[
\eta = q \log \eta_1 + (1 - q) \log \eta_2, 
\]

and \( \eta_1, \eta_2 \) are positive constants defined in Appendix A.3.

An investor that deviates for one period delays paying the investment cost, which accounts for the last term in (13). He or she also gains or loses from the deviation via the constant \( \eta \), which captures the impact on lifetime utility of bearing consumption risk during the single deviating period. The constant consists of the weighted average of \( \log \eta_1 \) and \( \log \eta_2 \).

These two constants are in fact the log returns on wealth in the two possible states: when the \( P \) asset persists and earns the high cash flow \( \log \eta_1 \) and when the \( P \) asset switches and earns the low cash flow \( \log \eta_2 \). Hence the constant \( \eta \) is the expected one-period log return on wealth from an optimal portfolio in the risk-free asset and the high cash-flow asset.
A symmetric equilibrium exists when a one-shot deviation is unprofitable. Ruling out a one-shot deviation involves comparing the lifetime utility from strictly following the symmetric equilibrium policy with the lifetime utility from deviating from the policy in one period but adhering to the policy from then on. Mathematically, a one-shot deviation is unprofitable and thus a symmetric equilibrium exists if and only if

\[ V_{\text{sym}}(W_t) \geq V_{\text{dev}}(W_t). \]

The next proposition reveals that a one-shot deviation is in fact always profitable, thus making a symmetric equilibrium always fail to exist.

**Proposition 3. (Symmetric equilibrium non-existence)** A symmetric equilibrium fails to exist in the economy. Specifically, there always exists a profitable deviation from the symmetric equilibrium policies.

A comparison of the two value functions \( V_{\text{sym}} \) and \( V_{\text{dev}} \) shows that the condition to rule out a profitable deviation is \( \kappa \leq -\frac{\eta}{1-\delta} \). Because the investment cost is strictly positive, the only way for this inequality to hold is if the optimal portfolio that invests only in the risk-free asset and the \( P \) asset in one period earned a negative expected return. Only then would an investor avoid deviating. Because the expected return on the risky portfolio must be positive, the inequality cannot hold, which prevents a symmetric equilibrium from existing.

### 4.5 An arbitrage

Because a symmetric equilibrium fails to exist, the natural alternative equilibria to study are asymmetric equilibria. In equilibria of this kind, investors adopt heterogeneous portfolio strategies despite ex ante homogeneity in preferences and investment opportunities. For such equilibria to exist, the risky asset prices \( (P_t, P_t) \) and interest rate \( R_{f,t} \) must adjust so that every investor is indifferent between the various portfolios.

However, when prices alter to guarantee the existence of an asymmetric equilibrium, an arbitrage forms in the financial markets. Recall from earlier the synthetic riskless claim generated by the \( \beta \)-dollar portfolio. That portfolio took \( \beta \) dollars and invested a fraction \( \sigma \) in the \( P \) asset and the remaining fraction in the \( \overline{P} \) asset. In the frictionless benchmark, the price of this portfolio matched the price of the risk-free asset, making \( \beta = \frac{1}{R_{f,t}} \).

In an asymmetric equilibrium, the equivalency in prices no longer holds. Creating the synthetic riskless claim requires investing in both risky asset markets and paying an investment cost \( \kappa \). To abstain from simply purchasing the risk-free asset at no cost and investing in the high cash-flow asset instead, an investor must receive compensation. Because the return on the synthetic riskless claim is fixed at \( R_{\beta} = \frac{1}{\beta} \), the investor is made indifferent to choosing the second strategy by earning a lower return on the risk-free asset. Hence the interest rate \( R_{f,t} \) in an asymmetric equilibrium must dip below \( R_{\beta} \).

But any difference in returns between the risk-free asset and the synthetic riskless claim is an arbitrage. With a lower return, the price \( P_{f,t} = \frac{1}{R_{f,t}} \) of the risk-free asset will be relatively
high, so an investor could short it (borrow), invest the proceeds in the \( \beta \)-dollar portfolio and generate a riskless profit worth \( (P_{f,t} - \beta) \).

If investors did not need to post collateral to borrow, this arbitrage strategy would force the price discrepancy to disappear. The interest rate would adjust so that \( R_{f,t} = \frac{1}{\beta} \), and the prices of the risky assets \( (\bar{P}_t, P_t) \) would settle to satisfy indifference between portfolio strategies. An asymmetric equilibrium could form without having an arbitrage.

A collateral constraint, however, restricts investors from borrowing without limit to close the price difference. Risky asset prices would still adjust to guarantee indifference, but an endogenous basis (a price gap) would form between the two risk-free investments. The size of the basis would be \( P_{f,t} - \beta \).

An asymmetric equilibrium with a margin requirement thus features an arbitrage that persists through time and a constraint that always binds. Because the collateral constraint embeds the minimum future return of the collateral, it creates an avenue for investor expectations in asymmetric equilibria to influence asset prices that has been absent so far. I next explore exactly what happens.

5 Asymmetric equilibria

The asymmetric equilibria I study are ones in which an endogenous fraction \( \pi_t \) of investors choose to invest only in the \( \bar{P} \) asset market. I label these “single market investors,” with their type denoted \( s \). Because these single market investors do not participate in the \( \bar{P} \) market, they do not pay the investment cost \( \kappa \).

The other fraction \( 1 - \pi_t \) of investors participate in both the \( \bar{P} \) and the \( P \) markets, pay the investment cost, and by doing so, they exploit the arbitrage between the synthetic riskless claim and the risk-free asset. I label these investors “arbitrageurs,” with type denoted \( a \). An alternative interpretation of \( \kappa \), therefore, is a cost of managing the arbitrage position. Prices in an asymmetric equilibrium are determined so that every investor is indifferent between being a single market investor and an arbitrageur.

Because the collateral constraint generates an endogenous basis between the \( \beta \)-dollar portfolio and the risk-free asset, there no longer is an equilibrium requirement that the prices of those investments match. Therefore, one can simplify the prices \( (\bar{P}_t, P_t) \) of the two risky assets and reduce notation.

5.1 Normalizing prices

The way to simplify prices is the following: recall that market clearing in the risk-free asset market implies \( \bar{P}_t + P_t = W + Y \). If the aggregate cash flow is set to \( D = (1 - \delta)(1 - Y) \), the risky asset prices sum to one. Therefore, the two risky asset prices can be denoted \( (P_t, 1 - P_t) \), where \( P_t \) is the price of the high cash-flow asset and \( 1 - P_t \) is the price of the low cash-flow asset. From now on, I use those prices.

Now, the “\( \bar{P} \)” asset is the risky asset with price \( P_t \), whereas the “\( 1 - \bar{P} \)” asset is the one with price \( 1 - P_t \). With their sum equaling one, the prices themselves are also the wealth shares
\((\sigma_t, 1 - \sigma_t)\). This connection makes the figures presented below for the asymmetric equilibria comparable to Figure 3 for the frictionless benchmark.\(^5\)

Similar to the frictionless case, asymmetric equilibria will be characterized by a deterministic difference equation \(P_{t+1} = \Phi (P_t)\). An equilibrium in period \(t\) will hence feature a price pair \((P_t, P_{t+1})\). Like before, investors will know the future price \(P_{t+1}\) in period \(t\), but not know which of the two risky assets will have it in the next period.

The optimal portfolios of both single market investors and arbitrageurs are presented next, followed by the interest rate and the indifference condition that pins down the equilibrium risky asset prices.

### 5.2 Single market investor portfolio

Portfolio policies of both types of investors can be determined by solving one-period problems that maximize next period wealth. The portfolio problem of a single market investor with starting wealth normalized to one is

\[
V_s^s (P_t, P_{t+1}) = \max_{\phi_{f,t}^s} E_t \left( \log W_{t+1}^s \right),
\]

where

\[
W_{t+1}^s = R_{P,t+1} + \phi_{f,t}^s (R_{f,t} - R_{P,t+1}),
\]

with \(\phi_{f,t}^s \in (-\infty, 1]\). The investor’s allocation to the \(P\) asset is \(1 - \phi_{f,t}^s\) and is subject to the collateral constraint

\[
(1 - \phi_{f,t}^s) R_{P,t+1} + \phi_{f,t}^s R_{f,t} \geq 0.
\]

The solution of the single market investor problem is in the following lemma.

**Lemma 2.** (Single market investor policy) The optimal investment policy for single market investors is

\[
\phi_{f,t}^s = (1 - q) \left( \frac{R_{P,t+1}}{R_{P,t+1} - R_{f,t}} \right) - q \left( \frac{R_{P,t+1}}{R_{f,t} - R_{P,t+1}} \right),
\]

and the single market investor’s collateral constraint never binds.

From the lemma, single market investors lend \((\phi_{f,t}^s > 0)\) if the risk-free rate exceeds the harmonic mean of the two possible returns on the \(P\) asset:

\[
\phi_{f,t}^s > 0 \iff R_{f,t} > \frac{1}{q \frac{1}{R_{P,t+1}} + (1 - q) \frac{1}{R_{f,t}}}. 
\]

Furthermore, the optimal strategy is to short the \(P\) asset if the risk-free rate exceeds the expected return:

\[
\phi_{f,t}^s > 1 \iff R_{f,t} > q R_{P,t+1} + (1 - q) R_{P,t+1}. 
\]

---

\(^5\)The price normalization also affects the price of the \(\beta\)-dollar portfolio. With the aggregate cash flow \(D\) specified above, aggregate wealth \(W = 1 - Y\). Therefore the price of the portfolio \(\beta = \frac{W + Y}{W + Y + D}\) becomes \(\frac{1}{1 + (1 - \beta)(1 - \gamma)}\).
5.3 Arbitrageur portfolio

An arbitrageur chooses an investment strategy to solve

\[ V^a(P_t, P_{t+1}) = \max_{\phi^a_{f,t}, \phi^a_{P,t}} E_t \left( \log W^a_{t+1} \right) - \kappa, \]

where

\[ W^a_{t+1} = \phi^a_{f,t} R_{f,t} + \left( 1 - \phi^a_{f,t} \right) \left( R_{1-P,t+1} + \phi^a_{P,t} \left( R_{P,t+1} - R_{1-P,t+1} \right) \right), \]

with \( \phi^a_{P,t} \in [0, 1] \). An arbitrageur’s strategy is subject to the collateral constraint

\[ \left( 1 - \phi^a_{f,t} \right) \phi^a_{P,t} R_{1-P,t+1} + \phi^a_{f,t} R_{f,t} \geq 0. \]

The arbitrageur can post the entire position in the \( P \) asset as collateral, and the minimum possible return on that position determines the amount he or she can borrow. The optimal investment policy for arbitrageurs is given in the following lemma.

**Lemma 3.** *(Arbitrageur investment policy)* For fixed leverage \( \left( 1 - \phi^a_{f,t} \right) \), the optimal investment share in the \( P \) asset for arbitrageurs is

\[ \phi^a_{P,t} = g(P_t, P_{t+1}) + h(P_t, P_{t+1}) R_{f,t} \left( \frac{\phi^a_{f,t}}{1 - \phi^a_{f,t}} \right), \tag{15} \]

where the functions

\[ g(P_t, P_{t+1}) = q \frac{R_{1-P,t+1}}{R_{1-P,t+1} - R_{P,t+1}} - \left( 1 - q \right) \frac{R_{1-P,t+1}}{R_{P,t+1} - R_{1-P,t+1}}, \]

\[ h(P_t, P_{t+1}) = \frac{q}{R_{1-P,t+1} - R_{P,t+1}} - \frac{1 - q}{R_{P,t+1} - R_{1-P,t+1}}. \]

Additionally, the collateral constraint for arbitrageurs always binds, making their constrained leverage

\[ 1 - \phi^a_{f,t} = \frac{R_{f,t} \left( 1 + hR_{P,t+1} \right)}{R_{f,t} \left( 1 + hR_{P,t+1} \right) - gR_{P,t+1}}. \tag{16} \]

Substituting (16) into (15) yields the arbitrageur’s constrained optimal investment share in the \( P \) asset:

\[ \phi^a_{P,t} = \frac{g(P_t, P_{t+1})}{1 + h(P_t, P_{t+1}) R_{f,t} R_{P,t+1}}. \tag{17} \]

5.4 Interest rate

With a fraction \( \pi_t \) of single market investors and the other fraction \( 1 - \pi_t \) of arbitrageurs, market clearing in the risk-free asset requires

\[ Y + W \left( \pi_t \phi^a_{f,t} + (1 - \pi_t) \phi^a_{f,t} \right) = 0. \]
This clearing condition implies that the fraction of single market investors must be

$$\pi_t = - \left( \frac{\frac{Y}{W} + \phi_a^g}{\phi_s^f - \phi_a^f} \right).$$  \hspace{1cm} (18)$$

In an asymmetric equilibrium, the fraction of single market investors must satisfy $$\pi_t \in [0, 1)$$.

All market participants could be arbitrageurs investing in both risky asset markets ($$\pi_t = 0$$), but it cannot be that all are single market investors ($$\pi_t = 1$$); otherwise, the $$1 - P$$ market cannot clear.

Risky asset market clearing requires:

$$P_t = W \left( \pi_t (1 - \phi_s^f) + (1 - \pi_t) (1 - \phi_a^g) \phi_a^P \right),$$

$$1 - P_t = W (1 - \pi_t) (1 - \phi_a^g) (1 - \phi_a^P).$$

The interest rate in the economy is determined using (18), the optimal portfolios of investors, and one of the risky-asset market-clearing conditions. The other risky asset market will clear from Walras’ Law. The next lemma presents the equilibrium interest rate.

**Lemma 4. (Interest rate, non-binding no-shorting constraint)** When the no-shorting constraint for investors does not bind, the equilibrium interest rate is a root of the following quadratic equation

$$a (P_t, P_{t+1}) R_{f,t}^2 + b (P_t, P_{t+1}) R_{f,t} + c (P_t, P_{t+1}) = 0, \hspace{1cm} (19)$$

where the functions $$a (P_t, P_{t+1})$$, $$b (P_t, P_{t+1})$$, and $$c (P_t, P_{t+1})$$ are defined in Appendix (A.7).

I find numerically that both roots of the quadratic in (19) are real. I choose the minimum of the two roots as the unique equilibrium interest rate of the economy. I do that because I assume the function $$R_{f,t} (\cdot)$$ is continuous in outside savings $$Y$$. If $$Y = 0$$, the above quadratic expression becomes linear and the interest rate is determined uniquely. For an infinitesimal amount of savings $$Y = \varepsilon$$ with $$\varepsilon > 0$$, the smaller root is arbitrarily close to the solution $$R_{f,t} (Y = 0)$$, whereas there is a discontinuity at the larger root. Therefore, I choose the minimum root to preserve continuity in $$R_{f,t} (\cdot)$$.

Lemma 5 presents the equilibrium interest rate when investor portfolios bind at different corners of the no-shorting region.

**Lemma 5. (Interest rate, binding no-shorting constraint)** The only case in which an equilibrium exists when the no-shorting constraint binds for either type of investor is when $$\phi_a^P = 0$$ and $$\phi_s^f < 0$$. The equilibrium interest rate in this case is the root of the following quadratic equation

$$a' R_{f,t}^2 + b' (P_t, P_{t+1}) R_{f,t} + c' (P_t, P_{t+1}) = 0 \hspace{1cm} (20)$$

where $$a'$$ is a constant and $$b' (P_t, P_{t+1})$$ and $$c' (P_t, P_{t+1})$$ are functions defined in Appendix (A.8).

When an asymmetric equilibrium exists with at least one type of investor’s portfolio binding
at the no-shorting constraint, arbitrageurs optimally choose not to hold the $P$ asset, while single market investors want to borrow. By not holding the collateralizable asset, arbitrageurs can obtain no leverage for their positions, so only single market investors can borrow the outside savings $Y$. For the same reason given earlier, I again choose the minimum root of the quadratic as the unique equilibrium interest rate. The other cases in which investor portfolios are at the corners cannot be equilibria; I explain why for each case in Appendix (A.8).

In both scenarios covered by the lemmas, the basis between the synthetic riskless claim and the risk-free asset is the difference in their prices. The price of the risk-free asset is the inverse of the minimum roots $R_{f,t}$ in the quadratic equations in the lemmas, whereas the price of the synthetic riskless claim is

$$\beta = \frac{1}{1+(1-\delta)(1-Y)}.$$  

Because the synthetic claim will have a higher return to compensate for the investment cost $\kappa$, the positive basis is

$$\frac{1}{R_{f,t}} - \beta.$$

5.5 Risky asset prices

One last condition determines the equilibrium price process $\{P_t\}_{t \in \mathbb{N}}$. Each period, the prices $\{P_t, P_{t+1}\}$ adjust to make investors indifferent between investing only in the $P$ asset market and exploiting the constrained arbitrage opportunity. This indifference condition amounts to

$$F(P_t, P_{t+1}) \equiv V^a(P_t, P_{t+1}) - V^s(P_t, P_{t+1}) = 0.$$  \hspace{1cm} (21)$$

The value functions of the two investors are

$$V^i(P_t, P_{t+1}) = q \log W_{t+1}^i(H) + (1-q) \log W_{t+1}^i(T) - \kappa 1_{\{i=a\}},$$

for $i \in \{s, a\}$. The first term of the value function is the (weighted) log utility of wealth if the state persists, whereas the second term is the (weighted) log utility of wealth if the state switches.

The indifference condition

$$F(P_t, P_{t+1}) = 0$$  \hspace{1cm} (22)$$

summarizes the equilibrium of the economy. For a fixed $P_t \in (0, 1)$, $F(\cdot)$ is a univariate function in $P_{t+1}$. The roots of this function for all $P_t \in (0, 1)$ amount to a discrete-time, deterministic, nonlinear dynamical system. If there exists no root for a given $P'_t$, no equilibrium exists for the price pair $(P'_t, P'_{t+1}) \forall P'_{t+1} \in (0, 1)$. If multiple roots exist for a given $P'_t$, the economy features multiple asymmetric equilibria.

6 Multi-valued dynamical system

Because the value $P_t$ is defined as the price of the asset yielding the high cash flow, $P_t \in (0.5, 1)$ by construction. To extract the equilibrium dynamical system, I solve numerically for the root(s) $P_{t+1}$ of (22) for each $P_t$ in a dense partition of the interval $I = [0.5, 1]$.

Over a variety of parameter sets, the resulting implicit function $P_{t+1} = \Phi(P_t)$ is multi-valued. Hence, for a given price $P_t$, more than one asset price $P_{t+1}$ satisfies the indifference
condition (22). The collection of pairs \((P_t, P_{t+1})\) are all equilibria for the economy in period \(t\), and each pair constitutes a rational expectations equilibrium. The next section presents a common example of the multi-valued dynamical system.

6.1 A common example

A common example of the equilibrium dynamical system is displayed in Figure 4. The figure depicts the pairs of prices \((P_t, P_{t+1})\) that satisfy the indifference condition \(F = 0\) in (22). The price pairs are on the solid black and gray curves. Together, these pairs of prices constitute the equilibrium dynamical system of the economy.

![Figure 4: Multi-valued Dynamical System](image)

*Notes:* Parameters used are \(\delta = 0.4\), \(\lambda = 0.9\), \(q = 0.7\), \(\kappa = 0.06\), and \(Y = 0.72\). The solid black and gray curves are price pairs \((P_t, P_{t+1})\) that satisfy the indifference condition \(F(P_t, P_{t+1}) = 0\). Price pairs on the solid black curves are sustainable along an equilibrium path, whereas those on the gray curves are not. The red line indicates a stochastic steady state \(P_t = P_{t+1} = P\). The steady state values in the figure are the blue dots at \(P = 0.57\) and \(P = 0.68\).

Although the price pairs on the solid black and dashed gray curves all satisfy (22), not all pairs are sustainable along an equilibrium price path of the economy. Some price pairs, if reached, lead to a divergent path outside the system. Exactly as it was in the frictionless benchmark, investors, anticipating the divergence, would not treat those price pairs as eligible on any possible equilibrium path. These prices must be trimmed from the dynamical system. The dashed gray curves are these trimmed price pairs. The solid black curves are the only
sustainable prices.

Just as in the linear system of the frictionless economy, the location at which the multi-valued system in the figure crosses the solid red $45^\circ$ line indicates a stochastic steady state. At that price $P_t = P_{t+1} = P$, which means the economy experiences risky asset prices that alternate randomly between $P$ and $1 - P$ each period. The blue dots in the figure depict the two steady state prices of $P = 0.57$ and $P = 0.68$. Because the system cuts the $45^\circ$ line from above at $P = 0.68$, that steady state is stable. The other steady state is unstable, as the system there cuts from below.

The multi-valued system in Figure 4 shares the same parameters as the linear system in Figure 3, save for $\kappa > 0$. The obvious difference between the dynamical systems of the two economies is that this one is multi-valued, whereas the other was linear. Another difference is that the multi-valued system has two stochastic steady states instead of just one. One of the steady states here is even stable, unlike the unique unstable steady state in the frictionless benchmark. That frictionless steady state under the parameters of Figure 4 (with $\kappa = 0$) would be $\sigma = P = 0.58$, which is around the same as the smaller-valued steady state in the multi-valued system.

The introduction of an arbitrage from the investment cost that binds the collateral constraint converts the linear system into a multi-valued one. Interestingly, an artifact of the linear system is still present in the figure via the linear portion of the multi-valued system. An additional non-linear part is appended. This portion includes a second set of future prices $P_{t+1}$ that also satisfy an equilibrium, which arises from the binding collateral constraint.

### 6.2 Leverage supply and demand

The intuition behind the multi-valued system and the existence of two steady states is illustrated in Figure 5.

The figure depicts arbitrageur leverage supply and leverage demand for the same parameters used in Figure 4. Prices are at their steady state values. I call the leverage supply the amount of leverage available to arbitrageurs from the binding collateral constraint. Supply is upward sloping in accordance with the minimum possible return of the collateral. As the price $1 - P$ increases, so too does the worst return the collateral can realize over one period, which increases the supply of leverage available from the collateral. Notice that for prices $1 - P$ that are low enough, the supply of leverage converges to the minimum of one. At such low prices, arbitrageurs prefer to hold only the $1 - P$ asset for its high expected return. By not holding the collateral, arbitrageurs have nothing to pledge in exchange for leverage.

I call leverage demand the amount of leverage that would make an arbitrageur indifferent to being a single market investor. Demand for leverage is increasing in the figure because the expected return on the $1 - P$ asset declines as the asset price rises. To make an arbitrageur indifferent to investing solely in the $P$ asset as a single market investor, he or she would need more leverage to compensate for the lower expected return.

An increasing supply and demand curve for arbitrageur leverage generates the multiple
Figure 5: Arbitrageur Leverage Supply and Demand

![Graph of Arbitrageur Leverage Supply and Demand](https://example.com/figure5.png)

Notes: Parameters used are $\delta = 0.4$, $\lambda = 0.9$, $q = 0.7$, $\kappa = 0.06$, and $Y = 0.72$. The x-axis represents prices at a possible stochastic steady state. The dashed green curve is the amount of leverage available from the collateral constraint (supply of leverage). The solid purple curve is the amount of leverage necessary to make arbitrageurs indifferent to being single market investors (demand for leverage). Supply and demand for leverage intersect at the two steady states of the system, represented by the blue dots at $1 - P = 0.32$ and $1 - P = 0.43$.

steady state equilibria. At the lower steady state value of $1 - P = 0.32$, expectations about the minimum value of the collateral are low, so leverage in constrained, creating low demand for the $1 - P$ asset, which confirms the low $1 - P$ price. Conversely, at the higher steady state value $1 - P = 0.43$, the collateral return is expected to be high, which frees up leverage, raises demand for the $1 - P$ asset, and confirms the higher $1 - P$ price.

The same economic force explains why the dynamical system is a multi-valued function rather than uniquely valued. In some region of the state space, the supply and demand for leverage intersect at a single price pair $(P_t, P_{t+1})$. In those regions, the dynamical system is single valued. In other regions, there are two intersecting price pairs, which makes the system multi-valued.

The domain of the system in Figure 4 does not extend above $P_t = 0.7$ because the expected return on the $1 - P$ asset at any higher price would be so large that an arbitrageur would have to save $(1 - \phi_{f,t}^a < 1)$ in order to be indifferent to a single market investment. However, the supply of leverage from the collateral constraint is bounded below by one in that region, so
demand and supply would not intersect.

7 Self-fulfilling prices

The presence of the binding collateral constraint in the economy generates a multi-valued dynamical system for equilibrium asset prices with multiple steady states. That characteristic of the equilibrium was missing from the frictionless economy which featured a linear system and a unique steady state. In the frictionless economy, a single equilibrium path existed. Risky asset prices alternated at random according only to the fundamental cash flow process \( \{X_t\} \).

A multi-valued system like the one in Figure 4, however, implies that multiple rational expectations equilibrium price paths are sustainable in the economy. While an equilibrium path is characterized by a fixed point between the risky market prices \((P_t, P_{t+1})\) and investor expectations of those prices, the fixed point is not unique in some regions. This multiplicity in the economy opens the door to an extrinsic source of variation that selects equilibrium paths.

The extrinsic process shift investor beliefs about future prices in ways that do not relate to changes in the fundamental process \( \{X_t\} \). While neither reflecting nor distorting beliefs about cash flows or discount rates, the shocks yet influence equilibrium expectations about prices and lead to aggregate price fluctuations. They are a public source of information, so in effect they correlate investor beliefs in favor of one equilibrium price over others.

A number of economic interpretations of this extrinsic variation are possible. They can represent investor sentiments, as in sudden shifts in overall market optimism or pessimism about the future prices of the risky assets. Another interpretation is that they capture an individual investor’s general uncertainty about the aggregate demand for either risky asset. An investor may be aware of his or her own personal demand via a portfolio allocation, but uncertain about the portfolio choices of others. If all investors share opinions about aggregate demand in the same direction, as in believing it to be high or low, these beliefs will be confirmed in the market price and self-fulfilling.

7.1 An extrinsic process

Along an equilibrium asset price path, up to two next period prices \( P_{t+1} \) are eligible in any period \( t \), with one strictly higher than the other. I insert an extrinsic process, which I denote \( \{\zeta_t\} \), that will influence the choice of the next period price if more than one is eligible. The process takes one of two values in a given period and the next-period value is perfectly foreseeable by investors. This means that investors rationally anticipate the next period price with probability one to occur when it does. Hence their beliefs that such asset price paths will take place are confirmed in equilibrium.

A value \( \zeta_t = 1 \) means investors are pessimistic about the \( 1 - P \) asset. Alternatively, they have poor expectations of its aggregate demand. Low expectations about the future value of the \( 1 - P \) asset in turn will constrict leverage because of the collateral requirement. Demand for the \( 1 - P \) asset will be suppressed and lead to a low future price \( 1 - P_{t+1} \). Hence this realization of the extrinsic process selects the lower \( 1 - P \) price. In Figure 4, that lower \( 1 - P \) price would
be on the nonlinear portion of the multi-valued system.

A value $\zeta_t = 0$ is the opposite: investors in aggregate are optimistic about the $1 - P$ asset, which raises expectations about its future value, allows greater leverage and demand, and reinforces the beliefs. In this case, the higher $1 - P_{t+1}$ price is chosen, which would be part of the linear portion of the dynamical system in the figure.

Realizations of this extrinsic process $\{\zeta_t\}$ generate a number of interesting price and leverage dynamics over a variety of model parameter sets. Sudden price crashes and booms, long price recoveries, and prices that overshoot and misfire in their trajectories toward a new steady state after an influx of new outside savings are all possible. In the next few sections I discuss and illustrate these phenomena.

7.2 Price crashes, booms, and long recoveries

In this section, I take the system depicted in Figure 4 and simulate a sample path of asset prices that could transpire from realizations of the extrinsic process $\{\zeta_t\}$ that shift investor expectations over time. The sample path of the $P$ asset price is given in Figure 6(a), and the corresponding path of the $1 - P$ asset price is in Figure 6(b).

There are two interpretations of the time series presented in the figures. The first interpretation is that they are the price paths of the original assets $A$ or $B$ if there were no cash flow shocks over the sample period displayed. An absence of cash flow shocks would imply $X_{t+1} = X_t$ over the sample period. In this case, any price movements can be considered a form of “excess” volatility, as no movement is attributable to changes in cash flows.

The second interpretation involves the presence of cash flow shocks. If the value of the state $X_t$ changes over the sample period, asset $A$ and asset $B$ would alternate their prices between the two paths in Figure 6(a) and Figure 6(b) according to the cash flow shock process. In this case, the paths presented would be interpreted as the prices of portfolios that always invested in either the high cash flow or low cash flow asset.

As seen in Figure 6(a), the price $P_t$ is converging to the stable steady state until investors anticipate a sudden 15% price drop, which then does occur in equilibrium. This price crash arrives after a realization $\zeta_t = 0$, which leads investors to suddenly become pessimistic about the $P$ asset (but optimistic about the $1 - P$ asset).

After the initial crash, the asset price recovers slowly toward the steady state almost to the price level before the crash until another drop. The pattern of prices crashes and long recoveries continues along this sample path. Because the $1 - P$ asset follows a path that is mirror to the $P$ asset, the $1 - P$ asset in this simulation displays repeated long price declines followed by booms, as seen in Figure 6(b).

While the price $1 - P_t$ is expected to decline, so too is the minimum return on the collateral, which constrains leverage for arbitrageurs, limits their demand, and thus reinforces the expected price decline. One can observe this relation in Figure 7, where the path of arbitrageur leverage $1 - \phi_{f,t}^\theta$ follows a similar pattern as the $1 - P$ asset price. When the $1 - P_t$ price is expected to boom, leverage expands by up to 80%, increasing demand for the $1 - P$ asset and
Figure 6: Price Crashes and Booms

(a) $P_t$ asset — Crashes and Recoveries

(b) $1 - P_t$ asset — Booms and Declines

Notes: Parameters used are $\delta = 0.4$, $\lambda = 0.9$, $q = 0.7$, $\kappa = 0.06$, and $Y = 0.72$. confirming the price boom.

Figure 7: Arbitrageur Leverage

Notes: Parameters used are $\delta = 0.4$, $\lambda = 0.9$, $q = 0.7$, $\kappa = 0.06$, and $Y = 0.72$.

The jumps in both asset prices occur when investors switch curves in the dynamical system.
before reaching the steady state according to the extrinsic process \( \{ \zeta_t \} \). An illustration of the sample price paths along the dynamical system is presented in Figure 8. Investors rationally anticipate this asset price path and expect these jumps to occur when they do. Their shared beliefs that such a path of prices will transpire is confirmed in equilibrium.

Figure 8: Price Crash and Recovery on the System

Notes: Parameters used are \( \delta = 0.4, \lambda = 0.9, q = 0.7, \kappa = 0.06, \) and \( Y = 0.72 \). The price path is the same as the first twenty periods depicted in Figure 6(a). The initial value of the path is depicted by the orange dot.

7.3 Price overshooting and misfiring

In this section, I present two variations of the model parameters in order to show that asset prices in the economy can either overshoot or misfire in their convergence to a new stochastic steady state upon a rise in the investment cost or outside savings.

Overshooting

Figure 9 presents the dynamical system when \( \delta = 0.4 \) and \( \lambda = 0.9 \) remain the same as before, but the persistence increases to \( q = 0.85 \) and the participation cost rises to \( \kappa = 0.19 \). The outside savings is \( Y = 0.64 \). The system now has three steady states. The lowest valued one is stable, while the larger two are unstable.
Figure 9: Equilibrium Dynamical System

Notes: Parameters used are \( \delta = 0.4, \lambda = 0.9, q = 0.85, \kappa = 0.19, \) and \( Y = 0.64. \) The solid black and gray dashed curves are price pairs \((P_t, P_{t+1})\) that satisfy the indifference condition \( F(P_t, P_{t+1}) = 0. \) Price pairs on the solid black curves are sustainable along an equilibrium path, whereas those on the gray dashed curves are not. The red line indicates a stochastic steady state \( P_t = P_{t+1} = P. \) The steady state values in the figure are the blue dots at \( P = 0.75, P = 0.82, \) and \( P = 0.85. \)

A small negative perturbation from the largest unstable steady state of \( P = 0.85 \) would initialize an asset price path that would eventually converge to the smaller stable steady state of \( P = 0.75. \) This small perturbation is triggered by a 3% increase in the investment cost from \( \kappa = 0.185 \) to the current system's value of \( \kappa = 0.19. \) An illustration of this equilibrium path on the dynamical system is given in Figure 10(a).

The price \( P_t \) descends along the bottom curve of the system toward the lower steady state, but overshoots it, initially falling below the steady state before rebounding toward convergence. The time path of the overshooting price \( P_t \) is displayed in Figure 10(b).

An increase in the participation cost \( \kappa \) increases the equilibrium prices \( P_t \) across the dynamical system in order to reduce the interest rate \( R_{f,t} \) and create a larger gap between the interest rate and the return on the synthetic riskless claim. The larger wedge enhances the arbitrage opportunity as compensation for the higher cost \( \kappa \) of exploiting it. However, as seen here, the price \( P \) at which the economy ultimately settles in steady state may be lower from which it began. An increase in the cost of the arbitrage investment (or the cost of gathering information on distressed assets) has the unexpected effect of decreasing the steady state price.
Notes: Parameters used are $\delta = 0.4$, $\lambda = 0.9$, $q = 0.85$, $\kappa = 0.19$, and $Y = 0.64$. The initial value of the path in the left panel is depicted by the orange dot.

$P$ rather than increasing it, because the higher cost changes the steady state.

**Misfiring**

Related to overshooting is what I call price “misfiring.” Here, an increase in the amount of outside savings $Y$ shifts the steady state to a higher price $1 - P$. On the path to the higher steady state, however, the price instead declines initially, to then have to undue this change by rising sharply thereafter in order to reach the higher steady state value.

Suppose the system begins as depicted in Figure 11(a). Parameters of the system are $\delta = 0.6$, $\lambda = 0.8$, $q = 0.85$, $\kappa = 0.08$, and $Y = 0.46$. All three steady states of this initial system are unstable.

Now suppose the economy begins at the middle steady state $P = 0.67$. A 4% increase in outside savings to $Y = 0.48$ is enough to change the system to the one depicted in Figure 11(b).

The lowest valued steady state of the system is now stable. The increase to savings $Y$ perturbs the economy positively away from the initial steady state of $P = 0.67$. A sample transition asset price path to the new steady state on the system which demonstrates the price misfiring is depicted in Figure 12(a).

This path on the system displays the price misfiring, initially moving away from the steady state, to then return the way it came toward convergence. The time series of the price of the $1 - P$ asset over this sample path is in Figure 12(b).

The $1 - P$ asset in the price misfiring at first declines in value before rebounding to the higher price. I focus on the $1 - P$ asset price path in order to show what happens to the allocation of leverage in the economy as prices misfire before transitioning to the new steady state. Leverage for both single market investors and arbitrageurs over the sample path is
Figure 11: Price Misfiring Dynamical Systems

Notes: (Left panel) Parameters used are $\delta = 0.6$, $\lambda = 0.8$, $q = 0.85$, $\kappa = 0.08$, and $Y = 0.46$. The solid black and gray dashed curves are price pairs $(P_t, P_{t+1})$ that satisfy the indifference condition $F(P_t, P_{t+1}) = 0$. Price pairs on the solid black curves are sustainable along an equilibrium path, whereas those on the gray dashed curves are not. The red line indicates a stochastic steady state $P_t = P_{t+1} = P$. The steady state values in the figure are the blue dots at $P = 0.62$, $P = 0.67$, and $P = 0.71$.

(Right panel) Parameters used are $\delta = 0.6$, $\lambda = 0.8$, $q = 0.85$, $\kappa = 0.08$, and $Y = 0.48$. The solid purple curves are the original system from the left panel. A 4% increase in outside savings $Y$ shifts the system to the solid black curves. The steady state values of the new system in the figure are the blue dots at $P = 0.62$, $P = 0.63$, and $P = 0.73$.

Figure 12: Price Misfiring Paths

Notes: Parameters used are $\delta = 0.6$, $\lambda = 0.8$, $q = 0.85$, $\kappa = 0.08$, and $Y = 0.48$. In the left panel, the initial value of the path is shown by the orange dot.
presented in Figure 13.

![Figure 13: Price Misfiring Leverage](image)

**Notes:** Parameters used are $\delta = 0.6$, $\lambda = 0.8$, $q = 0.85$, $\kappa = 0.08$, and $Y = 0.48$. Associated price path is the one depicted in Figure 12(a).

An increase in outside savings $Y$ reduces single market investor leverage and increases arbitrageur leverage in steady state. However, on the transition path to the new steady state, arbitrageur leverage initially does the opposite, declining along with the $1 - P_t$ price. Not until the $1 - P$ asset price rebounds does arbitrageur leverage sharply rise to the higher steady state level.

Single market investors initially are the ones who absorb the higher savings in the economy, even though the arbitrageurs ultimately take it as higher leverage at the new steady state. The price misfiring from the shocks to investor expectations distorts the allocation of leverage in the economy from the steady state allocation.

### 8 Leverage cycles

The tight link between the price $1 - P_t$ and arbitrageur leverage implies that any self-confirming expectations that generate cyclical prices can also generate cyclical leverage. I use the system from Figure 11(a) to simulate a path of such leverage cycles. The sample asset price path on the system is in Figure 14(a). The time path of arbitrageur leverage associated with this simulation is given in Figure 14(b).
Figure 14: Leverage Cycles

Notes: Parameters used are $\delta = 0.6$, $\lambda = 0.8$, $q = 0.85$, $\kappa = 0.08$, and $Y = 0.46$. The left panel depicts the price path on the dynamical system from the first forty periods of the associated leverage cycle in the right panel. The initial value of the path is shown by the orange dot. The top-half of the right panel depicts the evolution of arbitrageur leverage over time. The bottom-half of the right panel depicts the associated path of haircuts on the arbitrageur’s $P$ asset portfolio that is posted as collateral.

From the top-half of Figure 14(b), one can observe repeated periods of low leverage—when the price $1 - P_t$ declines and the collateral constraint tightens—followed by periods of expanded leverage—when $1 - P_t$ rebounds and arbitrageurs can borrow more off their pledged collateral. The leverage cycles are endogenous to the price expectations of investors.

The bottom-half of Figure 14(b) depicts the associated haircuts on the arbitrageur’s $P$ asset portfolio that is pledged as collateral. I define the haircut as

$$\text{Haircut} = \frac{\phi_{P,t}^a}{(1 - \phi_{f,t}^a) \phi_{P,t}^a}. \quad (23)$$

The numerator of (23) is the fraction of arbitrageur equity capital in the $P$ asset portfolio, whereas the denominator is the asset value of the position. The haircut is the equity-to-asset ratio of the pledged collateral. A high haircut implies that arbitrageurs can borrow less from the collateral and must finance a greater fraction of their portfolio with equity.

As seen in the bottom panel, haircuts mechanically move in opposite direction with arbitrageur leverage. The periods of high leverage are also those with low haircuts. At the peak of the leverage cycle, haircuts are seen to decrease by up to 40%, but then sharply rise to 100% at the trough. Leverage is seen to expand and shrink sharply as investors change expectations about the worst possible return of the collateral.

The leverage cycles stressed in Fostel and Geanakoplos (2008) and Geanakoplos (2009) originate from changes in the volatility of the payoffs to the collateralized asset after the
release of good or bad news about those payoffs. Here, leverage cycles can develop in the absence of news with no changes to the fundamentals in the economy. Sudden shifts in investor beliefs about the minimum return of the collateral are what generate the leverage cycles.

9 Conclusion

I present a model in which investor expectations about the future value of collateral can be self-fulfilling. When levered portfolio positions must be financed by collateral to prevent default, investor beliefs about the minimum return on that collateral dictate the amount of leverage available. Demand for the assets in the economy is influenced by this available leverage. If investors expect the value of the collateral to be high, the leverage extended will be high, creating high asset demand. The high demand reinforces the initial expectations of a high collateral value. Alternatively, a low expected collateral value constrains leverage, reduces asset demand, and confirms the beliefs. I show that excess volatility, price crashes and booms, long price recoveries, price overshooting and misfiring, as well as leverage cycles can transpire in equilibrium purely from shifts in investor expectations that are unrelated to shifts in fundamentals.
References


A Internet Appendix: Proofs

A.1 Proof of Proposition 1

In the frictionless economy, equilibrium portfolio policies of the representative investor consist of leverage $1 - \phi_{f,t}$ and shares $\phi_{P,t}$ and $\phi_{P,t}$ invested in the risky assets. Optimal leverage $1 - \phi_{f,t}$ must satisfy market clearing in the risk-free asset market:

$$Y + \phi_{f,t}W_t = 0.$$ 

Solving for $1 - \phi_{f,t}$ gives

$$1 - \phi_{f,t} = \frac{W + Y}{W},$$

which is the ratio of aggregate wealth to investor wealth. The risky asset portfolio policies must satisfy market clearing in the risky asset markets:

$$\phi_{P,t} = \frac{P_t}{W + Y},$$

$$\phi_{P,t} = \frac{P_t}{W + Y}.$$ 

With the wealth share satisfying $\sigma_t = \frac{P_t}{W + Y}$, the optimal risky portfolio is the share of aggregate wealth invested in each risky asset. Regarding prices, $R_{f,t} = \frac{1}{\delta}$ because the representative investor’s consumption is constant, and the three conditions that pin down equilibrium risky prices are

$$P_t = \sigma_t (W + Y),$$

$$E_t (R_{P,t+1}) = \frac{1}{\delta},$$

$$E_t (R_{P,t+1}) = \frac{1}{\delta}.$$ 

As discussed in the main text, the expected returns of the two risky assets are not conditioned on the state $X_t$, but simply conditioned on the period $t$. Breaking down the two conditional expectations gives

$$q \left( \frac{P_{t+1} + D}{P_t} \right) + (1 - q) \left( \frac{P_{t+1} + D}{P_t} \right) = \frac{1}{\delta},$$

$$q \left( \frac{P_{t+1} + D}{P_t} \right) + (1 - q) \left( \frac{P_{t+1} + D}{P_t} \right) = \frac{1}{\delta}.$$ 

Convert these two expected return conditions into expected payoff conditions by multiplying
both by the respective current prices. Then add both equations to get

\[ \overline{P}_{t+1} + \overline{P}_{t+1} + D = \frac{1}{\delta} (W + Y). \]

Next use the risky asset market clearing condition \( \overline{P}_{t+1} + \overline{P}_{t+1} = W + Y \) to get

\[ 1 + \frac{D}{W + Y} = \frac{1}{\delta}. \]  (24)

Now subtract the two expected payoff conditions to get

\[ \overline{P}_{t+1} - \overline{P}_{t+1} + q(D_D - D) + (1 - q)(D_D - D) = \frac{1}{\delta} (\overline{P}_t - \overline{P}_t). \]  (25)

Equation (24) pins down the equilibrium relation between aggregate cash flows \( D \) and aggregate wealth \( W + Y \) in the frictionless economy. Equation (25) pins down the equilibrium relation between prices in period \( t + 1 \) and prices in period \( t \).

Next, substitute \( D_D - D_D = (2\lambda - 1)D \) and the risky-asset market-clearing conditions for periods \( t \) and \( t + 1 \) into (25) to re-express the equation as

\[ \overline{P}_{t+1} = \frac{1}{\delta} \overline{P}_t - \left[ \frac{1}{2} \left( \frac{1 - \delta}{\delta} \right) (Y + W) + \left( q - \frac{1}{2} \right) \left( \lambda - \frac{1}{2} \right) D \right]. \]

Finally, substitute (24) and use the relation \( \sigma_t = \frac{\overline{P}_t}{W + Y} \) to get

\[ \sigma_{t+1} = \frac{1}{\delta} \sigma_t - \left( \frac{1 - \delta}{\delta} \right) \left( \frac{1}{2} + \left( q - \frac{1}{2} \right) \left( \lambda - \frac{1}{2} \right) \right). \]  (26)

Equation (26) is a linear dynamical system in the wealth share \( \sigma \) of the high cash-flow asset \( \overline{P} \). Let the initial value of the system be \( \sigma_{t=0} = \sigma_0 \). Because \( \frac{1}{2} + \left( q - \frac{1}{2} \right) \left( \lambda - \frac{1}{2} \right) > 0 \) and \( \frac{1}{\delta} > 1 \), the system has a positive steady state that is unique but unstable. The steady state is the fixed point of the system:

\[ \sigma = \frac{1}{2} + \left( q - \frac{1}{2} \right) \left( \lambda - \frac{1}{2} \right). \]

A.2 Proof of Lemma 1

Proposition 1 established that along the equilibrium path in a frictionless economy both prices and portfolios are constant. Hence the collateral constraint can be written without time subscripts

\[ (1 - \phi_f) \phi_{\overline{P}} R_{\overline{P}} + \phi_f R_f \geq 0. \]
Leverage of the representative investor 

\[ 1 - \phi_f = \frac{W + Y}{W}, \]

whereas the portfolio weight \( \phi_P = \sigma \). Finally, the risk-free rate is \( R_f = \frac{1}{\delta} \), and the minimum return on the high cash flow asset is 

\[
R_P = \frac{(1 - \sigma)(W + Y) + (1 - \lambda)D}{\sigma(W + Y)}.
\]

Making these substitutions into the inequality above and using the consumption market clearing condition \( D = (1 - \delta)W \) delivers the upper bound for \( \frac{Y}{W} \) presented in the lemma.

A.3 Proof of Proposition 2

The one-shot deviation of a typical investor \( i \) in period \( t \) consists of an allocation \( \phi_{f,t}^i \) in the risk-free asset and \( 1 - \phi_{f,t}^i \) in the \( P \) asset. Thereafter the investor elects the optimal portfolio in the symmetric equilibrium. The value function of such an investor is then

\[
V_{dev}(W_t^i) \equiv \max_{\phi_{f,t}^i} \log c_t^i + \delta E_t (V_{sym}(W_{t+1}^i)).
\]

The budget constraint for the investor in period \( t \) is

\[
W_{t+1}^i = (W_t^i - c_t) \left[ R_{P,t+1} + \phi_{f,t}^i \left( R_{f,t} - R_{P,t+1} \right) \right],
\]

and the investor must also obey the no-shorting constraint \( \phi_{f,t}^i \leq 1 \).

Optimal consumption in period \( t \) is \( c_t^i = (1 - \delta)W_t^i \). Therefore, the allocation problem of investor \( i \) in period \( t \) reduces to the one-period problem

\[
\max_{\phi_{f,t}^i} \delta E_t \log W_{t+1}^i,
\]

subject to the budget constraint

\[
W_{t+1}^i = \delta W_t \left[ R_{P,t+1} + \phi_{f,t}^i \left( R_{f,t} - R_{P,t+1} \right) \right]
\]

and the no-shorting constraint.

Asset prices in a symmetric equilibrium are constant and given in Proposition 1. Therefore, the time subscripts on the asset returns can be dropped in deriving the optimal policy. The interest rate \( R_f = \frac{1}{\delta} \), whereas the returns of the \( P \) when the state persists and when it switches, respectively, are

\[
\bar{R}_P = 1 + \frac{D}{\sigma(W + Y)},
\]

\[
\overline{R}_P = 1 - \frac{\sigma}{\sigma} + \frac{D}{\sigma(W + Y)}.
\]

with \( \sigma = \frac{1}{2} + (q - \frac{1}{2}) (\lambda - \frac{1}{2}) \). In a symmetric equilibrium, the interest rate also equals the
return on the synthetic riskless claim, which gives the relation
\[ \frac{1}{\delta} = 1 + \frac{D}{W + Y}. \]

The optimal portfolio policy for this investor is given in Lemma 2 and reprinted here
\[ \phi_{f,t} = (1 - q) \left( \frac{R_P}{R_P - R_f} \right) - q \left( \frac{R_P}{R_f - R_P} \right). \]

The investor faces two possible values of next period wealth. Let \( W_{i+1}^i (H) \) denote wealth when the state persists, and let \( W_{i+1}^i (T) \) be wealth when the state switches. Substituting the asset returns and the portfolio policy gives for next period’s wealth when the state persists
\[ W_{i+1}^i (H) = W_i \left[ \frac{2q (2D + (2q - 1) (W + Y))}{(2q + 1) D + 2 (2q - 1) (W + Y)} \right]. \]

Similarly, the next period wealth \( W_{i+1}^i (T) \) when the state switches is
\[ W_{i+1}^i (T) = W_i \left[ \frac{2 (1 - q) (2D + (2q - 1) (W + Y))}{D (3 - 2q)} \right]. \]

Let
\[ \eta_1 = \frac{2q (2D + (2q - 1) (W + Y))}{(2q + 1) D + 2 (2q - 1) (W + Y)}, \]
\[ \eta_2 = \frac{2 (1 - q) (2D + (2q - 1) (W + Y))}{D (3 - 2q)}. \]

Because investor \( i \)'s next-period wealth is positive, both constants must be positive. The maximal expected utility for the one-period problem is then
\[ q \log \eta_1 W_i^i + (1 - q) \log \eta_2 W_i^i. \]

Separating terms and simplifying gives
\[ \log W_i^i + \eta, \]
where
\[ \eta = q \log \eta_1 + (1 - q) \log \eta_2. \]

The value function from the symmetric equilibrium \( V_{sym} (W_{i+1}^i) = \log W_{i+1}^i - \kappa \), which equals \( \log W_i^i - \kappa \) because wealth is constant in that equilibrium. Hence the only change in the value function associated with a deviation is the addition of the constant \( \eta \) and a delayed investment cost \( \kappa \). This implies that
\[ V_{dev} (W_i^i) = \log W_i^i + \eta - \delta \kappa. \]
A.4 Proof of Proposition 3

A symmetric equilibrium exists if and only if a one-shot (one-period) deviation is unprofitable. The value function of a typical investor $i$ that chooses the symmetric equilibrium portfolio is

$$V_{\text{sym}}(W^i_t) = \log W^i_t - \kappa.$$ 

The value function associated with a one-shot deviation is (13) in Proposition 2. Mathematically, the deviation is unprofitable if and only if

$$V_{\text{sym}}(W^i_t) \geq V_{\text{dev}}(W^i_t).$$ 

Substituting the two value functions and re-arranging gives

$$\kappa \leq -\eta \frac{1}{1-\delta}.$$ 

The investment cost $\kappa$ is strictly positive. Therefore, for the above inequality to hold, the expected log return $\eta$ on the optimal portfolio that invests in the risk-free asset and the $P$ asset must be negative.

However, that could never be the case, as the investor would never optimally hold a negative expected return portfolio. A naive portfolio that invests one-hundred percent of wealth in the risk-free asset ($\phi^i_{f,t} = 1$) would bear no volatility and earn a guaranteed return $R_f$. The optimal portfolio must earn at least as much as this naive portfolio, which implies $\eta > 0$. Hence the inequality cannot hold.

A.5 Proof of Lemma 2

The portfolio decision problem of a single market investor is

$$V^s(P_t, P_{t+1}) = \max_{\phi^s_{f,t}} E_t \left[ \log W^s_{t+1} \right],$$

where

$$W^s_{t+1} = R_{P,t+1} + \phi^s_{f,t} (R_{f,t} - R_{P,t+1}),$$

and $\phi^s_{f,t} \in (\infty, 1]$ is the fraction invested in the risk-free asset. The single market investor is also subject to the collateral constraint

$$(1 - \phi^s_{f,t}) R_{P,t+1} + \phi^s_{f,t} R_{f,t} \geq 0.$$ 

For now assume the collateral constraint does not bind. I show later that it does not. The first order condition for optimality is

$$E_t \left[ \frac{R_{f,t} - R_{P,t+1}}{R_{P,t+1} + \phi^s_{f,t} (R_{f,t} - R_{P,t+1})} \right] = 0.$$
Separating the expectation into the two states gives
\[
q \left[ \frac{R_{f,t} - R_{P,t+1}}{R_{P,t+1} + \phi^s_{f,t} (R_{f,t} - R_{P,t+1})} \right] + (1 - q) \left[ \frac{R_{f,t} - R_{P,t+1}}{R_{P,t+1} + \phi^s_{f,t} (R_{f,t} - R_{P,t+1})} \right] = 0.
\]
This implies
\[
q \left[ \frac{R_{P,t+1} - R_{f,t}}{R_{P,t+1} + \phi^s_{f,t} (R_{f,t} - R_{P,t+1})} \right] = (1 - q) \left[ \frac{R_{f,t} - R_{P,t+1}}{R_{P,t+1} + \phi^s_{f,t} (R_{f,t} - R_{P,t+1})} \right]
q (R_{P,t+1} - R_{f,t}) \left[ \frac{R_{P,t+1} - R_{P,t+1}}{R_{P,t+1} + \phi^s_{f,t} (R_{f,t} - R_{P,t+1})} \right] = (1 - q) (R_{f,t} - R_{P,t+1}) \left[ R_{P,t+1} + \phi^s_{f,t} (R_{f,t} - R_{P,t+1}) \right]
\phi^s_{f,t} (R_{f,t} - R_{P,t+1}) (R_{P,t+1} - R_{f,t}) = (1 - q) (R_{f,t} - R_{P,t+1}) R_{P,t+1} - q (R_{P,t+1} - R_{f,t}) R_{P,t+1}
\phi^s_{f,t} = (1 - q) \left( \frac{R_{P,t+1}}{R_{P,t+1} - R_{f,t}} \right) - q \left( \frac{R_{P,t+1}}{R_{P,t+1} - R_{f,t}} \right).
\]

The single market investor lends ($\phi^s_{f,t} > 0$), if the interest rate $R_{f,t}$ satisfies
\[
(1 - q) \left( \frac{R_{P,t+1}}{R_{P,t+1} - R_{f,t}} \right) - q \left( \frac{R_{P,t+1}}{R_{f,t} - R_{P,t+1}} \right) > 0
(1 - q) R_{P,t+1} (R_{f,t} - R_{P,t+1}) > q R_{P,t+1} (R_{P,t+1} - R_{f,t})
R_{f,t} ((1 - q) R_{P,t+1} + q R_{P,t+1}) > R_{P,t+1} R_{P,t+1}
R_{f,t} > \frac{1}{q} \frac{1}{R_{P,t+1}} + \frac{1}{q}.
\]

Single market investors lend if the interest rate exceeds the harmonic mean return of the $P$ asset. Note that in the derivation I assumed $R_{P,t+1} < R_{f,t} < R_{P,t+1}$; otherwise, either the risk-free asset or the $P$ asset would strictly dominate the other as an investment, which cannot occur in equilibrium.

Single market investors short the $P$ asset ($\phi^s_{f,t} > 1$), if the interest rate satisfies
\[
(1 - q) \left( \frac{R_{P,t+1}}{R_{P,t+1} - R_{f,t}} \right) - q \left( \frac{R_{P,t+1}}{R_{f,t} - R_{P,t+1}} \right) > 1
(1 - q) R_{P,t+1} (R_{f,t} - R_{P,t+1}) - q R_{P,t+1} (R_{P,t+1} - R_{f,t}) > (R_{P,t+1} - R_{f,t}) (R_{f,t} - R_{P,t+1})
R_{f,t} (q R_{P,t+1} + (1 - q) R_{P,t+1}) - R_{P,t+1} R_{P,t+1} > R_{P,t+1} R_{f,t} - R_{P,t+1} R_{P,t+1} - R_{f,t} R_{f,t} + R_{f,t} R_{P,t+1}
R_{f,t} > q R_{P,t+1} + (1 - q) R_{P,t+1}.
\]

They will short the $P$ asset if the interest rate exceeds the expected arithmetic return of the $P$ asset.

I next show that the collateral constraint is never binding for single market investors. The collateral constraint for single market investors is
\[
(1 - \phi^s_{f,t}) R_{P,t+1} + \phi^s_{f,t} R_{f,t} \geq 0.
\]
A binding constraint would imply
\[ \phi^*_{f,t} = -\frac{R_{P,t+1}}{R_{f,t} - R_{P,t+1}}. \]

The collateral constraint will be slack for a single market investor if the optimal portfolio satisfies
\[ \phi^*_{s,t} > -\frac{R_{P,t+1}}{R_{f,t} - R_{P,t+1}}. \]

Substituting the single market investor’s optimal portfolio from (14) shows
\[
(1 - q) \left( \frac{R_{P,t+1}}{R_{f,t} - R_{P,t+1}} \right) - q \left( \frac{R_{P,t+1}}{R_{f,t} - R_{P,t+1}} \right) > -\frac{R_{P,t+1}}{R_{f,t} - R_{P,t+1}} \\
(1 - q) \left( \frac{R_{P,t+1}}{R_{f,t} - R_{P,t+1}} \right) > - (1 - q) \left( \frac{R_{P,t+1}}{R_{f,t} - R_{P,t+1}} \right),
\]

which is always true unless \( R_{f,t} = 0 \) or \( q = 1 \) or \( R_{P,t+1} = R_{P,t+1} \). So that an asset with price \( P_t \) has positive probability of switching to price \( 1 - P_{t+1} \) the following period, I do not consider the absorbing case \( q = 1 \). The condition \( R_{f,t} = 0 \) never occurs in equilibrium. The final condition would imply that the risky \( P \) asset yields the same return in either state—making it risk-free—which is a contradiction. Therefore, none of the three conditions can hold, meaning that single market investors will never face a binding collateral constraint.

A.6 Proof of Lemma 3

The portfolio problem of an arbitrageur is to pick a policy \( (\phi^a_{f,t}, \phi^a_{P,t}) \) that maximizes
\[
V^a (P_t, P_{t+1}) = \max_{\phi^a_{f,t}, \phi^a_{P,t}} E_t \left[ \log W^a_{t+1} \right] - \kappa,
\]
where
\[
W^a_{t+1} = \phi^a_{f,t} R_{f,t} + (1 - \phi^a_{f,t}) \left( R_{1-P_{t+1}} + \phi^a_{P,t} (R_{P,t+1} - R_{1-P_{t+1}}) \right),
\]
with \( \phi^a_{P,t} \in [0, 1] \). Because he or she faces an arbitrage, the chosen strategy is subject to a binding collateral constraint
\[
(1 - \phi^a_{f,t}) \phi^a_{P,t} R_{P,t+1} + \phi^a_{f,t} R_{f,t} = 0.
\]

The arbitrage would induce the investor to choose infinite leverage. Therefore, the choice variable \( 1 - \phi^a_{f,t} \) will be pinned down by the collateral constraint for an optimal risky asset allocation \( \phi^a_{P,t} \). The first order condition with respect to \( \phi^a_{P,t} \) is
\[
E_t \left[ \frac{R_{P,t+1} - R_{1-P_{t+1}}}{\phi^a_{f,t} R_{f,t} + (1 - \phi^a_{f,t}) \left( R_{1-P_{t+1}} + \phi^a_{P,t} (R_{P,t+1} - R_{1-P_{t+1}}) \right)} \right] = 0
\]
For the time being, I suppress the $t$ and $t+1$ subscripts to ease notation. Separate the expectation into the two states to get

$$q \frac{R_P - R_{1-P}}{R_f + \left(1 - \phi^q_f\right) \left(R_{1-P} + \phi^q_P \left(R_P - R_{1-P}\right)\right)} \quad = \quad \frac{(1-q) R_P - R_P}{R_f + \left(1 - \phi^q_f\right) \left(R_{1-P} + \phi^q_P \left(R_P - R_{1-P}\right)\right)}.$$

This implies

$$q (R_P - R_{1-P}) \left[\phi^q_P R_f + (1 - \phi^q_f) \left(R_{1-P} + \phi^q_P \left(R_P - R_{1-P}\right)\right)\right] \quad = \quad (1-q) (R_{1-P} - R_P) \left[\phi^q_P R_f + (1 - \phi^q_f) \left(R_{1-P} + \phi^q_P \left(R_P - R_{1-P}\right)\right)\right].$$

Separate terms involving $\phi^q_P$ to get

$$(1-q) \phi^q_P \left[q (R_P - R_{1-P}) \left(R_P - R_{1-P}\right) - (1-q) \left(R_{1-P} - R_P\right) \left(R_P - R_{1-P}\right)\right] \quad = \quad (1-q) \left(R_{1-P} - R_P\right) \left[R_f + (1 - \phi^q_f) \left(R_{1-P} - R_f\right)\right] - q \left(R_P - R_{1-P}\right) \left[R_f + (1 - \phi^q_f) \left(R_{1-P} - R_f\right)\right],$$

which implies

$$\phi^q_P = \frac{(1-q) (R_{1-P} - R_P) \left[R_f + (1 - \phi^q_f) \left(R_{1-P} - R_f\right)\right] - q \left(R_P - R_{1-P}\right) \left[R_f + (1 - \phi^q_f) \left(R_{1-P} - R_f\right)\right]}{q (R_P - R_{1-P}) \left(R_P - R_{1-P}\right) - (1-q) \left(R_{1-P} - R_P\right) \left(R_P - R_{1-P}\right)}.$$

Some simplification yields

$$\phi^q_P = - \left[ (1-q) \frac{R_f + (1 - \phi^q_f) \left(R_{1-P} - R_f\right)}{R_P - R_{1-P}} + q \frac{R_f + (1 - \phi^q_f) \left(R_{1-P} - R_f\right)}{R_P - R_{1-P}} \right].$$

Next, divide both sides by $1 - \phi^q_f$ to get

$$\phi^q_P = q \frac{R_{1-P}}{R_{1-P} - R_P} - (1-q) \frac{R_{1-P}}{R_P - R_{1-P}} + R_f \left(\frac{q}{R_{1-P} - R_P} \left(1 - \phi^q_f\right) - \frac{1-q}{R_P - R_{1-P}} \left(1 - \phi^q_f\right)\right).$$

Define the functions

$$g \equiv q \frac{R_{1-P}}{R_{1-P} - R_P} - (1-q) \frac{R_{1-P}}{R_P - R_{1-P}},$$

$$h \equiv \frac{q}{R_{1-P} - R_P} - \frac{1-q}{R_P - R_{1-P}}.$$
to re-write the optimal portfolio as

\[ \phi_P^0 = g - hR_f + \frac{hR_f}{1 - \phi_f^d}. \]

Some simplification gives

\[ \phi_P^0 = g + hR_f \left( \frac{\phi_f^d}{1 - \phi_f^d} \right), \]

which is (15).

Facing the prospect of infinite profit from the arbitrage opportunity, an arbitrageur will want to lever his or her portfolio as much as possible, making the collateral constraint always bind. Substituting the optimal portfolio from (15) into the collateral constraint (10) with equality yields the arbitrageur’s constrained leverage in (16). Substituting (16) into (15) gives (17).

A.7 Proof of Lemma 4

I derive the equilibrium interest rate in the cases where investor portfolios are (1) both interior, and (2) at the no-shorting boundary.

A.7.1 Interior investor portfolios

From Walras’ Law, I need only the risk-free asset market-clearing condition plus one risky-asset market-clearing condition to solve for the interest rate. I use the simpler $1 - P$ market clearing condition here. The two clearing conditions are

\[
0 = Y + W \left[ \phi_f^s \pi_t + \phi_f^a (1 - \pi_t) \right] \\
1 - P_t = W (1 - \pi_t) \left( 1 - \phi_f^a \right) \left( 1 - \phi_P^0 \right).
\]

Risk-free asset market clearing uniquely determines the fraction of arbitrageurs as

\[ 1 - \pi_t = \frac{\frac{Y}{W} + \phi_f^s}{\phi_f^s - \phi_f^a}. \]

Define the function

\[ z(P_t) \equiv \frac{1 - P_t}{W} \]

as the fraction of investor wealth in the $1 - P$ asset market. The single equation that determines the equilibrium interest rate is therefore

\[ z(P_t) = (1 - \pi_t) \left( 1 - \phi_f^a \right) \left( 1 - \phi_P^0 \right), \]

with the fraction $1 - \pi_t$ just expressed above and the investor portfolios defined in the text.

I separate the calculation of the interest rate into three steps: (1) express the fraction $1 - \pi_t$ in terms of returns, (2) express the levered arbitrageur portfolio in the $1 - P$ asset
\((1 - \phi_{f,t}^a) (1 - \phi_{P,t}^a)\) in terms of returns, and (3) combine the results to solve for the interest rate. For the time being, I suppress the \(t\) and \(t+1\) subscripts to ease notation.

**A.7.2 The fraction \(1 - \pi\) of arbitrageurs**

First, use the single market investor portfolio from (14) to express the numerator of \(1 - \pi\) as

\[
\frac{Y}{W} + \phi_s^f = \frac{Y}{W} + (1 - q) \left( \frac{\overline{R}_P}{\overline{R}_P - R_f} \right) - q \left( \frac{R_P}{R_f - \overline{R}_P} \right)
\]

\[
= \frac{Y}{W} \left( \overline{R}_P - R_f \right) \left( R_f - \overline{R}_P \right) + (1 - q) \overline{R}_P \left( R_f - \overline{R}_P \right) - qR_P \left( \overline{R}_P - R_f \right)
\]

\[
= \frac{Y}{W} \left( \overline{R}_P - R_f \right) \left( R_f - \overline{R}_P \right)
\]

Next, use (14) again and the arbitrageur’s borrowings from (16) to express the denominator of \(1 - \pi\) as

\[
\phi_s^f - \phi_f^a = \frac{(1 - q) \overline{R}_P \left( R_f - \overline{R}_P \right) - qR_P \left( \overline{R}_P - R_f \right)}{(\overline{R}_P - R_f) \left( R_f - \overline{R}_P \right)} + \frac{gR_P}{R_f (1 + hR_P) - gR_P}
\]

\[
= \frac{(1 - q) \overline{R}_P \left( R_f - \overline{R}_P \right) - qR_P \left( \overline{R}_P - R_f \right)}{(\overline{R}_P - R_f) \left( R_f - \overline{R}_P \right)} + \frac{\gamma}{\beta R_f - \gamma}
\]

\[
= \frac{(1 - q) \overline{R}_P \left( R_f - \overline{R}_P \right) \left( \beta R_f - \gamma \right) - qR_P \left( \overline{R}_P - R_f \right) \left( \beta R_f - \gamma \right) + \gamma \left( \overline{R}_P - R_f \right) \left( R_f - \overline{R}_P \right)}{(\overline{R}_P - R_f) \left( R_f - \overline{R}_P \right) \left( \beta R_f - \gamma \right)}
\]

where the temporary functions

\[
\gamma \equiv gR_P
\]

\[
\beta \equiv 1 + hR_P
\]

are defined to simplify notation.

Finally, combine the numerator and denominator to express \(1 - \pi\) as

\[
1 - \pi = \frac{(\beta R_f - \gamma) \left[ \frac{Y}{W} \left( \overline{R}_P - R_f \right) \left( R_f - \overline{R}_P \right) + (1 - q) \overline{R}_P \left( R_f - \overline{R}_P \right) - qR_P \left( \overline{R}_P - R_f \right) \right]}{(1 - q) \overline{R}_P \left( R_f - \overline{R}_P \right) \left( \beta R_f - \gamma \right) - qR_P \left( \overline{R}_P - R_f \right) \left( \beta R_f - \gamma \right) + \gamma \left( \overline{R}_P - R_f \right) \left( R_f - \overline{R}_P \right)}.
\]

**A.7.3 The levered arbitrageur portfolio**

Use (16) and (17) to express the levered portfolio as

\[
(1 - \phi_f^a) (1 - \phi_P^a) = \left( \frac{R_f (1 + hR_P)}{R_f (1 + hR_P) - gR_P} \right) \left( \frac{1 + hR_P - g}{1 + hR_P} \right)
\]

\[
= \frac{R_f (\beta - g)}{\beta R_f - \gamma}.
\]
A.7.4 Solving for the interest rate

Substitute the results from the previous two steps into the $1 - P$ market clearing condition to get

$$z = (1 - \pi) (1 - \phi_f^q) (1 - \phi_p^q)$$

$$= \frac{R_f (\beta - g) \left[ \frac{Y}{W} (\overline{R}_P - R_f) (R_f - R_P) + (1 - q) \overline{R}_P (R_f - R_P) - q \overline{R}_P (\overline{R}_P - R_f) \right]}{(1 - q) \overline{R}_P (R_f - R_P) (\beta R_f - \gamma) - q \overline{R}_P (\overline{R}_P - R_f) (\beta R_f - \gamma) + \gamma (\overline{R}_P - R_f) (R_f - R_P)}.$$  

Multiply both sides by the denominator to obtain

$$z \left[ (1 - q) \overline{R}_P (R_f - R_P) (\beta R_f - \gamma) - q \overline{R}_P (\overline{R}_P - R_f) (\beta R_f - \gamma) + \gamma (\overline{R}_P - R_f) (R_f - R_P) \right] = R_f (\beta - g) \left[ \frac{Y}{W} (\overline{R}_P - R_f) (R_f - R_P) + (1 - q) \overline{R}_P (R_f - R_P) - q \overline{R}_P (\overline{R}_P - R_f) \right].$$  

(27)

Expanding terms and isolating $R_f$ produces for the left-hand side of (27) the expression

$$z \left[ \beta ((1 - q) \overline{R}_P + q \overline{R}_P) - \gamma \right] R_f^2 - z \left[ \beta \overline{R}_P R_f - \gamma (q \overline{R}_P + (1 - q) R_f) \right] R_f.$$

(28)

Doing the same to the right-hand side of (27) gives

$$R_f (\beta - g) \left[ \left( - \frac{Y}{W} \right) R_f^2 + \left( \frac{Y}{W} (\overline{R}_P + R_f) + (1 - q) \overline{R}_P + q \overline{R}_P \right) R_f - \left( 1 + \frac{Y}{W} \right) \overline{R}_P R_f \right].$$  

(29)

Subtracting (29) from (28), replacing the temporary functions $\beta$ and $\gamma$ with their values, and restoring the time subscripts, gives the following quadratic for the interest rate $R_{f,t}$, which is equation (19) in the text:

$$a (P_t, P_{t+1}) R_{f,t}^2 + b (P_t, P_{t+1}) R_{f,t} + c (P_t, P_{t+1}) = 0.$$  

The coefficients are

$$a (P_t, P_{t+1}) \equiv \left( (1 + h \overline{R}_{P,t+1}) - g \right) \frac{Y}{W},$$

$$b (P_t, P_{t+1}) \equiv - \left( (1 + h \overline{R}_{P,t+1}) (1 - z) - g \right) (1 - q) \overline{R}_{P,t+1} + q \overline{R}_{P,t+1} + z g \overline{R}_{P,t+1} - a (P_t, P_{t+1}) (\overline{R}_{P,t+1} + \overline{R}_{P,t+1}),$$

$$c (P_t, P_{t+1}) \equiv \left[ (1 - g) \overline{R}_{P,t+1} + (1 - q) \overline{R}_{P,t+1} \right] z g \overline{R}_{P,t+1} + \left[ (1 + h \overline{R}_{P,t+1}) (1 - z) - g \right] \overline{R}_{P,t+1} \overline{R}_{P,t+1} + a (P_t, P_{t+1}) \overline{R}_{P,t+1} \overline{R}_{P,t+1}.$$  

A.8 Proof of Lemma 5

There are a few cases to consider if either the single market investor’s or arbitrageur’s portfolios are at the no-shorting bounds.

The first case is if $\phi_{P,t}^q = 1$ so that arbitrageurs optimally do not wish to hold the $1 - P$
asset. Arbitrageurs are the only investors in the economy who would hold the $1 - P$ asset. If they elected not to hold it, that market could not clear, so such a position cannot occur in equilibrium.

The second case is if $\phi^a_{t} = 0$ and $\phi^s_{f,t} \in [0, 1)$. Here, arbitrageurs do not want to hold the $P$ asset, which means they cannot post the asset as collateral and borrow. Single market investors neither want to borrow, so the risk-free asset market cannot clear as no investor borrows the outside savings $Y$. These positions cannot take place in equilibrium.

The third case is if $\phi^s_{f,t} = 1$ so that single market investors want to hold their entire portfolio in the $P$ asset and are up against the no-shorting constraint. Normally, single market investors are the marginal agents in the risk-free asset market and determine the interest rate $R_{f,t}$ since arbitrageurs are always constrained by the collateral requirement. But if single market investors are also constrained by the no-shorting requirement, then no agent is marginal in the risk-free asset market and the interest rate cannot be determined. This cannot occur in equilibrium, so $\phi^s_{f,t} = 1$ can be ruled out.

The fourth and final case is if $\phi^a_{t} = 0$ while $\phi^s_{f,t} < 0$. Arbitrageurs cannot obtain leverage ($\phi^a_{f,t} = 0$) because they do not wish to hold the collateral asset, while single market investors want to borrow. Here, the fraction of arbitrageurs is

$$1 - \pi_t = \frac{Y}{W} + \frac{\phi^s_{f,t}}{\phi^s_{f,t}},$$

and market clearing in the $1 - P$ market requires

$$z(P_t) = (1 - \pi_t) (1 - \phi^s_{f,t}) (1 - \phi^a_{t})$$

$$= 1 - \pi_t.$$

Using the single market investor optimal portfolio in (14) and isolating the interest rate produces the quadratic

$$a' R^2_{f,t} + b' (P_t, P_{t+1}) R_{f,t} + c' (P_t, P_{t+1}) = 0,$$

where

$$a' \equiv \frac{Y}{W},$$
$$b' (P_t, P_{t+1}) \equiv - \left[ (1 - z) ( (1 - q) R_{P,t+1} + q R_{P,t+1} ) + a' ( R_{P,t+1} + R_{P,t+1} ) \right],$$
$$c' (P_t, P_{t+1}) \equiv R_{P,t+1} R_{P,t+1} (1 + a' - z).$$

If investor portfolios satisfy this fourth case, the equilibrium interest rate is a root of this quadratic equation. Like before, I find numerically that both roots are real. I choose the minimum root as the unique equilibrium $R_{f,t}$ for the reason given in the main text.
B  Internet Appendix: Extensions

B.1  The effect of outside savings on asymmetric equilibria

From the analysis on asymmetric equilibria in the main text, positive outside savings $Y > 0$ can lead to multiple equilibria. In this section I describe how positive savings is actually necessary for the multiplicity: its absence leads to uniqueness. To see this, consider Figure 15 which presents the system with all parameters unchanged in Figure 4, save for $Y = 0$.

Figure 15: Equilibrium Dynamical System, $Y = 0$

![Figure 15: Equilibrium Dynamical System, $Y = 0$](image)

Notes: Parameters used are $\delta = 0.4$, $\lambda = 0.9$, $q = 0.7$, $\kappa = 0.06$, and $Y = 0$. The solid blue dot is the only price pair sustainable along an equilibrium path. The gray dashed curve represents price pairs $(P_t, P_{t+1})$ that satisfy the indifference condition $F(P_t, P_{t+1}) = 0$, but are on a path that would diverge outside the system. The red line indicates a stochastic steady state $P_t = P_{t+1} = P$. The steady state value in the figure is $P = 0.57$.

Cutting outside savings to zero has a profound effect on the system. By setting $Y = 0$, not only does the steady state become unique, but that steady state price becomes the only price sustainable along any equilibrium path. The gray curve in the figure represents all the price pairs that satisfy the indifference condition $F = 0$. However, if an asset price path in the economy began at any one of those prices, the path would diverge. Therefore, investors would not consider those prices as equilibria, and those prices must be trimmed from the system. Only the unique steady state remains as an equilibrium asset price. The economy would have to start and remain there (or jump there) on all equilibrium paths.
The elimination of the outside savings retains the smaller steady state value of $P = 0.57$ but loses the larger value of $P = 0.68$ from the earlier system. The reason why is the following. Setting $Y > 0$ allows all investors in the market to be levered. Without any outside savings, single market investors would have to save since arbitrageurs always borrow, as it would be sub-optimal for arbitrageurs to do otherwise. For a large price $P$, it becomes impossible for single market investors to be indifferent to an arbitrageur because the expected return on the constrained arbitrage portfolio position, particularly on the $1 - P$ asset, is so high. To meet the indifference condition for large prices $P$, single market investors would want to use leverage. Including the outside savings permits them to do so and achieves the second steady state equilibrium at $P = 0.68$. 