Risk-sensitive safety specifications for stochastic systems using Conditional Value-at-Risk*

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Abstract—This paper proposes a safety analysis method that facilitates a tunable balance between worst-case and risk-neutral perspectives. First, we define a risk-sensitive safe set to specify the degree of safety attained by a stochastic system. This set is defined as a sublevel set of the solution to an optimal control problem that is expressed using the Conditional Value-at-Risk (CVaR) measure. This problem does not satisfy Bellman’s Principle, thus our next contribution is to show how risk-sensitive safe sets can be under-approximated by the solution to a CVaR-Markov Decision Process. Our third and fourth contributions are to show that a value iteration algorithm solves the reduced problem and enables tractable policy synthesis for a class of linear systems. Fifth, we develop a realistic numerical example of a stormwater system to show that our approach can be applied to non-linear systems. Finally, we compare the CVaR criterion to the exponential disutility criterion. The latter allocates control effort evenly across the cost distribution to reduce variance, while the CVaR criterion focuses control effort on a given worst-case quantile—where it matters most for safety.

Index Terms—Conditional Value-at-Risk, Stochastic Optimal Control, Optimization, Linear Time-Varying Systems.

I. INTRODUCTION

CONTROL-theoretic formal verification methods for dynamic systems typically fall in the robust domain [1]–[5] or in the stochastic domain [6]–[9]. Robust methods for formal verification assume that uncertain disturbances lack probabilistic descriptions, live in bounded sets, and exhibit adversarial behavior. These assumptions are appropriate if probabilistic information about disturbances is not available, and if the conservative policy or safety specification that results from a pessimistic world view is useful in practice. However, when one considers formal verification as a design tool for safety-critical systems in the digital world today, it is reasonable to assume that simulation tools or sensor data are available to estimate probabilistic descriptions for disturbances. Moreover, it is reasonable to consider a more realistic world view: disturbances are not adversarial agents, but rare harmful outcomes are still possible in the long run.

Control-theoretic stochastic formal verification methods do assume that disturbances are probabilistic with non-adversarial [6] or adversarial [7]–[9] nature. These tools compute the probability of safety or performance using expected indicator cost functions. The expectation metric, however, does not address the possibility of distributions with large spread or the possibility of skewed distributions with “fat” harmful tails. Thus, there has been growing interest in the development of formal verification methods at the intersection of the robust and stochastic domains that specify robustness with respect to probability distributions [10] or utilize risk measure theory to quantify harmful tail costs [11], [23].

This recent work is closely related to the risk-sensitive optimal control paradigm pioneered by Peter Whittle [19]. Whittle showed that the expectation of an exponential disutility function is analogous to a mean-variance criterion, where the degree of disadvantage due to large variance can be tuned via a risk-sensitivity parameter [19]. The key assumption in this formulation is that the notion of risk in decision-making is sufficiently quantified through variance. In the case of asymmetric cost distributions skewed towards more harmful outcomes, however, quantifying risk in terms of variance may disregard realizations in the more harmful tail [20].

A strong theoretical basis for using the Conditional Value-at-Risk (CVaR) measure to explicitly address harmful tail costs has been developed since the early 2000s (e.g., see [21]). Intuitively, the Conditional Value-at-Risk of a continuous cost random variable is the expectation of the cost conditioned on the event that the cost takes on sufficiently large values. More precisely, as shown in Fig. 1, the Conditional Value-at-
Risk of a continuous cost random variable at level $\alpha$ is the expectation of the cost conditioned on the event that the cost exceeds its Value-at-Risk at level $\alpha$, which is the minimum cost in the $\alpha$% worst cases [22]. Conditional Value-at-Risk is a measure of the more harmful tail of a cost distribution, and managing this tail is paramount in safety-critical applications. (Outside the realm of formal verification, theoretical connections between robustness and risk measures, such as CVaR and exponential disutility, have been investigated [12], [13]. Concrete algorithms have been proposed to optimize policies for risk-sensitive Markov Decision Processes off-line [14], [15]. Scalable algorithms that use Q-learning techniques have been developed to extract approximate CVaR-optimal policies [16]. Also, a CVaR-constrained model predictive control (MPC) approach to risk-aware motion planning [17] and a risk-sensitive MPC framework for linear systems with stochastic multiplicative uncertainties [18] have been recently proposed.)

Statement of Contributions. First, we introduce the notion of a risk-sensitive safe set to specify the degree of safety of a stochastic dynamic system, which leverages the Conditional Value-at-Risk measure at a given risk-sensitivity level $\alpha$ [23]. Second, we show that risk-sensitive safe sets can be under-approximated by the solution to a Markov Decision Process (MDP), where the cost is assessed according to CVaR. This CVaR-MDP problem is time inconsistent, meaning that a policy that is optimal when viewed at time zero is not necessarily optimal when viewed at a later time, so dynamic programming cannot be directly applied (see discussion in [24]). However, a value iteration algorithm on the state space augmented by $Q$-learning techniques has been proposed to approximately solve the CVaR-MDP problem [13]. Proving that this algorithm solves the CVaR-MDP problem exactly for a class of linear systems is the third contribution of our paper. Fourth, for this class of systems, we show the existence of optimal history-dependent policies under a pre-commitment to certain risk-sensitivity level dynamics. These history-dependent policies are actually tractable, since the history is summarized in two parameters, the current state (which may be multi-dimensional) and the current risk-sensitivity level (which is one-dimensional). Fifth, we develop a realistic numerical example of a two-tank stormwater system to show that our approach can be applied in a non-linear setting. We implement the value iteration algorithm in order to estimate upper-approximations of risk-sensitive safe sets at various risk-sensitivity levels. The computation time of the algorithm on a 4-core machine demonstrates computational tractability for a realistic example. Finally, using the stormwater system and a thermostatically controlled load system, we compare the performance of the CVaR criterion to the performance of the standard risk-sensitive criterion, which is expressed in terms of an exponential disutility function [19]. We show empirically that the standard criterion can provide a policy that reduces the mean and variance of the cost of the state trajectory. Moreover, we show that reducing the mean and variance is not guaranteed to minimize the mean of the more harmful cost realizations. Fortunately, however, the CVaR criterion guarantees that this safety-critical tail risk will be minimized, if the cost distribution is continuous. While utilizing CVaR requires more involved implementation and theory than standard methods, one gains safety specifications that are more sensitive to rare high-consequence outcomes, which is an important development for formal verification of safety-critical systems.

II. BACKGROUND: NOTATION & CVaR

We use the following notation. If $X$ is any set and $f : X \rightarrow \mathbb{R}^n$ is bounded, then the uniform norm of $f$ is given by $\|f\|_{u} := \sup_{x \in X} |f(x)|$, where $|\cdot|$ is the Euclidean distance on $\mathbb{R}^n$. If $X$ is a metric space, then $B(X)$ is the Borel $\sigma$-algebra on $X$. If $A \subseteq B(X)$, then $A$ is a Borel set. If $1 \leq p \leq \infty$, then $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P})$ is the collection of functions $f : \Omega \rightarrow \mathbb{R}$, measurable relative to $\mathcal{F}$ and $B(\mathbb{R})$, such that $\|f\|_p := \left(\int_{\Omega} |f|^p d\mathbb{P}\right)^{1/p} < \infty$ (i.e., random variables with finite $p$th moment, where $\mathcal{F}$ is a collection of events). $f \in L^\infty$ indicates that $f$ is a bounded random variable. If $f$ is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then $\sigma(f)$ is the $\sigma$-algebra generated by $f$, and $\mathbb{E}(f) := \int_{\Omega} f d\mathbb{P} := \int_{\Omega} f(\omega) \mathbb{P}(d\omega)$ is the expected value of $f$. “Measurable” is short for Borel measurable, and “a.e.” stands for almost everywhere or almost every. “USC” denotes upper semi-continuous, and “LSC” denotes lower semi-continuous. Bold text signifies a deterministic quantity; e.g., $x_t = x$ indicates that the random variable $x_t$ takes on the value $x$.

There are various nearly equivalent definitions for Conditional Value-at-Risk (also called Average Value-at-Risk), and we use the following definition in this paper. Let $Y$ be a bounded random cost variable. The Conditional Value-at-Risk of $Y$ at risk-sensitivity level $\alpha \in [0, 1]$ is defined as:

$$\text{CVaR}_\alpha(Y) := \begin{cases} \frac{1}{\alpha} \int_{0}^{\alpha} \text{VaR}_{1-\tau}(Y) d\tau & \text{if } 0 < \alpha \leq 1 \\ \|Y\|_{\infty} & \text{if } \alpha = 0, \end{cases}$$

(1a)

where

$$\text{VaR}_\alpha(Y) := \inf\{y \in \mathbb{R} | \mathbb{P}(Y \leq y) \geq 1 - \alpha\}$$

(1b)

is the Value-at-Risk at level $\alpha$, and $\|Y\|_{\infty}$ is the essential supremum (upper bound) of $Y$ [22, Thm. 6.2], [26]. One can show that $\text{CVaR}_\alpha(Y) = \mathbb{E}(Y)$. If the cumulative distribution function of $Y$, $H_Y(y) := \mathbb{P}(Y \leq y)$, is continuous at $y = \text{VaR}_\alpha(Y)$, then $\text{CVaR}_\alpha(Y) = \mathbb{E}(Y|Y \geq \text{VaR}_\alpha(Y))$, which is illustrated in Fig. 1, and which explains the name “Conditional Value-at-Risk” [22, Thm. 6.2].
For $\alpha \in [0, 1]$, CVaR$_{\alpha}$ is a coherent risk functional on the space of bounded random variables. This means that for fixed $\alpha \in [0, 1]$, CVaR$_{\alpha} : L^\infty \to \mathbb{R}$ is convex, monotonic, translation equivariant (CVaR$_{\alpha}(Y + a) = CVaR_{\alpha}(Y) + a$ if $a \in \mathbb{R}$), and positively homogeneous (CVaR$_{\alpha}(\lambda Y) = \lambda CVaR_{\alpha}(Y)$ if $\lambda \geq 0$) [25], [26]. Moreover, since CVaR$_{\alpha}$ is a coherent risk functional, it is equivalent to an expectation maximized over a specific set of probability density functions [22, Thm. 6.4, Eqn. 6.40, Eqn. 6.70]. The above properties are mathematically useful and have intuitive interpretations. For example, the convexity of coherent risk functionals is consistent with the notion that the diversification of assets decreases risk. Moreover, CVaR$_{\alpha}$ may be considered a “robustified” expectation, since it is equivalent to a worst-case expectation with respect to certain perturbations in the assumed probability distribution [13].

III. PROBLEM FORMULATION

We use the Conditional Value-at-Risk measure to pose a safety specification problem for a given stochastic dynamic system over finite time. Suppose that the system dynamics take the form:

$$x_{t+1} = f_t(x_t, u_t, d_t), \quad t = 0, 1, \ldots, T - 1,$$

where $x_t \in X \subset \mathbb{R}^n$ is the state, $u_t \in U \subset \mathbb{R}^m$ is the control input, $d_t \in D \subset \mathbb{R}^d$ is the random disturbance, $f_t : X \times U \times D \to X$ is Borel measurable, and $X$, $U$, and $D$ are non-empty Borel sets. The disturbances $d_0, d_1, \ldots, d_{T - 1}$ are random variables defined on a probability space $(\Omega, F, \mathbb{P})$, so each $d_t$ is a function from $\Omega$ to $D$ that is measurable relative to $\mathcal{F}$ and $\mathcal{B}(D)$. Assume that $d_0, d_1, \ldots, d_{T - 1}$ are independent and identically distributed, where their common distribution is defined by the probability measure $\mathbb{P}$ and is independent of all states and actions. Moreover, suppose that $\Pi_0$ is a set of admissible control policies (to be specified precisely later), $K \subset X$ is a constraint set, and $g_K : \mathbb{R}^n \to \mathbb{R}$ is a bounded function that quantifies the extent of constraint violation (e.g., a clipped signed distance function from the boundary of $K$ [27, p. 8]). The risk-sensitive safe set at $(\alpha, r) \in [0, 1] \times \mathbb{R}$ is defined as:

$$S^r_{\alpha} := \{x \in X \mid W^r_{\alpha}(x, \alpha) \leq r\} \quad (3a)$$

where

$$W^r_{\alpha}(x, \alpha) := \inf_{\pi \in \Pi_0} CVaR^\pi_{\alpha}\left(\max_{t=0,1,\ldots,T} g_K(x_t) \mid x_0 = x\right) \quad (3b)$$

and the state trajectory $(x_0, x_1, \ldots, x_T)$ evolves under the policy $\pi \in \Pi_0$ according to $x_0 = x$ [23].

The risk-sensitive safe set definition is well-motivated for several reasons. First, the definition incorporates different risk-sensitivity levels and non-binary distance to the constraint set, and thereby generalizes the classic maximal probabilistic safe set from [6]. Specifically, let $\epsilon \in [0, 1]$ be a maximum tolerable probability of constraint violation, and choose $\alpha = 1, r = \epsilon$, and $g_K(x) = 1_K(x)$, where $1_K(x) = 1$ if $x \notin K$ and $1_K(x) = 0$ if $x \in K$. Then, the risk-sensitive safe set at $(\alpha, r) = (1, \epsilon)$ is equal to:

$$\left\{ x \in X \mid \inf_{\pi \in \Pi_0} \mathbb{E}^\pi \left[ \max_{t=0,1,\ldots,T} 1_K(x_t) \mid x_0 = x \right] \leq \epsilon \right\} \quad (4)$$

which is the maximal probabilistic safe set at the $\epsilon$-safety level [6] for a given stochastic system that evolves under policies in $\Pi_0$. Moreover, $S^r_{\alpha}$ encodes higher degrees of safety as $\alpha$ or $r$ decrease: $\alpha_1 \geq \alpha_2$ and $r_1 \geq r_2 \implies S^r_{\alpha_2} \subseteq S^r_{\alpha_1}$. Also, $S^r_{\alpha}$ specifies that the CVaR$_{\alpha}$ of the worst constraint violation of the state trajectory must be below a required threshold, whereas the safe set in [11] specifies that the CVaR$_{\alpha}$ of the constraint violation of $x_t$ must be below a required threshold for each $t$. So, $S^r_{\alpha}$ provides a risk-sensitive safety specification for the entire state trajectory. $S^r_{\alpha}$ is a well-motivated safety specification but is difficult to compute due to the CVaR and the maximum in the definition of $W^r_{\alpha}$. To facilitate this computation, we use the following method.

Theorem 1 (Reduction to CVaR-MDP [23]): Fix $\beta > 0$ and $\gamma > 0$. For any $(\alpha, r) \in [0, 1] \times \mathbb{R}$, define the set $U^r_{\alpha} \subset X$ as follows:

$$U^r_{\alpha} := \{x \in X \mid J^r_{\alpha}(x, \alpha) \leq \beta e^{\gamma r}\} \quad (5a)$$

where

$$J^r_{\alpha}(x, \alpha) := \inf_{\pi \in \Pi_0} CVaR^\pi_{\alpha}(C_{0:T} \mid x_0 = x) \quad (5b)$$

such that the state trajectory evolves under the policy $\pi$ according to the dynamics model (2) initialized at $x_0 = x$, $c_t : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a stage cost, and $c_T : \mathbb{R}^n \to \mathbb{R}$ is a terminal cost. If $c_t \equiv \beta e^{\gamma x}$ for all $t$, where $g_K : \mathbb{R}^n \to \mathbb{R}$ is bounded, then $U^r_{\alpha} \subseteq S^r_{\alpha}$. Moreover, the gap between $U^r_{\alpha}$ and $S^r_{\alpha}$ can be reduced by increasing $\gamma$.

Proof: Use the log-sum-exp relation in [31, Sec. 3.1.5] to show: for any $y \in \mathbb{R}^p$ and $\gamma > 0$,

$$\max_{i=1,2,\ldots,p} y_i \leq \frac{1}{\gamma} \log \left( \sum_{i=1}^p e^{\gamma y_i} \right) \leq \frac{\log p}{\gamma} + \max_{i=1,2,\ldots,p} y_i \quad (6)$$

So, as $\gamma \to \infty$, $\frac{1}{\gamma} \log \left( \sum_{i=1}^p e^{\gamma y_i} \right) \to \max_{i=1,2,\ldots,p} y_i$.

Further, for any $\alpha \in [0, 1]$ and any bounded positive random variable $Z$, the following relation holds:

$$CVaR_{\alpha}(\log(Z)) \leq \log \left( CVaR_{\alpha}(Z) \right) \quad (7)$$

since $CVaR_{\alpha}(Z) = \sup_{x \in \mathcal{A}} E_X(Z)$, where $\mathcal{A}$ is a specific set of probability density functions by [22, Thm. 6.4, Eqn. 6.40, Eqn. 6.70], and since $E_X(\log(Z)) \leq \log \left( E_X(Z) \right)$ for any $\xi \in \mathcal{A}$ by Jensen’s Inequality. For any $\alpha \in [0, 1]$, $x \in X$, and $\pi \in \Pi_0$, the following inequalities hold:

$$CVaR^\pi_{\alpha}\left( \max_{t=0,1,\ldots,T} g_K(x_t) \mid x_0 = x \right) \quad (8)$$

by the relations cited above each inequality symbol, and since CVaR$_{\alpha}$ is monotonic and positively homogeneous. Take $x \in U^r_{\alpha}$. For any $\epsilon > 0$, there exists $\pi_\epsilon \in \Pi_0$ such that:

$$0 < CVaR^\pi_{\alpha}\left( \sum_{t=0}^T e^{\gamma g_K(x_t)} \mid x_0 = x \right) \leq \epsilon + \inf_{\pi \in \Pi_0} CVaR^\pi_{\alpha}\left( \sum_{t=0}^T e^{\gamma g_K(x_t)} \mid x_0 = x \right) \quad (ii)$$

for the following reasons: (i) $\forall y \in X, e^{\gamma g_K(y)} > 0$, (ii) definition of infimum and $g_K$ is bounded, (iii) $x \in U^r_{\alpha}$,

$^1$ Abate et al. [6] used the formalism of stochastic hybrid systems, which we do not utilize in the current paper.
CVaR is positively homogeneous, and \( \beta > 0 \). Take the logarithm of both sides and divide by \( \gamma > 0 \) to obtain:

\[
\frac{1}{\gamma} \log \left( \text{CVaR}_T^{\gamma \alpha} \left( \sum_{t=0}^{T} e^{\gamma g(x_t)} \big| x_0 = x \right) \right) \leq \frac{1}{\gamma} \log \left( \epsilon + e^{\gamma r} \right).
\]

Use the definition of \( W_T^\alpha \) (3b) as an infimum to find:

\[
W_T^\alpha (x, \alpha) \leq \text{CVaR}_T^{\gamma \alpha} \left( \sum_{t=0}^{T} e^{\gamma g(x_t)} \big| x_0 = x \right).
\]

where the proper relative option is cited above each inequality symbol. Finally, since \( W_T^\alpha (x, \alpha) \leq \frac{1}{\gamma} \log (\epsilon + e^{\gamma r}) \) for any \( \epsilon > 0 \), take \( \epsilon \to 0 \) and apply continuity of the logarithm to obtain \( W_T^\alpha (x, \alpha) \leq r \). So, \( x \in S_T^\alpha \).

**Remark 1:** In Theorem 1, the parameter \( \beta \) is included to counter numerical issues that arise when \( \gamma \) is large.

Theorem 1 indicates that \( \mathcal{U} \) is an under-approximation of the risk-sensitive safe set \( S^\alpha \). The purpose of Theorem 1 is to approximate the maximum in the definition of \( S^\alpha \) in terms of a summation, since the latter is more amenable to computation due to translation equivariance (CVaR-\( \gamma \alpha \) + \( \gamma \alpha \)). However, computing the function \( J^\alpha \), as defined in Theorem 1, is still difficult because the Conditional Value-at-Risk measure is *time inconsistent* [28].

Intuitively, time inconsistency implies that a policy that is optimal when viewed at time zero is not necessarily optimal when viewed at a later time [24]. Thus, a standard dynamic programming value iteration algorithm, such as [30, Sec. 1.3], cannot be used to compute \( J^\alpha \). In the next section, taking inspiration from [13], [25], and [23], we prove that a value iteration algorithm on the augmented state space \( X := X \times [0,1] \) correctly computes \( J^\alpha \), by imposing additional structure on the problem.

**IV. CVaR Value Iteration Algorithm**

We consider a value iteration algorithm on \( X := X \times [0,1] \) that induces dynamics on the risk-sensitive level. The algorithm involves optimizing over a set of probability densities (i.e., weights) to determine the risk-sensitive level at a given time, so a desired risk-sensitive level is attained over the entire horizon. We specify this set of probability densities next and use \( \alpha_t \) to denote the risk-sensitive level at time \( t \).

**Definition 1 (Risk Envelope):** Fix \( t, x_t = x \in X, u_t = u \in U \), and \( \alpha_0 = \alpha \in [0,1] \). The risk envelope for time \( t \) is defined as follows:

\[
\mathcal{R}^\alpha_t (x, u) := \left\{ Z : \Omega \to \mathbb{R} \bigg| \frac{\sigma(f_t(x, u, d_t))}{\int_{\Omega} Z \mathbb{P} = 1, 0 \leq \alpha Z(\omega) \leq 1 \text{ for a.e. } \omega \in \Omega} \right\}.
\]

where \( \int_{\Omega} Z \mathbb{P} := \int_{\Omega} Z(\omega)\mathbb{P}(d\omega) \) is the expectation of \( Z \) with respect to the probability measure \( \mathbb{P} \). \( \Omega, \mathcal{F}, \mathbb{P} \) is the probability space upon which the random disturbance \( d_t \) is defined, and \( \sigma(f_t(x, u, d_t)) \) is a sub-\( \sigma \)-algebra of \( \mathcal{F} \).

The following algorithm, originally proposed by [13], defines the value function at time \( t \) in terms of a stage cost \( c_t \) and a worst-case weighted expected value function at time \( t + 1 \). Here, the meaning of worst-case is specified by optimizing over the risk envelope for time \( t \).

**Algorithm 1 (CVaR Value Iteration):** Define the functions \( J_{T+1}, \ldots, J_1, J_0 \) recursively as follows: \( \forall (x, \alpha) \in X \) and \( t = T-1, \ldots, 0 \),

\[
J_t(x, \alpha) := \inf_{u \in U} \left\{ c_t(x, u) + \frac{1}{1 - \alpha} \mathbb{E} \left[ \left( \max \left\{ -\alpha Z(\omega), \alpha \right\} \right) J_{t+1} \left( f_t(x, u, d_t(\omega)), \alpha Z(\omega) \right) \mathbb{P}(d\omega) \right] \right\},
\]

where the terminal condition \( J_0(x, \alpha) := c_T(x) \).

In this section, we will prove that \( J_0 \), as defined by (5b), is equivalent to \( J_0 \), as defined by Algorithm 1, by specifying additional structure on the problem, including linear time-varying (LTV) dynamics and history-dependent policies.

**Assumption 1 (Conditions on Control System):** The following conditions hold:

1) \( x_{t+1} = f_t(x_t, u_t, d_t) := A_t x_t + B_t u_t + E_t d_t \) for \( t = 0, 1, \ldots, T-1 \). The matrices \( A_t \in \mathbb{R}^{n \times n}, B_t \in \mathbb{R}^{n \times m}, \) and \( E_t \in \mathbb{R}^{n \times q} \) are given for each \( t \). The initial condition \( x_0 = x \) is deterministic.

2) The stage cost functions \( c_t : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) are bounded, convex, and continuous.

3) The action space \( U \subset \mathbb{R}^m \) is non-empty, compact, and convex.

Many practical control systems can be modeled by linear time-varying dynamics with compact, convex action spaces and convex, continuous stage (or terminal) costs. A concrete example can be found in Section VII-B. Moreover, Assumption 1 ensures that a minimax equality holds, and this minimax equality (which was not shown by [13]) is required to prove the validity of Algorithm 1. In the following definition, we specify the set of admissible control policies as being non-anticipatory, deterministic, and history-dependent.

**Definition 2 (Set of Admissible Policies):** The set of admissible policies for time \( t = 0, 1, \ldots, T-1 \) is defined as follows:

\[
\Pi_t := \left\{ \pi_t := (\mu_t, \mu_{t+1}, \ldots, \mu_{T-1}) \right\} \text{ such that each } \mu_k : H_{t+k} \to U \text{ is Borel measurable} \right\},
\]

where \( H_{t+k} := (X \times U)^{k-t} \times X \) is the set of histories from time \( t \) to time \( k \), whose elements take the form \( h_{t+k} := (x_t, u_t, \ldots, x_{k-1}, u_{k-1}, x_k) \).

**Remark 2:** We specify history-dependent policies above instead of standard Markov policies. This is necessary because the CVaR criterion (5b) does not satisfy certain critical properties that hold for standard expected cost functions, including: time consistency [28], [29], the principle of optimality [30, Sec. 1.3], and the tower rule [32, Prop. B.1 (c)].

The next definition specifies the CVaR-optimal cost-to-go at time \( t \) in terms of the set of history-dependent policies \( \Pi_t \).

**Definition 3 (CVaR-Optimal Cost-to-Go):** The CVaR-optimal cost-to-go function \( J^\alpha_t : X \to \mathbb{R} \) at \( t = 0, 1, \ldots, T-1 \) is defined as follows:

\[
J^\alpha_t (x, \alpha) := \inf_{\pi_t \in \Pi_t} \text{CVaR}_T^{\gamma \alpha} (C_{t:T} | x_t = x), \tag{13a}
\]
where $C_{t,T}$ is the random cumulative cost of the state trajectory for time $t$,
\begin{equation}
C_{t,T} := c_T(x_T) + \sum_{k=t}^{T-1} c_k(x_k, \mu_k(h_{t:k})),
\end{equation}
such that the state trajectory evolves under $\pi_{t} := (\mu_{t}, \mu_{t+1}, \ldots, \mu_{T-1}) \in \Pi_t$ via the LTV dynamics initialized at $x_t = x$. Moreover, we define the optimal terminal cost $J^*_T(x, \alpha) := c_T(x_T)$ for all $\alpha \in [0,1]$.

Since we have specified Assumption 1, the set of history-dependent policies $\Pi_t$, and the CVAR-optimal cost-to-go function $J^*_t$, we are ready to state the main result of this section.

**Theorem 2 (Validity of CVAR Value Iteration):** The value function $J_t$, as defined recursively by Algorithm 1, is equivalent to the CVAR-optimal cost-to-go function $J^*_t$, as defined by (13a), for $t = 0, 1, \ldots, T$.

Theorem 2 is significant because it allows us to compute the risk-sensitive safe set under-approximation $U^*_t$, defined by (5) using Algorithm 1. Our strategy to prove Theorem 2 leverages a representation theorem for the Conditional Value-at-Risk measure from [25] that is amenable to a value iteration approach. After showing how the representation theorem simplifies in the setting of a Markov Decision Process (Lemma 1), we will prove a critical minimax equality (Lemma 2) and then use these two lemmas to prove Theorem 2.

**Lemma 1 (Transfer Thm. 6 (iii) in [25] to MDP):** Fix $t$, $x_t = x \in X$, $u_t = u \in U$, and $\alpha_t = \alpha \in [0,1]$. For any $\pi \in \Pi_{t+1}$, the following equality holds:
\begin{equation}
\text{CVAR}_{\alpha}^\pi(C_{t+1:T} | x_t = x, u_t = u) = \sup_{Z \in R^\pi_{t+1}(x,u)} \int_{\Omega} Z(\omega) \cdot \text{CVAR}_{\alpha}^\pi(Z(\omega)(C_{t+1:T} | x_{t+1})) \mathbb{P}(d\omega),
\end{equation}
where $C_{t+1:T}$ is conditioned on $x_{t+1} = f_t(x, u, d_t(\omega))$ in the integral above, $C_{t+1:T}$ is the random cumulative cost of the state trajectory for time $t+1$ (13b), $R^\pi_{t+1}(x,u)$ is the risk envelope for time $t$ (10), and $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space upon which the random disturbance $d_t$ is defined.

**Proof:** Let $F_t := \sigma(h_{0:t})$, where $h_{0:t} := (x_0, u_0, \ldots, x_t, \mu_t, \mu_{t+1}, \ldots, \mu_{T-1}, x_{t+1})$ is the random history from time 0 to time $t$. Fix $u_t \in U$ and $\pi \in \Pi_{t+1}$. Apply [25, Thm. 6 (iii)] to the random cumulative cost $C_{t+1:T}$ of the state trajectory conditioned on $(u_t, \pi) \in U \times \Pi_{t+1}$ to obtain: 
\begin{equation}
\text{CVAR}_{\alpha}^\pi(C_{t+1:T} | F_t, u_t) = \text{ess sup}_{Z} \mathbb{E}(Z \cdot \text{CVAR}_{\alpha}^\pi(Z(C_{t+1:T} | F_{t+1}) | F_t, u_t)),
\end{equation}
where the essential supremum is taken over the set of measurable random variables $Z$ that satisfy $0 \leq \alpha Z \leq 1$ and $\mathbb{E}(Z | F_t, u_t) \equiv 1$ almost everywhere. We condition on $u_t \in U$ and $\pi \in \Pi_{t+1}$ above to fix the tree of possible outcomes, so [25, Thm. 6] applies, although the algorithm has no explicit notion of control. Further, since $(x_t, u_t, \alpha_t) = (x, u, \alpha)$ is given, $C_{t+1:T}$ is initialized at time $t+1$, and the system is Markov, we can simplify (15) by removing the dependency on the history prior to time $t$ as follows:
\begin{equation}
\text{CVAR}_{\alpha}^\pi(C_{t+1:T} | x_t = x, u_t = u) = \sup_{Z} \mathbb{E}(Z(x_{t+1}) \text{CVAR}_{\alpha}^\pi(Z(x_{t+1})(C_{t+1:T} | x_{t+1}) | x_t = x, u_t = u)),
\end{equation}
where the supremum is taken over the set of Borel measurable functions $Z : X \to \mathbb{R}$ with $0 \leq \alpha Z(x_{t+1}) \leq 1$ for almost every $x_{t+1}$ and $\mathbb{E}(Z(x_{t+1}) | x_t = x, u_t = u) = 1$. The expectations above are taken with respect to the probability distribution of $x_{t+1}$ conditioned on $(x_t, u_t) = (x, u)$. Note that $x_{t+1}$ is a function from $\Omega$ to $X$ through $d_t$, and the disturbances are independent with a common distribution defined by the probability measure $\mathbb{P}$. Thus, for any Borel measurable function $g : X \to \mathbb{R}$, we have $\mathbb{E}(g(x_{t+1}) | x_t = x, u_t = u) = \mathbb{E}(g(f_t(x, u, d_t(\omega))) := \int_{\Omega} g(f_t(x, u, d_t(\omega)))\mathbb{P}(d\omega)$, where $\hat{g} : \Omega \to \mathbb{R}$ is $\sigma(f_t(x, u, d_t))$-measurable; see [33, Thm. 6.4.2 (c), p. 251] and [32, Eqn. 3:4:2, p. 31]. So, the desired result holds.

Having shown Lemma 1, we proceed to Lemma 2. In Lemma 2, we will show a critical minimax equality that uses Sion’s Minimax Theorem and two intermediary results in Appendix I (Lemmas 4 and 5). The intermediary results specify properties of certain functions using Assumption 1 as well as properties of the risk envelope to allow the application of Sion’s Minimax Theorem. Next, we state and prove Lemma 2.

**Lemma 2 (Minimax Equality):** Fix $t$, $x_t = x \in X$, $u_t = u \in U$, and $\alpha_t = \alpha \in [0,1]$. Define $H : R^\pi_{t}(x,u) \times \Pi_{t+1} \to \mathbb{R}$ as follows:
\begin{equation}
H(Z, \pi) := \int_{\Omega} Z(\omega) \cdot \text{CVAR}_{\alpha}^{\pi}(Z(\omega)(C_{t+1:T} | x_{t+1})) \mathbb{P}(d\omega),
\end{equation}
where $C_{t+1:T}$ is conditioned on $x_{t+1} = f_t(x, u, d_t(\omega))$ (16).

This equivalence means that we can compute the risk-sensitive safe set under-approximation $U^*_t$, as defined by (5), using Algorithm 1. Next, we prove Theorem 2 by induction.

**Proof:** $J^*_T = J_T$ by definition. Assume $J^*_{t+1} = J_{t+1}$ for
some $t$. Take $(x, \alpha) \in X$. Then, $J^*_t(x, \alpha)$ equals the following:

\[
J^*_t(x, \alpha) = \inf_{u \in U} \left\{ c_t(x, u) + \inf_{\pi \in P_t} \text{CVaR}_{\alpha_t}^\pi (C_{t+1:T} | x_t = x, u_t = u) \right\}
\]

\[
= \inf_{u \in U} \left\{ c_t(x, u) + \inf_{\pi \in P_t} \sup_{Z \in \mathcal{R}^+_t(x, u)} H^\alpha_t(Z, \pi, x, u) \right\}
\]

\[
= \inf_{u \in U} \left\{ c_t(x, u) + \sup_{Z \in \mathcal{R}^+_t(x, u)} \inf_{\pi \in P_t} H^\alpha_t(Z, \pi, x, u) \right\}
\]

where

\[
H^\alpha_t(Z, \pi, x, u) := \int Z(\omega) \cdot \text{CVaR}_{\alpha(Z(\omega))}^\pi (C_{t+1:T} | x_{t+1}) \mathbb{P}(d\omega),
\]

such that $C_{t+1:T}$ is conditioned on $x_{t+1} = f_t(x, u, d_t(\omega)) := A_t x + B_t u + E_t d_t(\omega)$ in the integral above, and $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space upon which the random disturbance $d_t$ is defined. We justify each equality: (i) equivalence between minimization over $\mu_t : H_{t:T} \rightarrow U$ and $u \in U$ when $x_t = x$ is given, and CVaR is translation equivariant; (ii) Lemma 1; and (iii) Lemma 2. Moreover, the following equalities hold:

\[
\inf_{\pi \in P_t} H^\alpha_t(Z, \pi, x, u) = \int_{\Omega} Z(\omega) \cdot \inf_{\pi \in P_t} \text{CVaR}_{\alpha(Z(\omega))}^\pi (C_{t+1:T} | x_{t+1}) \mathbb{P}(d\omega),
\]

where we justify each step: (iv) interchangeability assertion from the proof of [25, Thm. 18, p. 162]; (v) definition of $J^*_t$ (13a), and $C_{t+1:T}$ is conditioned on $x_{t+1} = f_t(x, u, d_t(\omega))$ in the previous line; and (vi) induction hypothesis $J^*_{t+1} = J_{t+1}$. The above steps together with (11) show that $J^*_t = J_t$, which completes the induction.

We have shown that $J^*_t$, as defined by (13a), is equal to $J_t$, as defined by Algorithm 1, for $t = 0, 1, \ldots, T - 1$, under the conditions specified in Assumption 1. In the next section, we prove the existence of tractable policies that attain $J^*_t$, if we pre-commit to the dynamics of the risk-sensitivity level induced by Algorithm 1.

V. SYNTHESIS OF PRE-COMMITMENT POLICIES

Our approach to the synthesis of optimal policies involves restricting the optimization space of history-dependent policies to policies that commit to specific risk-sensitivity level dynamics. Such policies are called pre-commitment policies. This restriction to pre-commitment policies is well-motivated for two reasons. First, due to time inconsistency, a Conditional Value-at-Risk cost function is not amenable to dynamic programming unless the entire history is recorded [25]. Second, computing history-dependent policies is intractable generally, since substantial memory and computation time are required. However, we can overcome these two challenges by restricting the optimization space to policies that commit to risk-sensitivity level dynamics that summarize the history.

The purpose of this section is to synthesize tractable policies that attain the CVaR-optimal cost-to-go function $J^*_t$ (13a) under mild restrictions on the optimization space of history-dependent policies $\Pi_t$. We define a control law at time $k = t$ on the augmented state space $X := X \times [0, 1]$ that is a function of the current state $x_k$ and the current risk-sensitivity level $\alpha_k$. Accordingly, this law is called a $X$-Markov control law. The current state $x_k \in X$ satisfies the linear time-varying dynamics specified by Assumption 1. The current risk-sensitivity level $\alpha_k \in [0, 1]$ satisfies particular dynamics induced by Algorithm 1. Under these dynamics, $\alpha_k$ is a one-dimensional parameter that summarizes the higher-dimensional prior history $(x_1, u_1, x_{t+1}, u_{t+1}, \ldots, x_k, u_k) \in (X \times U)^k$.

This section is structured as follows. First, we specify a regularity condition on the risk envelope to facilitate policy synthesis (Assumption 2). Second, we define $X$-Markov control laws and $X$-Markov policies (Definition 4). We prove the existence of $X$-Markov control laws by invoking Assumptions 1 and 2. Third, we define the dynamics of the risk-sensitivity level induced by Algorithm 1 (Definition 5). Finally, in Theorem 3, we show that $X$-Markov policies attain the CVaR-optimal cost-to-go function (13a), if we pre-commit to the risk-sensitivity level dynamics induced by Algorithm 1.

A. Regularity Condition on Risk Envelope

Here, we specify and justify a regularity condition on the risk envelope defined by (10).

Assumption 2 (Regularity Condition on Risk Envelope): The set-valued mapping $(x, \alpha, u) \mapsto R^*_2(x, u)$ is lower semi-continuous on $X \times U$.

Assumption 2 is useful in general, since it guarantees the existence of policies that optimize certain performance criteria. For example, in the classic setting of games against nature, where stochastic control systems are affected by adversarial players with unknown distributions, the set of admissible actions of the adversary (player 2) is assumed to be lower semi-continuous in the state and action of the controller (player 1) [38, Assumption 3.1 (g), p. 1632]. Also, Assumption 2 resembles conditions in the risk-averse dynamic programming literature. Ruszczyński [37] assumes that the stochastic transition kernel is continuous in the control and the risk envelope is lower semi-continuous in the probability measure, implying that the risk envelope is lower semi-continuous in the control by composition [37, Thm. 2 (i) (iii)]. The conditions in [37] are required for a time-consistent problem, whereas the condition in Assumption 2 is required for a time-inconsistent problem.

Assumption 2 is a measurable selection condition for time-inconsistent CVaR-MDP problems that guarantees the existence of a $X$-Markov control law.3 In the next subsection, we define a $X$-Markov control law in terms of an argument that minimizes an objective function over a compact set. To ensure that a minimum argument exists, the objective function must be lower semi-continuous. Assumption 2 specifies a sufficient condition to guarantee that the objective function is indeed lower semi-continuous.

---

3The condition specified in Assumption 2 is equivalent to the following statement: If $(x_n, \alpha_n, u_n) \rightarrow (x, \alpha, u)$ in $X \times U$ and $Z \in R^*_2(x, u)$, then there exist $Z_n \in R^*_{2n}(x_n, u_n)$ s.t. $Z_n \rightarrow Z$ [32, Prop. D.2, p. 182].
B. Existence of $\mathbb{X}$-Markov Policies

Recall that $X := X \times [0, 1]$ is the state space augmented by the space of risk-sensitivity levels. Here, we define $\mathbb{X}$-Markov control laws as Borel measurable mappings from $X$ to $U$ and $\mathbb{X}$-Markov policies as time-based sequences of such mappings. Then, we prove the existence of $\mathbb{X}$-Markov control laws using the conditions specified in Assumptions 1 and 2.

Definition 4 (\textit{$\mathbb{X}$-Markov Control Law, Policy}): Fix $t$. A $\mathbb{X}$-Markov control law at time $t$ is a Borel measurable function $\mu_t^* : X \to U$ that satisfies the following:
\[
\mu_t^*(x, \alpha) \in \text{arg min}_{u \in U} \left\{ c_t(x, u) + \sup_{Z \in \mathcal{R}_t^u(x, u)} G_t^u(Z, x, u) \right\},
\]

such that $(x_t, \alpha_t) = (x, \alpha)$ is the state and risk-sensitivity level at time $t$, and $G_t^u(Z, x, u)$ is defined as:
\[
G_t^u(Z, x, u) := \int_{\Omega} Z(\omega) \cdot J_{t+1}^u \left( f_t(x, u, d_t(\omega)), \alpha Z(\omega) \right) \mathbb{P}(d\omega),
\]

where $f_t(x, u, d_t(\omega)) := A x + B_t u + E d_t(\omega)$, $x_t = x$ is the state at time $t$, $u_t = u$ is the control at time $t$, and $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space upon which the random disturbance $d_t$ is defined. A sequence of $\mathbb{X}$-Markov control laws $\pi_t^* := (\mu_t^*, \mu_{t+1}^*, \ldots, \mu_{T-1}^*)$ is a $\mathbb{X}$-Markov policy for time $t$.

The arguments in Lemmas 3, 6, and 7 together show the existence of $\mathbb{X}$-Markov control laws, by invoking Assumptions 1 and 2. Lemma 3 specifies that the objective function in (18) is lower semi-continuous in $u$ to ensure the existence of a $\mathbb{X}$-Markov control law, if $G_t^u(Z, x, u)$ is lower semi-continuous in $(Z, u)$. Lemma 6 specifies that $G_t^u(Z, x, u)$ is lower semi-continuous in $(Z, u)$, if $J_{t+1}^u$ is lower semi-continuous (Appendix II). Finally, Lemma 7 specifies that $J_{t}^u$ is lower semi-continuous for each $t$ (Appendix II). We provide the statements and proofs of Lemmas 6 and 7 in Appendix II, while we provide the statement and proof of Lemma 3 next.

Lemma 3 (Existence of $\mathbb{X}$-Markov Control Laws): Fix $t$, and define $v_t : X \times U \to \mathbb{R}$ as follows:
\[
v_t(x, \alpha, u) := c_t(x, u) + \sup_{Z \in \mathcal{R}_t^u(x, u)} G_t^u(Z, x, u),
\]

where $\mathcal{R}_t^u(x, u)$ is given by (10), and $G_t^u(Z, x, u)$ is given by (19). Then, there exists a Borel measurable function $\mu_t^* : X \to U$ that satisfies the following statement:
\[
v_t(x, \alpha, \mu_t^*(x, \alpha)) = \text{inf}_{u \in U} v_t(x, \alpha, u) = \text{min}_{u \in U} v_t(x, \alpha, u).
\]

Proof: Since $U$ is compact, it suffices to show that $v_t(x, \alpha, \cdot)$ is lower semi-continuous on $U$ for every $(x, \alpha) \in X$ by [32, Prop. D.5 (a)]. Since the sum of lower semi-continuous functions is lower semi-continuous, it suffices to show that $u \mapsto \psi_t^u(x) := \sup_{Z \in \mathcal{R}_t^u(x, u)} G_t^u(Z, x, u)$ is lower semi-continuous on $U$ for every $(x, \alpha) \in X$. Since $u \mapsto \mathcal{R}_t^u(x, u)$ is a lower semi-continuous set-valued mapping by Assumption 2, for any $\{u^{(j)}\} \subset U$ converging to a point $u \in U$ and for any $Z \in \mathcal{R}_t^u(x, u)$, there exists $Z^{(j)} \in \mathcal{R}_t^{u^{(j)}}(x, u^{(j)})$ such that $Z^{(j)} \to Z$. If $G_t^u(Z, x, u)$ is lower semi-continuous in $(Z, u)$, then one can show:
\[
\lim_{j \to \infty} \psi_t^u(x^{(j)}) \geq \lim_{j \to \infty} G_t^u(Z^{(j)}, x, u^{(j)}) \geq G_t^u(Z, x, u),
\]

by the definition of supremum and lower semi-continuity, respectively. Since the above inequalities hold for any $Z \in \mathcal{R}_t^u(x, u)$, and since the supremum is the lower semi-continuity bound,
\[
\liminf_{j \to \infty} \psi_t^u(x^{(j)}) \geq \sup_{Z \in \mathcal{R}_t^u(x, u)} G_t^u(Z, x, u) =: \psi_t^u(x),
\]

showing that $\psi_t^u(x)$ is lower semi-continuous on $U$, the desired result. (See [38, Lemma 3.2 (a)] for the original proof.) So, it suffices to show that $G_t^u(Z, x, u)$ is lower semi-continuous in $(Z, u)$, which is shown in Lemma 6 (Appendix II).

Next, we specify the dynamics of the risk-sensitivity level so that $\mathbb{X}$-Markov policies under these dynamics are history-dependent.

C. Risk-Sensitivity Level Dynamics

Here, we define the risk-sensitivity level dynamics in terms of a member of the risk envelope that maximizes a weighted CVaR-optimal cost-to-go in expectation.

Definition 5 (Risk-Sensitivity Level Dynamics): Fix $t$, $x_t = x \in U$, and $\alpha_t = \alpha \in [0, 1]$. Let $Z_t^* \in \mathcal{R}_t^u(x, u)$ satisfy the following statement:
\[
Z_t^* \in \arg \max_{Z \in \mathcal{R}_t^u(x, u)} G_t^u(Z, x, u),
\]

where $G_t^u(Z, x, u)$ is defined by (19). Then, the risk-sensitivity level at time $t + 1$ is given by $\alpha_{t+1} = \alpha \cdot Z_t^*$. Please note the following remarks concerning the risk-sensitivity level dynamics defined above.

Remark 3: Lemma 5 and Lemma 8 ensure that $Z_t^* \in \mathcal{R}_t^u(x, u)$ is well-defined, since the supremum of an upper semi-continuous map on a compact topological space is attained [33, Thm. A6.3]. Lemma 5 specifies that the risk envelope $R_t^u(x, u)$ is compact in $L^2 := L^2(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the weak topology (Appendix I). Lemma 8 specifies that $G_t^u(Z, x, u)$ is upper semi-continuous in the relative weak topology on $\mathcal{R}_t^u(x, u) \subset L^2$ (Appendix II).

Remark 4: The risk-sensitivity level at time $t + 1$, $\alpha_{t+1} = \alpha \cdot Z_t^*$, is a $\sigma(f_t(x, u, d_t))$-measurable random variable, since $Z_t^*$ is. If $\omega \in \Omega$ is fixed (which implies that $d_t(\omega)$ occurs at time $t$), then the risk-sensitivity level that occurs at time $t + 1$ is given by $\alpha_{t+1}(\omega) = \alpha \cdot Z_t^*(\omega)$.

Remark 5: $\mathbb{X}$-Markov policies (Definition 4) under the dynamics of the risk-sensitivity level (Definition 5) are history-dependent policies. Specifically, a $\mathbb{X}$-Markov policy for time $t$ is a function of the augmented state trajectory $(x_t, \alpha_t, x_{t+1}, \alpha_{t+1}, \ldots, x_T, \alpha_T)$ by Definition 4, where $\alpha_k$ is a function of $(\alpha_t, x_t, u_t, \ldots, x_{k-1}, u_{k-1})$ for $k = t + 1, t + 2, \ldots, T$ by Definition 5 and [33, Thm. 6.4.2 (c), p. 251].

D. Optimality of Pre-Commitment $\mathbb{X}$-Markov Policies

The CVaR-optimal cost-to-go $J_t^*$ (13a) is an infimum over a set of history-dependent policies $\Pi_t$ (Definition 2). The final theorem of this paper indicates that $\mathbb{X}$-Markov policies (Definition 4) attain $J_t^*$, if we pre-commit to the risk-sensitivity level dynamics given by Definition 5. This is a powerful practical result because history-dependent policies are generally intractable due to extensive memory requirements, but the following theorem enables tractable computations.

Theorem 3 (Optimal Pre-Commitment Policies): Fix $t$. Let $\pi_t^* := (\mu_t^*, \mu_{t+1}^*, \ldots, \mu_{T-1})$ be a $\mathbb{X}$-Markov policy for time
where $C_{t:T} = c_T(x_T) + \sum_{k=t}^{T-1} c_k(x_k, \mu^*_k(x_k, \alpha_k))$, and $(x_t, \alpha_t, \ldots, x_T, \alpha_T)$ satisfies the linear time-varying dynamics (Assumption 1) and the risk-sensitivity level dynamics (Definition 5) under $\pi^*_t$ initialized at $(x_t, \alpha_t) = (x, \alpha)$.

**Proof:** We proceed by induction. Since $J^*_t(x_t, \alpha_T) = c_T(x_T)$ does not depend on the control when $(x_T, \alpha_T) = (x, \alpha)$ is given, the base case is $t = T - 1$. In this case, $\pi^*_{T-1} = \mu^*_T$, where $w_{t-1} = \mu^*_T(x, \alpha)$. Then, the following equalities hold:

\[
\text{CVaR}_{\alpha}^{\pi^*_T - 1}(C_{T-1:T} | x_{T-1} = x) - c_{T-1}(x, \mu^*_T(x, \alpha))
\]

\[
= \text{CVaR}_{\alpha}(c_T(x_T) | x_{T-1} = x, w_{T-1} = \mu^*_T(x, \alpha))
\]

\[
= \sup_{\alpha} \int_{\Omega} \{ Z(\omega) \cdot c_T(x_T(\omega)) \} P(\omega) d\omega,
\]

where $\alpha = \frac{1}{\alpha_1} \cdot \frac{1}{\alpha_2}$ which shows the base case. Now, for some $t$, consider a $\mathcal{X}$-Markov policy $\pi^*_t$, and assume the following: \( \forall (x, \alpha) \in \mathcal{X}, \)

\[
J^*_t(x, \alpha) = \text{CVaR}_{\alpha}^{\pi^*_t}(C_{t+1:T} | x_{t+1} = \bar{x}),
\]

where $(x_{t+1}, \alpha_{t+1}, u_{t+1}, \ldots, x_{T-1}, w_{T-1}, u_{T-1}, x_T, \alpha_T)$ satisfies the linear time-varying dynamics (Assumption 1) and the risk-sensitivity level dynamics (Definition 5) under $\pi^*_t$ with the initialization $(x_{t+1}, \alpha_{t+1}) = (\bar{x}, \alpha)$. Consider $\pi^*_t = (\mu^*_t, \pi^*_t)$, where $\mu^*_t$ is a $\mathcal{X}$-Markov control law. Then, for any $(x_t, \alpha_t) = (x, \alpha) \in \mathcal{X}$, the following equalities hold:

\[
\text{CVaR}_{\alpha}^{\pi^*_t}(C_{t:T} | x_t = x)
\]

\[
= \inf_{u \in \mathcal{U}} \left\{ c_T(x, u) + \sup_{Z \in \mathcal{R}^a} G_{T-1}(Z, x, u) \right\}
\]

\[
= J^*_t(x, \alpha),
\]

which shows the base case. Now, for some $t$, consider a $\mathcal{X}$-Markov policy $\pi^*_t$, and assume the following: \( \forall (x, \alpha) \in \mathcal{X}, \)

\[
J^*_t(x, \alpha) = \text{CVaR}_{\alpha}^{\pi^*_t}(C_{t+1:T} | x_{t+1} = \bar{x}),
\]

where $(x_{t+1}, \alpha_{t+1}, u_{t+1}, \ldots, x_{T-1}, w_{T-1}, u_{T-1}, x_T, \alpha_T)$ satisfies the linear time-varying dynamics (Assumption 1) and the risk-sensitivity level dynamics (Definition 5) under $\pi^*_t$ with the initialization $(x_{t+1}, \alpha_{t+1}) = (\bar{x}, \alpha)$.

**VI. NUMERICAL EXAMPLE: RISK-SENSITIVE SAFE SETS**

To illustrate our approach in a realistic setting, we provide a numerical example of a two-tank gravity-driven stormwater system with an automated valve (Fig. 2, Table I).\(^4\) Consider the following non-linear discrete-time dynamics model: $x_{t+1} = x_t + F(x_t, u_t, d_t) \cdot \Delta \tau$ for $t = 0, 1, \ldots, T - 1$, where

\[
F(x, u, d) := \begin{bmatrix}
\frac{d - g_{valve}(x, u)}{\alpha_1} \\
\frac{d + g_{valve}(x, u) - q_{drain}(x)}{\alpha_2}
\end{bmatrix}^T,
\]

$q_{valve}(x, u) := u \cdot \pi r_2^2 \cdot \text{sgn}(h(x)) \cdot \sqrt{2gh(x)}$

$h(x) := \max(x_1 - Z_1, 0) - \max(x_2 - Z_1, 0)$

$q_{drain}(x) := \begin{cases}
C_\alpha \pi r_2^2 \sqrt{2gh(x)} \cdot \cos(\theta) & \text{if } x_2 \geq Z_2 \\
0 & \text{otherwise}
\end{cases}$

and $x_t = [x_{t1}, x_{t2}]^T \in \mathbb{R}^2$ is the state vector, $x_{t1} \in \mathbb{R}$ is the water level of tank 1, $u_t \in [0, 1]$ is the valve setting (closed to open), $d_t \in \mathbb{R}$ is the random surface runoff (disturbance), and $\Delta \tau$ is the duration of $[t, t+1]$. The constraint set $K := [0, K_1] \times [0, K_2]$ specifies the maximum water levels in the tanks. The stage cost $g(x_t) := \max(x_{t1} - K_1, x_{t2} - K_2, 0)$ is the maximum level of overflow when the system occupies state $x_t$. We identified a finite probability distribution for

\[^4\]Here, the stormwater system is two-dimensional, whereas the system in our prior work [23] is one-dimensional. Moreover, an optimal policy for the system here is not known *a priori*, whereas an optimal policy for the system in [23] is known *a priori*.\]
\(d_t\) using the first three empirical moments of time-averaged runoff samples (Table II). The runoff samples were obtained by simulating a design storm in PCSWMM (Computational Hydraulics International), which extends USEPA’s Stormwater Management Model [39], [40].

We approximated \(\{U^*_\alpha\}\) by approximating \(J^*_\alpha\) on a grid of states and risk-sensitivity levels via Algorithm 1, and by utilizing the risk-sensitivity level interpolation approach proposed by [13], multi-linear interpolation of the state space, and uniform discretization of the action space. The values of \(\beta\) and \(\gamma\) from Theorem 1 were chosen empirically, where the magnitude of \(\gamma\) was constrained by the limitations of numerical solvers to manage differently scaled constraints. The computation time of the value functions \(J_0, J_1, \ldots, J_{T-1}\) and an optimal pre-commitment policy \(\pi^*_0 = (\mu^*_0, \mu^*_1, \ldots, \mu^*_{T-1})\) was about 230 hours for a three-dimensional grid of 50,490 nodes over \(T = 48\) time points when run on a 4-core machine.\(^5\)

Algorithm 1 was run serially on the state space grid, however at a given time point \(t\), the computations at each state are independent and can be run in parallel to reduce computation time. We approximated \(\{S^*_\alpha\}\) by performing 100,000 Monte Carlo simulations of \(\max\{q_k(x_t) : k = 0, 1, \ldots, T\}\) initialized at each \(x_0 = x\) in the state space grid under the policy computed by Algorithm 1, and by utilizing a consistent CVaR estimator [22, p. 300].

Numerical approximations of \(U^*_\alpha\) and \(S^*_\alpha\) (denoted by \(\hat{U}^*_\alpha\) and \(\hat{S}^*_\alpha\), respectively) for the stormwater system are shown in Fig. 3. Although the correctness of Algorithm 1 and the existence of optimal policies were formally proven for linear systems, we found that \(\hat{U}^*_\alpha\) provides an under-approximation of \(S^*_\alpha\) when \(\beta e^{\gamma r}\) is sufficiently large, although Theorems 2 and 3 were proven for linear systems. Approximations of \(J_0(\cdot, \alpha)\) and \(W_0^*(\cdot, \alpha)\) are also provided. \(U^*_\alpha\) is the \(\beta e^{\gamma r}\)-sublevel set of \(J^*_\alpha\) and \(W^*_\alpha\) is the \(\beta e^{\gamma r}\)-sublevel set of \(S^*_\alpha\).

\(^5\)The three-dimensional grid consists of 15 risk-sensitivity levels, 51 values of \(x_1\), and 66 values of \(x_2\) for a total of 50,490 nodes. To compute the value functions and an optimal pre-commitment policy, we ran Algorithm 1 in a cluster computing session that was allocated 4 CPU cores running at 2.8 GHz. We used the Tufts Linux Research Cluster (Medford, MA) running MATLAB (The Mathworks, Inc.) with MOSEK [44] and CVX [41], [42].

![Fig. 2. To illustrate our approach in a realistic setting, we develop a numerical example of a non-linear, two-tank stormwater system with an automated valve, where water flows by gravity between the tanks and can flow in either direction.](image)

![Fig. 3. Approximate contours of \(U^*_\alpha\) and \(S^*_\alpha\) (denoted by \(\partial U^*_\alpha\) and \(\partial S^*_\alpha\), respectively) are shown for \(\alpha\in\{0.99, 0.05, 0.01\}\) and \(r\in\{1.25, 1.5\}\) for the (non-linear) stormwater system. These results indicate that our approach can be applied in a non-linear setting. \(U^*_\alpha\) provides an under-approximation of \(S^*_\alpha\) when \(\beta e^{\gamma r}\) is sufficiently large.\(^6\) These results indicate that Algorithm 1 is tractable on a realistic stochastic system.]()

**VII. CVaR versus Exponential Disutility**

Here we compare the Conditional Value-at-Risk criterion, \(J^*_\alpha\) (13a), to the standard risk-sensitive criterion, which is expressed in terms of an exponential disutility function [19].

\(^6\)We found that \(\hat{U}^*_\alpha\) provides an under-approximation of \(S^*_\alpha\) when \(\beta e^{\gamma r}\) is on the order of \(10^{-4}\) or greater for \(\alpha\in\{0.99, 0.05, 0.01\}\). When \(\beta e^{\gamma r}\) is too small, numerical inaccuracies in the approximation of \(U^*_\alpha\) may be amplified by the transformation from \(J_0 \leq \beta e^{\gamma r}\) to \(\frac{1}{\gamma} \log(J_0/\beta) \leq r\). The numerical stability of the computations requires a careful selection of \((\beta, \gamma)\) that balances the desire to make \(\gamma\) as large as possible with the understanding that \(\beta\) cannot be made arbitrarily small without introducing additional numerical issues.
The latter was developed to encode a preference for mean-variance-sensitive controllers by Peter Whittle in the early 1990s [19]. We provide relevant background from his seminal work [19] before showing our numerical results.

The (risk-averse) exponential disutility criterion $V^*_0 : X \times (-1, 0) \to \mathbb{R}$ is defined as follows:

$$V^*_0(x, \theta) := \inf \frac{2}{\theta} \log \mathbb{E}^\pi(e^{-\frac{1}{\theta}C_{0:T}|x_0 = x})$$

where the infimum is taken over a set of policies, $\theta \in (-1, 0)$ is the risk-sensitivity level, $x_0 = x \in X$ is the initial condition, and $C_{0:T}$ is the random cost of the state trajectory [19, Eqn. 1.10, Eqn. 1.11]. Equality $(i)$ above holds since $0 < |\theta| < 1$. $\mathbb{E}^\pi(\cdot)$ and $\mathbb{V}^\pi(\cdot)$ denote expectation and variance under the policy $\pi$, respectively. The range of $\theta$ encodes the risk-averse perspective that high variance is disadvantageous. If $\theta$ is closer to $-1$, then $V^*_0(x, \theta)$ is more risk-averse; i.e., high variance is penalized more. However, if $\theta$ is closer to $0$, then $V^*_0(x, \theta)$ is more risk-neutral; i.e., high variance is penalized less.

If $C_{0:T} := c_T(x_t) + \sum_{t=0}^{T-1} c_t(x_t, u_t)$, then the following algorithm computes $V^*_0(x, \theta)$ and an optimal Markov policy under standard assumptions; see Remark 6, [19], and [30].

Algorithm 2 (Exp. Disutility Value Iteration [19], [30]):

Fix $\theta \in (-1, 0)$. Define the functions $V^0_{T-1}, \ldots, V^0_0, V^0_\theta$ recursively as follows: $\forall x \in X \text{ and } t = T - 1, \ldots, 0$,

$$V^0_t(x) := \inf_{u \in U} \left\{ c_t(x,u) + \frac{2}{\theta} \log \mathbb{E}^\pi(e^{-\frac{1}{\theta}V^0_{t+1}(f_t(x,u,d_t)))}) \right\}$$

(24)

with the terminal condition $V^0_{T}(x) := c_T(x)$, where the expectation is taken with respect to the distribution of $d_t$.

Remark 6: One can show that $V^0_{T}(x, \theta) = V^0_\theta$. An optimal action at time $t$ when the system occupies the state $x_t = x$ is one that minimizes the objective function in (24). An optimal action is guaranteed to exist, if the objective in (24) is lower semi-continuous on $U$ and $U$ is compact.

Next, we use the stormwater system and a pedagogical thermal system to evaluate the CVaR criterion $J^\pi_0(x, \alpha, \theta)$ relative to the exponential disutility criterion $V^0(x, \theta)$ at different risk-sensitivity levels. The values $(\alpha, \theta) = (0.99, -0.01)$ encode a risk-neutral perspective, whereas the values $(\alpha, \theta) = (0.01, -0.99)$ encode a worst-case perspective. The values $(\alpha, \theta) = (0.05, -0.95)$ strike a balance between risk-neutral and worst-case by specifying a typical level of risk aversion.

A. Stormwater System Example

We sampled $C_{0:T} = \sum_{t=0}^{T} \max(x_{t+1} - K_1, x_{t+1} - K_2, 0)$ one million times from $x_0 \in [0, 1]^7$ using a policy synthesized with respect to the CVaR criterion for various $\alpha$ (Fig. 4, top row). We repeated this procedure for the exponential disutility criterion for various $\theta$ (Fig. 4, bottom row). The system that we consider is a realistic stormwater system that relies on gravity and a difference in hydraulic head to move water between the tanks. While the gravity-constrained maximum flow rate between the tanks can be modulated by the automated valve, its control authority is limited when compared with other active control options, such as pumps. Hence, when using the CVaR criterion, the CVaR$^{0.01}$, VaR$^{0.01}$, mean, and variance are only reduced by about 0.02, 0.02, 0.01, and 0.004, respectively, under a worst-case perspective versus a risk-neutral perspective (Fig. 4, top row). Despite this limited control authority, however, it is remarkable that similar reductions in these empirical statistics can be attained just by using the CVaR criterion instead of the exponential disutility criterion at a typical level of risk aversion (Fig. 4, center column).

B. TCL System Example

Now, consider the following discrete-time thermostatically controlled load (TCL) system:

$$x_{t+1} = a x_t + (1-a) (b - R P u_t) + d_t, \quad t = 0, 1, \ldots, T - 1$$

which we adopted from [10], and which was first developed by [43]. In the above model, $x_t \in \mathbb{R}$ is the temperature ($^\circ\text{C}$) of the thermal mass, $u_t \in [0, 1]$ is a continuous control input from no power to full power, and $d_t \in \mathbb{R}$ is a random disturbance due to environmental uncertainty (Table III). The stage cost $g_k(x_t) := \max(x_t - 21, 20 - x_t)$ quantifies the extent of constraint violation of the state $x_t$ with respect to the constraint set $K = [20, 21] ^\circ\text{C}$.

We discretized the action space $U = [0, 1]$ and the state space $X = [18, 23] ^\circ\text{C}$ uniformly at a resolution of 0.1 and used linear interpolation of $X$ to implement Algorithms 1 and 2. We discretized the risk-sensitivity level space non-uniformly with more points near the levels of interest $\alpha \in \{0.99, 0.05, 0.01\}$ and utilized the risk-sensitivity level interpolation approach proposed by [13].

From two distinct initial conditions, we sampled $C_{0:T} = \sum_{t=0}^{T} \max(x_t - 21, 20 - x_t)$ one million times using a policy synthesized with respect to the CVaR criterion for various $\alpha$, and we repeated this procedure for the exponential disutility criterion for various $\theta$. We utilized the heavy-tailed finite probability distribution for $d_t$ called “Original” in Fig. 5 to train the policies and to generate the histograms of $C_{0:T}$ shown in Fig. 6 and Fig. 7. Further, we used the distribution for $d_t$ called “Perturbed” in Fig. 5 to generate the histograms of $C_{0:T}$ in Fig. 8 to assess robustness to distribution estimation error.

Our results show that the CVaR criterion minimizes the mean of higher-consequence outcomes, evident by reduced weight on the right-hand tail of the empirical distribution of $C_{0:T}$ as the degree of risk aversion increases (Fig. 6, top row, left to right). The exponential disutility criterion penalizes the mean and variance of $C_{0:T}$, and the empirical variance becomes smaller as the degree of risk aversion increases (Fig. 6, top row, right to left).
bottom row, left to right). While the exponential disutility criterion penalizes variance, this criterion is not guaranteed to minimize the mean of higher-consequence outcomes in the setting of asymmetric cost distributions (see estimated values of CVaR_{0.01}, top vs. bottom row, Fig. 6). Notably, at a typical level of risk aversion, the empirical variance of C_{0:T} is smaller under the CVaR criterion compared to the exponential disutility criterion (Fig. 6, center column).

Our results indicate that the initial condition of the system impacts the relative advantage of using the CVaR criterion versus the exponential disutility criterion for policy synthesis. The probability distribution of the disturbance in the TCL example represents a setting where the thermal mass is exposed to random temperature perturbations with a positive bias (Fig. 5), and the control input u_t is only able to provide heat. Thus, if the system is initialized at the center of the constraint set, the controller has limited authority to avoid high temperatures, regardless of the chosen criterion (CVaR or exponential disutility), as shown in Fig. 7. However, if the system is initialized at a cooler temperature where the controller has more authority, there is a substantial advantage to employing the CVaR criterion rather than the exponential disutility criterion to reduce the mean of high-consequence outcomes (Fig. 6). In addition, the CVaR criterion continues to offer advantages over the exponential disutility criterion when the cumulative cost C_{0:T} is sampled under a perturbed probability distribution for the disturbance (Fig. 8).

VIII. CONCLUSIONS

The CVaR criterion has the substantial advantage of improved interpretability compared to the exponential disutility criterion. The CVaR criterion at the risk-sensitivity level \( \alpha \) is guaranteed to minimize the mean of the \( \alpha \)-fraction of worst-case outcomes, if the outcome distribution is continuous. However, the risk-sensitivity level \( \theta \) of the exponential disutility criterion does not map to a precise reduction in variance.

Our results suggest that the Conditional Value-at-Risk criterion may be preferable to the exponential disutility criterion under the following conditions: the distribution of the cumulative cost of the state trajectory is asymmetric (e.g., as in Fig. 4 or Fig. 6); the optimizer wishes to penalize tail risk; and the system has adequate control authority. Otherwise, it may be preferable to utilize the exponential disutility criterion at a single risk-sensitivity level due to more straightforward implementation, faster computation, and milder assumptions.

While optimizing with respect to Conditional Value-at-Risk is more involved than standard dynamic programming, one gains safety specifications that are more sensitive to rare high-consequence outcomes, which is an important development for formal verification. Future steps to extend this work to a larger class of systems include leveraging novel approximate dynamic programming approaches [45] and transformations between non-linear and linear dynamics [46].

APPENDIX I

Here we show two results that are required to prove the minimax equality (Lemma 2). Lemma 4 states that an expected CVaR of the cost of the state trajectory is convex and continuous on the policy space and is continuous on the risk envelope. Lemma 5 specifies important topological properties of the risk envelope.

Remark 7: Recall that \((\Omega, \mathcal{F}, \mathbb{P})\) is the probability space upon which the random disturbance \( d_t \) is defined. The risk envelope \( R^g_{\alpha} (\mathbf{x}, \mathbf{u}) \) is a subset of \( L^2 (\Omega, \sigma(f_t (\mathbf{x}, \mathbf{u}, d_t)), \mathbb{P}) \), since it is a collection of \( \mathbb{P} \)-integrable, \( \sigma(f_t (\mathbf{x}, \mathbf{u}, d_t)) \)-measurable bounded functions on \( \Omega \) by definition (10). “Bounded” means bounded in \( || \cdot ||_\infty \), which implies bounded in \( || \cdot ||_2 \). We study properties of the risk envelope in \( L^2 := L^2 (\Omega, \sigma(f_t (\mathbf{x}, \mathbf{u}, d_t)), \mathbb{P}) \), since \( L^2 \) is reflexive and equals its dual, which simplifies proofs.

Lemma 4 (Convexity and Continuity): Fix \( t, x_t = x \in X, u_t = u \in U \), and \( \alpha_t = \alpha \in [0, 1] \). Define \( H : R^g_{\alpha} (\mathbf{x}, \mathbf{u}) \times \Pi_{t+1} \rightarrow \mathbb{R} \) as follows:

\[
H(Z, \pi) := \int_{\Omega} Z(\omega) \cdot CVaR_{\alpha} (\mathcal{Z}(\omega)) (C_{t+1:T} | x_{t+1}) \mathbb{P}(d\omega),
\]

where \( C_{t+1:T} \) is conditioned on \( x_{t+1} = f_t (\mathbf{x}, \mathbf{u}, d_t) (\omega) := A_t x_t + B_t u_t + E_t d_t (\omega) \) in the integral above. Then, the following properties hold:

1) \( H(Z, \cdot) \) is convex and continuous in the norm topology on \( \Pi_{t+1} \) for any \( Z \in R^g_{\alpha} (\mathbf{x}, \mathbf{u}) \).
2) \( H(\cdot, \pi) \) is Lipschitz continuous in the relative norm topology on \( R^g_{\alpha} (\mathbf{x}, \mathbf{u}) \subset L^2 := L^2 (\Omega, \sigma(f_t (\mathbf{x}, \mathbf{u}, d_t)), \mathbb{P}) \) for any \( \pi \in \Pi_{t+1} \).

Proof: \( C_{t+1:T} \) is convex on \( \Pi_{t+1} \) since \( C_{t+1:T} \) is a convex function, the state trajectory is affine on \( \Pi_{t+1} \), and the composition of a convex function with an affine function is convex. Further, \( H(Z, \cdot) \) is convex on \( \Pi_{t+1} \) for any \( Z \in R^g_{\alpha} (\mathbf{x}, \mathbf{u}) \) because \( C_{t+1:T} \) is convex on \( \Pi_{t+1} \). Conditional Value-at-Risk is a convex functional, \( Z(\omega) \geq 0 \) for almost every \( \omega \in \Omega \), and expectation is a linear functional.

Define the product norm on \( \Pi_{t+1} \) as the maximum uniform norm, i.e., \( ||\pi|| := \max (||\mu_k||_u : k = t + 1, \ldots, T - 1) \). Take \( \{\pi^n\} \) in \( \Pi_{t+1} \) converging to \( \pi \in \Pi_{t+1} \). Fix \( r > 0 \), define \( \gamma_n := \frac{||\pi - \pi^n||_u}{r + ||\pi - \pi^n||_u} \), and define \( \tilde{\pi}^n \) in \( \Pi_{t+1} \) so that \( \pi = \gamma_n \tilde{\pi}^n + (1 - \gamma_n) \pi^n \). Fix \( Z \in R^g_{\alpha} (\mathbf{x}, \mathbf{u}) \). Since \( H(Z, \cdot) \) is convex on \( \Pi_{t+1} \),

\[
H(Z, \pi) \leq \gamma_n H(Z, \tilde{\pi}^n) + (1 - \gamma_n) H(Z, \pi^n).
\]

As \( n \rightarrow \infty \), \( \gamma_n H(Z, \tilde{\pi}^n) \rightarrow 0 \), since \( H(Z, \cdot) \) is bounded and \( \gamma_n \rightarrow 0 \). So, \( H(Z, \pi) \leq \lim inf H(Z, \pi^n) \), which shows lower semi-continuity. To show upper semi-continuity, i.e., \( \lim sup \ H(Z, \pi^n) \leq H(Z, \pi) \), define \( \tilde{\pi}^n \) in \( \Pi_{t+1} \) so that \( \pi^n = \gamma_n \tilde{\pi}^n + (1 - \gamma_n) \pi \), use convexity to obtain

\[
H(Z, \pi^n) \leq \gamma_n H(Z, \tilde{\pi}^n) + (1 - \gamma_n) H(Z, \pi),
\]

and take the limit superior as \( n \rightarrow \infty \).

Now, fix \( \pi \in \Pi_{t+1} \) and \( \omega \in \Omega \). Use the definition of CVaR (1) as an integral over VaR if \( \alpha Z(\omega) > 0 \) and as an essential supremum if \( \alpha Z(\omega) = 0 \) to show that \( Z(\omega) \rightarrow Z(\omega) \cdot CVaR_{\alpha} (\mathcal{Z}(\omega)) (C_{t+1:T} | x_{t+1}) = f_t (\mathbf{x}, \mathbf{u}, d_t) (\omega) \) is Lipschitz continuous on \( \mathbb{R} \) for almost every \( \omega \in \Omega \); set the Lipschitz constant equal to the essential supremum of \( C_{t+1:T} \). Since \( \int_{\Omega} |Z(\omega)| \mathbb{P}(d\omega) := ||Z||_1 \leq ||Z||_2 \) for any \( \Omega \in R^g_{\alpha} (\mathbf{x}, \mathbf{u}) \), \( H(\cdot, \pi) \) is Lipschitz continuous in the relative norm topology on \( R^g_{\alpha} (\mathbf{x}, \mathbf{u}) \subset L^2 \) for any \( \pi \in \Pi_{t+1} \).
$x_0 = [0, 1]^T$

Fig. 4. Histograms of $C_{0:T} = \sum_{t=0}^{T} \max (x_{1t} - K_1, x_{2t} - K_2, 0)$ for the stormwater system initialized at $x_0 = [0 \text{ ft}, 1 \text{ ft}]^T$ are shown under different policies. The system has limited control authority, so when using the CVaR criterion for policy synthesis, the empirical statistics (CVaR$_{0.01}$, Var$_{0.01}$, mean, variance) are only slightly reduced under a worst-case perspective versus a risk-neutral perspective (top row, right vs. left). Despite this limited control authority, however, it is remarkable that similar reductions can be attained by using the CVaR criterion for policy synthesis at a nearly worst case ($\theta$ fixed) with respect to the CVaR criterion $J^*_{\alpha, \theta}$ for a fixed $\alpha$. Bottom row: A policy was synthesized with respect to the exponential distribution criterion $V^*_{\alpha, \theta}$ for a fixed $\theta$. Left column: nearly risk neutral ($\alpha, \theta = (0.99, -0.01)$). Center column: typical risk aversion ($\alpha, \theta = (0.05, -0.95)$). Right column: nearly worst case ($\alpha, \theta = (0.01, -0.99)$). One million samples of $C_{0:T}$ are shown in each histogram, and relevant empirical statistics of these samples are displayed. The red circle marks the empirical mean, and each green circle marks the empirical mean plus/minus one/two standard deviations. The same pseudorandom sequence of disturbance realizations was used for each histogram.

Fig. 5. Probability distributions for the disturbance of the thermostatically controlled load (TCL) system are shown. The horizontal axis shows their associated probabilities are shown on the vertical axis. We trained the possible (discrete) realizations of the disturbance in degrees Celsius. Domains and variables have bounded second moments. So, $R^\alpha_t(x, u)$ is bounded in $|| \cdot ||_2$, $R^\alpha_t(x, u)$ is a convex subset of $L^2$ by inspection. Since only bounded costs are evaluated in our setting, for any $\alpha \in [0, 1]$, CVaR$_\alpha$ is a real-valued coherent risk measure on $L^2$. Moreover, for any $\alpha \in [0, 1]$ and $Y \in L^2$ such that $Y$ is bounded, the following equality holds by [22, Thm. 6.4, Eqn. 6.40, Eqn. 6.70]: CVaR$_\alpha(Y) = \sup_{Z \in R^\alpha_t(x, u)} \int \mathbb{E}_t Z(\omega) Y(\omega) \mathbb{P}(d\omega)$. So, $R^\alpha_t(x, u)$ is closed in the weak* topology on $L^2$ by [22, Thm. 6.6, p. 264]. Finally, a bounded and weakly* closed set is weakly* compact by the Banach-Alaoglu Theorem [22, Thm. 7.70, p. 401]. Thus, $R^\alpha_t(x, u)$ is compact in the weak* topology on $L^2$. Since $L^2$ is reflexive, the weak and the weak* topologies on $L^2$ coincide, hence $R^\alpha_t(x, u)$ is compact in the weak topology on $L^2$.

Remark 8: $L^2$ endowed with the weak topology is a locally convex topological vector space, since the weak topology in a topological vector space is locally convex [33, p. 161].

APPENDIX II

Here we provide three results that facilitate the synthesis of optimal pre-commitment policies (Section V). Lemmas 6 and 7 are required to prove the existence of $\mathbb{X}$-Markov control laws. Lemma 8 is critical for ensuring that the risk-sensitivity level...
dynamics are well-defined. These results require the following definition repeated from (19) for convenience:

\[ G_t^\alpha(Z, x, u) := \int Z(\omega) \cdot J_{t+1}^\alpha(f_t(x, u, d_t(\omega)), \alpha Z(\omega))\mathcal{P}(d\omega), \]

where \((x, \alpha) \in X, u \in U, Z \in \mathcal{R}_T^I(x, u), f_t(x, u, d_t(\omega)) := A_t x + B_t u + E_t d_t(\omega),\) and \((\Omega, \mathcal{F}, \mathcal{P})\) is the probability space upon which the random disturbance \(d_t\) is defined.

**Lemma 6** \((G_t^\alpha)\) is LSC in \((Z, u))\): For any \((x, \alpha) \in X, G_t^\alpha(Z, x, u)\) is lower semi-continuous in \((Z, u).\)

**Proof:** Fix \(t\) and \((x_t, \alpha_t) = (x, \alpha) \in X). Since \(u \mapsto f_t(x, u, d_t(\omega))\) is continuous for almost every \(\omega \in \Omega\) by Assumption 1, \((Z(\omega), u) \mapsto (f_t(x, u, d_t(\omega)), \alpha Z(\omega))\) is continuous for almost every \(\omega \in \Omega\). If \(J_{t+1}^\alpha\) is lower semi-continuous on \(X,\) then \((Z(\omega), u) \mapsto J_{t+1}^\alpha(f_t(x, u, d_t(\omega)), \alpha Z(\omega))\) would be lower semi-continuous for almost every \(\omega \in \Omega,\) since a lower semi-continuous function composed with a continuous function is lower semi-continuous. Then, \((Z(\omega), u) \mapsto Z(\omega) \cdot J_{t+1}^\alpha(f_t(x, u, d_t(\omega)), \alpha Z(\omega))\) would be lower semi-continuous for almost every \(\omega \in \Omega,\) since \(Z(\omega)\) is nonnegative and bounded for almost every \(\omega \in \Omega.\) (The product of a nonnegative, continuous bounded function and a lower semi-continuous bounded function is lower semi-continuous.) Since expectation preserves lower semi-continuity, \(G_t^\alpha(Z, x, u)\) would be lower semi-continuous in \((Z, u)\) to complete the proof. So, it suffices to show that \(J_{t+1}^\alpha \) is lower semi-continuous on \(X,\) which is shown in Lemma 7 below.

**Lemma 7** (Lower Semi-Continuity of \(J_t^\alpha\)): Fix \(t. J_t^\alpha,\) as defined in (13a), is lower semi-continuous on \(X := X \times [0, 1].\)

**Proof:** Recall that each \(J_t^\alpha\) is bounded since the stage costs and the terminal cost are bounded. It suffices to show that \(J_t^\alpha,\) as defined in (11), is lower semi-continuous on \(X\) by Theorem 2. We proceed by induction. Let \((x, \alpha) \in X, J_t^\alpha(x, \alpha) = cr(x),\) which is continuous. Now, assume \(J_{t+1}^\alpha\) is lower semi-continuous on \(X\) for some \(t.\) Since \(U\) is compact, \(c_t\) is continuous, and lower semi-continuity is preserved through summation, it suffices to show that the following (bounded) map \((x, \alpha, u) \mapsto \sup_{Z \in \mathcal{R}_T^I(x, u)} G_t^\alpha(Z, x, u)\) is lower semi-continuous on \(X \times U\) by [32, Prop. D.5 (b)], where \(G_t^\alpha(Z, x, u) := \int_0^1 Z(\omega) \cdot J_{t+1}^\alpha(f_t(x, u, d_t(\omega)), \alpha Z(\omega))d\omega\) by Theorem 2. Since \((x, \alpha, u) \mapsto \mathcal{R}_T^I(x, u)\) is a lower semi-continuous set-valued mapping by Assumption 2, if \(G_t^\alpha(Z, x, u)\) is lower semi-continuous in \((x, \alpha, u),\) then the desired result holds by [38, Lemma 3.2 (a)]. Since \((x, u) \mapsto f_t(x, u, d_t(\omega))\) is continuous for almost every \(\omega \in \Omega\) by Assumption 1, \((Z(\omega), x, \alpha, u) \mapsto (f_t(x, u, d_t(\omega)), \alpha Z(\omega))\) is continuous for almost every \(\omega \in \Omega.\) Since \(J_{t+1}^\alpha\) is lower semi-continuous on \(X\) by the induction hypothesis, \((Z(\omega), x, \alpha, u) \mapsto J_{t+1}^\alpha(f_t(x, u, d_t(\omega)), \alpha Z(\omega))\) is lower semi-continuous for
and bounded for a.e. \( \omega \) almost every continuity, the proof is complete.


deviations.

in each histogram, and relevant empirical statistics of these samples are displayed. The red circle marks the empirical mean, and each green circle marks the empirical mean plus/minus one/two standard deviations.

are shown. The red circle marks the empirical mean, and each green circle marks the empirical mean plus/minus one/two standard deviations.

almost every \( \omega \in \Omega \). Moreover, since \( Z(\omega) \) is nonnegative and bounded for a.e. \( \omega \in \Omega \), \( Z(\omega), x, \alpha, u \mapsto Z(\omega) \cdot J_{t+1}^\alpha(f_t(x, u, d_t(\omega)), \alpha Z(\omega)) \) is lower semi-continuous for almost every \( \omega \in \Omega \). Since expectation preserves lower semi-continuity, the proof is complete.

**Lemma 8** (\( G_t^\alpha \) is USC on Risk Envelope): Fix \( t, x_t = x, u_t = u \), and \( \alpha_t = \alpha \). \( G_t^\alpha(\cdot, x; u) \), as defined by (19), is upper semi-continuous in the relative weak topology on \( R_t^\alpha(x, u) \subset L^2(\Omega, \sigma(f_t(x, u, d_t(\omega))) \mathbb{P}) \).

**Proof:** It suffices to show that \( G_t^\alpha(\cdot, x; u) \) is real-valued concave and upper semi-continuous in the relative norm topology on \( R_t^\alpha(x, u) \subset L^2 \) by [35, Prop. 2.10]. \( G_t^\alpha(\cdot, x; u) \) is real-valued, since \( Z \in R_t^\alpha(x, u) \) and \( J_{t+1}^\alpha \) are bounded. \( G_t^\alpha(\cdot, x; u) \) is concave on \( R_t^\alpha(x, u) \) by [25, Thm. 12] applied to a random cost, since the pointwise infimum of concave functions is concave, by the definition of \( J_{t+1}^\alpha \), and by linearity of expectation. It suffices to show that \( G_t^\alpha(\cdot, x; u) \) is Lipschitz continuous in the relative norm topology on \( R_t^\alpha(x, u) \subset L^2 \), as this implies upper semi-continuous in this topology.

One can show that for any \( \pi \in \Pi_{t+1}, Z(\omega) \mapsto Z(\omega) \cdot \text{CVaR}_{\alpha}^\pi Z(\omega)(C_{t+1}^\alpha|x_{t+1} = f_t(x, u, d_t(\omega))) \) is Lipschitz continuous on \( \mathbb{R} \) for almost every \( \omega \in \Omega \). It follows (after a few steps) that \( Z(\omega) \mapsto Z(\omega) \cdot J_{t+1}^\alpha(f_t(x, u, d_t(\omega)), \alpha Z(\omega)) \) is Lipschitz continuous on \( \mathbb{R} \) for almost every \( \omega \in \Omega \), by using the definition of \( J_{t+1}^\alpha \) as an infimum over the policy space \( \Pi_{t+1} \). Then, take the expectation with respect to \( \mathbb{P} \), and use \( |E(\cdot)| \leq E(|\cdot|) \leq ||\cdot||_2 \) to complete the proof.

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