# Chapter 5

# Second-order Cone and Robust Models

*Second-order cone programming* (SOCP) is a generalization of linear and quadratic programming that allows for affine combinations of variables to be constrained inside a special convex set, a *second-order cone*. The SOCP model includes as special cases problems with convex quadratic objective and constraints. SOCP models are particularly useful in geometry problems, approximation problems, and probabilistic problems.

## 5.1 Geometry of Cones

- **Cone**. A set of points $C \in \mathbb{R}^n$ is called a *cone* if

$$\begin{aligned} \alpha x \in C, \forall x \in C, \alpha \geq 0 \\ x + y \in C, \forall x \in C, y \in C \end{aligned}$$

  This is similar to a subspace, but instead of $\alpha \in \mathbb{R}$, here $\alpha > 0$.

- **Example**. Simple examples: $|x| \leq y,\ y \geq 0$. See Figure 5.1.

- **Slice**. A slice of a cone is its intersection with a subspace (e.g. linear constraint). It can be a polyhedral, ellipsoidal, or something else.

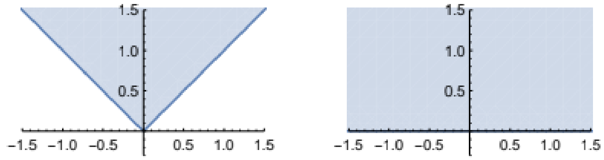


Figure 5.1: Examples of a cone

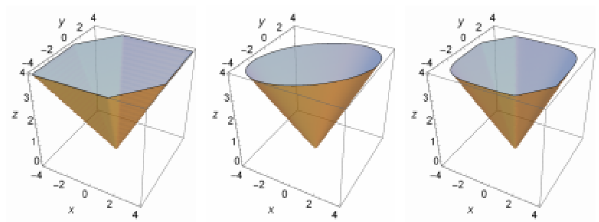

Figure 5.2: Examples of intersection of convex cones and subspace

- **Polyhedral cone**. By adding one dimension to the polyhedron $Ax \leq b$, $x \in \mathbb{R}^n$, we can get a polyhedral cone in $\mathbb{R}^{n+1}$. A polyhedral cone in $(x, t) \in \mathbb{R}^{n+1}$ is

$$\{Ax \leq bt, t \geq 0\}$$

The slice $t = 1$ is the original polyhedron. See Figure 5.3

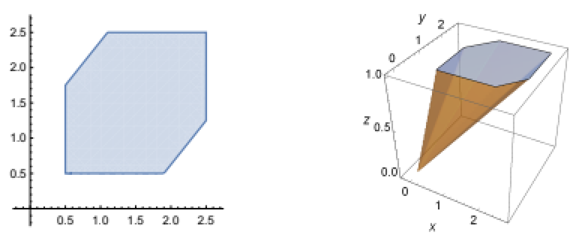


Figure 5.3: Polyhedron and polyhedral cone

- **Ellipsoidal cone**. A ellipsoid $x^T Px + q^T x + r \leq 0, P \succ 0, x \in \mathbb{R}^n$ can be represented by $\|Ax + b\| \leq c$. By adding a dimension, we can get a ellipsoidal cone

$$\{\|Ax + bt\| \leq ct\}$$

in $(x, t) \in \mathbb{R}^{n+1}$. The slice $t = 1$ is the original ellipsoid. See Figure 5.4.

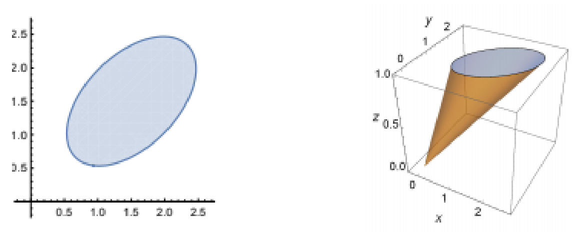


Figure 5.4: Ellipsoid and ellipsoidal cone

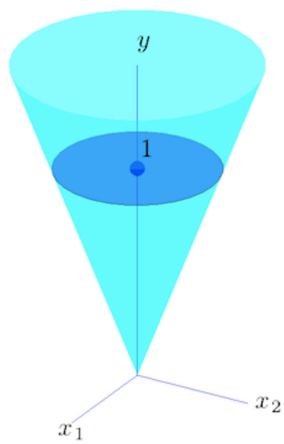


Figure 5.5: SOC in $\mathbb{R}^3$

## 5.2 Second-order Cone Programs

- **Second-order cone (SOC) in** $\mathbb{R}^3$. The SOC in $\mathbb{R}^3$ is the set of vectors $(x_1, x_2, y)$ such that $\sqrt{x_1^2 + x_2^2} \leq y$. Horizontal sections of this set at level $y = 1$ is is the circle of center $(0, 0, 1)$ and radius one (in dark blue).

- $(n+1)$-**dimensional SOC**. The *second-order cone* in $\mathbb{R}^{n+1}$ is defined as

$$\mathcal{K}_n = \left\{ (x, t), x \in \mathbb{R}^n, t \in \mathbb{R} : \|x\|_2 \leq t \right\}$$

- **Convex cone**. An SOC is a convex cone. The set $\mathcal{K}_n$ is convex, since it can be expressed as the intersection of (infinite) half-spcaes:

$$\mathcal{K}_n = \bigcap_{u:\|u\|_2 \leq 1} \left\{ (x, t), x \in \mathbb{R}^n, t \in \mathbb{R} : x^T u \leq t \right\}$$

It is also a *cone*, since for any $z \in \mathcal{K}_n$ it holds that $\alpha z \in \mathcal{K}_n, \forall \alpha \geq 0$.

### 5.2.1 The Rotated Second-order Cone

The *rotated second-order cone* in $\mathbb{R}^{n+2}$ is the set

$$\mathcal{K}_n^r = \left\{(x, y, z), x \in \mathbb{R}^n, y \in \mathbb{R}, z \in \mathbb{R} : x^T x \leq 2yz, y \geq 0, z \geq 0\right\}$$

It can be expressed as a linear transformation (a **rotation**) of the (plain) second-order cone in $\mathbb{R}^{n+2}$

$$\|x\|_2^2 \leq 2yz,\ y \geq 0,\ z \geq 0 \Leftrightarrow \left\| \begin{bmatrix} x \\ \frac{1}{\sqrt{2}}(y-z) \end{bmatrix} \right\|_2 \leq \frac{1}{\sqrt{2}}(y+z)$$

Pick $w = (x, \frac{y-z}{\sqrt{2}}), t = \frac{y+z}{\sqrt{2}}$. The two sets of variables are related by a rotation matrix $R$.

**Proof**.

$$\begin{aligned} \left\| \begin{bmatrix} x \\ \frac{1}{\sqrt{2}}(y-z) \end{bmatrix} \right\|_2 &\leq \frac{1}{\sqrt{2}}(y+z) \\ \|x\|_2^2 + \frac{1}{2}(y-z)^2 &\leq \frac{1}{2}(y+z)^2 \\ \|x\|_2^2 + \frac{1}{2}(y^2 - 2yz + z^2) &\leq \frac{1}{2}(y^2 + 2yz + z^2) \\ \|x\|_2^2 &\leq 2yz \end{aligned}$$