Homotopy Theory

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Contents

0 Introduction

These notes are taken from a course in homotopy theory taught by Dr. Nicholas Kuhn at the University of Virginia in the fall of 2017. The main topics covered include:

- Mapping spaces and their topologies
- Cofibrations and fibrations
- Puppe sequences, Verdier's lemmas
- The first Whitehead theorem
- $\lim^1$ and mapping telescopes
- Standard theorems in homotopy theory, including:
  - Blakers-Massey theorem
  - Hurewicz theorem
  - the second Whitehead theorem
  - Freudenthal suspension theorem
- Brown representability and Eilenberg-MacLane spaces
- Steenrod operations
- The Serre spectral sequence.

The companion texts for the course are May's *A Concise Course in Algebraic Topology* and May-Ponto's follow-up *More Concise Algebraic Topology.*

Recall that two maps $f, g : X \to Y$ between topological spaces are said to be *homotopic*, written $f \simeq g$, if there exists a continuous map $H : X \times I \to Y$ (where $I = [0, 1]$) such that $H_0 = f$ and $H_1 = g$. Then $\simeq$ is an equivalence relation and we denote the set of equivalence classes of maps $X \to Y$ under $\simeq$ by $[X, Y]$.

Lemma 0.0.1. *If $W \xrightarrow{e} X \xrightarrow{f,g} Y \xrightarrow{Z}$ are maps and $f \simeq g$, then $h \circ f \simeq h \circ g$ and $f \circ e \simeq g \circ e$.*

This lemma implies that we may form the *homotopy category* $h(\texttt{Top})$ whose objects are topological spaces and whose Hom sets are $[X, Y]$. Similarly, one can define $h(\texttt{Top}_*)$ from the category $\texttt{Top}_*$ of pointed topological spaces; the constructions make sense for (reasonable) subcategories of $\texttt{Top}$ and $\texttt{Top}_*$. Homotopy theory studies these and related categories using algebraic topology. For example, the Homotopy Axiom for a homology theory H says that each H_n factors through $h(\texttt{Top})$:

$$\begin{tikzcd} \texttt{Top} \arrow[rr, "H_n"] \arrow[dr] & & \texttt{AbGps} \\ & h(\texttt{Top}) \arrow[ur] & \end{tikzcd}$$

Definition. *A* **homotopy functor** *on the category of topological spaces is a functor $T : \texttt{Top} \to \texttt{Sets}$ that factors through the homotopy category $h(\texttt{Top})$.*

Two of the basic problems studied in homotopy theory are:

(a) For a space X, study the functor $[X, -] : h(\texttt{Top}) \to \texttt{Sets}$. For example, setting $X = S^n$ and studying $[S^n, -]$ is basically the study of homotopy groups $\pi_n(-) = [S^n, -]_*$.

(b) For a space Y, study the contravariant functor $[-, Y] : h(\texttt{Top}) \to \texttt{Sets}$. For example, $H^2(X; \mathbb{Z}) \cong [X, \mathbb{C}P^\infty]$. Another important example is if $\mathrm{Vect}_n(X)$ is the set of isomorphism classes of n-dimensional vector bundles over X, then there is a space $BO(n)$ (called a classifying space) such that $\mathrm{Vect}_n(X) \cong [X, BO(n)]$.

Homotopy theory also provides methods of calculation that are useful in different areas of topology, including:

- If a homotopy functor $T : h(\texttt{Top}) \to \texttt{Sets}$ actually takes values in a category with more algebraic structure, such as $\texttt{AbGps}, \texttt{Rings}, \texttt{Vec}_k$, then we can apply algebraic techniques to study T.
- Long exact sequences allow for efficient computation.
- "Local-to-global" properties allow one to study a characteristic of a space X by studying it on simpler subspaces. For example, the Mayer-Vietoris sequence and van Kampen's theorem exhibit this type of property. They have a common generalization in the Blakers-Massey theorem.
- Stable invariants play an important role in homotopy theory. For a simple example, recall that if ΣX is the suspension of X, then $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(\Sigma X)$.

0.1 Some Point-Set Topology

In this section, we review two important concepts from general topology: the compact-open topology and the category of compactly generated spaces.

Suppose X and Y are topological spaces and define

$$\begin{aligned} Y^X &= \{f : X \to Y\} \\ C(X, Y) &= \{f : X \to Y \mid f \text{ is continuous}\}. \end{aligned}$$

The following construction, due to Ralph Fox (1947), puts a topology on $C(X, Y)$ with certain nice formal properties which we will explain afterward.

Definition. *Suppose $C \subseteq X$ is compact and $U \subseteq Y$ is open. Define*

$$\langle C, U \rangle = \{f \in C(X, Y) \mid f(C) \subseteq U\}.$$

Then the **compact-open topology** *on $C(X, Y)$ is defined to be the topology generated by the subbasis $\{\langle C, U \rangle \mid C \subseteq X \text{ compact}, U \subseteq Y \text{ open}\}$. We denote the resulting topological space by* $\mathrm{Map}(X, Y)$.

Example 0.1.1. If X is a discrete space, $\mathrm{Map}(X,Y) = Y^X$ since all functions are continuous. In particular, $\mathrm{Map}(X,Y) \cong \prod_{x\in X} Y$ with the usual product topology.

Suppose X, Y and Z are sets and $F : X \times Y \to Z$ is a function. This induces a function $\widehat{F} : X \to Z^Y$, $\widehat{F}(x)(y) = F(x,y)$.

Theorem 0.1.2. *For any X, Y, Z, the assignment $F \mapsto \widehat{F}$ induces a bijection $Z^{X\times Y} \cong (Z^Y)^X$.*

If X, Y, Z are topological spaces, we can ask about the subsets $\mathrm{Map}(X \times Y, Z) \subseteq Z^{X\times Y}$ and $\mathrm{Map}(X, \mathrm{Map}(Y,Z)) \subseteq (Z^Y)^X$. In particular, we can ask:

(i) if the map $Z^{X\times Y} \to (Z^Y)^X$ (which is a bijection) induces a map $\mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y,Z))$ – if so, it is injective of course;

(ii) whether this map is also a bijection;

(iii) whether the map is continuous;

(iv) and if so, whether it is also a homeomorphism.

Lemma 0.1.3. *If $F : X \times Y \to Z$ is continuous, then for each $x \in X$, $\widehat{F}(x) : Y \to Z$ is continuous.*

Proof. $\widehat{F}(x)$ is the composition $Y \xhookrightarrow{i_x} X \times Y \xrightarrow{F} Z$, where $i_x : y \mapsto (x,y)$. □

Proposition 0.1.4. *If $F : X \times Y \to Z$ is continuous, then $\widehat{F}$ is continuous. In particular, the assignment $F \mapsto \widehat{F}$ restricts to an injection* $\mathrm{Map}(X \times Y, Z) \hookrightarrow \mathrm{Map}(X, \mathrm{Map}(Y,Z))$.

Proof. To show $\widehat{F} : X \to \mathrm{Map}(Y,Z)$ is continuous, i.e. that open sets pull back to open sets, it suffices to check this on a subbasis. In other words, it suffices to show if $C \subseteq X$ is compact and $U \subseteq Y$ is open, then $\widehat{F}^{-1}(\langle C, U\rangle)$ is open in X. First note that

$$x \in \widehat{F}^{-1}(\langle C, U\rangle) \iff \{x\} \times C \subseteq F^{-1}(U).$$

Since C is compact, the tube lemma implies that there is an open neighborhood V of x in X such that $V \times C \subseteq F^{-1}(U)$. That is, $V \subseteq \widehat{F}^{-1}(\langle C, U\rangle)$. Therefore $\widehat{F}^{-1}(\langle C, U\rangle)$ is open. □

It turns out that the converse to this statement, i.e. the surjectivity of the map $\mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y,Z))$, only holds with certain extra conditions. For any spaces Y and Z, let $\varepsilon_{Y,Z} : \mathrm{Map}(Y,Z) \times Y \to Z$ be the *evaluation map*, $\varepsilon_{Y,Z}(f,y) = f(y)$. Then it's easy to check that $\widehat{\varepsilon}_{Y,Z}$ is the identity on $\mathrm{Map}(Y,Z)$.

Lemma 0.1.5. *Suppose $\varepsilon_{Y,Z}$ is continuous. Then if $\widehat{F} : X \to \mathrm{Map}(Y,Z)$ is continuous, so is $F : X \times Y \to Z$.*

Proof. F is the composition $X \times Y \xrightarrow{\widehat{F}\times id_Y} \mathrm{Map}(Y,Z) \times Y \xrightarrow{\varepsilon_{Y,Z}} Z$ and each piece is continuous. Therefore F is continuous. □

Proposition 0.1.6. *If Y is locally compact and Hausdorff, then for all spaces Z, $\varepsilon_{Y,Z}$ is continuous.*

Corollary 0.1.7. *If Y is locally compact and Hausdorff, then the map* $\mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z)), F \mapsto \widehat{F}$ *is a bijection.*

In other words, the functors $- \times Y$ and $\mathrm{Map}(Y, -)$ are an adjoint pair from the category `Top` to itself. Similar proofs to those above give the following easy results using the evaluation map.

Lemma 0.1.8. *If $\varepsilon_{X\times Y,Z}$ is continuous, then* $\mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z))$ *is continuous.*

Lemma 0.1.9. *If $\varepsilon_{X,\mathrm{Map}(Y,Z)}$ is continuous, then the inverse of $Map(X\times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z))$ is also continuous.*

Lemma 0.1.10. *If $\varepsilon_{X,Y}$ and $\varepsilon_{X,Z}$ are both continuous, then*

$$\begin{aligned} \mathrm{Map}(Y, Z) \times \mathrm{Map}(X, Y) &\longrightarrow \mathrm{Map}(X, Z) \\ (f, g) &\longmapsto f \circ g \end{aligned}$$

is continuous.

Lemma 0.1.11. *If $\varepsilon_{X,Y}$ and $\varepsilon_{X,Z}$ are continuous, then* $\mathrm{Map}(X, Y \times Z) \to \mathrm{Map}(X, Y) \times \mathrm{Map}(X, Z)$ *is a homeomorphism.*

Corollary 0.1.12. *If X and Y are locally compact, Hausdorff spaces, then the map* $\mathrm{Map}(X\times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z))$ *is a homeomorphism.*

Remark. To explain the prominence of the evaluation maps in these proofs about $\mathrm{Map}(X, Y)$, one may prove that the compact-open topology is the coarsest topology on $C(X, Y)$ for which the evaluation $\varepsilon_{X,Y} : C(X, Y) \times X \to Y$ is a continuous map.

In 1967, Steenrod introduced the following class of topological spaces.

Definition. *A space X is* **compactly generated** *if X is Hausdorff and a subset $C \subseteq X$ is closed if and only if $C \cap K$ is closed in K for all compact subspaces $K \subseteq X$.*

Example 0.1.13. Every locally compact, Hausdorff space is compactly generated.

Example 0.1.14. If X is metrizable, then X is compactly generated.

Proposition 0.1.15. *Let X and Y be Hausdorff spaces. Then*

(a) If X is compactly generated, then a map $f : X \to Y$ is continuous if and only if $f|_K$ is continuous for all $K \subseteq X$ compact.

(b) If $q : X \to Y$ is a quotient map and X is compactly generated, then Y is compactly generated.

(c) *Suppose $X = \bigcup X_\alpha$ has the coherent (direct limit) topology, i.e. $C \subseteq X$ is closed if and only if $C \cap X_\alpha$ is closed in X_α for all α. Then if each X_α is compactly generated, so is X.*

Remark. There are some inherent failures in restricting to the class of compactly generated spaces. For example:

- If X and Y are compactly generated, then $X \times Y$ is not always compactly generated. However, if X is compactly generated and Y is locally compact and Hausdorff, then $X \times Y$ is compactly generated.

- If X is compactly generated and $A \subseteq X$ is a subspace, then A need not be compactly generated. For example, let $X = \mathbb{R}$ and $A = \mathbb{Q} \subseteq \mathbb{R}$ with the subspace topology. Then $\mathbb{R}$ is locally compact and Hausdorff and therefore compactly generated by Example 0.1.13, but $\mathbb{Q}$ is not compactly generated.

- Even if X and Y are compactly generated, $\mathrm{Map}(X, Y)$ is not always compactly generated.

Let $\mathtt{HTop}$ be the category of Hausdorff topological spaces and let $\mathcal{K}$ be the category of compactly generated spaces. Steenrod defines the following functor between these categories.

Definition. *The* **k-functor** *is the assignment $k : \mathtt{HTop} \to \mathcal{K}$ sending X to $k(X) = X$ as a set with the topology given by declaring $C \subseteq X$ to be closed if and only if $C \cap K$ is closed in K for all compact subspaces $K \subseteq X$.*

Lemma 0.1.16. *For any Hausdorff topological space X, $k(X)$ is compactly generated.*

Lemma 0.1.17. *For all Hausdorff X, the identity on X induces a continuous map $X \to k(X)$.*

Theorem 0.1.18. *The k-functor $k : \mathtt{HTop} \to \mathcal{K}$ has a left adjoint given by the inclusion $\mathcal{K} \hookrightarrow \mathtt{HTop}$.*

To remedy the failures in the above remark, we make the following definitions standard for spaces in $\mathcal{K}$:

- For a subset $A \subseteq X$ where $X \in \mathcal{K}$, we regard A as a subspace of X by viewing $A = k(A)$. Thus a subspace of a compactly generated space does not in general have the subspace topology.

- Products in $\mathcal{K}$ are given by $X \times_{\mathcal{K}} Y = k(X \times Y)$. In the future we will simply write $X \times Y$.

- The morphisms $\mathrm{Hom}(X, Y) = \mathrm{Map}(X, Y)$ do not in general form a compactly generated space, so we set $\mathrm{Hom}_{\mathcal{K}}(X, Y) = k(\mathrm{Hom}(X, Y)$. In the future we will denote this by $\mathrm{Map}(X, Y)$.

Proposition 0.1.19. *For all compactly generated spaces X and Y, the evaluation map $\varepsilon_{X,Y} : \mathrm{Map}(X, Y) \times X \to Y$ is continuous.*

In particular, the previous results for $\varepsilon_{X,Y}$ hold in $\mathcal{K}$, notably the homeomorphism

$$\mathrm{Map}(X \times Y, Z) \xrightarrow{\sim} \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

from Corollary 0.1.12.

Remark. Since $[0,1]$ and any n-simplex Δ_n are compactly generated, for any space X,

- $X \times [0,1]$ is compactly generated if X is compactly generated;
- $\mathrm{Map}(\Delta_n, X) = \mathrm{Map}(\Delta_n, k(X))$.

Therefore we do not lose any information with the homotopy functors $H_\bullet$ or $[Y, -]$, since $H_\bullet(k(X)) = H_\bullet(X)$ and $[Y, k(X)] = [Y, X]$.

Example 0.1.20. Let $I = [0,1]$ and $X, Y \in \mathcal{K}$. Then $\mathrm{Map}(I \times X, Y) = [X, Y]$ and $\mathrm{Map}(I, \mathrm{Map}(X, Y))$ coincides with the set of path components of $\mathrm{Map}(X, Y)$. So the bijection in Corollary 0.1.7 identifies $[X, Y] = \pi_0(\mathrm{Map}(X, Y))$.

0.2 Based Categories

Let $\mathtt{Top}_*$ be the category of *based spaces*, with objects (X, x_0) where X is a topological space and $x_0 \in X$ is a point, and with based maps as morphisms:

$$\mathrm{Map}_*(X, Y) = \{f \in \mathrm{Map}(X, Y) \mid f(x_0) = y_0\}.$$

Note that $\mathrm{Map}_*(X, Y)$ is itself based: the constant map $X \to Y$ sending every $x \in X$ to y_0 is continuous. We say two based maps $f, g : (X, x_0) \to (Y, y_0)$ are *based homotopic* if there exists a continuous map $H : X \times I \to Y$ such that $H_0 = f, H_1 = g$ and $H_t(x_0) = y_0$ for all $t \in I$. Denote by $[X, Y]_*$ the set of based homotopy equivalence classes of based maps $f : X \to Y$.

Remark. The category of based spaces is related to the ordinary category of topological spaces as follows. For any space X, let $X_+ = X \coprod *$ where $*$ is a basepoint disjoint from X that is both open and closed in X_+. Then for any based space Y, $[X_+, Y]_* = [X, Y]$. Further, the natural map $\pi : X_+ \to X$ induces a morphism $[X, Y]_* \to [X_+, Y], f \mapsto f \circ \pi$ whenever X, Y are both based.

Lemma 0.2.1. *For any topological spaces X, Y and points $x_0 \in X, y_0 \in Y$, there is a pullback diagram*

$$\begin{tikzcd} \mathrm{Map}_*(X, Y) \arrow[r] \arrow[d] & \mathrm{Map}(X, Y) \arrow[d, "e_{x_0}"] \\ \{y_0\} \arrow[r] & Y \end{tikzcd}$$

where e_{x_0} is the evaluation map $f \mapsto f(x_0)$.

Definition. *For a collection of based spaces* $\{(X_\alpha, x_\alpha)\}$*, their* **wedge product** *is the quotient space*

$$\bigvee X_\alpha = \left(\coprod X_\alpha\right)/(x_\alpha \sim x_\beta).$$

Lemma 0.2.2. *Wedge product is a pushout in* `Top`*:*

$$\begin{tikzcd} \coprod\{x_\alpha\} \arrow[r] \arrow[d] & \coprod X_\alpha \arrow[d] \\ \{*\} \arrow[r] & \bigvee X_\alpha \end{tikzcd}$$

The notion of products in the based category $\mathtt{Top}_*$ is defined as follows.

Definition. *The* **smash product** *of two based spaces* $X, Y \in \mathtt{Top}_*$ *is the space*

$$X \wedge Y = (X \times Y)/(X \vee Y).$$

In analogy with Corollary 0.1.7, we have:

Proposition 0.2.3. *For any based spaces* X, Y *and* Z*, the natural map*

$$\mathrm{Map}_*(X \wedge Y, Z) \longrightarrow \mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z)), \quad F \mapsto \widehat{F}$$

is a bijection.

View the interval $I = [0, 1]$ as a based space with basepoint 0. Using previous notation, let I_+ be the interval with a disjoint basepoint $*$.

Definition. *Let* $f, g : X \to Y$ *be two based maps. A* **based homotopy** *from* f *to* g *is a based map* $H : X \wedge I_+ \to Y$ *satisfying* $H_0 = f$ *and* $H_1 = g$*.*

By Proposition 0.2.3, a based homotopy is equivalent to a map $\widehat{H} : X \to \mathrm{Map}(I_+, Y)$. Each of these definitions coincides with the 'usual' notion of based homotopy, i.e. a continuous map $H : X \times I \to Y$ with $H_0 = f, H_1 = g$ and $H_t(x_0) = y_0$ for every t.

Definition. *The* **cone** *on a based space* X *is the smash product* $CX = X \wedge I$*.*

Definition. *The* **path space** *on a based space* X *is the space* $PX = \mathrm{Map}_*(I, X)$*.*

Corollary 0.2.4. *For any based spaces* X, Y*, there is a bijection* $\mathrm{Map}_*(CX, Y) = \mathrm{Map}_*(X, PY)$*.*

Proof. Apply Proposition 0.2.3. □

Definition. *For a based space* X*, the* **suspension** *of* X *is the smash product* $\Sigma X = X \wedge S^1$*.*

Definition. *For a based space* X*, the* **loop space** *of* X *is the space* $\Omega X = \mathrm{Map}_*(S^1, X)$*.*

Corollary 0.2.5. *For any based spaces* X, Y*, there is a bijection* $\mathrm{Map}_*(\Sigma X, Y) = \mathrm{Map}_*(X, \Omega Y)$*.*

Proof. Apply Proposition 0.2.3. □

Lemma 0.2.6. *Let X be a based space. Then*

(a) The suspension ΣX is a pushout:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X \end{tikzcd}$$

(b) Dually, the loop space ΩX is a pullback:

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & PX \arrow[d] \\ PX \arrow[r] & X \end{tikzcd}$$

0.3 H-Spaces and Co-H-Spaces

Suppose S is a set. We say S has a *product with unit* if there is a basepoint $e \in S$ and a set map $m : S \times S \to S$ such that $m(s, e) = s = m(e, s)$ for all $s \in S$. We describe the following situations in $\texttt{Top}_*$:

(1) For fixed Y, when does $[X, Y]$ have a natural product with unit?

(2) For fixed X, when does $[X, Y]$ have a natural product with unit?

By natural, we mean something different in each case. For (1), naturality says that for any map $f : X' \to X$, there is a commutative diagram

$$\begin{tikzcd} {[X,Y] \times [X,Y]} \arrow[r] \arrow[d, "f^* \times f^*"'] & {[X,Y]} \arrow[d, "f^*"] \\ {[X',Y] \times [X',Y]} \arrow[r] & {[X',Y]} \end{tikzcd}$$

where $f^*(h) = h \circ f$ for any $h : X \to Y$. On the other hand, naturality in situation (2) means for any map $g : Y \to Y'$, there is a commutative diagram

$$\begin{tikzcd} {[X,Y] \times [X,Y]} \arrow[r] \arrow[d, "g_* \times g_*"'] & {[X,Y]} \arrow[d, "g_*"] \\ {[X,Y'] \times [X,Y']} \arrow[r] & {[X,Y']} \end{tikzcd}$$

where $g_*(h) = g \circ h$ for any $h : X \to Y$.

Example 0.3.1. If Y is a topological group, then every $[X, Y]$ has a product given by $(fg)(x) = f(x)g(x)$ for all $f, g : X \to Y$. This is an example of situation (1).

Example 0.3.2. Similarly, when $Y = \Omega Z$ is a loop space, $[X, \Omega Z]$ has a product given by concatenation of loops: if $\alpha, \beta : X \to \Omega Z$ then $(\alpha\beta)(x) = \alpha * \beta(x)$ where

$$\alpha * \beta(x,t) = \begin{cases} \alpha(x, 2t), & 0 \le t \le \frac{1}{2} \\ \beta(x, 2t-1), & \frac{1}{2} < t \le 1. \end{cases}$$

Fix Y and suppose that for all X, there exists a natural product with unit $m_X : [X, Y] \times [X, Y] \to [X, Y]$. Then for each X, there is a canonical identification $[X, Y \times Y] = [X, Y] \times [X, Y]$. Define m'_X to be the resulting map $[X, Y \times Y] \to [X, Y]$. For the specific case $X = Y \times Y$, the identity is a distinguished map $1_{Y\times Y} \in [Y \times Y, Y \times Y]$. We write $m = m'_{Y\times Y}(1_{Y\times Y}) \in [Y \times Y, Y]$.

Lemma 0.3.3. *For any X, the diagram*

$$\begin{tikzcd}
{[X, Y \times Y]} \arrow[dr, "m_*"] \arrow[dd, equal] & \\
& {[X, Y]} \\
{[X, Y] \times [X, Y]} \arrow[ur, "m_X"'] &
\end{tikzcd}$$

commutes.

Proof. For any $h : X \to Y \times Y$, we must show that $m_*(h) = m'_X(h)$. Consider the diagram

$$\begin{tikzcd}
{[X, Y \times Y]} \arrow[d, equal] \arrow[dr, "m_*"] & \\
{[X, Y] \times [X, Y]} \arrow[r, "m_X"'] & {[X, Y]} \\
{[Y \times Y, Y] \times [Y \times Y, Y]} \arrow[u, "h^* \times h^*"] \arrow[r, "m_{Y\times Y}"] \arrow[d, equal] & {[Y \times Y, Y]} \arrow[u, "h^*"'] \\
{[Y \times Y, Y \times Y]} \arrow[ur, "m'_{Y\times Y}"'] &
\end{tikzcd}$$

Then the bottom triangle commutes by definition of $m'_{Y\times Y}$ and the square commutes by naturality of m_X. Moreover, if we take $1_{Y\times Y} \in [Y \times Y, Y \times Y] = [Y \times Y, Y] \times [Y \times Y, Y]$ and apply $h^* \times h^*$, we get precisely $h \in [X, Y \times Y]$. Going around the diagram a different direction, $h^* \circ m'_{Y\times Y}(1_{Y\times Y}) = h^*(m) = m \circ h = m_*(h)$. So it follows that $m'_X(h) = m_*(h)$, i.e. the top triangle commutes. $\square$

The following result says that natural products with unit m_X on every $[X,Y]$ are completely determined by this product map $m : Y \times Y \to Y$.

Proposition 0.3.4. *Fix Y. Then $m_X : [X,Y] \times [X,Y] \to [X,Y]$ is a natural product with unit if and only if the diagram*

$$\begin{tikzcd} Y \vee Y \arrow[dd, hook] \arrow[dr, "\nabla"] & \\ & Y \\ Y \times Y \arrow[ur, "m"'] & \end{tikzcd}$$

commutes up to homotopy, where $\nabla : Y \vee Y \to Y$ is the 'fold map', i.e. the identity on each component.

Proof. ($\Longrightarrow$) If m_X is a product with unit, let $\pi_1, \pi_2 : Y \times Y \to Y$ be the two coordinate projections and $i_1, i_2 : Y \hookrightarrow Y \times Y$ the coordinate inclusions. Then by naturality,

$$m_X(\pi_1, \pi_2) \circ i_1 = m_X(\pi_1 \circ i_1, \pi_2 \circ i_1) = m_X(1, u) = 1$$

up to homotopy, where u is the unit of m_X. Likewise, $m_X(\pi_1, \pi_2) \circ i_2 = 1$ up to homotopy. It follows that $Y \vee Y \hookrightarrow Y \times Y \xrightarrow{m_X(\pi_1,\pi_2)} Y$ is homotopic to ∇.

($\Longleftarrow$) Conversely, suppose m makes the above diagram homotopy commute. Then the product on $[X,Y]$ can be written $m_X(h_1, h_2) = m \circ (h_1 \times h_2) \circ \Delta$ where $\Delta : x \mapsto (x,x)$ is the usual diagonal map on X. Then for any $h : X \to Y$ and the constant map $c : X \to, x \mapsto y_0$,

$$m_X(c,h)(x) = m \circ (c \times h) \circ \Delta(x) = m \circ (c(x), h(x)) = m \circ (y_0, h(x)) \simeq h(x)$$

by hypothesis. Similarly $m_X(h,c) = h$ so m_X is a product with unit. Naturality is a similar chase. $\square$

Definition. *We say Y is an* **H-space** *if there is a map $m : Y \times Y \to Y$ making the diagram*

$$\begin{tikzcd} Y \vee Y \arrow[dd, hook] \arrow[dr, "\nabla"] & \\ & Y \\ Y \times Y \arrow[ur, "m"'] & \end{tikzcd}$$

commute up to homotopy.

There is a dual notion:

Definition. *We say X is a* **co-H-space** *if there is a map $p : X \to X \vee X$ making the diagram*

$$\begin{tikzcd} & X \vee X \arrow[dd, hook] \\ X \arrow[ur, "p"] \arrow[dr, "\Delta"'] & \\ & X \times X \end{tikzcd}$$

commute up to homotopy.

Proposition 0.3.5. *Fix X. Then there is a natural product with unit $p^Y : [X,Y]\times[X,Y] \to [X,Y]$ for all Y if and only if X is a co-H-space.*

Proof. Dual to the proof of Proposition 0.3.4. □

Lemma 0.3.6. *For any space Z, the suspension ΣZ is a co-H-space.*

Proof. First, note that S^1 is a co-H-space by Proposition 0.3.5, since $[S^1, Y] = \pi_1(Y)$ admits a group structure which is natural in Y. More generally, for any X, Y, Z, we have a natural identification $[X \wedge Z, Y] \cong [X, \mathrm{Map}_*(Z, Y)]$ so it follows that if X is a co-H-space, then so is $X \wedge Z$. In particular, the case $X = S^1$ implies ΣZ is a co-H-space for all Z. □

Theorem 0.3.7. *Suppose that X is a co-H-space and Y is an H-space. Then the two associated products on $[X,Y]$ are, in fact, the same. Moreover, this product is associative and commutative.*

Proof. For any $h_1, h_2 : X \to Y$, let $h_1 \cdot h_2 = m_X(h_1, h_2)$ be their product in $[X,Y]$ coming from the H-space structure on Y and let $h_1 * h_2 = p^Y(h_1, h_2)$ be their product coming from the co-H-space structure on X. Consider the diagram

$$\begin{tikzcd}
X \arrow[r, "p"] \arrow[dr, "\Delta"'] & X \vee X \arrow[r, "h_1 \vee h_2"] \arrow[d] & Y \vee Y \arrow[d] \arrow[dr, "\nabla"] & \\
 & X \times X \arrow[r, "h_1 \times h_2"'] & Y \times Y \arrow[r, "m"'] & Y
\end{tikzcd}$$

By hypothesis, the triangles homotopy commute and of course the square commutes on the nose. However, from Proposition 0.3.4, the right triangle encodes the product on $[X,Y]$ via $h_1 \cdot h_2 = m \circ (h_1 \times h_2) \circ \Delta$, while the dual version in Proposition 0.3.5 shows that the left triangle gives $h_1 * h_2 = \nabla \circ (h_1 \vee h_2) \circ p$. Hence $h_1 \cdot h_2$ equals $h_1 * h_2$ up to homotopy.

To show the product is associative and commutative, we first show that it is a homomorphism of sets with product, i.e. for any $f, g, h, j \in [X,Y]$, $(fg)(hj) = (fh)(gj)$. This follows from the commutative diagram

$$\begin{tikzcd}
{[X,Y] \times [X,Y] \times [X,Y] \times [X,Y]} \arrow[r, "(p^Y \vee p^Y) \circ \mathrm{flip}"] \arrow[d, "m_X \times m_X"'] & {[X \vee X, Y] \times [X \vee X, Y]} \arrow[d, "m_X \vee m_X"] \\
{[X, Y \times Y] \times [X, Y \times Y]} \arrow[d, "p^{Y \times Y}"'] & {[X \vee X, Y]} \arrow[d, "p^*"] \\
{[X, Y \times Y]} \arrow[r, "m_*"'] & {[X,Y]}
\end{tikzcd}$$

Here, flip interchanges the second and third copies of $[X, Y]$. The diagram commutes by naturality of m_X, p^Y and the fact that the product structures on $[X, Y]$ are the same.

Finally, take $f, g, h \in [X, Y]$ and let e be the unit. Then by the above, $f(gh) = (fe)(gh) = (fg)(eh) = (fg)h$ and $fg = (ef)(ge) = (eg)(fe) = gf$. Therefore the product is associative and commutative. $\square$

Example 0.3.8. One can show (e.g. using the ring structures on $H^\bullet(S^n)$ and $H^\bullet(S^n \times S^n)$) that if S^n is an H-space, then n must be odd. In fact:

Theorem 0.3.9 (Adams)**.** *S^n is an H-space if and only if $n = 1, 3, 7$.*

1 Cofibration and Fibration

Cofibrations are, in a simple sense, maps $i : A \to X$ that are 'nice inclusions'. To motivate this, recall that the excision theorem from homology theory says: for a subspace $Z \subseteq X$ and $A \subseteq X$ such that $\overline{Z} \subseteq \mathrm{Int}(A)$, the inclusion of pairs $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces an isomorphism

$$H_\bullet(X \setminus Z, A \setminus Z) \xrightarrow{\sim} H_\bullet(X, A).$$

This is equivalent to the Mayer-Vietoris sequence by the following argument. Let $B = X \setminus Z$ and $A \subseteq X$ such that $A \setminus Z = A \cap B$. Then $\overline{Z} \subseteq \mathrm{Int}(A)$ if and only if $\mathrm{Int}(A) \cup \mathrm{Int}(B) = X$, and if this holds, the excision isomorphism is $H_\bullet(B, A \cap B) \xrightarrow{\sim} H_\bullet(X, A)$.

The condition $A \cup B = X$ even suffices for the Mayer-Vietoris sequence when some 'nice' conditions are assumed. One version of 'nice' is the condition that (X, A) is a *good pair*, sometimes called a *collared pair*: there exists an open set $U \subseteq X$ such that $A \subseteq U \subseteq X$ and $A \hookrightarrow U$ is a deformation retract.

One familiar situation arises when X is equal to A with an n-cell attached, or more explicitly, X is the pushout of the following diagram:

$$\begin{tikzcd}
S^{n-1} \arrow[r, "f"] \arrow[d, hook, "i"'] & A \arrow[d, hook, "j"] \\
D^n \arrow[r] & X = A \cup_f D^n
\end{tikzcd}$$

Then the map $(D^n, S^{n-1}) \to (X, A)$ induces an isomorphism $H_\bullet(D^n, S^{n-1}) \xrightarrow{\sim} H_\bullet(X, A)$, and the former is computable so (X, A) is a really nice pair in this case.

A key fact that relates all of these conditions in the above example is that (D^n, S^{n-1}) is a collared pair and this implies (X, A) is also a collared pair. This property of passing to pushouts will be an important characteristic of cofibrations.

1.1 Cofibrations

Definition. *A map $i : A \to X$ is called a* **cofibration** *if it satisfies the* **homotopy extension property** *(abbreviated HEP), which says that given the natural inclusion $i_0 : A \hookrightarrow A \times I$, a map $h : X \to Y$ and a homotopy $K : A \times I \to Y$, there exist a homotopy $H : X \times I \to Y$ making the following diagram commute:*

$$\begin{tikzcd}
A \arrow[r, hook, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d] \arrow[ddr, bend left, "K"] & \\
X \arrow[r, hook] \arrow[drr, bend right, "h"'] & X \times I & \\
& & Y
\end{tikzcd}$$

Example 1.1.1. Let $i : A \hookrightarrow X$ be an honest embedding and let Y be the subspace $X \times \{0\} \cup A \times I \subseteq X \times I$, with $h : X \hookrightarrow Y$ and $K : A \times I \hookrightarrow Y$ the usual inclusions. Then if $i : A \hookrightarrow X$ is a cofibration, this just means the induced map $H : X \times I \to Y$ is a retraction. This statement holds even when $i : A \to X$ is any cofibration (though we will see that every fibration is an inclusion of a closed subspace) and with $X \times \{0\} \cup A \times I$ replaced by the pushout of the diagram

$$\begin{tikzcd} A \arrow[r, hook, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

In fact, the converse is true as well:

Lemma 1.1.2. *For any $i : A \to X$, let Y be the pushout of i and the map $A \hookrightarrow A \times I$. Then the following are equivalent:*

(1) i satisfies the HEP.

(2) There exists a map $r : X \times I \to Y$ which satisfies $r \circ j = id_Y$, where $j : Y \to X \times I$ is the map induced by pushout.

Example 1.1.3. The canonical embedding $S^{n-1} \hookrightarrow D^n$ is a cofibration since $D^n \times I$ retracts onto $D^n \times \{0\} \cup S^{n-1} \times I$ by "punching the can in". More generally, for any space X, the natural inclusion $X \hookrightarrow CX$ induces such a retract and hence is a cofibration.

Remark. The homotopy extension property for $i : A \to X$ can be rephrased in the following way (it is equivalent by Corollary 0.1.12): for any $h : X \to Y$ and $\widehat{K} : A \to \mathrm{Map}(I, Y)$, there is a map $\widehat{H} : X \to \mathrm{Map}(I, Y)$ making the diagram

$$\begin{tikzcd} A \arrow[r, "\widehat{K}"] \arrow[d, "i"'] & \mathrm{Map}(I, Y) \arrow[d, "e_0"] \\ X \arrow[r, "h"'] \arrow[ur, dashed, "\widehat{H}"] & Y \end{tikzcd}$$

commute, where e_0 is the evaluation map $g \mapsto g(0)$.

Lemma 1.1.4. *Suppose Z is Hausdorff. Given a pair $j : Y \to Z$ and $r : Z \to Y$ with $r \circ j = id_Y$, j is injective and $j(Y)$ is closed in Z.*

Proof. First, $r \circ j$ restricts to $Y \xrightarrow{j} j(Y) \xrightarrow{r} Y$ which is still the identity on Y, hence a bijection with continuous inverse. It follows that j is a homeomorphism onto $j(Y)$. To show the image is closed, let $e = j \circ r : Z \xrightarrow{r} Y \xrightarrow{j} Z$ and consider the map $E : Z \to Z \times Z$

sending $z \mapsto (z, e(z))$. Also let $\Delta(Z) \subseteq Z \times Z$ be the usual diagonal subspace. Then $E^{-1}(\Delta(Z)) = \{z \in Z \mid e(z) = z\} = j(Y)$, but since Z is Hausdorff, $\Delta(Z)$ is closed and thus so is $E^{-1}(\Delta(Z))$. $\square$

If r, j is such a pair of maps, it now makes sense to call j a closed embedding and r a retraction (onto Y). From now on we assume all spaces are Hausdorff.

Proposition 1.1.5. *Every cofibration $i : A \to X$ is a closed embedding.*

Proof. Let P be the pushout of $A \to A \times I$ and $A \to X$. By Lemma 1.1.4, the induced map $j : P \to X \times I$ is a closed embedding. It follows that $X \times \{1\} \cup P \to X \times \{1\}$ is a closed embedding, but this is precisely $i : A \to X$. $\square$

Proposition 1.1.6. *Cofibrations satisfy the following properties:*

(a) Suppose Y is the pushout of the following diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ X \arrow[r] & Y \end{tikzcd}$$

Then if i is a cofibration, so is j.

(b) If $\{i_\alpha : A_\alpha \to X_\alpha\}$ is a collection of cofibrations, then $\coprod i_\alpha : \coprod A_\alpha \to \coprod X_\alpha$ is a cofibration.

(c) The composite of cofibrations is a cofibration.

(d) Suppose $A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} A_3 \xrightarrow{i_3} \cdots$ is a coherent sequence of cofibrations and $A = \varinjlim(A_j, i_j)$. Then each induced map $A_j \to A$ is a cofibration.

Definition. *The* **unbased cone** *of X is the space $CX = (X \times I)/(X \times \{0\})$.*

Proposition 1.1.7. *The inclusion $i_1 : X \hookrightarrow CX, x \mapsto (x, 1)$ is a cofibration.*

Proof. Define $r : CX \times I \to CX \times \{0\} \cup X \times \{1\} \times I$ by

$$r([x, s], t) = \begin{cases} \left(\left[x, \frac{2s}{2-t}\right], 0\right), & t < 2 - 2s \\ \left([x, 1], \frac{2s-2+t}{s}\right), & t \geq 2 - 2s. \end{cases}$$

Then r is a retraction. Apply Lemma 1.1.2. $\square$

Thus we can justify the claim in Example 1.1.3.

Corollary 1.1.8. *$S^{n-1} \hookrightarrow D^n$ is a cofibration for any n.*

We say (X, A) is a *relative CW-complex* if $X = \bigcup X_n$ for a sequence of skeleta $X_0 \to X_1 \to X_2 \to \cdots$ and $A = X_0$.

Corollary 1.1.9. *If (X, A) is a relative CW-complex, the $A \hookrightarrow X$ is a cofibration.*

Proof. By definition, each skeleton X_n is obtained from X_{n-1} by attaching n-cells, which can be viewed as a pushout diagram

$$\begin{tikzcd} \coprod_\alpha S^{n-1} \arrow[r, "\coprod f_\alpha"] \arrow[d, hook] & X_{n-1} \arrow[d] \\ \coprod_\alpha D^n \arrow[r] & X_n \end{tikzcd}$$

Then by Corollary 1.1.8, each $f_\alpha : S^{n-1} \to D^n$ is a cofibration, and applying parts of Proposition 1.1.6 says that:

- $\coprod S^{n-1} \to \coprod D^n$ is a cofibration by (b);
- $X_{n-1} \to X_n$ is a cofibration by (a);
- Each $X_n \to X$ is a cofibration by (d);

In particular, $A = X_0 \to X$ is a cofibration. □

Proposition 1.1.10. *If $i : A \to X$ is a cofibration and A is contractible, then the quotient map $X \to X/A$ is a homotopy equivalence.*

Proof. Since A is contractible, there is a homotopy $K : A \times I \to A$ with $K_0 = id_A$ and K_1 constant. Thus since i is a cofibration, $H_0 : X \times \{0\} \to X$ lifts to a homotopy $H : X \times I \to X$ making the following diagram commute:

$$\begin{tikzcd} A \times I \arrow[r, "K"] \arrow[d] & A \arrow[d, "i"] \\ X \times I \arrow[r, "H"] & X \end{tikzcd}$$

This descends to a map $\overline{H} : X/A \times I \to X/A$ by the universal property of the quotient map $p : X \to X/A$. Notice that H_1 sends A to a point, so $\overline{H}_1$ lifts to X:

$$\begin{tikzcd} X \times \{1\} \arrow[r] \arrow[d] & X \times I \arrow[r] \arrow[d] & X \arrow[d] \\ X/A \times \{1\} \arrow[r] \arrow[urr, dashed, "\overline{H}_1"] & X/A \times I \arrow[r] & X/A \end{tikzcd}$$

Then $\overline{H}$ is a homotopy $id_{X/A} \simeq p \circ \overline{H}_1$ and H is a homotopy $id_X \simeq \overline{H}_1 \circ q$. Hence p is a homotopy equivalence. $\square$

Corollary 1.1.11. *Suppose $i : A \to X$ is a cofibration and $X \cup CA$ is the following pushout:*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & CA \arrow[d, "j"] \\ X \arrow[r] & X \cup CA \end{tikzcd}$$

Then $X \cup CA \to X \cup CA/CA$ is a homotopy equivalence.

Proof. It's easy to see that $X \cup CA/CA$ is exactly X/A. Then the previous result applies. $\square$

Proposition 1.1.12. *Suppose $i : A \to X$ and $j : B \to Y$ are cofibrations and $X \times B \cup A \times Y$ is the pushout of the diagram*

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d] & A \times Y \arrow[d] \\ X \times B \arrow[r] & X \times B \cup A \times Y \end{tikzcd}$$

Then the induced map $X \times B \cup A \times Y \to X \times Y$ is a cofibration.

Definition. *For $i : A \to X$ and $j : B \to Y$ as above, the map $X \times B \cup A \times Y \to X \times Y$ is called the* **pushout product** *of i and j, written $i \square j$.*

Definition. *A basepoint $x_0 \in X$ is* **nondegenerate** *if $\{x_0\} \hookrightarrow X$ is a cofibration.*

Corollary 1.1.13. *If X and Y have nondegenerate basepoints, then the natural inclusion $X \vee Y \to X \times Y$ is a cofibration.*

Corollary 1.1.14. *Suppose Y is an H-space with nondegenerate basepoint. Then the multiplication map $m : Y \times Y \to Y$ from Proposition 0.3.4 makes the diagram*

$$\begin{tikzcd} Y \vee Y \arrow[d, hook] \arrow[dr, "\nabla"] & \\ Y \times Y \arrow[r, "m"'] & Y \end{tikzcd}$$

commute directly, not just up to homotopy.

Definition. *A pair of spaces (X, A) is a* **neighborhood deformation retract**, *or* **NDR pair**, *if there exists a pair of maps $u : X \to I$ and $h : X \times I \to X$ satisfying:*

(i) $u^{-1}(0) = A$.

(ii) $h_0 = id_X$.

(iii) $h_t|_A = id_A$ for all $t \in I$.

(iv) If $U = u^{-1}([0,1))$, then $h_t(U) \subseteq A$ for all $t \in I$.

Proposition 1.1.15. *(X, A) is an NDR pair if and only if $A \hookrightarrow X$ is a cofibration.*

Next, suppose we have a pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ X \arrow[r] & Y \end{tikzcd}$$

This can be viewed as a map of pairs $(X, A) \to (Y, B)$ and there is an induced map on homology $H_\bullet(X, A) \to H_\bullet(Y, B)$. We will show that when one of the maps is a cofibration, the map on homology is actually an isomorphism. We begin with a special case that is well-known for 'nice pairs' (X, A).

Lemma 1.1.16. *Suppose $i : A \to X$ is a cofibration. Then there is an isomorphism $H_\bullet(X, A) \to H_\bullet(X/A, A/A) = \widetilde{H}_\bullet(X/A)$.*

Proof. Note that X/A can be viewed as a pushout:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & * \arrow[d] \\ X \arrow[r] & X/A \end{tikzcd}$$

Let CA be the cone on A and consider the diagram

$$\begin{tikzcd} H_\bullet\left(X \cup A \times \left(\frac{1}{2}, 1\right]\right) \arrow[r] \arrow[d] & H_\bullet(X, A) \arrow[d] \\ H_\bullet(X \cup CA, CA) \arrow[r] & H_\bullet(X/A, A/A) \end{tikzcd}$$

The top arrow is an isomorphism by the homotopy axiom and the left arrow is an isomorphism by excision. Further, since i is a cofibration, $CA \to X \cup CA$ is a cofibration and $X \cup CA \to X/A$ is a homotopy equivalence by Corollary 1.1.11. Hence the bottom row is also an isomorphism, so this proves $H_\bullet(X, A) \to H_\bullet(X/A, A/A)$ is an isomorphism. $\square$

Lemma 1.1.17. *If*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is a pushout diagram then the induced map $X/A \to Y/B$ is a homeomorphism.

Proof. The pushout of a pushout is a pushout, so the result follows from the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Y/B \end{tikzcd}$$

□

Theorem 1.1.18. *If $i : A \to X$ is a cofibration, $f : A \to B$ is any map and Y is the pushout of i and f,*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ X \arrow[r] & Y \end{tikzcd}$$

then the induced map on homology $H_\bullet(X, A) \to H_\bullet(Y, B)$ is an isomorphism.

Proof. Consider the diagram

$$\begin{tikzcd} H_\bullet(X, A) \arrow[r] \arrow[d] & H_\bullet(Y, B) \arrow[d] \\ H_\bullet(X/A, A/A) \arrow[r] & H_\bullet(Y/B, B/B) \end{tikzcd}$$

Then the left and right arrows are isomorphisms by Lemma 1.1.16, while the bottom row is an isomorphism Lemma 1.1.17. Hence $H_\bullet(X, A) \to H_\bullet(Y, B)$ is an isomorphism. □

Fix arbitrary maps $i : A \to X$ and $f : A \to B$ and denote the pushout by $P(f)$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "i(f)"] \\ X \arrow[r] & P(f) \end{tikzcd}$$

A natural question to ask is: if $f, g : A \to B$ are homotopic, then are $P(f)$ and $P(g)$ necessarily homotopy equivalent? The answer, as it turns out, is no in general:

Example 1.1.19. Consider the trivial map $i : S^1 \to *$ and two maps $f, g : S^1 \to D^2$, where f is the natural inclusion $S^1 = \partial D^2$ and g is a constant map. Then $P(f) = S^2$, while $P(g) = D^2$ and these are clearly not homotopy equivalent spaces.

Lemma 1.1.20. *Let $i : A \to X$ be a cofibration and let $f : A \to B$ and $F : A \times I \to B$ be maps such that $F_0 = f$ and*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ X \arrow[r, "\tilde{f}"'] & P(f) \end{tikzcd} \qquad \begin{tikzcd} A \times I \arrow[r, "F"] \arrow[d, "i \times 1"'] & B \arrow[d, "J"] \\ X \times I \arrow[r, "\widetilde{F}"'] & P(F) \end{tikzcd}$$

are pushout squares. Then the induced map $K : P(f) \to P(F)$ is a homotopy equivalence under B.

Proof. We have a diagram

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[d, "i"'] \arrow[rr, bend left, "f"] & A \times I \arrow[r, "F"] \arrow[d, "i \times 1"] & B \arrow[dd, bend left, "j"] \\ X \arrow[r] \arrow[drr, bend right, "\tilde{f}"'] & X \times I \arrow[dr, dashed, "\overline{F}"] & \\ & & P(f) \end{tikzcd}$$

Since i is a cofibration, there exists a map $\overline{F} : X \times I \to P(f)$ making the above commute. Now the right square may be viewed as a pushout diagram:

$$\begin{tikzcd} A \times I \arrow[r, "F"] \arrow[d, "i \times 1"'] & B \arrow[d, "J"] \arrow[ddr, bend left, "j"] & \\ X \times I \arrow[r, "\widetilde{F}"] \arrow[drr, bend right, "\tilde{f}"'] & P(F) \arrow[dr, dashed, "r"] & \\ & & P(f) \end{tikzcd}$$

In particular, we get a map $r : P(F) \to P(f)$ which satisfies $r \circ K = id_{P(f)}$. We finish by showing $K \circ r \simeq id_{P(F)}$. Define a set

$$U = \{0\} \times I \cup I \times \{0\} \cup I \times \{1\} \subseteq I \times I.$$

(As a subset of the square $I \times I$, this set is equal to the union of the bottom, left and right sides.) Now we have a diagram in which the right square is a pushout:

$$\begin{tikzcd}
A \times U \arrow[r] \arrow[d] & A \times I \times I \arrow[r] \arrow[d] & B \times I \arrow[d] \arrow[ddr, bend left, "J"] & \\
X \times U \arrow[r] \arrow[drrr, bend right] & X \times I \times I \arrow[r] \arrow[drr, dashed, "H'"'] & P(F) \times I \arrow[dr, dashed, "H"] & \\
 & & & P(F)
\end{tikzcd}$$

Here, the map $X \times U \to P(F)$ is defined on $X \times \{0\} \times I$ by $\tilde{f} \times 1$, on $X \times I \times \{0\}$ by $\tilde{F}$, and on $X \times I \times \{1\}$ by $K \times \overline{F}$. Since i is a cofibration, $i \times 1 : A \times I \to X \times I$ is also a cofibration (this is obvious from the HEP). Therefore the map $X \times U \to P(F)$ just described induces $H' : X \times I \times I \to P(F)$, and since the right square in the diagram above is a pushout, this induces $H : P(F) \times I \to P(F)$. By construction, $H_0 = id_{P(F)}$ and $H_1 = K \circ r$ so we're done. □

Theorem 1.1.21. *If $i : A \to X$ is a cofibration and $f, g : A \to B$ are homotopic maps, then $P(f)$ and $P(g)$ are* homotopy equivalent under B, *i.e. there is a commutative diagram*

$$\begin{tikzcd}
 & B \arrow[dl, "i(f)"'] \arrow[dr, "i(g)"] & \\
P(f) \arrow[rr, shift left, "a"] & & P(g) \arrow[ll, shift left, "b"]
\end{tikzcd}$$

and homotopies $H : P(f) \times I \to P(f)$, with $H_0 = id_{P(f)}$ and $H_1 = b \circ a$; and $K : P(g) \times I \to P(g)$, with $K_0 = id_{P(g)}$ and $K_1 = a \circ b$; such that H and K also commute with $i(f)$ and $i(g)$.

Proof. If $F : A \times I \to B$ is the homotopy, with $F_0 = f$ and $F_1 = g$, then Lemma 1.1.20 gives a diagram

$$\begin{tikzcd}
 & B \arrow[dl, "i(f)"'] \arrow[d] \arrow[dr, "i(g)"] & \\
P(f) \arrow[r] & P(F) & P(g) \arrow[l]
\end{tikzcd}$$

in which each triangle is a homotopy equivalence under B. Therefore $P(f)$ and $P(g)$ are homotopy equivalent under B. □

Corollary 1.1.22. *Suppose Z is obtained from a space Y by attaching an n-cell via an attaching map $f : S^{n-1} \to Y$ which is nullhomotopic. Then Z is homotopy equivalent to $Y \vee S^n$ under Y.*

Proof. The wedge product $Y \vee S^n$ and $Z = Y \cup_f D^n$ can each be viewed as a pushout:

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ D^n \arrow[r] & Y \cup_f D^n \end{tikzcd} \qquad \begin{tikzcd} S^{n-1} \arrow[r, "c"] \arrow[d] & Y \arrow[d] \\ D^n \arrow[r] & Y \vee S^n \end{tikzcd}$$

where c is a constant map, but by hypothesis, f and c are homotopic. □

Definition. *The* **homotopy pushout** *of two maps $f : X \to Y$ and $g : X \to Z$ is the pushout $P^h(f, g)$ of the diagram*

$$\begin{tikzcd} X \coprod X \arrow[r, "f \coprod g"] \arrow[d, "i_0 \times i_1"'] & Y \coprod Z \arrow[d] \\ X \times I \arrow[r] & P^h(f, g) \end{tikzcd}$$

Corollary 1.1.23. *If $f, f' : X \to Y$ are homotopic and $g, g' : X \to Z$ are homotopic, then the homotopy pushouts $P^h(f, g)$ and $P^h(f', g')$ are homotopy equivalent.*

Proof. By Proposition 1.1.6(b), $i_0 \times i_1 : X \times X \to X \times I$ is a cofibration so apply Theorem 1.1.21. □

Observe that for any maps $f : X \to Y$ and $g : X \to Z$, there is a natural map $\pi : P^h(f, g) \to P(f, g)$ given by the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & X \times Y \arrow[d] \arrow[ddr, bend left] & \\ Z \arrow[r] \arrow[drr, bend right] & P^h(f, g) \arrow[dr, dashed, "\pi"] & \\ & & P(f, g) \end{tikzcd}$$

In fact, π can be viewed as a quotient map.

Proposition 1.1.24. *If either f or g is a cofibration, the induced map $\pi : P^h(f, g) \to P(f, g)$ is a homotopy equivalence.*

Definition. *The* **mapping cylinder** *of $f : X \to Y$ is the pushout of the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i_1"'] & Y \arrow[d] \\ X \times [0,1] \arrow[r] & M(f) \end{tikzcd}$$

Theorem 1.1.25. *Every map $f : X \to Y$ factors through a cofibration $\tilde{f}$,*

$$\begin{tikzcd} & M \arrow[dl, "\tilde{f}"'] \arrow[dr, "p"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

such that p is a homotopy equivalence.

Proof. Take M to be the mapping cylinder $M(f)$. □

1.2 Cofibration Sequences

Suppose $[X, Y]$ is the based space of homotopy classes of based maps from X to Y, two based spaces. Given such a map $f : X \to Y$, there is an induced map

$$\begin{aligned} f^* : [Y, W] &\longrightarrow [X, W] \\ g &\longmapsto g \circ f \end{aligned}$$

for any W, which is natural in W. Consider the "kernel" of f, i.e. the set

$$\ker f^* = \{g \in [Y, W] \mid g \circ f \text{ is nullhomotopic}\}.$$

Note that $g \circ f$ being nullhomotopic is equivalent to the existence of a based homotopy $H : X \times I \to W$ such that $H_0 = c$ is a constant map and $H_1 = g \circ f$, which in turn is equivalent to the existence of $\widetilde{H} : CX \to W$ such that $\widetilde{H}_1 = g \circ f$, i.e. $g \circ f$ is equal to $X \xrightarrow{i_1} CX \xrightarrow{\widetilde{H}} W$.

Now let $C(f)$ be the pushout of the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i_1"'] & Y \arrow[d, "i(f)"] \\ CX \arrow[r] & C(f) \end{tikzcd}$$

Then $g \circ f$ is nullhomotopic if and only if g factors through $i(f) : Y \to C(f)$, i.e. the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook, "i_1"'] & Y \arrow[d, "i(f)"] \arrow[ddr, bend left, "g"] & \\
CX \arrow[r] \arrow[drr, bend right, "\widetilde{H}"'] & C(f) \arrow[dr, dashed] & \\
 & & W
\end{tikzcd}$$

In other words, $[C(f), W]$ can be identified with the kernel of f^*. To be precise:

Lemma 1.2.1. *For any $f : X \to Y$ and any W, there is an exact sequence of based sets*

$$[C(f), W] \xrightarrow{i(f)^*} [Y, W] \xrightarrow{f^*} [X, W].$$

Definition. *The set $C(f)$ is called the* **(homotopy) cofibre** *of f, or sometimes the* **mapping cone** *of f.*

In analogy with the kernel-cokernel sequence in algebra, one has $C(f) \simeq Y/f(X)$.

Proposition 1.2.2. *Let X and Y be spaces. Then*

(a) If $f, g : X \to Y$ are homotopic, then $C(f)$ and $C(g)$ are homotopy equivalent under Y.

(b) If $j : A \to X$ is a cofibration, then $C(j)$ and X/A are homotopic under X.

Proof. (a) follows from the diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "i(f)"'] \arrow[dr, "i(g)"] & \\
C(f) \arrow[rr, shift left] & & C(g) \arrow[ll, shift left]
\end{tikzcd}$$

and Theorem 1.1.21.

(b) likewise follows from the diagram

$$\begin{tikzcd}
 & X \arrow[dl, "i(j)"'] \arrow[dr] & \\
C(j) \arrow[rr, shift left] & & X/A \arrow[ll, shift left]
\end{tikzcd}$$

$\square$

Proposition 1.2.3. *For any map $f : X \to Y$, there are natural long exact sequences in reduced homology and cohomology*

$$\cdots \to \widetilde{H}_n(X) \xrightarrow{f_*} \widetilde{H}_n(Y) \xrightarrow{i(f)_*} \widetilde{H}_n(C(f)) \xrightarrow{\delta} \widetilde{H}_{n-1}(X) \to \cdots$$
$$\cdots \to H^n(C(f)) \xrightarrow{i(f)^*} H^n(Y) \xrightarrow{f^*} H^n(X) \xrightarrow{\delta} H^{n+1}(C(f)) \to \cdots$$

Proof. Since $i(f)$ is a cofibration (by Proposition 1.1.6), it is an embedding by Proposition 1.1.5 so $(C(f), Y)$ is a pair of spaces and there is a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{H}_n(Y) \arrow[r] & \widetilde{H}_n(C(f)) \arrow[r] & H_n(C(f), Y) \arrow[r] & \widetilde{H}_{n-1}(Y) \arrow[r] & \cdots \\
\cdots \arrow[r] & \widetilde{H}_n(X) \arrow[r] \arrow[u, "f_*"] & \widetilde{H}_n(CX) \arrow[r] \arrow[u] & H_n(CX, X) \arrow[r] \arrow[u, "\cong"] & \widetilde{H}_{n-1}(X) \arrow[r] \arrow[u, "f_*"] & \cdots
\end{tikzcd}$$

where the top row is the long exact sequence for $(C(f), Y)$, the bottom row is the long exact sequence for the pair (CX, X), and the vertical column is an isomorphism by Theorem 1.1.18. Then CX is contractible, so $\widetilde{H}_n(CX) = 0$ and so by exactness we get an isomorphism $H_n(CX, X) \xrightarrow{\sim} \widetilde{H}_{n-1}(X)$. This constructs the desired exact sequence. The proof for cohomology is analogous. □

Consider the second iteration of the cone construction:

$$X \xrightarrow{f} Y \xrightarrow{i(f)} C(f) \xrightarrow{i^2(f)} C(i(f)).$$

Since $i(f)$ is a cofibration, the natural quotient map

$$q : C(i(f)) \longrightarrow C(i(f))/CY = C(f)/Y = CX/X = \Sigma X$$

is a homotopy equivalence by Proposition 1.1.10. Let $\pi(f) = q \circ i^2(f) : C(f) \to \Sigma X$.

Lemma 1.2.4. *For any map $f : X \to Y$ and any space W,*

$$[\Sigma X, W] \xrightarrow{\pi(f)^*} [C(f), W] \xrightarrow{i(f)^*} [Y, W] \xrightarrow{f^*} [X, W]$$

is an exact sequence of sets.

Iterating again, we get a sequence

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "i(f)"] & C(f) \arrow[r, "i^2(f)"] & C(i(f)) \arrow[r] \arrow[d, "q"] & C(i^2(f)) \arrow[d] \\
 & & & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y
\end{tikzcd}$$

where $-\Sigma f$ is the map $\Sigma X = X \wedge S^1 \to Y \wedge S^1 = \Sigma Y$ induced by $z \mapsto \bar{z}$ on S^1. This construction continues inductively, constructing the so-called long exact sequence in homotopy.

Theorem 1.2.5 (Puppe). *For any map $f : X \to Y$ and any space W, there is a long exact sequence of pointed sets which is natural in W:*

$$\cdots \to [\Sigma^n C(f), W] \to [\Sigma^n Y, W] \to [\Sigma^n X, W] \to [\Sigma^{n-1} C(f), W] \to \cdots$$

$$\cdots \to [\Sigma C(f), W] \xrightarrow{(\Sigma i(f))^*} [\Sigma Y, W] \xrightarrow{(\Sigma f)^*} [\Sigma X, W] \xrightarrow{\pi(f)^*} [C(f), W] \xrightarrow{i(f)^*} [Y, W] \xrightarrow{f^*} [X, W].$$

Corollary 1.2.6. *For any $f : X \to Y$ and W, there is a natural long exact sequence*

$$\cdots \to [C(f), \Omega^n W] \to [Y, \Omega^n W] \to [X, \Omega^n W] \to [C(f), \Omega^{n-1} W] \to \cdots$$

Proof. Apply Corollary 0.2.5. □

Proposition 1.2.7. *Given a map of pairs*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

which commutes up to homotopy, there exists a map $\gamma : C(f) \to C(f')$ and a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "i(f)"] \arrow[d, "\beta"] & C(f) \arrow[r, "\pi(f)"] \arrow[d, dashed, "\gamma"] & \Sigma X \arrow[d, "\Sigma\alpha"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "i(f')"] & C(f') \arrow[r, "\pi(f')"] & \Sigma X' \end{tikzcd}$$

in which the middle square commutes and the right square commutes up to homotopy.

Suppose we have a composition $X \xrightarrow{g} Y \xrightarrow{f} Z$. Then up to homotopy equivalence, we can think of $C(g)$ as $Y/g(X)$ and $C(f)$ as $Z/f(Y)$, and also $C(f \circ g)$ as $Z/f \circ g(X)$. In analogy with algebra, we might hope to have Z/X homotopy equivalent to $(Z/X)/(Y/X)$, e.g. if we suppose $X \subseteq Y \subseteq Z$ are subspaces, for simplicity. Indeed, we have:

Proposition 1.2.8. *For any maps $X \xrightarrow{g} Y \xrightarrow{f} Z$, the maps $C(g) \xrightarrow{h} C(f \circ g) \to C(h)$ induce a homotopy equivalent between $C(f)$ and $C(h)$.*

Proof. This follows from the diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "f \circ g"'] & Y \arrow[r] \arrow[d, "f"] & C(g) \arrow[d] \\ & Z \arrow[r] \arrow[d] & C(f \circ g) \arrow[d] \\ & C(f) \arrow[r] & C(h) \end{tikzcd}$$

where the top row is the cofibre sequence for $g : X \to Y$, the left column is the cofibre sequence for $f : Y \to Z$, the diagonal-to-middle row is the cofibre sequence for $f \circ g : X \to Z$ and the right column comes from naturality. $\square$

As a result, we can identify $C(f)$ as the cofibre of the map $C(g) \to C(f \circ g)$.

Proposition 1.2.9. *For any based map $f : X \to Y$,*

(a) The group $[\Sigma X, W]$ acts on $[C(f), W]$ via the map $\pi(f)^* : [\Sigma X, W] \to [C(f), W]$ for any space W.

(b) $\pi(f)^*$ is a map of $[\Sigma X, W]$-sets.

(c) For any maps $\alpha, \beta : C(f) \to W$, $i(f)^*[\alpha] = i(f)^*[\beta]$ if and only if α and β are in the same orbit of the $[\Sigma X, W]$-action on $[C(f), W]$.

Recall from Section 0.2 that for a based space X, the space $X_+ = X \coprod \{x_0\}$ induces an equivalence between the based maps $[X, Y]_*$ and the unbased maps $[X_+, Y]$ for any Y. This induces a cofibration sequence $S^0 \to X_+ \xrightarrow{p} X \to S^1$. Then for any connected Y, we get an exact sequence of sets

$$\pi_1(Y) = [S^1, Y]_* \to [X, Y]_* \to [X_+, Y] \to [S^0, Y] = *$$

Corollary 1.2.10. *For any based spaces X and Y, $\pi_1(Y)$ acts on $[X, Y]_*$ and the orbits are in one-to-one correspondence with the unbased homotopy classes of maps $X \to Y$.*

Corollary 1.2.11. *For any connected space Y, $\pi_1(Y)$ acts on $[S^1_+, Y]$ by conjugation.*

Cofibration sequences have many applications to the study of homotopy groups of spheres, $\pi_k(S^n)$.

Proposition 1.2.12. *For all X and Y, the map $d : X \wedge Y \to \Sigma(X \vee Y)$ is nullhomotopic.*

Proof. Consider the cofibration sequence

$$X \vee Y \xrightarrow{i} X \times Y \xrightarrow{p} X \wedge Y \xrightarrow{d} \Sigma(X \vee Y) \xrightarrow{\Sigma i} \Sigma(X \times Y) \xrightarrow{\Sigma p} \Sigma(X \wedge Y).$$

Then to prove d is nullhomotopic, it is equivalent to prove the map $(\Sigma i)^* : [\Sigma(X \times Y), W] \to [\Sigma(X \vee Y), W]$ is surjective for any space W. Indeed, this follows from the Puppe sequence

$$\cdots \to [\Sigma(X \wedge Y), W] \xrightarrow{(\Sigma p)^*} [\Sigma(X \times Y), W] \xrightarrow{(\Sigma i)^*} [\Sigma(X \vee Y), W] \xrightarrow{d^*} [X \wedge Y, W] \to \cdots$$

To show $(\Sigma i)^*$ is surjective, note that $\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y$, so we can split $[\Sigma(X \vee Y), W] = [\Sigma X, W] \times [\Sigma Y, W]$. If $X \xrightarrow{i_X} X \times Y \xrightarrow{\pi_X} X$ are the natural inclusion and projection for X (and i_Y, π_Y are the same for Y), then $(\Sigma i)^*$ can be viewed as

$$(i_X^*, i_Y^*) : [\Sigma(X \times Y), W] \longrightarrow [\Sigma X, W] \times [\Sigma Y, W].$$

Consider the diagram of *groups*

$$\begin{tikzcd}
& {[\Sigma X, W]} \arrow[dl, "\pi_X^*"'] \arrow[d, hook] \\
{[\Sigma(X \times Y), W]} \arrow[r, "(i_X^*, i_Y^*)"] & {[\Sigma X, W] \times [\Sigma Y, W]} \\
& {[\Sigma Y, W]} \arrow[ul, "\pi_Y^*"] \arrow[u, hook]
\end{tikzcd}$$

Then by group theory, (i_X^*, i_Y^*) is onto. $\square$

Example 1.2.13. Let $X = Y = S^n$ and consider the map $f_{n,n} : S^{2n-1} \to S^n \vee S^n$ which attaches a $2n$-cell to $S^n \vee S^n$ to form $S^n \times S^n$. If $\nabla : S^n \vee S^n \to S^n$ is the folding map, we obtain a class $[i_n, i_n] := [\nabla \circ f_{n,n}] \in \pi_{2n-1}(S^n)$, called the *Whitehead product*. Consider the cofibration sequence

$$S^{2n-1} \xrightarrow{f_{n,n}} S^n \vee S^n \to S^n \times S^n \to S^{2n} \xrightarrow{d} \Sigma(S^n \vee S^n).$$

Then $d = \Sigma f_{n,n}$ so by Proposition 1.2.12, $\Sigma[i_n, i_n] : S^{2n} \to S^{n+1}$ is nullhomotopic.

Recall from Example 0.3.8 that S^n may only be an H-space if n is odd (and moreover, this only happens when $n = 1, 3, 7$ by Adams' theorem). Consider the pushout

$$\begin{tikzcd}
S^n \wedge S^n \arrow[r, "\nabla"] \arrow[d] & S^n \arrow[d] \\
S^n \times S^n \arrow[r] & J
\end{tikzcd}$$

Then by definition of the pushout, there exists an H-space structure $m : S^n \times S^n \to S^n$ if and only if there exists a retract $r : J \to S^n$. Composing with the attaching map $f_{n,n}$, we get a pushout diagram

$$\begin{tikzcd}
S^{2n-1} \arrow[r, "{[i_n, i_n]}"] \arrow[d] & S^n \arrow[d] \\
D^{2n} \arrow[r] & J
\end{tikzcd}$$

so J retracts onto S^n if and only if there exists a map $D^{2n} \to S^n$ which induces a nullhomotopy of $[i_n, i_n]$. In other words, S^n admits an H-space structure precisely when $[i_n, i_n]$ is nullhomotopic. Thus in general, $[i_n, i_n] \in \pi_{2n-1}(S^n)$ is not null (e.g. when n is even), but we showed above that $\Sigma[i_n, i_n] = 0$ in $\pi_{2n}(S^{n+1})$.

Proposition 1.2.14. *There is a homotopy equivalence between $\Sigma(X \vee Y \vee (X \wedge Y))$ and $\Sigma(X \times Y)$.*

Corollary 1.2.15. *The exact sequence*

$$0 \to [\Sigma(X \wedge Y), W] \xrightarrow{(\Sigma p)^*} [\Sigma(X \times Y), W] \xrightarrow{(\Sigma i)^*} [\Sigma(X \vee Y), W] \to 0$$

splits.

1.3 Fibrations

In this section we discuss the dual notion to cofibrations. Recall that the homotopy extension property defining a cofibration $i : A \to X$ can be stated as a lifting problem:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & \mathrm{Map}(I, Y) \arrow[d, "e_0"] \\
X \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

We use this to define fibrations.

Definition. *A map $p : E \to B$ is a* **fibration** *if for any maps $\tilde{h} : W \to E$ and $H : W \times I \to B$ such that $p \circ \tilde{h} = H_0$, there exists a lift $\widetilde{H} : W \times I \to E$ of H making the following diagram commute:*

$$\begin{tikzcd}
W \arrow[r, "\tilde{h}"] \arrow[d, hook, "i_0"'] & E \arrow[d, "p"] \\
W \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & B
\end{tikzcd}$$

Example 1.3.1. It follows from the above diagrams that if $i : A \hookrightarrow X$ is a cofibration, then for all Z, the adjoint $i^* : \mathrm{Map}(X, Z) \to \mathrm{Map}(A, Z)$ is a fibration. For example, if $\{x_0\} \hookrightarrow X$ is the inclusion of a nondegenerate basepoint of X, then the evaluation map $e_{x_0} : \mathrm{Map}(X, Z) \to Z$ is a fibration for every Z.

Example 1.3.2. Given any spaces B and F, the projection $B \times F \to B$ is a fibration. Indeed, given a diagram

$$\begin{tikzcd}
W \arrow[r, "{(h_B, h_F)}"] \arrow[d, hook] & B \times F \arrow[d, "p"] \\
W \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & B
\end{tikzcd}$$

there is a lift $\widetilde{H} = (H, \tilde{h}_F)$, where $\tilde{h}_F$ is the composition $W \times I \to W \xrightarrow{h_F} F$.

Example 1.3.3. Suppose $p : Y \to X$ is a covering map of (reasonable) spaces. Given a diagram

$$\begin{tikzcd} W \arrow[r, "\tilde{h}"] \arrow[d, hook] & Y \arrow[d, "p"] \\ W \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & X \end{tikzcd}$$

with W path-connected (in general, one may define $\widetilde{H}$ on each path component), pick $w_0 \in W$ and set $y_0 = \tilde{h}(w_0) \in Y$ and $x_0 = p(y_0) \in X$. Then lifting H to $\widetilde{H}$ is equivalent to $H_*(\pi_1(W \times I, (w_0, 0))) \subseteq p_*(\pi_1(Y, y_0))$ but since $(i_0)_* : \pi_1(W, w_0) \to \pi_1(W \times I, (w_0, 0))$ is induced by a homotopy equivalence and the above square commutes, this inclusion is guaranteed. Hence p is a fibration and the lift $\widetilde{H}$ is always *unique.* This is not true of a general fibration.

We will see that every fibration is essentially equivalent to one of these examples.

Definition. *A map $p : E \to B$ is a* **(locally trivial) fibre bundle** *with fibre F if there is an open cover $\{U_\alpha\}$ of B and homeomorphisms $U_\alpha \times F \to p^{-1}(U_\alpha)$ making the following diagram commute for each U_α:*

$$\begin{tikzcd} U_\alpha \times F \arrow[rr, "\sim"] \arrow[dr] & & p^{-1}(U_\alpha) \arrow[dl, "p"] \\ & U_\alpha & \end{tikzcd}$$

It will follow from a result in Section 1.5 that every fibre bundle is a fibration.

Example 1.3.4. When F is a discrete space, a bundle with fibre F is a covering space.

Proposition 1.3.5. *Let $p : E \to B$ be a fibration. Then*

(a) For any map $f : B' \to B$, the pullback $f^*p : P \to B'$,

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f^*p"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

is a fibration.

(b) *For any fibrations* $\{p_\alpha : E_\alpha \to B_\alpha\}$*, the product* $\prod p_\alpha : \prod E_\alpha \to \prod B_\alpha$ *is a fibration.*

(c) *If* $p' : E' \to E$ *is a fibration, then* $p \circ p' : E' \to B$ *is also a fibration.*

(d) *The induced map* $p_* : \mathrm{Map}(X, E) \to \mathrm{Map}(X, B)$ *is a fibration for all* X*.*

Proof. Dual to the proof of Proposition 1.1.6. □

Theorem 1.3.6. *Suppose* B *is path-connected and* $p : E \to B$ *is a fibration. Then*

(1) *All of the fibres* $E_x := p^{-1}(x)$ *are homotopy equivalent.*

(2) *Every choice of path* α *in* B *from* x *to* y *determines a homotopy class of homotopy equivalences* $\alpha_* : E_x \to E_y$ *depending only on the homotopy class of* α *rel endpoints.*

(3) *Under the above, concatenation of paths corresponds to composition of homotopy equivalences. In other words, there is a well-defined homomorphism of groups*

$$\begin{aligned} \pi_1(B, x) &\longrightarrow \{\textit{homotopy classes of self homotopy equivalences of } E_x\} \\ [\alpha] &\longmapsto (\alpha^{-1})_*. \end{aligned}$$

Proof. Take a path α from x to y in B. Then the inclusion $E_x \hookrightarrow E$ induces the following diagram:

$$\begin{tikzcd} E_x \times \{0\} \arrow[r, hook] \arrow[d, hook] & E \arrow[d, "p"] \\ E_x \times [0,1] \arrow[r, "G"'] \arrow[ur, dashed, "\widetilde{G}"] & B \end{tikzcd}$$

where $G(e, t) = \alpha(t)$ for all $t \in [0, 1]$. Since p is a fibration, we get a lift $\widetilde{G}$. At $t = 0$, $\widetilde{G}_0 : E_x \times \{0\} \to E$ is just the inclusion of the fibre $E_x \hookrightarrow E$. On the other hand, for any t, $p \circ \widetilde{G}_t$ is the constant map at $\alpha(t)$ so in particular at $t = 1$, $\widetilde{G}_1$ gives a map $E_x \to E_{\alpha(1)} = E_y$. Set $\alpha_* = [\widetilde{G}_1]$. To check α_* is well-defined, suppose $\alpha' : [0, 1] \to B$ is another path homotopic rel endpoints to α. Set $H = \alpha' \circ \mathrm{proj}_{[0,1]}$ where $\mathrm{proj}_{[0,1]} : E_x \times [0, 1] \to [0, 1]$ is the second coordinate projection. Then using the homotopy lifting property on the diagram

$$\begin{tikzcd} E_x \times \{0\} \arrow[r, hook] \arrow[d, hook] & E \arrow[d, "p"] \\ E_x \times [0,1] \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & B \end{tikzcd}$$

we get a map $\widetilde{H} : E_x \times [0, 1] \to E$ and, as above, a map $\widetilde{H}_1 : E_x \to E_y$. One then constructs a homotopy from $\widetilde{G} \to \widetilde{H}$ using that $\alpha \simeq \alpha'$ rel endpoints; this then induces a homotopy $\widetilde{G}_1 \to \widetilde{H}_1$. Hence α_* is well-defined and (2) is proved.

It is clear that for paths α, β in B such that $\beta(0) = \alpha(1)$, we have $(\alpha * \beta)_* = \beta_* \circ \alpha_*$. Thus when $\beta = \alpha^{-1}$, $\beta_* \circ \alpha_* = (c_x)_*$, where c_x is the constant path at x. Since $(c_x)_* = [id_{E_x}]$, we have that $\beta_* = (\alpha^{-1})_*$ is a homotopy inverse of α_*. Hence α_* is a homotopy equivalence, so using path-connectedness we see that all fibres are homotopy equivalent, proving (1).

Finally, it is routine to prove the homotopy classes of homotopy equivalences $E_x \to E_x$ form a group under composition. Then for (3), the above shows that $(\alpha * \beta)_* = \beta_* \circ \alpha_*$ and the trivial class goes to the homotopy class of the identity map $E_x \to E_x$, so $[\alpha] \mapsto (\alpha^{-1})_*$ is a homomorphism. □

As with cofibrations, there is a notion of fibration in the based category $\texttt{Top}_*$. For a based space (X, x_0), let

$$PX = \{\alpha : I \to X \mid \alpha(0) = x_0\}$$

denote the (based) path space of X, as in Section 0.2.

Lemma 1.3.7. *For any X, PX is contractible.*

Proof. This follows easily from the fact that $I = [0, 1]$ is contractible. □

Lemma 1.3.8. *For any X, the endpoint map $PX \to X, \alpha \mapsto \alpha(1)$ is a fibration.*

Proof. To prove p satisfies the homotopy lifting property, we need to complete the following diagram for any space W:

$$\begin{tikzcd}
W \times \{0\} \arrow[r, "g"] \arrow[d, hook] & PY \arrow[d, "p"] \\
W \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & Y
\end{tikzcd}$$

For $w \in W$, $g(w) \in PY$ is a map such that $p \circ g(w) = H(w, 0)$. That is, $g(w)$ is a path ending at the starting point of the homotopy $H(w, -)$. To lift, just continue this path by defining

$$\widetilde{H}(w, s)(t) = \begin{cases} g(a)((1+s)t), & 0 \le t \le \frac{1}{1+s} \\ H(a, (1+s)t - 1), & \frac{1}{1+s} < t \le 1. \end{cases}$$

Then $\widetilde{H}$ is continuous, $\widetilde{H}(w, 0) = w(a)$ and $p \circ \widetilde{H}(w, s)(-) = \widetilde{H}(w, s)(1) = H(w, s)$. Hence $p : PY \to Y$ is a fibration. □

Notice that the fibres of the endpoint fibration $p : PY \to Y$ are, up to homotopy, the loop space ΩY.

Definition. *A* **map of fibrations** *between $p : E \to B$ and $p' : E' \to B'$ is a pair of maps $f : B \to B'$ and $\tilde{f} : E \to E'$ making the following diagram commute:*

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

Definition. *A* **fibre homotopy** *between maps of fibrations* $(f_0, \tilde{f}_0)$ *and* $(f_1, \tilde{f}_1)$ *is a pair of homotopies* $(H, \widetilde{H})$ *such that* H *is a homotopy* $f_0 \to f_1$, $\widetilde{H}$ *is a homotopy* $\tilde{f}_0 \to \tilde{f}_1$ *and the following diagram commutes:*

$$\begin{tikzcd} E \times I \arrow[r, "\widetilde{H}"] \arrow[d, "p \times id"'] & E' \arrow[d, "p'"] \\ B \times I \arrow[r, "H"'] & B' \end{tikzcd}$$

Two fibrations $p : E \to B$ *and* $q : E' \to B$ *over the same base are said to be* **fibre homotopy equivalent** *if there exist maps of fibrations*

$$\begin{tikzcd} E \arrow[rr, shift left, "f"] \arrow[dr, "p"'] & & E' \arrow[ll, shift left, "g"] \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

such that $f \circ g$ *and* $g \circ f$ *are fibre homotopic to the identity.*

We next make rigorous the idea that 'every map is a fibration' (up to homotopy), just as Theorem 1.1.25 showed that every map was a cofibration up to homotopy. Suppose $f : X \to Y$ is continuous.

Definition. *The* **mapping path space** *of* f *is the pullback fibration* $P_f := f^*(Y^I)$ *along the starting point fibration* $q : Y^I \to Y, \alpha \mapsto \alpha(0)$. *That is,* $P_f = \{(x, \alpha) \in X \times Y^I \mid \alpha(0) = f(x)\}$ *and there is a commutative diagram*

$$\begin{tikzcd} P_f \arrow[r] \arrow[d, "f^*q"'] & Y^I \arrow[d, "q"] \\ X \arrow[r] & Y \end{tikzcd}$$

Definition. *The* **mapping path fibration** *of* $f : X \to Y$ *is the map* $p_f : P_f \to Y$ *given by* $p(x, \alpha) = \alpha(1)$, *that is, the restriction of the endpoint fibration.*

Theorem 1.3.9. *For any continuous $f : X \to Y$,*

(1) There is a homotopy equivalence $h : X \to P_f$ such that the diagram

$$\begin{tikzcd} & P_f \arrow[dr, "p_f"] & \\ X \arrow[ur, "h"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

commutes.

(2) $p_f : P_f \to Y$ is a fibration.

(3) If f is a fibration, then h is a fibre homotopy equivalence.

Proof. (1) Define $h(x) = (x, c_{f(x)})$ where $c_{f(x)}$ is the constant path (in Y) at $f(x)$. Then the projection $\pi : P_f \to X$ is obviously a homotopy inverse to h, since $\pi \circ h(x) = \pi(x, c_{f(x)}) = x$; and $h \circ \pi \simeq id$ via $F((x, \alpha), s) = (x, \alpha_s)$, where $\alpha_s(t) = \alpha(st)$.

(2) We must complete the following diagram for any space A:

$$\begin{tikzcd} A \times \{0\} \arrow[r, "g"] \arrow[d, hook] & P_f \arrow[d, "p_f"] \\ A \times [0,1] \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & Y \end{tikzcd}$$

For $a \in A$, we have $g(a) = (g_1(a), g_2(a))$ where $g_1(a) \in X$ and $g_2(a)$ is a path in Y starting at $f(g_1(a))$ and ending at $H(a, 0)$. Continue this path to get the desired lift by setting $\widetilde{H}(a, s)(t) = (g_1(a), \widetilde{H}_2(a, s)(t))$, where

$$\widetilde{H}_2(a, s)(t) = \begin{cases} g_2(a)((1+s)t), & 0 \le t \le \frac{1}{1+s} \\ H(a, (1+s)t - 1), & \frac{1}{1+s} < t \le 1. \end{cases}$$

(3) Note that $\pi : P_f \to X$ is not a fibration map a priori. To fix this, define $\gamma : P_f \times I \to Y$ by $\gamma(x, \alpha, t) = \alpha(t)$. Then we have a diagram

$$\begin{tikzcd} P_f \times \{0\} \arrow[r, "\pi"] \arrow[d, hook] & X \arrow[d, "f"] \\ P_f \times I \arrow[r, "\gamma"'] \arrow[ur, dashed, "\tilde{\gamma}"] & Y \end{tikzcd}$$

which commutes by definition of P_f, so there exists a lift $\tilde{\gamma}$ since f is a fibration. Define $g : P_f \to X$ by $g(x, \alpha) = \tilde{\gamma}(x, \alpha, 1)$. Then the diagram

$$\begin{tikzcd} P_f \arrow[rr, shift left, "g"] \arrow[dr, "p_f"'] & & X \arrow[ll, shift left, "h"] \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

commutes by construction and g is a fibre homotopy inverse of f. □

1.4 Fibration Sequences

Recall that in Section 1.2, we constructed a cofibre $C(f)$ for every map $f : X \to Y$ such that the sequence $X \to Y \to C(f)$ induced an exact sequence of sets $[C(f), W] \to [Y, W] \to [X, W]$ for every W. Analogously, we define:

Definition. *The* **homotopy fibre** $F(f)$ *of a map* $f : Y \to X$ *is the pullback of the diagram*

$$\begin{tikzcd} F(f) \arrow[r] \arrow[d, "p(f)"'] & PX \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

Explicitly, $F(f) = \{(y, \alpha) \mid y \in Y, \alpha : I \to X, \alpha(0) = x_0, \alpha(1) = f(y)\}$ *where* x_0 *is the basepoint of* X.

Proposition 1.4.1. *For all* $f : Y \to X$ *and spaces* W,

$$[W, F(f)] \xrightarrow{p(f)_*} [W, Y] \xrightarrow{f_*} [W, X]$$

is an exact sequence of sets.

Proof. Dual to Lemma 1.2.1. □

Recall (Proposition 1.2.2) that if $i : A \hookrightarrow X$ is a cofibration, then $C(i) \to X/A$ is a homotopy equivalence. There is a dual notion for fibrations. Suppose $p : Y \to X$ is a fibration and let $F = p^{-1}(x_0)$ be the fibre of a basepoint $x_0 \in X$. Then there is a diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F(p) \arrow[r] & Y \arrow[d, "p"] \\ \{x_0\} \arrow[rr] & & X \end{tikzcd}$$

where $F \to F(p)$ is the map $y \mapsto (y, c_{x_0})$ for the constant map c_{x_0} at x_0.

Proposition 1.4.2. *If $p : Y \to X$ is a fibration, then $F = p^{-1}(x_0) \to F(p)$ is a homotopy equivalence.*

Lemma 1.4.3. *For any map $f : Y \to X$, $p(f) : F(f) \to Y$ is a fibration.*

Thus we can continue to build our homotopy sequence as in Section 1.2:

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega Y \arrow[r] & F(p(f)) \arrow[r, "p^2(f)"] & F(f) \arrow[r, "p(f)"] & Y \arrow[r, "f"] & X \\
& & p(f)^{-1}(c_{y_0}) = \Omega X \arrow[u, "\simeq"] \arrow[ur, "i(f)"'] & & &
\end{tikzcd}$$

Theorem 1.4.4 (Puppe Fibration Sequence)**.** *For any map $f : Y \to X$, there is a long exact sequence of sets for any W:*

$$\cdots \to [W, \Omega F(f)] \to [W, \Omega Y] \to [W, \Omega X] \to [W, F(f)] \to [W, Y] \to [W, X].$$

Corollary 1.4.5. *Suppose $p : E \to B$ is a based fibration with fibre $F = p^{-1}(b_0)$. Then*

$$\cdots \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B$$

is a sequence of fibrations in which every map (but the last) is an inclusion of a fibre in a fibration. Moreover, for any W there is a long exact sequence of sets

$$\cdots \to [W, \Omega F] \to [W, \Omega E] \to [W, \Omega B] \to [W, F] \to [W, E] \to [W, B].$$

Corollary 1.4.6 (Homotopy Long Exact Sequence)**.** *For any based fibration $p : E \to B$ with homotopy fibre F, there is a long exact sequence of groups*

$$\cdots \to \pi_2(F) \to \pi_2(E) \to \pi_2(B) \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to \pi_0(F) \to \pi_0(E) \to \pi_0(B).$$

Corollary 1.4.7. *If $p : Y \to X$ is a covering map, then $p_* : \pi_n(Y) \to \pi_n(X)$ is an isomorphism for all $n > 1$.*

Proof. For any covering space, the fibre $F \hookrightarrow Y$ is a discrete set, so $\pi_n(F) = 0$ for all $n > 0$. Apply the long exact sequence of homotopy groups. □

Corollary 1.4.8. *For any spaces B, F and any $n \geq 0$, there is an isomorphism $\pi_n(B \times F) \cong \pi_n(B) \times \pi_n(F)$.*

Proof. Apply the long exact sequence to the trivial fibration $F \hookrightarrow B \times F \to B$. □

1.5 Fibrations and Bundles

In this section we prove that every fibre bundle is a fibration.

Definition. *A map $p : E \to B$ is a* **local fibration** *if there exists an open cover $\{U_\alpha\}$ of B such that each $p^{-1}(U_\alpha) \to U_\alpha$ is a fibration.*

Theorem 1.5.1 (Local-to-Global)**.** *If B is a paracompact space then any local fibration $p : E \to B$ is a fibration.*

Proof. (Sketch) Given a diagram

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\widetilde{H}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & B \end{tikzcd}$$

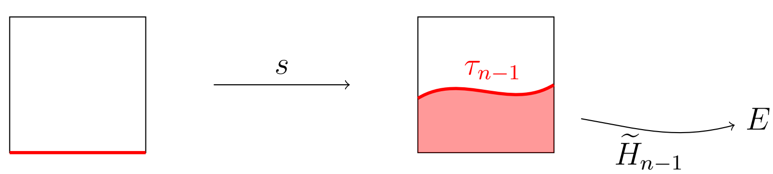
we want to construct an extension $\widetilde{H}$ of $\widetilde{H}_0$. Assume $\{U_\alpha\}$ is a cover such that p is a fibration on each $p^{-1}(U_\alpha)$. Since B is paracompact, there exists a partition of unity $\{\varphi_\alpha\}$ subordinate to the U_α. We may assume $\{U_\alpha\}$ is countable, so that we have $U_1, U_2, \ldots$ For each $n \geq 1$, set $\tau_n = \varphi_1 + \ldots + \varphi_n$; also set $\tau_0 = 0$. Now define

$$X_n = \{(x,t) \in X \times I \mid t \leq \tau_n(x)\}.$$

Then $X_0 = X \times \{0\}$ and we have a chain of subspaces

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq \bigcup_{n=1}^{\infty} X_n = X \times I.$$

Set $\widetilde{H}_0 = H_0$. We construct an extension $\widetilde{H}_n : X_n \to E$ for each n inductively. Given $\widetilde{H}_{n-1}$, consider the map $s : X \times I \to X \times I$ given by $(x,t) \mapsto (x, \min(\tau_{n-1}(x) + t, 1))$.



We may restrict to an open set $V_n \subseteq X \times I$ such that $H \circ s(V_n) \subseteq U_n$. Then the fibration property on U_n gives a lift $K : V_n \to E$ in the following diagram:

$$\begin{tikzcd} V_n \cap X \times \{0\} \arrow[rr] \arrow[d, hook] & & \Gamma(\tau_n) \arrow[r] & E \arrow[d, "p"] \\ V_n \arrow[r] \arrow[urrr, dashed, "K"] & X \times I \arrow[r, "s"] & X \times I \arrow[r] & B \end{tikzcd}$$

where $\Gamma(\tau_n)$ is the graph of τ_n. Then define the extension $\widetilde{H}_n : X_n \to E$ by

$$\widetilde{H}_n(x,t) = \begin{cases} \widetilde{H}_{n-1}(x,t), & \text{if } (x,t) \in X_{n-1} \\ K(x,t), & \text{if } (x,t) \in X_n \smallsetminus X_{n-1}. \end{cases}$$

It is easy to check that $\widetilde{H}_n$ has the desired properties, so we are done by induction. $\square$

Recall that $p : E \to B$ is a (locally trivial) fibre bundle with fibre F if there exists an open cover $\{U_\alpha\}$ of B and homeomorphisms

$$\begin{tikzcd} U_\alpha \times F \arrow[rr, "\sim"] \arrow[dr] & & p^{-1}(U_\alpha) \arrow[dl, "p"] \\ & U_\alpha & \end{tikzcd}$$

Corollary 1.5.2. *Every fibre bundle is a fibration.*

Proof. Each $U_\alpha \times F \to U_\alpha$ is a fibration by Example 1.3.2, so p is a local fibration. Apply Theorem 1.5.1. $\square$

Example 1.5.3. If G is a discrete group acting freely and properly discontinuously on a space X and H is any subgroup of G, then $X/H \to X/G$ is a covering map. Conversely, when Y is a nice enough space and X is its universal cover, then $G = \pi_1(Y)$ acts freely and properly discontinuously on X and every cover of X has the form $X/H \to X/G = Y$ for some subgroup $H \leq G$. One might hope that a similar result would hold if G is a group object in a different category. For example, if G is a Lie group acting smoothly, freely and properly discontinuously on a manifold M, then for all *closed* subgroups $H \leq G$, $M/H \to M/G$ is a fibre bundle with fibre G/H.

Example 1.5.4. View S^{2n+1} as the set of unit vectors in $\mathbb{C}^{n+1}$ and let S^1 be the circle group of complex numbers with modulus 1. Then S^1 acts freely and properly discontinuously on S^{2n+1} via $\lambda(x_0, \ldots, x_n) = (\lambda x_0, \ldots, \lambda x_n)$ for any $\lambda \in S^1$. This determines a fibration

$$S^1 \to S^{2n+1} \to \mathbb{C}P^n.$$

In particular, when $n = 1$, we get the *Hopf fibration* $S^1 \to S^3 \to S^2 = \mathbb{C}P^1$. Applying Corollary 1.4.6, we get the following information:

$$\begin{aligned} \pi_n(S^3) &\cong \pi_n(S^2) \quad \text{for } n \geq 3 \\ \text{and} \quad \pi_2(S^2) &\cong \pi_1(S^1) \cong \mathbb{Z}. \end{aligned}$$

A *vector bundle* is a fibre bundle $p : E \to B$ in which each fibre $p^{-1}(b)$ is a (real) vector space of dimension n and the local trivializations of the bundle, $h_i : U_i \times \mathbb{R}^n \to p^{-1}(U_i)$, induce vector space isomorphisms

$$\{x\} \times \mathbb{R}^n \xrightarrow{h_i} p^{-1}(x) \xrightarrow{h_j^{-1}} \{x\} \times \mathbb{R}^n$$

whenever $x \in U_i \cap U_j$. We say two vector bundles $E \xrightarrow{p} B$ and $E' \xrightarrow{p'} B$ are isomorphic if there is a homeomorphism over B

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

inducing linear isomorphisms on each fibre. A bundle is called *trivial* if it is isomorphic to a product bundle of the form $B \times \mathbb{R}^n \to B$.

Example 1.5.5. The *tangent bundle* to a smooth manifold is a vector bundle $TM \to M$ whose fibres are the tangent spaces T_xM at each $x \in M$.

For a map $f : B' \to B$ and any bundle $p : E \to B$, we define the *pullback bundle* $f^*p : f^*E \to B'$ by setting $f^*E = \{(x, y) \in B' \times E \mid f(x) = p(y)\}$ and taking f^*p to be the natural projection.

For a fixed space B, let $\mathrm{Vect}_n(B)$ be the set of all isomorphism classes of n-dimensional vector bundles on B.

Theorem 1.5.6. *For any n,*

(1) $\mathrm{Vect}_n(-)$ *is a contravariant functor under pullback.*

(2) If $f, g : X \to B$ are homotopic maps, then f^*E and g^*E are homotopy equivalent for any bundle $E \to B$.

That is, $\mathrm{Vect}_n(-)$ is a homotopy functor, so it factors through the homotopy category: $\mathrm{Vect}_n(-) : h(\mathtt{Top})^{op} \to \mathtt{Sets}$.

Example 1.5.7. Recall that for a smooth n-manifold M, a map $f : M \to \mathbb{R}^\ell$ is an immersion if $d_xf : T_xM \to \mathbb{R}^\ell$ is a one-to-one linear map for all $x \in M$. An important problem in geometric topology is to find the smallest ℓ so that there exists an immersion $f : M \to \mathbb{R}^\ell$.

Let $\mathrm{Gr}_n(\mathbb{R}^\ell)$ be the nth Grassmannian of $\mathbb{R}^\ell$ and set

$$\mathrm{Gr}_n = \mathrm{Gr}_n(\mathbb{R}^\infty) = \bigcup_{\ell=1}^{\infty} \mathrm{Gr}_n(\mathbb{R}^\ell).$$

Then one can define a *canonical bundle* $\gamma_\ell^n : E_n(\mathbb{R}^\ell) \to \mathrm{Gr}_n(\mathbb{R}^\ell)$ by $E_n(\mathbb{R}^\ell) = \{(V, v) \in \mathrm{Gr}_n(\mathbb{R}^\ell) \times \mathbb{R}^\ell \mid v \in V\}$. This extends to a so-called *universal bundle* $\gamma^n : E_n \to \mathrm{Gr}_n$ which has the property that for all paracompact spaces X,

$$\begin{aligned} [X, \mathrm{Gr}_n] &\longrightarrow \mathrm{Vect}_n(X) \\ f &\longmapsto f^*E_n \end{aligned}$$

is an isomorphism. Now if $f : M \to \mathbb{R}^\ell$ is an immersion, there is a natural map

$$\begin{aligned} \tau_f : M &\longrightarrow \mathrm{Gr}_n(\mathbb{R}^\ell) \\ x &\longmapsto d_xf(T_xM) \end{aligned}$$

such that $TM \cong \tau_f^* E_n(\mathbb{R}^\ell)$ as bundles over M. It turns out that if an immersion $f : M \to \mathbb{R}^\infty$ exists, the homotopy class of τ_f is independent of f. Let τ_M denote the homotopy class of τ_f for any immersion $f : M \to \mathbb{R}^\infty$.

Suppose M admits an immersion $f : M \to \mathbb{R}^\ell$ for some ℓ. Then the diagram

$$\begin{tikzcd} & \mathrm{Gr}_n(\mathbb{R}^\ell) \arrow[d, "i_\ell"] \\ M \arrow[ur, "\tau_f"] \arrow[r, "\tau_M"'] & \mathrm{Gr}_n \end{tikzcd}$$

commutes up to homotopy. Applying $\mathbb{Z}/2\mathbb{Z}$-cohomology, we get

$$\begin{tikzcd} & H^\bullet(\mathrm{Gr}_n(\mathbb{R}^\ell); \mathbb{Z}/2\mathbb{Z}) \arrow[dl, "\tau_f^*"'] \\ H^\bullet(M; \mathbb{Z}/2\mathbb{Z}) & H^\bullet(\mathrm{Gr}_n; \mathbb{Z}/2\mathbb{Z}) \arrow[l, "\tau_M^*"] \arrow[u, "i_\ell^*"'] \end{tikzcd}$$

which implies $\ker i_\ell^* \subseteq \ker \tau_M^*$. In fact, the converse holds as well, so M admits an immersion into $\mathbb{R}^\ell$ if and only if $\ker i_\ell^* \subseteq \ker \tau_M^*$. This provides us with an algebraic obstruction to check rather than a topological one.

1.6 Serre Fibrations

Consider the defining diagram for a map $p : E \to B$ to be a fibration:

$$\begin{tikzcd} X \times \{0\} \arrow[r] \arrow[d, hook, "i"'] & E \arrow[d, "p"] \\ X \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

Some nice features of the map i are that it's both a cofibration and a homotopy equivalence. This generalizes as follows.

Theorem 1.6.1 (Strøm). *Suppose $i : A \to X$ is a cofibration, $p : E \to B$ is a fibration and there is a pair of maps*

$$\begin{tikzcd} A \arrow[r, "f_0"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r, "f_1"'] \arrow[ur, dashed, "f"] & B \end{tikzcd}$$

Then if either i or p is a homotopy equivalence, there exists an $f : X \to E$ making the diagram commute.

To prove Strøm's theorem, we need:

Lemma 1.6.2. *If $i : A \to X$ is a cofibration and a homotopy equivalence, then there are maps $r : X \to A, s : X \to X \times I$ and $H : X \times I \to X$ making the diagram*

$$\begin{tikzcd}
A \arrow[rr, "id"] \arrow[dr, "i"] \arrow[ddd, "i"'] & & A \arrow[ddd, "i"] \\
& X \arrow[ur, "r"'] \arrow[d, hook] & \\
& X \times I \arrow[dr, "H"] & \\
X \arrow[ur, "s"] \arrow[rr, "id"'] & & X
\end{tikzcd}$$

commute.

Proof. Since i is a cofibration, by Lemma 1.1.2 there exists a $u : X \to I$ such that $u^{-1}(0) = A$. One can modify this to give a deformation retract $r : X \to A$ together with a homotopy $H' : X \times I \to X$ satisfying $H'_0 = i \circ r, H'_1 = id_X$ and $H'_t|_A = id_A$ for all $t \in I$. Define $s : X \to X \times I$ by $s(x) = (x, u(x))$ and $H : X \times I \to X$ by

$$H(x,t) = \begin{cases} H'\left(x, \frac{t}{u(x)}\right), & \text{if } t < u(x) \\ H'(x,1), & \text{if } t \geq u(x). \end{cases}$$

Then r, s and H make the diagram commute by construction, but we must check H is continuous. Let $C = \{(x,t) \in X \times I \mid t \leq u(x)\}$. Then we may write H' as the composition $X \times I \xrightarrow{K} C \xrightarrow{H|_C} X$, where $K(x,t) = (x, tu(x))$, and since H' and K are continuous, so must be H. $\square$

Proof of Strøm's Theorem. Suppose i is a homotopy equivalence. Then Lemma 1.6.2 gives maps r, s, H and a diagram:

$$\begin{tikzcd}
A \arrow[rr, "id"] \arrow[dr, "i"] \arrow[ddd, "i"'] & & A \arrow[r, "f_0"] & E \arrow[ddd, "p"] \\
& X \arrow[ur, "r"'] \arrow[d, hook] & & \\
& X \times I \arrow[dr, "H"] \arrow[uurr, dashed, "\tilde{f}_0"] & & \\
X \arrow[ur, "s"] \arrow[rr, "id"'] & & X \arrow[r, "f_1"'] & B
\end{tikzcd}$$

Letting $f = s \circ \tilde{f}_0$ gives the desired lift. The proof when p is a homotopy equivalence is similar. $\square$

The following is a useful variant on fibrations defined by Serre.

Definition. *A map $p : E \to B$ is a* **Serre fibration** *if the diagram*

$$\begin{tikzcd} D^k \times \{0\} \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ D^k \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

can be completed for any disk D^k.

A fibration in the usual sense is sometimes referred to as a *Hurewicz fibration* to distinguish from Serre fibrations. The following is an analogy of Strøm's theorem for Serre fibrations.

Theorem 1.6.3. *Suppose $i : A \to X$ is a relative CW-complex, $p : E \to B$ is a Serre fibration and there is a pair of maps*

$$\begin{tikzcd} A \arrow[r, "f_0"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r, "f_1"'] \arrow[ur, dashed, "f"] & B \end{tikzcd}$$

Then if either i or p is a weak homotopy equivalence, there exists an $f : X \to E$ making the diagram commute.

2 Cellular Theory

2.1 Relative CW-Complexes

Recall that X is a CW-complex if there is a chain of subspaces $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ such that $X = \bigcup_{n=0}^{\infty} X_n$, X_0 is discrete and, inductively, each X_n is the pushout of

$$\begin{tikzcd} \coprod S^{n-1} \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ \coprod D^n \arrow[r] & X_n \end{tikzcd}$$

Definition. *A pair of spaces (X, A) is a* **relative CW-pair** *if there is a chain of subspaces*

$$A =: X_{-1} \subseteq X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$$

such that $X = \bigcup_{n=0}^{\infty} X_n$ and, inductively, each X_n is the pushout of

$$\begin{tikzcd} \coprod S^{n-1} \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ \coprod D^n \arrow[r] & X_n \end{tikzcd}$$

where $S^{-1} = \varnothing$.

Definition. *A* **subcomplex** *of a CW-complex X is a subspace $A \subseteq X$ and a CW-structure $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq \bigcup_{n=0}^{\infty} A_n = A$ such that for each cell $D^n \to A$, the composition $D^n \to A \hookrightarrow X$ is a cell of X.*

In other words, a subcomplex of a CW-complex is just a union of some collection of cells of the CW-complex. If $A \subseteq X$ is a subcomplex, then (X, A) is a relative CW-pair.

Recall that a map $f : X \to Y$ between CW-complexes is *cellular* if $f(X_n) \subseteq Y_n$ for all n. The following results are basic.

Lemma 2.1.1. *If (X, A) is a relative CW-pair then X/A is a CW-complex with a 0-cell corresponding to A and an n-cell for each relative n-cell of (X, A).*

Lemma 2.1.2. *Suppose $\{X_i\}$ are CW-complexes with specified 0-cells $x_i \in X_i$. Then $X = \bigvee X_i$ is a CW-complex and each $X_i \subseteq X$ is a subcomplex.*

Lemma 2.1.3. *Suppose $A \subseteq X$ is a subcomplex, Y is a CW-complex and $f : A \to Y$ is a cellular map. Then the pushout $Y \cup_f X$ is a CW-complex having Y as a subcomplex. Moreover, $(Y \cup_f A)/Y \cong X/A$ as CW-complexes.*

Lemma 2.1.4. *If $X_0 \to X_1 \to X_2 \to \cdots$ is a sequence of CW-complexes in which each $X_i \to X_{i+1}$ is an inclusion of a subcomplex, then $X = \varinjlim X_i$ is a CW-complex having each X_i as a subcomplex.*

Lemma 2.1.5. *If X and Y are CW-complexes, then $X \times Y$ with the product topology is a CW-complex with an n-cell $\sigma \times \tau$ for each p-cell $\sigma \subset X$ and q-cell $\tau \subset Y$, where $p + q = n$.*

Example 2.1.6. For any CW-complex X, the product $X \times I$ is a CW-complex containing a subcomplex $X \times \{0, 1\} = X \coprod X$

2.2 Whitehead's First Theorem

Definition. *Let (X, x) and (Y, y) be based space. A based map $f : X \to Y$ is a* **weak equivalence** *if $f_* : \pi_0(X) \to \pi_0(Y)$ is a bijection and $f_* : \pi_k(X, x) \to \pi_k(Y, y)$ is an isomorphism for all $k \geq 1$.*

Whitehead's theorem says that any weak equivalence between CW-complexes is a homotopy equivalence. To prove this, we need some technical results that are useful in their own right.

Lemma 2.2.1. *If $f : X \to Y$ is a weak equivalence and $F(f)$ is its homotopy fibre, then $\pi_k(F(f)) = 0$ for all k.*

Proof. Apply the homotopy long exact sequence (Corollary 1.4.6). □

For a map $f : X \to Y$ with homotopy fibre $F = F(f)$, we make the following observations:

- Any map $S^{n-1} \to F$ corresponds to a diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "g"] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\ D^n \arrow[r, "h"'] & Y \end{tikzcd}$$

  which commutes. This gives a natural description of $\pi_{n-1}(F)$.

- The map $\pi_n(Y) \to \pi_{n-1}(F)$ in the long exact sequence of homotopy groups (Corollary 1.4.6) corresponds to the map sending the class of $\bar{h} : S^n \to Y$ to the class of $\pi_{n-1}(F)$ represented by the diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "c"] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\ D^n \arrow[r, "h"'] & Y \end{tikzcd}$$

where $c = c_x$ is constant and h is $D^n \to D^n/S^{n-1} = S^n \xrightarrow{\bar{h}} Y$. In a similar manner, $\pi_{n-1}(F) \to \pi_{n-1}(X)$ corresponds to sending a diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "g"] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\ D^n \arrow[r, "h"'] & Y \end{tikzcd}$$

to the class $[g]$.

- $\pi_{n-1}(F) = 0$ is equivalent to completing such a diagram to a diagram

$$\begin{tikzcd} S^{n-1} \arrow[rr, "g"] \arrow[dr] \arrow[ddd, "i"'] & & X \arrow[ddd, "f"] \\ & CS^{n-1} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\ & CD^n \arrow[dr, dashed, "H"] & \\ D^n \arrow[ur] \arrow[rr, "h"'] & & Y \end{tikzcd}$$

This is stated slightly differently in the following lemma.

Lemma 2.2.2. *Suppose $f : X \to Y$ is a map with homotopy fibre F. Then $\pi_{n-1}(F) = 0$ if and only if each diagram*

$$\begin{tikzcd} S^{n-1} \times \{1\} \arrow[r, "g"] \arrow[d, hook] & X \arrow[d, "f"] \\ S^{n-1} \times I \cup D^n \cup \{0\} \arrow[r, "h"'] & Y \end{tikzcd}$$

can be completed to a diagram

$$\begin{tikzcd}
S^{n-1} \times \{1\} \arrow[rr, "g"] \arrow[dr, hook] \arrow[ddd, hook] & & X \arrow[ddd, "f"] \\
& D^n \times \{1\} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& D^n \times I \arrow[dr, dashed, "H"] & \\
S^{n-1} \times I \cup D^n \times \{0\} \arrow[ur, hook] \arrow[rr, "h"'] & & Y
\end{tikzcd}$$

Proof. For any disk we have a homeomorphism $CD^n \cong D^n \times I$ which sends the cone point to the center of $D^n \times \{1\}$, D^n to $S^{n-1} \times I \cup D^n \times \{0\}$, S^{n-1} to $S^{n-1} \times \{1\}$ and CS^{n-1} to $D^n \times \{1\}$. Thus the statement follows from the last observation above. $\square$

This generalizes to the so-called homotopy extension and lifting property, or HELP for short.

Theorem 2.2.3 (Homotopy Extension and Lifting Property)**.** *Suppose (X, A) is a relative CW-pair and $f : Y \to Z$ is a weak equivalence. Then every diagram*

$$\begin{tikzcd}
A \times \{1\} \arrow[r, "g"] \arrow[d, hook] & Y \arrow[d, "f"] \\
A \times I \cup X \cup \{0\} \arrow[r, "h"'] & Z
\end{tikzcd}$$

can be completed to a diagram

$$\begin{tikzcd}
A \times \{1\} \arrow[rr, "g"] \arrow[dr, hook] \arrow[ddd, hook] & & Y \arrow[ddd, "f"] \\
& X \times \{1\} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& X \times I \arrow[dr, dashed, "H"] & \\
A \times I \cup X \times \{0\} \arrow[ur, hook] \arrow[rr, "h"'] & & Z
\end{tikzcd}$$

Proof. We construct $G_n : X_n \times \{1\} \to Y$ and $H_n : X_n \times I \to Z$ inductively. The base case is trivial, and for the inductive step, one can extend cell-by-cell using Lemma 2.2.2 applied to the diagram

$$\begin{tikzcd}
X_{n-1} \times \{1\} \arrow[rr, "G_{n-1}"] \arrow[dr, hook] \arrow[ddd, hook] & & Y \arrow[ddd, "f"] \\
& X_n \times \{1\} \arrow[ur, dashed, "G_n"] \arrow[d, hook] & \\
& X_n \times I \arrow[dr, dashed, "H_n"] & \\
X_{n-1} \times I \cup X_n \times \{0\} \arrow[ur, hook] \arrow[rr, "H_{n-1} \cup h"'] & & Z
\end{tikzcd}$$

$\square$

Lemma 2.2.4. *For any weak equivalence $f : Y \to Z$ and any CW-complex X, the induced map $f_* : [X, Y] \to [X, Z]$ is a bijection.*

Proof. For surjectivity, consider the relative CW-pair $(X, \varnothing)$. Then for every $h : X \to Z$ there is a diagram

$$\begin{tikzcd}
* \arrow[rr] \arrow[dr, hook] \arrow[ddd, hook] & & Y \arrow[ddd, "f"] \\
& X \times \{1\} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& X \times I \arrow[dr, dashed, "H"] & \\
X \times \{0\} \arrow[ur, hook] \arrow[rr, "h"'] & & Z
\end{tikzcd}$$

By HELP, there is a map $G : X = X \times \{1\} \to Y$ and a homotopy $H : X \times I \to Z$ satisfying $H_0 = h$ and $H_1 = f \circ G$. Therefore $[h] = f_*[G]$. Similarly, we can show injectivity by analyzing the relative CW-pair $(X \times I, X \times \{0, 1\})$ using HELP. $\square$

Theorem 2.2.5 (First Whitehead Theorem)**.** *If $f : X \to Y$ is a weak equivalence between CW-complexes, then f is a homotopy equivalence.*

Proof. Take $X = Z$. Then by Lemma 2.2.4 there is a map $g : Z \to Y$ which is unique up to homotopy and satisfies $f \circ g \simeq 1_Z$. On the other hand, letting $X = Y$ and applying Lemma 2.2.4 shows that $f_* : [Y, Y] \to [Y, Z]$ is a bijection, so

$$f_*[g \circ f] = [f \circ (g \circ f)] = [(f \circ g) \circ f] = [id_Z \circ f] = [f] = f_*[id_Y].$$

Hence $g \circ f \simeq id_Y$ so f is a homotopy equivalence with homotopy inverse g. $\square$

Corollary 2.2.6. *If X is a CW-complex such that $\pi_k(X) = 0$ for every k, then X is contractible.*

Proof. Apply Whitehead's theorem to the map $X \to *$. $\square$

3 Higher Homotopy Groups

In this chapter we study the homotopy groups $\pi_n(X) = [S^n, X]$. Recall that $\pi_n(X)$ is a group if $n \geq 1$ and is abelian if $n \geq 2$.

When (X, A) is a pair of based spaces, the inclusion $A \to X$ has homotopy fibre F which can be interpreted as the relative path space $P(X, A) = \{\gamma : I \to X \mid \gamma(0) = x_0, \gamma(1) \in A\}$.

Definition. *For a pair (X, A), the n***th relative homotopy group** *is $\pi_n(X, A) := \pi_{n-1}(P(X, A))$.*

It follows Corollary 0.2.5 that $\pi_n(X, A) = \pi_{n-1}(P(X, A)) = \ldots = \pi_0(\Omega^{n-1}P(X, A))$. In particular, $\pi_n(X, A)$ is a group if $n \geq 2$ and an abelian group if $n \geq 3$.

Remark. The relative homotopy groups may equivalently be defined by homotopy classes of maps of pairs $\pi_n(X, A) = [(D^n, S^{n-1}), (X, A)]$.

Proposition 3.0.1. *For all pairs (X, A), there is a long exact sequence*

$$\cdots \to \pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A) \to \cdots$$

Proof. This is just Corollary 1.4.6. □

3.1 n-Connectedness

In the last chapter, we defined a weak equivalence to be a map $f : X \to Y$ which induces an isomorphism $\pi_k(X) \to \pi_k(Y)$ for all k. In this section, we give a bounded version of weak equivalence which satisfies an analogue of Whitehead's theorem.

Definition. *A space X is n-***connected** *if $\pi_k(X) = 0$ for all $k \leq n$. We interpret 0-connected to mean that X is path-connected.*

Example 3.1.1. Saying a space is 1-connected is the same as saying it is simply connected.

Definition. *A map $f : X \to Y$ is an n-***equivalence***, or is an n-***connected map***, if the homotopy fibre $F(f)$ is $(n-1)$-connected.*

Lemma 3.1.2. *For a map $f : X \to Y$ with homotopy fibre $F = F(f)$, the following are equivalent:*

(1) F is $(n-1)$-connected, i.e. f is n-connected.

(2) The induced map $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for all $k < n$ and is surjective for $k = n$.

(3) $(Y, f(X))$ is an n-connected pair, i.e. $\pi_k(Y, f(X)) = 0$ for all $k \leq n$.

Proof. (1) $\iff$ (2) follows from the homotopy long exact sequence in Corollary 1.4.6.

(2) $\iff$ (3) follows from Proposition 3.0.1. □

Theorem 3.1.3 (Whitehead). *If $f : Y \to Z$ is n-connected then for any CW-complex X, the induced map $f_* : [X, Y] \to [X, Z]$ is an isomorphism if $n < \dim X$ and is surjective if $n = \dim X$.*

Proof. This follows from the proof of Lemma 2.2.4. □

Corollary 3.1.4. *If X and Y are CW-complexes of dimensions $\dim X, \dim Y < n$ and $f : X \to Y$ is an n-equivalence, then f is a homotopy equivalence.*

Suppose X and Y are CW-complexes with n-skeleta X_n and Y_n, respectively. Recall that the nth cellular chain group of X is defined as $C_n^{cell}(X) = H_n(X_n, X_{n-1})$. If $f : X \to Y$ is a cellular map, then there is an induced map

$$f_* : C_\bullet^{cell}(X) \longrightarrow C_\bullet^{cell}(Y)$$

but this need not be true for general f. However, the cellular approximation theorem says that an arbitrary map $f : X \to Y$ is homotopic to a cellular map, and any two cellular maps that are homotopic are in fact homotopic via a cellular homotopy. We generalize this in the theorem below, after the following lemma.

Lemma 3.1.5. *If Z is obtained from Y by attaching cells of dimension greater than n, then $\pi_k(Z, Y) = 0$ for all $k \leq n$, i.e. (Z, Y) is n-connected.*

Proof. (Sketch) We can reduce to the case of attaching a single cell, $Z = Y \cup_\alpha D^r$ for an attaching map $\alpha : S^{r-1} \to Y$, where $r \geq n + 1$. Then by the remark in the introduction, $\pi_n(Z, Y)$ corresponds to pairs of maps

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ D^n \arrow[r] & Y \cup D^r \end{tikzcd}$$

The result then follows using simplicial (or smooth) approximation, Sard's theorem to bound the image of the pair and then a retraction of pairs. □

Theorem 3.1.6. *Suppose $f : (X, A) \to (Y, B)$ is a map of relative CW-pairs. Then f is homotopic rel A to a cellular map of pairs.*

Proof. Assume X_0 is equal to the union of A and some discrete set of points, and likewise for $B \subseteq Y_0$, so that there are sequences of subcomplexes $A \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X$ and $B \subseteq Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y$. By induction, we may assume there is a map $g_{n-1} : X_{n-1} \to Y_{n-1}$ and a homotopy $H : X_{n-1} \times I \to Y$ such that $H_0 = f$ and $H_1 = g_{n-1}$. Consider the diagram

$$\begin{tikzcd}
X_{n-1} \times \{1\} \arrow[rr, "g_{n-1}"] \arrow[dr, hook] \arrow[ddd, hook] & & Y_{n-1} \arrow[ddd, hook] \\
& X_n \times \{1\} \arrow[ur, dashed, "g_n"] \arrow[d, hook] & \\
& X_n \times I \arrow[dr, dashed, "\widetilde{H}"] & \\
X_{n-1} \times I \cup X_n \times \{0\} \arrow[ur, hook] \arrow[rr, "H \cup f"'] & & Y
\end{tikzcd}$$

By HELP (Theorem 2.2.3), there exist maps $g_n : X_n = X_n \times \{1\} \to Y_{n-1}$ and $\widetilde{H} : X_n \times I \to Y$ completing the diagram as shown, as long as $\pi_k(Y, Y_n) = 0$ for all $k \leq n$. However, by Lemma 3.1.5 this holds. Thus g_n and $\widetilde{H}$ extend g_{n-1} and H to the n-skeleton. By induction, we can extend to all of (X, A). □

3.2 The Blakers-Massey Theorem

Consider the homotopy groups as functors

$$\begin{aligned} \pi_n : \texttt{Top} \times \texttt{Top} &\longrightarrow \texttt{Groups} \\ (X, A) &\longmapsto \pi_n(X, A). \end{aligned}$$

It is known that $\pi_n(-,-)$ are homotopy functors, and by Proposition 3.0.1 we know there is a long exact sequence $\cdots \to \pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A) \to \cdots$ for any pair (X, A). These are almost all of the Eilenberg-Steenrod axioms for a homology theory (that π_n satisfies the dimension axiom is obvious), but excision is missing. In fact, excision fails in general as the following example shows.

Example 3.2.1. We know that $\pi_k(S^1) = \mathbb{Z}$ if $k = 1$ and is 0 otherwise. If the excision axiom held for homotopy groups, then the embedding $S^1 \hookrightarrow S^2$ would imply $\pi_{k+1}(S^2) \cong \pi_k(S^1)$ for all k, but this is false since e.g. $\pi_3(S^2) \neq 0$ by Example 1.2.13.

However, the Seifert-van Kampen theorem gives an excision-type result when the spaces involved satisfy certain conditions (path-connected with contractible intersection). The Blakers-Massey theorem generalizes this considerably.

Theorem 3.2.2 (Blakers-Massey)**.** *Suppose $A, B \subseteq X$ are subsets and $m, n \geq 0$ are integers such that either $m \geq 1$ or $n \geq 1$. If $(A, A \cap B)$ is an m-connected pair and $(B, A \cap B)$ is an n-connected pair, then the map of pairs $(A, A \cap B) \to (A \cup B, B)$ is $(m+n)$-connected.*

Proof. See May (called the Homotopy Excision Theorem) or tom Dieck. □

Corollary 3.2.3. *For $A, B \subseteq X$ such that $(A, A \cap B)$ is m-connected and $(B, A \cap B)$ is n-connected, there is a long exact sequence*

$$0 \to \pi_{m+n}(A \cap B) \to \pi_{m+n}(A) \oplus \pi_{m+n}(B) \to \pi_{m+n}(A \cup B) \to \pi_{m+n-1}(A \cap B) \to \cdots$$

Example 3.2.4. Let X be a space and consider the suspension diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X \end{tikzcd}$$

Then (CX, X) is m-connected if and only if X is $(m-1)$-connected. This means that if X is $(m-1)$-connected, the map $\pi_k(CX, X) \to \pi_k(\Sigma X, CX)$ is an isomorphism for $k < 2m$ and is surjective for $k = 2m$, by Theorem 3.2.2. But since CX is contractible, the long exact sequence implies $\pi_k(CX, X) \cong \pi_{k-1}(X)$ and $\pi_k(\Sigma X, CX) \cong \pi_k(\Sigma X)$ for all k. In fact, the resulting isomorphism $\pi_{k-1}(X) \to \pi_k(\Sigma X)$ is just the map induced by suspension, which proves the following important result.

Corollary 3.2.5 (Freudenthal Suspension Theorem)**.** *Suppose X is $(m-1)$-connected. Then the suspension functor $\Sigma : X \to \Sigma X$ induces $\pi_{k-1}(X) \to \pi_k(\Sigma X)$ which is an isomorphism for $k < 2m$ and a surjection for $k = 2m$.*

Note that the map $\pi_k(X) \to \pi_{k+1}(\Sigma X) \cong \pi_k(\Omega\Sigma X)$ induced by $X \to \Omega\Sigma X$ is the adjoint to the identity $id : \Sigma X \to \Sigma X$.

Corollary 3.2.6. *If X is $(n-1)$-connected, then the map $[K, X] \to [\Sigma K, \Sigma X]$ is a bijection if K is a CW-complex of dimension less than $2n-1$ and is a surjection if K is a CW-complex of dimension equal to $2n-1$.*

Proof. Apply Freudenthal's suspension theorem and Whitehead's theorem (2.2.5). ☐

Corollary 3.2.7. *Let $n \geq 1$. Then $\pi_k(S^n) = 0$ if $k < n$ and $\pi_n(S^n) \cong \mathbb{Z}$.*

Proof. This is true for $n = 1$ by standard computations. To induct, assume S^n is $(n-1)$-connected. Then by Freudenthal's suspension theorem, $\pi_k(S^n) \to \pi_{k+1}(S^{n+1})$ is an isomorphism for all $k < 2n-1$ and is surjective for $k = 2n-1$. This directly implies both statements. ☐

Corollary 3.2.8. *The map $\pi_n(S^n) \to H_n(S^n)$ is an isomorphism for all $n \geq 1$.*

Corollary 3.2.9. *The Hopf map $\mathcal{H} : \pi_3(S^2) \to \mathbb{Z}$ is an isomorphism.*

Proof. By Example 1.5.4, $\pi_3(S^2) \cong \pi_3(S^3)$ but Corollary 3.2.7 shows that $\pi_3(S^3) \cong \mathbb{Z}$. ☐

Corollary 3.2.10. *The Whitehead product $[i_2, i_2]$ (see Example 1.2.13) is homotopic to $2\eta : S^3 \to S^2$.*

Proof. This follows from the previous corollary and the fact that the Hopf map takes $[i_2, i_2]$ to $2 \in \mathbb{Z}$. ☐

More generally, if

$$\begin{tikzcd} C \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

is an "excisive pair", e.g. $C = A \cap B$ and $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$ as in the situation above, or the diagram is a pushout of CW-complexes, or the diagram is a homotopy pushout, then we have a similar statement.

Theorem 3.2.11. *If $((A, C), (X, B))$ is an excisive pair, (A, C) is m-connected and (B, C) is n-connected, then the map of pairs $(A, C) \to (X, B)$ is an $(m + n)$-equivalence.*

Example 3.2.12. $X = A \vee B$ fits into a pushout diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & A \vee B \end{tikzcd}$$

If A is m-connected and B is n-connected, the long exact sequence

$$\cdots \to \pi_k(B) \to \pi_k(A \vee B) \to \pi_k(A \vee B, B) \to \pi_{k-1}(B) \to \cdots$$

is split by the map induced by the retraction $r : A \vee B \to B$. This implies that $\pi_k(A \vee B) \cong \pi_k(B) \oplus \pi_k(A \vee B, B)$ for all k. On the other hand, Corollary 1.4.8 shows that $\pi_k(A \times B) \cong \pi_k(A) \oplus \pi_k(B)$ for all $k \geq 2$, so by the Blakers-Massey theorem (3.2.2 or 3.2.11), we get an isomorphism $\pi_k(A) \cong \pi_k(A \vee B, B)$ for all $k \leq m+n$. In particular, $\pi_k(A \vee B) \to \pi_k(A \times B)$ is an isomorphism for all $k \leq m + n$.

If A is a CW-complex obtained by gluing cells of dimension at least m to a 0-cell $*$, and B is the same thing with dimension at least n, then $A \times B$ can be viewed as $A \vee B$ together with cells glued on of dimension at least $m + n$. Thus the Blakers-Massey theorem in some way measures the difference between the product and wedge product of two spaces (at least up to homotopy).

Corollary 3.2.13. *Let $i : A \to X$ be a map. If A is m-connected and (X, A) is n-connected, then $(X, A) \to (X/A, *)$ is an $(m + n + 1)$-equivalence.*

Proof. By Theorem 1.1.25 and Proposition 1.2.2(b), $C(i)$ is homotopy equivalent to X/A. Consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & CA \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r] & C(i) \arrow[r] & X/A \end{tikzcd}$$

Since A is m-connected, (CA, A) is $(m+1)$-connected. Thus two applications of Theorem 3.2.11 give the result. □

For a map $f : Y \to Z$, we constructed a homotopy fibre $F(f)$ and a homotopy cofibre $C(f)$, giving a sequence $F(f) \to Y \to Z \to C(f)$. These can be compared via the diagram

$$\begin{tikzcd} \Omega Z \arrow[r] \arrow[d, equal] & F(f) \arrow[r] \arrow[d, dashed] & Y \arrow[r, "f"] \arrow[d] & Z \arrow[d] \\ \Omega C(f) \arrow[r, equal] & \Omega C(f) \arrow[r] & * \arrow[r] & C(f) \end{tikzcd}$$

in which the right square commutes up to homotopy and there is an interesting map $F(f) \to \Omega C(f)$ filling in the middle square.

Corollary 3.2.14. *If Y is m-connected and $f : Y \to Z$ is an n-equivalence, then the above map $F(f) \to \Omega C(f)$ is an $(m+n)$-equivalence.*

Proof. There is a diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & CY \arrow[d] \\ M_f \arrow[r] & C(f) \end{tikzcd}$$

where M_f is the mapping cylinder of f. Then by Corollary 3.2.13, $\pi_k(M_f, Y) \to \pi_k(C(f))$ is an isomorphism for $k < m+n+1$ and is surjective for $k = m+n+1$. But by Theorem 1.1.25 and Lemma 3.1.2, $\pi_k(M_f, Y) \cong \pi_k(Z, Y) \cong \pi_{k-1}(F(f))$, so the result follows. □

Example 3.2.15. By Freudenthal's suspension theorem (Corollary 3.2.5), the suspension functor induces a sequence

$$\mathbb{Z} \cong \pi_3(S^2) \twoheadrightarrow \pi_4(S^3) \xrightarrow{\sim} \pi_5(S^4) \xrightarrow{\cong} \cdots$$

where $\pi_3(S^2) = \langle \eta \rangle$ for the Hopf fibration η (by Corollary 3.2.9). Since $\Sigma[i_2, i_2]$ is nullhomotopic (Example 1.2.13), Corollary 3.2.10 implies that 2η lies in the kernel of $\pi_3(S^2) \to \pi_4(S^3)$. Thus $\pi_4(S^3)$ (and all subsequent $\pi_{n+1}(S^n)$ groups) is either 0 or $\mathbb{Z}/2\mathbb{Z}$ according to whether $\Sigma\eta$ is null or not.

Consider the cofibration sequence $S^3 \xrightarrow{\eta} S^2 \to C(\eta) = \mathbb{C}P^2$. Note that x generates $H^2(\mathbb{C}P^2; \mathbb{Z})$ if and only if x^2 generates $H^4(\mathbb{C}P^2; \mathbb{Z})$ – which would imply $\mathcal{H}(\eta) = 1$. With mod 2 coefficients, we can apply the Steenrod square

$$\mathrm{Sq}^2 : H^n(X; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^{n+2}(X; \mathbb{Z}/2\mathbb{Z})$$

which, usefully, commutes with suspension. Thus if η were nullhomotopic, we would have $C(\Sigma\eta)$ homotopy equivalent to $S^3 \vee S^5$ but we know that $C(\Sigma\eta) \cong \Sigma C(\eta) = \Sigma\mathbb{C}P^2$. By naturality of Sq^2, if $C(\Sigma\eta)$ were indeed homotopic to $S^3 \vee S^5$, the map

$$\mathrm{Sq}^2 : H^3(C(\Sigma\eta); \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^5(C(\Sigma\eta); \mathbb{Z}/2\mathbb{Z})$$

would be 0. However, in the commutative diagram

$$\begin{tikzcd}
H^3(\Sigma\mathbb{C}P^2;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\mathrm{Sq}^2"] \arrow[d, "\cong"'] & H^5(\Sigma\mathbb{C}P^2;\mathbb{Z}/2\mathbb{Z}) \arrow[d, "\cong"] \\
H^2(\mathbb{C}P^2;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\mathrm{Sq}^2"] & H^4(\mathbb{C}P^2;\mathbb{Z}/2\mathbb{Z})
\end{tikzcd}$$

the columns are isomorphisms (a classic corollary of the Mayer-Vietoris sequence) and the bottom row is an isomorphism since $\mathrm{Sq}^2(x) = x^2 \neq 0$ when x is the generator of $H^2(\mathbb{C}P^2;\mathbb{Z}/2\mathbb{Z})$. This proves:

Theorem 3.2.16. *For $n \geq 3$, $\pi_{n+1}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$.*

Here is a reformulation of the Blakers-Massey theorem (either Theorem 3.2.2 or 3.2.11). Suppose

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d] \\
A \arrow[r] & X
\end{tikzcd}$$

is a homotopy pushout (as in Section 1.1). Form the homotopy pullback

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "k"'] & B \arrow[d, "\ell"] \\
A \arrow[r] & X
\end{tikzcd}$$

This determines a map $h : C \to P$ which completes the diagram

$$\begin{tikzcd}
C \arrow[dr, dashed, "h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& P \arrow[r] \arrow[d] & B \arrow[d] \\
& A \arrow[r] & X
\end{tikzcd}$$

Theorem 3.2.17. *For A, B, C, X, P as above, if $f : C \to A$ is an m-equivalence and $g : C \to B$ is an n-equivalence, then $h : C \to P$ is an $(m + n - 1)$-equivalence.*

Proof. We prove that the statement is equivalent to Theorem 3.2.11. Taking homotopy fibres of f, k and ℓ, we get a diagram

$$\begin{tikzcd}
F(f) \arrow[r] \arrow[d] & F(k) \arrow[r, "\simeq"] \arrow[d] & F(\ell) \arrow[d] \\
C \arrow[r] \arrow[d, "f"] & P \arrow[r] \arrow[d, "k"] & B \arrow[d, "\ell"] \\
A \arrow[r, equal] & A \arrow[r] & X
\end{tikzcd}$$

where $F(k) \to F(\ell)$ is a homotopy equivalence since the lower right square is a homotopy pullback. Applying the long exact sequence in homotopy (Corollary 1.4.6) to the left and middle columns gives us a diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\pi_r(A) \arrow[r, equal] \arrow[d] & \pi_r(A) \arrow[d] \\
\pi_{r-1}(F(f)) \arrow[r, "\alpha"] \arrow[d] & \pi_{r-1}(F(k)) \arrow[d] \\
\pi_{r-1}(C) \arrow[r, "\beta"] \arrow[d] & \pi_{r-1}(P) \arrow[d] \\
\pi_{r-1}(A) \arrow[r, equal] \arrow[d] & \pi_{r-1}(A) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

By the Five Lemma, α is an isomorphism for $r < m+n$ and is surjective for $r = m+n$ *precisely* when β is the same. Finally, it's easy to see that this property for α is equivalent to Theorem 3.2.11 and the property for β is equivalent to the statement of the theorem. $\square$

A second reformulation of the Blakers-Massey theorem makes the analogy with the excision theorem for homology more apparent.

Lemma 3.2.18. *Suppose we have a map of long exact sequences of groups*

$$\begin{tikzcd}
\cdots \arrow[r] & A'_n \arrow[r] \arrow[d, "\alpha"] & B'_n \arrow[r] \arrow[d, "\beta"] & C'_n \arrow[r] \arrow[d, "\gamma"] & A'_{n-1} \arrow[r] \arrow[d, "\alpha"] & \cdots \\
\cdots \arrow[r] & A_n \arrow[r] & B_n \arrow[r] & C_n \arrow[r] & A_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

such that $\gamma : C'_\bullet \to C_\bullet$ is an isomorphism. Then there is a long exact sequence

$$\cdots \to A'_n \to B'_n \oplus A_n \to B_n \to A'_{n-1} \to \cdots$$

Example 3.2.19. Using the long exact sequences of the pairs $(A, A \cap B)$ and $(B, A \cup B)$, with γ the excision isomorphism, one obtains the Mayer-Vietoris sequence. (In fact, excision and Mayer-Vietoris are equivalent.)

Theorem 3.2.20. *If*

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & B \arrow[d] \\
A \arrow[r] & X
\end{tikzcd}$$

is a homotopy pullback, then there is a long exact sequence

$$\cdots \to \pi_r(P) \to \pi_r(B) \oplus \pi_r(A) \to \pi_r(X) \to \pi_{r-1}(P) \to \cdots$$

Corollary 3.2.21. *If*

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d] \\
A \arrow[r] & X
\end{tikzcd}$$

is a homotopy pushout such that f is an m-equivalence and g is an n-equivalence, then there is a long exact sequence

$$\pi_{m+n-1}(C) \to \pi_{m+n-1}(B) \oplus \pi_{m+n-1}(A) \to \pi_{m+n-1}(X) \to \pi_{m+n-2}(C) \to \cdots$$

3.3 The Hurewicz Theorem

Let (X, x_0) be a based space and take $\alpha : S^n \to X$. This induces a map on homology groups $\alpha_* : H_\bullet(S^n) \to H_\bullet(X)$. We know that $H_n(S^n) \cong \mathbb{Z}$; fix a generator $u_n \in H_n(S^n)$. Then the assignment $\alpha \mapsto \alpha_*(u_n)$ determines a well-defined map

$$h_X : \pi_n(X, x_0) \longrightarrow H_n(X)$$

called the *Hurewicz map*.

Lemma 3.3.1. *The Hurewicz map induces a natural transformation*

$$h : \pi_n(-) \longrightarrow H_n(-)$$

of functors of groups for every $n \geq 1$.

In particular, each $h_X : \pi_n(X, x_0) \to H_n(X)$ is a group homomorphism. From algebraic topology, we know that $h_X : \pi_1(X, x_0) \to H_1(X)$ is surjective with kernel $[\pi_1(X, x_0), \pi_1(X, x_0)]$, so that $H_1(X) \cong \pi_1(X, x_0)^{ab}$. When $n \geq 2$, we will prove that h is an isomorphism in certain degrees.

First, we need the following technical result, called the CW Resolution Theorem, or sometimes the CW Approximation Theorem.

Theorem 3.3.2 (CW Resolution)**.** *For every space X, there is a CW-complex K and a weak equivalence $g : K \to X$. Furthermore, if X is $(n-1)$-connected ($n \geq 1$), then such a K exists consisting of one 0-cell and all other cells of dimension n or higher.*

Proof. (Sketch) We construct the CW-complex $K = \bigcup_{n=0}^{\infty} K_n$ and maps $g_n : K_n \to X$ inductively, such that g_n is an n-equivalence. We may suppose X is path-connected, or repeat the subsequent process on each path component. Let K_0 be a point and $g_0 : K_0 \hookrightarrow X$ the inclusion of any basepoint. Inductively, suppose $K_0 \subset K_1 \subset \cdots \subset K_n$ and $g_r : K_r \to X$ have been constructed for $0 \leq r \leq n$ so that g_r is an r-equivalence. Consider $\ker(g_n)_*$, the kernel of the induced map $(g_n)_* : \pi_n(K_n) \to \pi_n(X)$. We may choose maps $\{f_\alpha : S^n \to K_n\}_\alpha$ generating $\ker(g_n)_*$ which are the attaching maps for the n-skeleton: $\vee f_\alpha : \bigvee_\alpha S^n \to K_n$. Let K'_{n+1} be the pushout of the diagram

$$\begin{tikzcd}
\bigvee_\alpha S^n \arrow[r, "\vee f_\alpha"] \arrow[d, hook] & K_n \arrow[d] \\
\bigvee_\alpha D^{n+1} \arrow[r] & K'_{n+1}
\end{tikzcd}$$

Then composing with g_n, we get a map $g'_{n+1} : K'_{n+1} \to X$ completing the diagram

$$\begin{tikzcd}
\bigvee_\alpha S^n \arrow[r, "\vee f_\alpha"] \arrow[d, hook] & K_n \arrow[d] \arrow[ddr, bend left, "g_n"] & \\
\bigvee_\alpha D^{n+1} \arrow[r] \arrow[drr, bend right] & K'_{n+1} \arrow[dr, dashed, "g'_{n+1}"] & \\
 & & X
\end{tikzcd}$$

Since $\{f_\alpha\}$ were chosen to generate $\ker(g_n)_*$, we can factor $(g_n)_*$ through a surjection and g'_{n+1}:

$$\begin{tikzcd}
\ker(g_n)_* \arrow[r] & \pi_n(K_n) \arrow[r, "(g_n)_*"] \arrow[dr] & \pi_n(X) \\
& & \pi_n(K'_{n+1}) \arrow[u, "(g'_{n+1})_*"']
\end{tikzcd}$$

It follows that g'_{n+1} is an isomorphism. Finally, $(g'_{n+1})_* : \pi_{n+1}(K'_{n+1}) \to \pi_{n+1}(X)$ is not guaranteed to be onto, but we remedy this by setting $K_{n+1} = \bigvee_\alpha S^n \vee K'_{n+1}$ and defining $g_{n+1} : K_{n+1} \to X$ by $g_n \circ \vee f_\alpha$ on $\bigvee_\alpha S^n$ and g'_{n+1} on K'_{n+1}. By construction, g_{n+1} is an $(n+1)$-equivalence so we are done by induction. □

Lemma 3.3.3. *If $n > 1$ then $\pi_n\left(\bigvee_\alpha S^n\right) \cong \bigoplus_\alpha \mathbb{Z}$.*

Proof. There is a natural map

$$\bigoplus_\alpha \mathbb{Z} \cong \bigoplus_\alpha \pi_n(S^n) \to \pi_n\left(\bigvee_\alpha S^n\right)$$

induced by the αth inclusion $S^n \hookrightarrow \bigvee_\alpha S^n$ on each copy of S^n. For an arbitrary space X, consider the wedge $Y = X \vee S^n$. By Example 3.2.12, the above sequence restricts to an isomorphism $\pi_n(Y) \cong \pi_n(X) \oplus \pi_n(S^n)$, so inductively (or passing to the colimit), $\pi_n\left(\bigvee_\alpha S^n\right) \cong \bigoplus_\alpha \mathbb{Z}$. □

Theorem 3.3.4 (Hurewicz)**.** *Suppose X is simply connected. Then the conditions*

(1) $\pi_k(X) = 0$ for all $k < n$

(2) $H_k(X) = 0$ for all $k < n$

are equivalent, and when either holds, the Hurewicz map $h : \pi_n(X) \to H_n(X)$ is an isomorphism and $h : \pi_{n+1}(X) \to H_{n+1}(X)$ is surjective.

Proof. Suppose (1) $\implies$ (2) holds (for any n) and implies that $h : \pi_n(X) \to H_n(X)$ is an isomorphism (for any n such that the conditions hold). We deduce (2) $\implies$ (1) from this assumption. Suppose (1) is false, so there is a smallest $m < n$ such that $\pi_m(X) \neq 0$. Then $\pi_k(X) = 0$ for $k < m$, so $h : \pi_m(X) \to H_m(X)$ is an isomorphism by hypothesis, but then $H_m(X) \neq 0$, a contradiction. Hence (2) $\implies$ (1) holds in the presence of the assumption at the beginning, so it's enough to show that implication is always valid.

Assume $\pi_k(X) = 0$ for all $k < n$. By Theorem 3.3.2, we may assume $X = X_{n+1}$ is the cofibre of attaching $\vee f_\alpha : \bigvee_\alpha S^n \to \bigvee_\alpha D^{n+1}$ along $\bigvee_\alpha S^n \subseteq \bigvee_\alpha D^{n+1}$. That is, X is the pushout of the diagram

$$\begin{tikzcd}
\bigvee_\alpha S^n \arrow[r, "\vee f_\alpha"] \arrow[d, hook] & \bigvee_\alpha S^n \arrow[d] \\
\bigvee_\alpha D^{n+1} \arrow[r] & X
\end{tikzcd}$$

Applying π_n and H_n, we get a commutative diagram (since h is natural by Lemma 3.3.1):

$$\begin{tikzcd}
\pi_n\left(\bigvee_\alpha S^n\right) \arrow[r] \arrow[d, "h"] & \pi_n\left(\bigvee_\alpha S^n\right) \arrow[r, "j"] \arrow[d, "h"] & \pi_n(X) \arrow[d, "h"] & \\
H_n\left(\bigvee_\alpha S^n\right) \arrow[r] & H_n\left(\bigvee_\alpha S^n\right) \arrow[r] & H_n(X) \arrow[r] & 0
\end{tikzcd}$$

Note that the left and middle vertical arrows are isomorphisms by Lemma 3.3.3. Thus it remains to show j is onto and apply the Five Lemma to see that $h : \pi_n(X) \to H_n(X)$ is an isomorphism. In the long exact sequence in relative homotopy,

$$\cdots \to \pi_{n+1}\left(X, \bigvee_\alpha S^n\right) \to \pi_n\left(\bigvee_\alpha S^n\right) \xrightarrow{j} \pi_n(X) \to \pi_n\left(X, \bigvee_\alpha S^n\right) \to \cdots$$

we have $\pi_n\left(X, \bigvee_\alpha S^n\right) \cong \pi_n\left(\bigvee_\alpha D^{n+1}, \bigvee_\alpha S^n\right) = 0$ by Corollary 3.2.21 but the latter is 0 by definition of the relative homotopy group. Hence j is onto, so $h : \pi_n(X) \to H_n(X)$ as desired. A similar proof shows that (2) holds, so we are done. $\square$

Corollary 3.3.5 (Relative Hurewicz Theorem)**.** *For any pair (X, A), the conditions*

(1) $\pi_k(X, A) = 0$ for all $k < n$

(2) $H_k(X, A) = 0$ for all $k < n$

are equivalent, and when either holds, the relative Hurewicz map $h : \pi_n(X, A) \to H_n(X, A)$ is an isomorphism and $h : \pi_{n+1}(X, A) \to H_{n+1}(X, A)$ is surjective.

Proof. Up to homotopy equivalence, we may assume $A \hookrightarrow X$ is a cofibration (Theorem 1.1.25). Consider the commutative diagram

$$\begin{tikzcd}
\pi_k(X, A) \arrow[r, "h"] \arrow[d] & H_k(X, A) \arrow[d] \\
\pi_k(X/A) \arrow[r, "h"] & H_k(X/A)
\end{tikzcd}$$

The vertical arrows are isomorphisms by Corollary 3.2.13 and Lemma 1.1.16, respectively. The corollary then follows from the ordinary Hurewicz theorem. $\square$

Corollary 3.3.6 (Second Whitehead Theorem)**.** *Suppose X and Y are simply connected CW-complexes and $f : X \to Y$ is any map. Then $f_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism if and only if $f_* : H_n(X) \to H_n(Y)$ is an isomorphism.*

Proof. Replacing Y by the mapping cylinder M_f, we may assume f is a cofibration (Theorem 1.1.25). In particular, f is an inclusion by Proposition 1.1.5. Then f inducing an isomorphism on π_n is equivalent to $\pi_n(Y, X) = 0$, and likewise f inducing an isomorphism on H_n is equivalent to $H_n(Y, X) = 0$. Hence the result follows from the relative Hurewicz theorem. $\square$

For each space X, fix a choice of a CW-complex ΓX and a resolution $g_X : \Gamma X \to X$ by Theorem 3.3.2. Then for any map $f : X \to Y$, there is a diagram

$$\begin{tikzcd} \Gamma X \arrow[r, dashed] \arrow[d, "g_X"'] & \Gamma Y \arrow[d, "g_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Then by the first Whitehead theorem (Theorem 2.2.5), $(g_Y)_* : [\Gamma X, \Gamma Y] \to [\Gamma X, Y]$ is a bijection, so there exists a map $\Gamma f : \Gamma X \to \Gamma Y$ which makes the diagram above commute up to homotopy. Such a Γf is unique up to homotopy, so in fact we have defined a functor

$$\Gamma : h(\texttt{Top}) \longrightarrow h(\texttt{Top})$$

$$X \longrightarrow \Gamma X$$

$$(X \xrightarrow{f} Y) \rightsquigarrow (\Gamma X \xrightarrow{\Gamma f} \Gamma Y).$$

Corollary 3.3.7. *If $f : X \to Y$ is a weak equivalence then $f_* : H_\bullet(X) \to H_\bullet(Y)$ is an isomorphism.*

Proof. By the above, we may transfer the question to $\Gamma X \to \Gamma Y$, and then the result follows from the second Whitehead theorem (Corollary 3.3.6). $\square$

3.4 Brown Representability

Let $\texttt{Top}_*$ and $\texttt{Set}_*$ be the categories of based topological spaces and based sets, respectively. Each space $Y \in \texttt{Top}_*$ defines a contravariant functor

$$\begin{aligned} h_Y : \texttt{Top}_* &\longrightarrow \texttt{Set}_* \\ X &\longmapsto [X, Y]. \end{aligned}$$

Definition. *A contravariant functor $F : \texttt{Top}_* \to \texttt{Set}_*$ is said to be* **representable** *if it is naturally isomorphic to h_Y for some $Y \in \texttt{Top}_*$.*

Remark. Suppose $F \cong h_Y$. By Yoneda's lemma, any bijection $[X, Y] \cong F(X)$ must be of the form $f \mapsto f^*(u)$ for some $u \in F(Y)$, where f^* is the pullback in $\texttt{Set}_*$.

The most obvious question is: what conditions on a functor $F : \texttt{Top}_* \to \texttt{Set}_*$ guarantee that F is representable? Certainly the following conditions are necessary:

- F must be a homotopy functor, i.e. if f and g are homotopic maps then $F(f) = F(g)$.
- For any collection of based spaces $\{X_\alpha\}$, the natural map $F\left(\bigvee X_\alpha\right) \to \prod F(X_\alpha)$ must be a bijection.

- F must satisfy some type of Mayer-Vietoris principle, like the Blakers-Massey theorem for homotopy groups.

The Brown representability theorem shows that in fact, these conditions are sufficient as well.

Theorem 3.4.1 (Brown Representability). *Suppose $F : \mathtt{Top}_* \to \mathtt{Set}_*$ is a contravariant functor satisfying:*

(1) F is a homotopy functor.

(2) For any $\{X_\alpha\}$, $F\left(\bigvee X_\alpha\right) \to \prod F(X_\alpha)$ is bijective.

(3) For any homotopy pushout

$$\begin{tikzcd} C \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & P \end{tikzcd}$$

the corresponding diagram

$$\begin{tikzcd} F(P) \arrow[dr, "\alpha"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & F(A) \times_{F(C)} F(B) \arrow[r] \arrow[d] & F(B) \arrow[d] \\ & F(A) \arrow[r] & F(C) \end{tikzcd}$$

is a weak pullback, i.e. α is onto. (Here, $F(A) \times_{F(C)} F(B)$ is the fibre product of $F(A)$ and $F(B)$ over $F(C)$.)

Then F is representable.

To prove this, we need the following lemma.

Lemma 3.4.2. *Suppose F is a functor satisfying the above conditions. Given any CW-complex Z and an element $z \in F(Z)$, there exists a CW-complex Y, a map $i : Y \to Z$ and some $u \in F(Y)$ such that Y is obtained from Z by attaching cells and $i^*(u) = z$. Moreover, the map*

$$\begin{aligned} \theta_u : \pi_n(Y) &\longrightarrow F(S^n) \\ f &\longmapsto f^*(u) \end{aligned}$$

is a bijection for every n.

Proof. We construct Y as the direct limit of a sequence $Z \xrightarrow{i} Y_0 \xrightarrow{i_0} Y_1 \xrightarrow{i_1} Y_2 \xrightarrow{i_2} \cdots$, along with elements $u_k \in F(Y_k)$ such that

(i) For each k, $(i_k)_*(u_{k+1}) = u_k$ and $(i_k \circ \cdots \circ i_0)^*(u_{k+1}) = z$.

(ii) The induced maps $(i_k)_* : [S^n, Y_k] \to [S^n, Y_{k+1}]$ are compatible with the θ_{u_k}:

$$\begin{tikzcd}
{[S^n, Y_0]} \arrow[r, "(i_0)_*"] \arrow[dr, "\theta_{u_0}"'] & {[S^n, Y_1]} \arrow[r, "(i_1)_*"] \arrow[d, "\theta_{u_1}"] & {[S^n, Y_2]} \arrow[r] \arrow[dl, "\theta_{u_2}"] & \cdots \\
& F(S^n) & &
\end{tikzcd}$$

and each θ_{u_k} is onto with $\ker \theta_{u_k} \subseteq \ker(i_k)_*$.

Given these properties, condition (3) will imply for $Y = \varinjlim Y_k$ that there exists $u \in F(Y)$ with $u|_{Y_k} = u_k$ for each k. Thus the map $[S^n, Y] = \varinjlim [S^n, Y_k] \xrightarrow{\theta_u} F(S^n)$ will be a bijection. To construct the sequence $Y_0 \to Y_1 \to \cdots$, start with

$$Y_0 = Z \vee \bigvee_{n \geq 0} \bigvee_{x \in F(S^n)} S^n.$$

Then by condition (2),

$$F(Y_0) = F(Z) \times \prod_{n \geq 0} \prod_{x \in F(S^n)} F(S^n)$$

so we may choose $u_0 \in F(Y_0)$ to be the element corresponding to $(z, x, x, x, \ldots)$ in the above product. Inductively, given Y_k and $u_k \in F(Y_k)$, let Y_{k+1} be the following pushout:

$$\begin{tikzcd}
\bigvee_{n \geq 0} \bigvee_{f \in \ker \theta_{u_k}} S^n \arrow[r] \arrow[d] & Y_k \arrow[d] \\
\bigvee_{n \geq 0} \bigvee_{f \in \ker \theta_{u_k}} D^{n+1} \arrow[r] & Y_{k+1}
\end{tikzcd}$$

By condition (3), we get a commutative diagram

$$\begin{tikzcd}
{[S^n, Y_k]} \arrow[r, "(i_k)_*"] \arrow[dr, "\theta_{u_k}"'] & {[S^n, Y_{k+1}]} \arrow[d, "\theta_{u_{k+1}}"] \\
& F(S^n)
\end{tikzcd}$$

so we are done by induction. $\square$

Proof of Brown Representability. Apply Lemma 3.4.2 to $Z = *$ to produce a space Y and $u \in F(Y)$ such that $\theta_u : \pi_n(Y) \to F(S^n)$ is an isomorphism for all n. We claim that $\theta_u : [X, Y] \to F(X), f \mapsto f^*(u)$ is an isomorphism for all X, and hence induces a natural isomorphism $h_Y \cong F$.

To show θ_u is onto, pick $x \in F(X)$. Then applying F to the diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \vee Y \end{tikzcd}$$

and using condition (2) on F gives a commutative diagram in $\texttt{Set}_*$:

$$\begin{tikzcd} F(*) & F(X) \arrow[l] \\ F(Y) \arrow[u] & F(X) \times F(Y) \arrow[l] \arrow[u] \end{tikzcd}$$

Then for $Z = X \vee Y$ and $z = (x, u) \in F(X) \times F(Y) = F(X \vee Y)$, Lemma 3.4.2 gives a space Y', a map $i : X \vee Y \to Y'$ and $u' \in F(Y')$ satisfying $i^*(u') = (x, u)$ and an isomorphism $\theta_{u'} : \pi_n(Y') \to F(S^n)$ for all n. Note that the composition $k : Y \to X \vee Y \xrightarrow{i} Y'$ is then a weak homotopy equivalence, so the diagram

$$\begin{tikzcd} {[X, Y']} \arrow[dr, "\theta_{u'}"'] & & {[X, Y]} \arrow[ll, "k^*"'] \arrow[dl, "\theta_u"] \\ & F(X) & \end{tikzcd}$$

commutes and hence k^* is a bijection. Hence there is some $f \in [X, Y]$ such that $\theta_u(f) = x$.

To prove θ_u is one-to-one, suppose $\theta_u(f) = x = \theta_u(g)$ for $f, g : X \to Y$. Then we have a pushout

$$\begin{tikzcd} X \vee X \arrow[r, "i_0 \vee i_1"] \arrow[d, "f \vee g"'] & X \times I \arrow[d, "\beta"] \\ Y \arrow[r, "\alpha"] & P \end{tikzcd}$$

By condition (3), there exists $z \in F(P)$ such that $\alpha^*(z) = u$ and $\beta^*(z) = x$. Applying Lemma 3.4.2 to $Z = P$ and this z, we get $Y', u' \in F(Y')$ and $i : P \to Y'$ such that $i^*(u') = z$. Hence $(i \circ \alpha)^*(u') = u$, so $i \circ \alpha$ is a homotopy equivalence $Y \to P \to Y'$; denote by q its homotopy inverse $Y' \to Y$. Then $H = q \circ \beta : X \times I \to Y' \to Y$ is a homotopy from f to g, so θ_u is one-to-one. This completes the proof. $\square$

Example 3.4.3. Let F be the functor assigning X to $F(X) = \mathrm{Vect}_n(X)$, the set of n-dimensional vector bundles over X up to isomorphism. Then Vect_n is a homotopy functor (Theorem 1.5.6) and it's easy to prove it satsifies the product condition. Moreover, the weak pullback condition can be seen by extending bundles over unions. Hence by Brown representability, there is some space G_n for each n such that $\mathrm{Vect}_n(X) \cong [X, G_n]$ for all X. In fact, we saw in Example 1.5.7 that G_n may be taken to be the nth infinite Grassmannian, $\mathrm{Gr}_n = \bigcup_{\ell=1}^{\infty} \mathrm{Gr}_n(\mathbb{R}^\ell)$.

3.5 Eilenberg-Maclane Spaces

Let G be an abelian group and fix $n \geq 0$. Consider the cohomology functor with coefficients in G:

$$\begin{aligned} \widetilde{H}^n(-;G) : \mathtt{Top}_* &\longrightarrow \mathtt{Set}_* \\ X &\longmapsto \widetilde{H}^n(X;G). \end{aligned}$$

Then $\widetilde{H}^n(-;G)$ satisfies the three conditions in the Brown representability theorem (3.4.1), so there exists a space $K(G,n)$ such that $\widetilde{H}^n(-;G)$ is naturally isomorphic to $h_{K(G,n)}$, i.e. there is a class $u \in \widetilde{H}^n(K(G,n);G)$ such that $\theta_u : [X, K(G,n)] \to \widetilde{H}^n(X;G)$ is a bijection for all X.

Definition. *Such a space $K(G,n)$ is called an* **Eilenberg-Maclane space** *of type (G,n), and the class $u \in \widetilde{H}^n(K(G,n);G)$ is called a* **fundamental class**.

Proposition 3.5.1. *For any Eilenberg-Maclane space $K(G,n)$ and any $k \geq 0$,*

$$\pi_k(K(G,n)) = \begin{cases} G, & k = n \\ 0, & k \neq n. \end{cases}$$

Proof. This follows from Lemma 3.4.2: $\pi_k(K(G,n)) \cong \widetilde{H}^n(S^n)$. □

Proposition 3.5.2. *For any abelian group G and natural number n, an Eilenberg-Maclane space $K(G,n)$ is unique up to homotopy equivalence.*

Proof. Suppose K is another space satisfying $\pi_n(K) = G$ and $\pi_k(K) = 0$ for $k \neq n$. Then by the Hurewicz theorem (3.3.4), $H_n(K;G) = G$ and $H_{n-1}(K;G) = 0$. By the universal coefficient theorem, $H^n(K;G) \cong \mathrm{Hom}(G,G)$ so we may choose $u \in H^n(K;G)$ corresponding to the identity $1_G \in \mathrm{Hom}(G,G)$. Then $\theta_u : \pi_k(K) \to H^n(S^k;G)$ is an isomorphism for all k, so it follows from Lemma 3.4.2 that $\theta_u : [X,K] \to H^n(X;G)$ is an isomorphism for all spaces X. Hence $h_K \cong h_{K(G,n)}$ as functors, so by Yoneda's lemma, K is homotopy equivalent to $K(G,n)$ □

Corollary 3.5.3. *Every Eilenberg-Maclane space is an H-space.*

Proof. By Brown representability (Theorem 3.4.1), $[-, K(G,n)] \cong \widetilde{H}^n(-;G)$ is a functor $\mathtt{Top}_* \to \mathtt{AbGps}$, so by Proposition 0.3.4, $K(G,n)$ is an H-space. □

Corollary 3.5.4. *For every abelian group G and $n \geq 0$, there is a natural isomorphism*

$$\widetilde{H}^n(-;G) \cong \widetilde{H}^{n+1}(\Sigma -;G).$$

Proof. For every space X, we have $\widetilde{H}^n(X;G) \cong [X, K(G,n)]$ and $\widetilde{H}^{n+1}(\Sigma X;G) \cong [\Sigma X, K(G,n+1)]$ by Brown representability (Theorem 3.4.1). Also, $[\Sigma X, K(G,n+1)] = [X, \Omega K(G,n+1)]$ by Corollary 0.2.5 and $K(G,n) \to \Omega K(G,n+1)$ is a homotopy equivalence by uniqueness of Eilenberg-Maclane spaces. Hence $\widetilde{H}^n(X;G) \cong \widetilde{H}^{n+1}(\Sigma X;G)$ and this isomorphism is natural since each isomorphism above is natural. □

Example 3.5.5. Let $G = \mathbb{Z}$. For $n = 0, 1, 2$, we have:

$$\begin{aligned} K(\mathbb{Z},0) &= \mathbb{Z} \\ K(\mathbb{Z},1) &= S^1 \\ K(\mathbb{Z},2) &= \mathbb{C}P^\infty. \end{aligned}$$

Each of these can be verified by computing the ordinary cohomology groups of the spaces on the right, and noting that $K(G,n)$ is unique up to homotopy equivalence (Prop. 3.5.2). In particular, the identification $K(\mathbb{Z},2) = \mathbb{C}P^\infty$ is borne out by an isomorphism

$$\begin{aligned} [X, \mathbb{C}P^\infty] &\longrightarrow H^2(X;\mathbb{Z}) \\ L &\longmapsto c_1(L) \end{aligned}$$

which sends a line bundle L over X (this is the complex version of Example 3.4.3) to the first *Chern class* $c_1(L)$ of the bundle.

Example 3.5.6. For $G = \mathbb{Z}/2\mathbb{Z}$, we have $K(\mathbb{Z}/2\mathbb{Z},1) = \mathbb{R}P^\infty$, again by computing the homology groups of $\mathbb{Z}/2\mathbb{Z}$. Here, the representability is encoded by the isomorphism

$$\begin{aligned} [X, \mathbb{R}P^\infty] &\longrightarrow H^1(X;\mathbb{Z}/2\mathbb{Z}) \\ L &\longmapsto \omega_1(L) \end{aligned}$$

where L is a line bundle (again, see Example 3.4.3) and $\omega_1(L)$ is the first *Stiefel-Whitney class* of L.

Theorem 3.5.7. *For any two abelian groups G and G' and any $n \geq 0$, there are bijections*

$$[K(G,n), K(G',n)]_* \longleftrightarrow [K(G,n), K(G',n)] \longleftrightarrow \mathrm{Hom}(G,G').$$

Proof. Send a map $f : K(G,n) \to K(G',n)$ to the induced map $f_* : \pi_n(K(G,n)) \to \pi_n(K(G',n))$ which by definition is a map $G \to G'$. Then by the universal coefficient theorem,

$$[K(G,n), K(G',n)] \cong \widetilde{H}^n(K(G,n);G') \cong \mathrm{Hom}(H_n(K(G,n);\mathbb{Z}), G') = \mathrm{Hom}(G,G').$$

□

Definition. *For integers n, m and abelian groups G, G', a* **cohomology operation** *of type (n, G, m, G') is a natural transformation*

$$\theta : H^n(-; G) \longrightarrow H^m(-; G').$$

Suppose θ is a cohomology operation. Then applying it to the fundamental class of $K(G, n)$ determines a class $\theta(u) \in \widetilde{H}^m(K(G, n); G') = [K(G, n), K(G', m)]$. Further, if $O(n, G, m, G')$ represents the set of all cohomology operations of this type, then evaluation on u induces a bijection

$$O(n, G, m, G') \longleftrightarrow \widetilde{H}^m(K(G, n); G') = [K(G, n), K(G', m)].$$

Example 3.5.8. It's easy to compute $K(\mathbb{Z}/2\mathbb{Z}, 1) = \mathbb{R}P^\infty$ by considering the CW-structure of $\mathbb{R}P^\infty$. One can in fact prove that $H^\bullet(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[x]$, the polynomial ring in one variable with $\mathbb{Z}/2\mathbb{Z}$-coefficients generated by the fundamental class $x \in H^1(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z})$. It turns out that $x^2 \in H^2(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z})$ represents the "Steenrod square" cohomology operation,

$$\begin{aligned} H^1(X; \mathbb{Z}/2\mathbb{Z}) &\longrightarrow H^2(X; \mathbb{Z}/2\mathbb{Z}) \\ \alpha &\longmapsto \alpha^2 = \alpha \cup \alpha. \end{aligned}$$

3.6 Infinite Symmetric Products

Recall that a *monoid* in a (tensor) category (with unit $*$) is an object X with distinguished morphism $m : X \otimes X \to X$ and $e : * \to X$ satisfying the usual associativity and identity axioms of a set monoid.

Definition. *Let (X, x_0) be a based space. The* **James construction** *on X, $J(X)$, is the free monoid in $\mathtt{Top}_*$ generated by X. Explicitly,*

$$J(X) := \coprod_{n=1}^{\infty} X^n / \sim$$

where $(x_1, \ldots, x_k, x_0, x_{k+1}, \ldots, x_{n-1}) \in X^n$ is equivalent to $(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{n-1}) \in X^{n-1}$. The monoidal structure $J(X) \times J(X) \to J(X)$ is induced by identifying $X^m \times X^n = X^{m+n}$, which is obviously associative.

The James construction is universal in the following sense.

Proposition 3.6.1. *If M is a monoid in $\mathtt{Top}_*$ and $f : X \to M$ is any map of based spaces, then there is a unique map $\bar{f} : J(X) \to M$ making the diagram commute:*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook] & & M \\ & J(X) \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

For any X, $J(X)$ has a natural filtration

$$X = J_1 X \subseteq J_2 X \subseteq J_3 X \subseteq \cdots$$

where $J_n X = \coprod_{m=1}^{n} X^m / \sim$. (We actually saw $J = J_2 X$ in Example 1.2.13, where $X = S^n$.) Moreover, for each $n \geq 2$,

$$J_n X / J_{n-1} X \cong \underbrace{X \wedge \cdots \wedge X}_{n}.$$

Recall that the identity map $X \to X$ induces, by adjointness, a natural map $X \to \Omega\Sigma X$ which makes $\Omega\Sigma X$ into an H-space – that is, $\Omega\Sigma X$ is an "associative monoid up to homotopy", but may not be truly associative. It turns out that one can replace $\Omega\Sigma X$ with a space to which it is homotopy equivalent and that is itself an associative monoid in $\mathtt{Top}_*$. Then by Proposition 3.6.1, there is a morphism $J(X) \to \Omega\Sigma X$ commuting with the maps $X \to J(X)$ and $X \to \Omega\Sigma X$.

Theorem 3.6.2 (James)**.** *If X is path-connected, then $J(X) \to \Omega\Sigma X$ is a homotopy equivalence.*

By the Freudenthal suspension theorem (3.2.5), the map $X \to \Omega\Sigma X$ induces a map $\pi_n(X) \to \pi_{n+1}(\Sigma X)$, so James' theorem allows us to view this as a map $\pi_n(X) \to \pi_n(J(X))$. Then the filtration of $J(X)$ makes this map between fundamental groups easier to study.

In their 1958 paper *Quasifaserungen und Unendliche Symmetrische Produkte*, Dold and Thom gave a similar construction to James' construction.

Definition. *For a based space (X, x_0), the* **infinite symmetric product** *on X is the free* commutative *monoid $SP^\infty(X)$ in $\mathtt{Top}_*$ generated by X. Explicitly,*

$$SP^\infty(X) := \coprod_{n=1}^{\infty} X^n / \approx$$

where $\approx$ is the equivalence relation generated by $\sim$ from the James construction as well as $(x_1, \ldots, x_n) \approx (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for every element σ of the symmetric group Σ_n.

Again, $SP^\infty(X)$ has a natural filtration

$$X = SP^1(X) \subseteq SP^2(X) \subseteq SP^3(X) \subseteq \cdots$$

where $SP^n(X) = X^n/\Sigma_n$; this is sometimes called the *nth symmetric product* of X. Here, we also have

$$SP^n(X)/SP^{n-1}(X) \cong (X \wedge \cdots \wedge X)/\Sigma_n.$$

Example 3.6.3. For any $n \geq 2$, $SP^2(S^2) \cong \mathbb{C}P^n$.

Example 3.6.4. For each $n \geq 1$, $SP^n(\mathbb{C}) = \mathbb{C}^n/\Sigma_n$, but notice that $\mathbb{C}^n/\Sigma_n \cong \mathbb{C}^n$ via the isomorphism $[z_1, \ldots, z_n] \mapsto (a_0, \ldots, a_{n-1})$, where $\prod_{i=1}^{n}(z - z_i) = \sum_{j=0}^{n} a_j z^j$. One can also compute that $SP^n(\mathbb{C} \smallsetminus \{0\}) \cong (\mathbb{C} \smallsetminus \{0\}) \times \mathbb{C}^{n-1}$.

Lemma 3.6.5. *Let $SP^\infty : \mathtt{Top}_* \to \mathtt{Top}_*$ be the assignment $X \mapsto SP^\infty(X)$. Then*

(1) SP^∞ is a functor.

(2) If $f, g : X \to Y$ are homotopic then $SP^\infty(f)$ and $SP^\infty(g)$ are homotopic.

(3) The map $S^1 \to SP^\infty(S^1)$ is a homotopy equivalence.

(4) SP^∞ takes homotopy pushout squares to homotopy pullback squares.

Theorem 3.6.6 (Dold-Thom)**.** *There is a natural isomorphism $\pi_n(SP^\infty(X)) \cong \widetilde{H}_n(X)$ for all based spaces X.*

Corollary 3.6.7. *For each $n \geq 1$, $SP^\infty(S^n) = K(\mathbb{Z}, n)$.*

It turns out that the natural inclusion $X \hookrightarrow SP^\infty(X)$ induces the Hurewicz map (Section 3.3)

$$h : \pi_n(X) \longrightarrow \pi_n(SP^\infty) \cong \widetilde{H}_n(X).$$

This allows one to filter h using the filtration on SP^∞ and study it in more detail.

Definition. *Let A be an abelian group and $n \geq 0$ be an integer. A* **Moore space** *of type (A, n) is a space $M(A, n)$ which satisfies*

$$\widetilde{H}_k(M(A,n);\mathbb{Z}) = \begin{cases} A, & k = n \\ 0, & k \neq n. \end{cases}$$

Example 3.6.8. For each $n \geq 0$, $M(\mathbb{Z}, n) = S^n$ and by additivity, $M(\mathbb{Z}^k, n) = \bigwedge_k S^n$.

Example 3.6.9. $M(\mathbb{Z}/2\mathbb{Z}, 1) = \mathbb{R}P^2$ and in general, $M(\mathbb{Z}/2\mathbb{Z}, n) = \Sigma^{n-1}\mathbb{R}P^2$.

Moore spaces may be viewed as an analogue of Eilenberg-Maclane spaces. This is made precise in the following lemma.

Lemma 3.6.10. *For any abelian group A and any integer $n \geq 0$, $SP^\infty(M(A, n)) = K(A, n)$.*

Theorem 3.6.11. *If M is any commutative monoid in $\mathtt{Top}_*$, then M is homotopy equivalent to the infinite product of Eilenberg-Maclane spaces*

$$\prod_{n=1}^{\infty} K(\pi_n(M), n).$$

In particular, $SP^\infty(X) = \prod_{n=1}^\infty K\left(\widetilde{H}_n(X), n\right)$.

Proof. (Sketch) For each n, there is a map $f_n : M(\pi_n(M), n) \to M$ which induces an isomorphism on π_n. Since M is a monoid, this extends to a map $\bar{f}_n$ completing the following diagram:

$$\begin{tikzcd} M(\pi_n(M), n) \arrow[rr, "f_n"] \arrow[dr, hook] & & M \\ & SP^\infty(M(\pi_n(M), n)) \arrow[ur, "\bar{f}_n"'] & \end{tikzcd}$$

Taking infinite products gives us a map

$$\prod_{n=1}^{\infty} SP^{\infty}(M(\pi_n(M), n)) \to SP^{\infty}\left(\bigvee_{n=1}^{\infty} M(\pi_n(M), n)\right) \xrightarrow{\bigvee \bar{f}_n} M$$

which induces an isomorphism on homotopy groups. Now apply Lemma 3.6.10 and Whitehead's first theorem (2.2.5) to get a homotopy equivalence $\prod_{n=1}^{\infty} K(\pi_n(M), n) \to M$. $\square$

4 Algebraic Constructions

4.1 The Derived Functor $\lim^1$

Let (A_n, α_n) be a direct system of abelian groups. Then the direct limit, $\varinjlim A_n$, can be defined explicitly as the coequalizer of the following pair of maps:

$$\bigoplus_{n=1}^{\infty} A_n \underset{\mathrm{id}}{\overset{\bigoplus \alpha_n}{\rightrightarrows}} \bigoplus_{n=1}^{\infty} A_n \longrightarrow \varinjlim A_n$$

In fact, the direct limit (also sometimes called a *colimit* and written $\operatorname{colim} A_n$) fits into a short exact sequence:

$$0 \to \bigoplus_{n=1}^{\infty} A_n \xrightarrow{\bigoplus(1-\alpha_n)} \bigoplus_{n=1}^{\infty} A_n \to \varinjlim A_n \to 0. \tag{1}$$

That is, $\varinjlim A_n$ can be written down explicitly as a cokernel:

$$\varinjlim A_n = \left(\bigoplus_{n=1}^{\infty} A_n \right) / \{x - \alpha_n(x) \mid x \in A_n\}.$$

Reversing the arrows, for an inverse system of abelian groups (A_n, α_n), the inverse limit $\varprojlim A_n$, also referred to as the (projective) limit, can be defined as the equalizer of the following pair of maps:

$$\varprojlim A_n \longrightarrow \prod_{n=1}^{\infty} A_n \underset{\mathrm{id}}{\overset{\prod \alpha_n}{\rightrightarrows}} \prod_{n=1}^{\infty} A_n$$

In contrast with the direct limit, however, the inverse limit does not always fit into a short exact sequence like (1). In general, $\varinjlim$ is a left exact functor, that is, there is always an exact sequence

$$0 \to \varprojlim A_n \to \prod_{n=1}^{\infty} A_n \xrightarrow{\prod(1-\alpha_n)} \prod_{n=1}^{\infty} A_n,$$

but the sequence may not be exact on the right. Instead, there is a (right) derived functor which repairs the failure of exactness.

Definition. *For an inverse system of abelian groups (A_n, α_n), $\lim^1 A_n$ is defined to be the cokernel of the map $\prod(1-\alpha_n) : \prod_{n=1}^{\infty} A_n \to \prod_{n=1}^{\infty} A_n$.*

Lemma 4.1.1. *There is an exact sequence of abelian groups*

$$0 \to \varprojlim A_n \to \prod_{n=1}^{\infty} A_n \xrightarrow{\prod(1-\alpha_n)} \prod_{n=1}^{\infty} A_n \to \lim{}^1 A_n \to 0.$$

Lemma 4.1.2. *For an inverse system* (A_n, α_n),

(a) If every α_n *is an epimorphism, then* $\lim^1 A_n = 0$.

(b) If every $\alpha_n = 0$, *then* $\lim^1 A_n = 0$.

Proposition 4.1.3. *Let* $(A_n, \alpha_n), (B_n, \beta_n), (C_n, \gamma_n)$ *be inverse systems of abelian groups and* $0 \to A_n \to B_n \to C_n \to 0$ *a short exact sequence of inverse systems. Then there is an exact sequence*

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to \lim{}^1 A_n \to \lim{}^1 B_n \to \lim{}^1 C_n \to 0.$$

Proof. Use Lemma 4.1.1 and the Snake Lemma. □

Example 4.1.4. For a prime integer p, the *p-adic integers* $\mathbb{Z}_p$ can be defined in many ways, but one of the definitions is as the limit of an inverse system:

$$\mathbb{Z}_p = \lim\{\mathbb{Z}/p\mathbb{Z} \leftarrow \mathbb{Z}/p^2\mathbb{Z} \leftarrow \mathbb{Z}/p^3\mathbb{Z} \leftarrow \cdots\}.$$

Consider the short exact sequence of inverse systems

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "p^3"] \arrow[d, "p"] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathbb{Z}/p^3\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "p^2"] \arrow[d, "p"] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathbb{Z}/p^2\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "p"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/p\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Applying Proposition 4.1.3 and Lemma 4.1.2, we get an exact sequence

$$0 \to \lim\{\mathbb{Z} \xleftarrow{p} \mathbb{Z} \xleftarrow{p} \cdots\} \to \mathbb{Z} \to \mathbb{Z}_p \to \lim{}^1\{Z \xleftarrow{p} \mathbb{Z} \xleftarrow{p} \cdots\} \to 0 \to 0 \to 0.$$

In particular, since $\mathbb{Z} \to \mathbb{Z}_p$ is injective, we get

$$\lim\{\mathbb{Z} \xleftarrow{p} \mathbb{Z} \xleftarrow{p} \cdots\} = 0 \quad \text{and} \quad \lim{}^1\{\mathbb{Z} \xleftarrow{p} \mathbb{Z} \xleftarrow{p} \cdots\} \cong \mathbb{Z}_p/\mathbb{Z}.$$

This shows that in general, $\lim^1$ need not vanish.

Example 4.1.5. The profinite completion of the integers $\widehat{\mathbb{Z}}$ is the limit of the inverse system $\{\mathbb{Z}/n\mathbb{Z}\}_{n\geq 2}$ partially ordered by $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ if $m \mid n$. One can show that $\widehat{\mathbb{Z}}$ is the limit of a linearly ordered sequence:

$$\widehat{\mathbb{Z}} = \lim\{\mathbb{Z}/2\mathbb{Z} \leftarrow \mathbb{Z}/6\mathbb{Z} \leftarrow \mathbb{Z}/24\mathbb{Z} \leftarrow \mathbb{Z}/120\mathbb{Z} \leftarrow \cdots\}.$$

Then a similar proof as in Example 4.1.4 shows that $\lim^1\{\mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{3} \mathbb{Z} \xleftarrow{4} \cdots\} = \widehat{\mathbb{Z}}/\mathbb{Z}$.

The following demonstrates the interaction between $\varinjlim$ and the derived functors of Hom and $\varprojlim$.

Proposition 4.1.6. *Suppose (A_n, α_n) is a direct system of flat abelian groups. Then for all B, $(\mathrm{Hom}(A_n, B), \alpha_n^*)$ is an inverse system of abelian groups and there is an isomorphism*

$$\lim{}^1 \mathrm{Hom}(A_n, B) \cong \mathrm{Ext}^1\left(\varinjlim A_n, B\right).$$

Proof. Set $A = \varinjlim A_n$. Applying $\mathrm{Hom}(-, B)$ to the short exact sequence (1), we get a long exact sequence in Ext groups:

$$\begin{aligned} 0 \to \mathrm{Hom}(A, B) &\to \mathrm{Hom}\left(\bigoplus_{n=1}^{\infty} A_n, B\right) \to \mathrm{Hom}\left(\bigoplus_{n=1}^{\infty} A_n, B\right) \\ \to \mathrm{Ext}^1(A, B) &\to \mathrm{Ext}^1\left(\bigoplus_{n=1}^{\infty} A_n, B\right) \to \mathrm{Ext}^1\left(\bigoplus_{n=1}^{\infty} A_n, B\right) \to \cdots \end{aligned}$$

Note that $\mathrm{Hom}(A, B) = \mathrm{Hom}\left(\varinjlim A_n, B\right) \cong \varprojlim \mathrm{Hom}(A_n, B)$, $\mathrm{Hom}\left(\bigoplus A_n, B\right) \cong \prod \mathrm{Hom}(A_n, B)$ and $\mathrm{Ext}^1\left(\bigoplus A_n, B\right) \cong \prod \mathrm{Ext}^1(A_n, B)$. Moreover, by the flatness assumption $\mathrm{Ext}^1(A_n, B) = 0$ for all n. Hence the sequence above becomes

$$0 \to \varprojlim \mathrm{Hom}(A_n, B) \to \prod_{n=1}^{\infty} \mathrm{Hom}(A_n, B) \to \prod_{n=1}^{\infty} \mathrm{Hom}(A_n, B) \to \mathrm{Ext}^1(A, B) \to 0.$$

Thus Lemma 4.1.1 shows that $\mathrm{Ext}^1(A, B) \cong \lim^1 \mathrm{Hom}(A_n, B)$. $\square$

Example 4.1.7. It's easy to see that $\mathbb{Q} = \lim\{\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{4} \cdots\}$. Then by Proposition 4.1.6,

$$\begin{aligned} \mathrm{Ext}^1(\mathbb{Q}, \mathbb{Z}) &\cong \lim{}^1\{\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{2} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{3} \cdots\} \\ &= \lim{}^1\{\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \cdots\} \\ &= \widehat{\mathbb{Z}}/\mathbb{Z}, \end{aligned}$$

using Example 4.1.5.

Definition. *Let (A_n, α_n) be an inverse system of abelian groups and for each $j > n$, write $\alpha_n^j = \alpha_n \alpha_{n+1} \cdots \alpha_{j-1} : A_j \to A_n$. Set $A_n^j = \mathrm{im}\, \alpha_n^j$. Then (A_n, α_n) is said to satisfy the* **Mittag-Leffler condition** *if for each n, there exists an N such that $A_n^j = A_n^N$ for all $j > N$, that is, if the sequences of images of α_n^j eventually stabilize.*

Example 4.1.8. The Mittag-Leffler condition clearly holds for (A_n, α_n) when every α_n is an epimorphism, or when all the A_n are finite abelian groups. We saw in Lemma 4.1.2 that in the former case, $\lim^1 A_n$ vanishes. The following theorem generalizes this to all inverse systems satisfying the Mittag-Leffler condition.

Theorem 4.1.9. *For an inverse system of abelian groups (A_n, α_n), either*

(1) (A_n, α_n) satisfies the Mittag-Leffler condition and $\lim^1 A_n = 0$; or

(2) $\lim^1 A_n$ is an uncountable divisible group.

4.2 Mapping Telescopes

Definition. *For a pair of maps $f, g : X \to Y$, the* **homotopy coequalizer** *(or* **mapping torus***) of f and g is the homotopy pushout $T(f,g)$ of the maps $f \vee g : X \vee X \to Y$ and $\nabla : X \vee X \to Y$, where ∇ is the 'fold map' of Proposition 0.3.4. Explicitly,*

$$\begin{array}{ccc} (X \vee X)\coprod(X \vee X) & \xrightarrow{(f\vee g)\coprod\nabla} & Y \coprod Y \\ {\scriptstyle i_0 \times i_1}\downarrow & & \downarrow \\ X \vee X \times I & \longrightarrow & T(f,g) \end{array}$$

Definition. *Let (X_n, f_n) be a direct system of topological spaces. Then the* **mapping telescope** *(or* **homotopy colimit***) of (X_n, f_n) is the homotopy coequalizer $\mathrm{Tel}(X_n)$ of the maps $\bigvee f_n : \bigvee X_n \to \bigvee X_n$ and $id : \bigvee X_n \to \bigvee X_n$.*

Proposition 4.2.1. *For any direct system (X_n, f_n) of spaces, the mapping telescope can be written as a union of subspaces*

$$Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \cdots \subseteq \bigcup_{n=1}^{\infty} Y_n = \mathrm{Tel}(X_n)$$

such that there are deformation retractions $Y_n \to X_n$ compatible with the f_n. In particular, there is a weak equivalence

$$\mathrm{Tel}(X_n) \to \varinjlim X_n.$$

Proposition 4.2.2. *Let (X_n, f_n) be a direct system and Z any space. Then*

$$[\mathrm{Tel}(X_n), Z] \longrightarrow \varinjlim [X_n, Z]$$

is a surjective map of pointed sets with kernel $\lim^1 [\Sigma X_n, Z]$.

Proposition 4.2.3. *If each $f_n : X_n \to X_{n+1}$ is a cofibration, then the map $\mathrm{Tel}(X_n) \to \varinjlim X_n$ is a homotopy equivalence.*

We now give an application of telescopes to phantom maps, using the properties of $\lim^1$ studied in Section 4.1.

Definition. *For a CW-complex $X = \bigcup_{n=1}^{\infty} X_n$, a map $f : X \to Y$ is called a* **phantom map** *if $f|_{X_n}$ is nullhomotopic for every $n \geq 1$.*

A priori, it is not clear if such maps even exist in general. However, one has:

Corollary 4.2.4. *Let X be a CW-complex. Then the set of homotopy classes of phantom maps $X \to Y$ is naturally isomorphic to $\lim^1 [\Sigma X_n, Y]$.*

Proof. This follows from Propositions 4.2.2 and 4.2.3. □

The following example, due to Gray, shows that there are uncountably many phantom maps $\mathbb{C}P^\infty \to S^3$.

Example 4.2.5. (Gray) Let $X = \mathbb{C}P^\infty = \bigcup_{n=1}^\infty \mathbb{C}P^n$ and $Y = S^3$, so that Y is a topological group, in particular an H-space, and therefore each $[\Sigma\mathbb{C}P^n, S^3]$ is an abelian group by Theorem 0.3.7. Then Corollary 4.2.4 identifies $\lim^1 [\Sigma\mathbb{C}P^n, S^3]$ as the subgroup of phantom maps in $[\mathbb{C}P^\infty, S^3]$. We claim that this is a finitely generated group. Consider the cofibration sequence

$$S^{2n+1} \to \mathbb{C}P^n \to \mathbb{C}P^{n+1} \to S^{2n+2} \to \Sigma\mathbb{C}P^n \to \Sigma\mathbb{C}P^{n+1} \to S^{2n+3} \to \cdots$$

Applying $[-, S^3]$ gives a sequence of abelian groups

$$\cdots \to \pi_{2n+3}(S^3) \to [\Sigma\mathbb{C}P^{n+1}, S^3] \to [\Sigma\mathbb{C}P^n, S^3] \to \pi_{2n+2}(S^3) \to \cdots \tag{2}$$

which is exact by Theorem 1.2.5. We will prove in Section 4.6 that $\pi_n(S^3)$ is a finite group for all $n > 3$. Assuming this, the long exact sequence above and induction on n imply that each $[\Sigma\mathbb{C}P^n, S^3]$ is finitely generated.

To show that there are uncountably many phantom maps $\mathbb{C}P^\infty \to S^3$, our goal is to prove that the inverse system $[\Sigma\mathbb{C}P^n, S^3]$ does not satisfy the Mittag-Leffler condition and apply Theorem 4.1.9. In fact, we will show that for each n, there is a map $g_n : \Sigma\mathbb{C}P^n \to S^3$ that does not extend to a map $\Sigma\mathbb{C}P^\infty \to S^3$. Suppose $g_n : \Sigma\mathbb{C}P^n \to S^3$ has been constructed; let d_n be its degree on the bottom cell of $\Sigma\mathbb{C}P^n$. Consider the composition $S^{2n+2} \to \Sigma\mathbb{C}P^n \xrightarrow{g_n} S^3$; let its order in the finite group $\pi_{2n+3}(S^3)$ be denoted a. Then we have a commutative diagram

$$\begin{tikzcd}
S^{2n+2} \arrow[r, "\gamma"] \arrow[d, "a"] & \Sigma\mathbb{C}P^n \arrow[r] \arrow[d, "a"] & \Sigma\mathbb{C}P^{n+1} \arrow[d, dashed, "g_{n+1}"] \\
S^{2n+2} \arrow[r, "\gamma"] & \Sigma\mathbb{C}P^n \arrow[r, "g_n"] & S^3
\end{tikzcd}$$

(Here, $a : \Sigma\mathbb{C}P^n \to \Sigma\mathbb{C}P^n$ is the map induced on $\Sigma\mathbb{C}P^n = S^1 \wedge \mathbb{C}P^n$ by the degree a map on S^1.) Note that by definition, a annihilates $g_n \circ \gamma$ along the bottom row, so $g_n \circ \gamma \circ a$ is nullhomotopic. By commutativity, so is $g_n \circ a \circ \gamma$, i.e. $\gamma_*(g_n \circ a) = 0$ in $\pi_{2n+2}(S^3)$. Now exactness of sequence (1) implies that there is some $g_{n+1} \in [\mathbb{C}P^{n+1}, S^3]$ mapping to $g_n \circ a$. By construction, the degree of g_{n+1} on the bottom cell of $\Sigma\mathbb{C}P^{n+1}$ is ad_n. This shows that for each n, there is a map $g_n : \Sigma\mathbb{C}P^n \to S^3$ which is not nullhomotopic as a map $\Sigma\mathbb{C}P^2 \hookrightarrow \Sigma\mathbb{C}P^3 \to S^3$.

In general, let $g : \Sigma\mathbb{C}P^\infty \to S^3$ be any map, say of degree m on the bottom cell $\Sigma\mathbb{C}P^2$. Let p be a prime not dividing m and let

$$\mathcal{P} : H^n(X; \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{n+2(p-1)}(X; \mathbb{Z}/p\mathbb{Z})$$

denote the mod p Steenrod operation for a space X, constructed for any p in a similar fashion to the $p = 2$ case. As with mod 2 Steenrod operations, $\mathcal{P}$ is the pth power map on degree 2 homology and commutes with the suspension isomorphism $\sigma : H^n(X; \mathbb{Z}/p\mathbb{Z}) \to$

$H^{n+1}(X;\mathbb{Z}/p\mathbb{Z})$ in every dimension n. For $X = \mathbb{C}P^\infty$, we have $H^\bullet(\mathbb{C}P^\infty;\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}[y]$ for a generator $y \in H^2(\mathbb{C}P^\infty;\mathbb{Z}/p\mathbb{Z})$. Then $\mathcal{P}(y) = y^p \neq 0$, so if x is a generator in $H^3(\mathbb{C}P^\infty;\mathbb{Z}/p\mathbb{Z})$, we have $\mathcal{P}(x) = \mathcal{P}(\sigma y) = \sigma\mathcal{P}(y) = \sigma y^p \neq 0$. The composite $\Sigma\mathbb{C}P^2 \hookrightarrow \Sigma\mathbb{C}P^\infty \to S^3$ induces a map on cohomology,

$$\mathbb{Z}/p\mathbb{Z} = H^3(S^3;\mathbb{Z}/p\mathbb{Z}) \xrightarrow{g^*} H^3(\Sigma\mathbb{C}P^\infty;\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} H^3(S^3;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$$

which is just given by multiplication by a unit in $\mathbb{Z}/p\mathbb{Z}$, so it is an isomorphism. In particular, if $x \in \operatorname{im} g^*$ then the above shows $\mathcal{P}(x) \in \operatorname{im} g^* \subseteq H^{2p+1}(\Sigma\mathbb{C}P^\infty;\mathbb{Z}/p\mathbb{Z})$ which is 0 for p large enough, contradicting $\mathcal{P}(x) \neq 0$. Thus $m = 0$, meaning $\Sigma\mathbb{C}P^2 \hookrightarrow \Sigma\mathbb{C}P^\infty \xrightarrow{g} S^3$ is nullhomotopic for any map g.

Altogether, this proves that our constructed maps $g_n : \Sigma\mathbb{C}P^n \to S^3$ do not extend to all of $\Sigma\mathbb{C}P^\infty$, so Theorem 4.1.9 implies that $\lim^1[\Sigma\mathbb{C}P^n, S^3]$ is an uncountable (divisible abelian) group. Hence by Corollary 4.2.4, we have shown:

Corollary 4.2.6. *There are an uncountable number of phantom maps $\Sigma\mathbb{C}P^\infty \to S^3$.*

4.3 Postnikov Towers

Cellular theory and CW-complexes are constructed from the building blocks $S^n, n \geq 0$, using attachments coming from pushout diagrams:

$$\begin{tikzcd} \bigvee S^n \arrow[r] \arrow[d] & X_n \arrow[d] \\ \bigvee D^{n+1} \arrow[r] & X_{n+1} \end{tikzcd}$$

In this setting, X_{n+1} is the homotopy cofibre of $\bigvee S^n \to X_n$ and this gives a nice space $X = \varinjlim X_n$.

There is a dual construction in which each X_n is (homotopy equivalent to) an Eilenberg-Maclane space $K(A,n)$ for some group A and such that X_{n+1} is the homotopy fibre of a map $X_n \to K(A, n+2)$. In fact, every 'nice' space X is of the form $X = \varprojlim X_n$ for such a sequence of X_n. Explicitly:

Theorem 4.3.1. *Suppose X is simply connected. Then there exists a sequence of spaces $X_2 \leftarrow X_3 \leftarrow X_4 \leftarrow \cdots$ and maps $f_i : X \to X_i$ such that each $X_n \to X_{n-1}$ is a fibration, $X_2 = K(\pi_2(X), 2)$ and there is a commutative diagram*

$$
\begin{tikzcd}
 & X \arrow[dl, "f_2"'] \arrow[d, "f_3"] \arrow[dr, "f_4"] & & \\
X_2 \arrow[d, "k_2"'] & X_3 \arrow[l] \arrow[d, "k_3"'] & X_4 \arrow[l] \arrow[d, "k_4"'] & \cdots \arrow[l] \\
K(\pi_3(X),4) & K(\pi_4(X),5) & K(\pi_5(X),6) &
\end{tikzcd}
$$

Moreover, each f_n induces an isomorphism $\pi_k(X) \to \pi_k(X_n)$ for $k \leq n$, $\pi_k(X_n) = 0$ for $k > n$ and each $X_{n+1} \to X_n \to K(\pi_{n+1}(X), n+2)$ is a fibration sequence.

Note that each map $k_n : X_n \to K(\pi_{n+1}(X), n+2)$ may be viewed as a class $k_n \in H^{n+2}(X_n; \pi_{n+1}(X))$.

Proof. Assume inductively that $X_2, \dots, X_n$ and the associated maps have been constructed. The idea in constructing the X_{n+1} is to "kill off" the higher homotopy groups of X. Let $\mathcal{P}_{n+1} : \mathtt{Top}_* \to \mathtt{Top}_*$ be the functor sending a space X to the cofibre of $\wedge f : \bigwedge_f S^{n+1} \to X$, where the wedge product is over all $f \in \mathrm{Map}(S^{n+1}, X)$. Then for each $X \in \mathtt{Top}_*$, $\mathcal{P}_{n+1}(X)$ is a pushout:

$$
\begin{tikzcd}
\bigvee_{f\in\mathrm{Map}(S^{n+1},X)} S^{n+1} \arrow[r] \arrow[d] & X \arrow[d] \\
\bigvee_{f\in\mathrm{Map}(S^{n+1},X)} D^{n+2} \arrow[r] & \mathcal{P}_{n+1}(X)
\end{tikzcd}
$$

Then the Blakers-Massey theorem (3.2.11) implies $\pi_{n+1}(\mathcal{P}_{n+1}(X)) = 0$ and $\pi_i(X) \to \pi_i(\mathcal{P}_{n+1}(X))$ is an isomorphism for $i \leq n$. Thus the direct limit

$$X_{n+1} = \varinjlim(X \to \mathcal{P}_{n+1}(X) \to \mathcal{P}_{n+2}(\mathcal{P}_{n+1}(X)) \to \cdots)$$

is well-defined. This gives a map $f_{n+1} : X \to X_{n+1}$. We now want to show that k_{n+1} is defined:

$$k_{n+1} : X_{n+1} \to K(\pi_{n+2}(X), n+3).$$

By the comment preceding this proof, such a k_n is an element of $H^{n+2}(X_n; \pi_{n+1}(X))$. By the long exact sequence in homotopy for the pair (X_n, X_{n+1}),

$$\pi_i(X_n, X_{n+1}) = \begin{cases} \pi_{n+1}(X), & i = n+2 \\ 0, & i \neq n+2. \end{cases}$$

When X is simply connected, the relative Hurewicz theorem (Corollary 3.3.5) implies

$$H_{n+2}(X_n, X_{n+1}) \cong \pi_{n+1}(X).$$

By the universal coefficient theorem, $H^{n+2}(X_n, X_{n+1}; \pi_{n+1}(X)) \cong \mathrm{Hom}(\pi_{n+1}(X), \pi_{n+1}(X))$ so there is an element $u \in H^{n+2}(X_n, X_{n+1}; \pi_{n+1}(X))$ corresponding to the identity $1_{\pi_{n+1}(X)} \in \mathrm{Hom}(\pi_{n+1}(X), \pi_{n+1}(X))$. Take k_{n+1} to be the image of u under the map

$$H^{n+2}(X_n, X_{n+1}; \pi_{n+1}(X)) \longrightarrow H^{n+2}(X_n; \pi_{n+1}(X))$$

in the long exact sequence for (X_n, X_{n+1}). One now checks that the desired properties of X_{n+1} and k_{n+1} are met. □

Corollary 4.3.2. *If $X_2 \leftarrow X_3 \leftarrow \cdots$ is the Postnikov tower of X, then X_{n+1} is the pullback along k_n of the path space fibration over $K(\pi_{n+1}(X), n+2)$. In particular, we have a fibration sequence*

$$K(\pi_{n+1}(X), n+1) \to X_{n+1} \to X_n \xrightarrow{k_n} K(\pi_{n+1}(X), n+2).$$

Definition. *For X simply connected, a tower $X_2 \leftarrow X_3 \leftarrow \cdots$ as in the theorem is called a* **Postnikov tower** *for X.*

Proposition 4.3.3. *For space X, let $X_2 \leftarrow X_3 \leftarrow \cdots$ be its Postnikov tower and call $X\langle n\rangle$ the fibre of the map $f_n : X \to X_n$. Then*

(1) The assignment $X \mapsto (X_2 \leftarrow X_3 \leftarrow \cdots)$ is a functor on the homotopy category $h(\mathtt{Top}_*)$.

(2) For each $n \geq 2$, $X\langle n\rangle$ is an n-connected cover of X.

Suppose X and Y are spaces, with X simply connected, and $X = \varprojlim X_n$ is a Postnikov tower for X. Given a map $g_n : Y \to X_n$ for some n, a natural question to ask is when g_n lifts to some g_{n+1}:

$$\begin{tikzcd} & X_{n+1} \arrow[d] & \\ Y \arrow[r, "g_n"'] \arrow[ur, dashed, "g_{n+1}"] & X_n \arrow[r, "k_n"'] & K(\pi_{n+1}(X), n+2) \end{tikzcd}$$

By obstruction theory, there exists such a lift g_{n+1} precisely when $k_n g_n$ is trivial in $H^{n+2}(Y; \pi_{n+1}(X))$.

Example 4.3.4. If Y is a CW-complex with only even-degree cells and $\pi_{2n+1}(X) = 0$ for all n, then $k_n g_n = 0$ will always hold in homology, so such a map $g_n : Y \to X_n$ will in fact lift to $g : Y \to X$.

Proposition 4.3.5. *Suppose Y is a CW-complex with only even degree cells and $\pi_{2n+1}(X) = 0$ for all n. Then for any $g_2 \in H^2(Y; \pi_2(X))$, there exists a map $g : Y \to X$ inducing g_2 on degree 2 cohomology.*

Example 4.3.6. This holds when $Y = \mathbb{C}P^\infty, \Omega S^3$ or G/T for G a compact Lie group with maximal torus T, for a few examples.

Example 4.3.7. Let $X = BU$ be the universal classifying space for complex vector bundles. Then by Bott periodicity, $\pi_n(BU) = \mathbb{Z}$ when n is even and $\pi_n(BU)$ is torsion when n is odd, so Proposition 4.3.5 applies. In particular, any $g_2 \in H^2(Y; \pi_n(BU))$ is induced by a map $g : Y \to BU$.

4.4 Goodwillie Towers

Recall that the infinite symmetric product functor $SP^\infty : \mathtt{Top}_* \to \mathtt{Top}_*$ takes homotopy pushout squares to homotopy pullback squares (this is Lemma 3.6.5). Also, the Dold-Thom theorem (3.6.6) shows that the collection of functors $\pi_n(SP^\infty(-))$ satisfies the axioms of a homology theory.

Definition. *A homotopy functor $F : \mathtt{Top}_* \to \mathtt{Top}_*$ is* **linear** *(or* **polynomial of degree** 1*) if F takes pushouts to pullbacks.*

Example 4.4.1. Note that, contrary to what the terminology might suggest, the identity functor is *not* a linear functor.

Theorem 4.4.2. *For every functor $F : \mathtt{Top}_* \to \mathtt{Top}_*$, there is a natural transformation $F \to P_1F$ where P_1F is a linear functor.*

Goodwillie used this theorem as a jumping off point to define "higher degree polynomial functors"

$$P_1F \leftarrow P_2F \leftarrow P_3F \leftarrow \cdots$$

that are compatible with the natural transformations $F \to P_n$. Moreover, he showed that for each X, the homotopy cofibres $Q_n(X)$,

$$\begin{tikzcd}
 & F(X) \arrow[dl] \arrow[d] \arrow[dr] & & \\
P_1F(X) \arrow[d] & P_2F(X) \arrow[l] \arrow[d] & P_3F(X) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
Q_1(X) & Q_2(X) & Q_3(X) &
\end{tikzcd}$$

are infinite loop spaces.

4.5 Localization of a Topological Space

Let A be a finitely generated abelian group. Then A can be written

$$A = A_{p_1} \oplus A_{p_2} \oplus \cdots \oplus A_{p_k} \oplus \mathbb{Z}^r$$

where $r \in \mathbb{N}$, $p_1, \ldots, p_k$ are prime integers and A_{p_i} is a Sylow p_i-subgroup of A. For each prime p dividing the order of A_{tors}, A_p may be viewed as the localization of A at the prime p: $A_p = A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)} = \left\{\frac{a}{b} : p \nmid b\right\} \subseteq \mathbb{Q}$. The goal of topological localization is to find a space $X_{(p)}$ for each prime p satisfying

$$\pi_n(X_{(p)}) = \pi_n(X) \oplus_{\mathbb{Z}} \mathbb{Z}_{(p)}$$

for each $n \geq 2$. It will even follow that

$$H_\bullet(X_{(p)}) = H_\bullet(X) \oplus \mathbb{Z}_{(p)}.$$

Let T be a set of prime numbers in $\mathbb{Z}$ and define the *localization at* T:

$$\mathbb{Z}_T := \left\{\frac{m}{n} \in \mathbb{Q} : (p, n) = 1 \text{ for all } p \in T\right\}.$$

Example 4.5.1. Some important examples of localizations at sets of primes are:

$$\mathbb{Z}_\varnothing = \mathbb{Q}, \quad \mathbb{Z}_{\operatorname{Spec}\mathbb{Z}} = \mathbb{Z}, \quad \mathbb{Z}_{\{p\}} = \mathbb{Z}_{(p)}.$$

(Here, $\operatorname{Spec}\mathbb{Z}$ denotes the set of all prime integers.)

Fix a set of primes T.

Definition. *An abelian group A is said to be T-**local** if A is a $\mathbb{Z}_T$-module, i.e. if for all $p \notin T$, multiplication by p gives an isomorphism $A \to A$.*

Example 4.5.2. When $T = \varnothing$, so $\mathbb{Z}_T = \mathbb{Q}$, we call a T-local abelian group a *rational abelian group*. When $T = \{p\}$ consists of a single prime, T-local groups are called *p-local* for short.

Definition. *For any abelian group A, the **localization** of A at T is the $\mathbb{Z}_T$-module $A_T = A \otimes_{\mathbb{Z}} \mathbb{Z}_T$.*

Lemma 4.5.3. *For all sets of primes T and abelian groups A,*

(a) A_T is T-local and the induced map $A \to A_T$ is universal with respect to T-local abelian groups.

(b) The map $A \to A_T$ is an isomorphism if and only if A is T-local.

(c) The functor $(\cdot)^T$: `AbGps` $\to$ `AbGps` which sends $A \mapsto A_T$ is exact. (Equivalently, $\mathbb{Z}_T$ is flat.)

(d) If $0 \to A \to B \to C \to 0$ is a short exact sequence of abelian groups then B is T-local if and only if A and C are both T-local.

This has the following topological consequences.

Lemma 4.5.4. *For any space X, the homology ring $\widetilde{H}_\bullet(X)$ is T-local if and only if for all $p \notin T$, $\widetilde{H}_\bullet(X;\mathbb{Z}/p\mathbb{Z}) = 0$.*

Proof. By the homology axioms, the short exact sequence $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ induces a long exact sequence

$$\cdots \to \widetilde{H}_n(X;\mathbb{Z}) \xrightarrow{p} \widetilde{H}_n(X;\mathbb{Z}) \to \widetilde{H}_n(X;\mathbb{Z}/p\mathbb{Z}) \to \widetilde{H}_{n-1}(X;\mathbb{Z}) \to \cdots$$

which implies the result. □

Proposition 4.5.5. *Assume $F \to E \xrightarrow{q} B$ is a fibration such that $\pi_1(F), \pi_1(E)$ and $\pi_1(B)$ are all abelian. For any X, let $\pi_\bullet(X) = \bigoplus_{n\geq 1} \pi_n(X)$. Then*

(1) If two of $\pi_\bullet(F), \pi_\bullet(E), \pi_\bullet(B)$ are T-local, then so is the third.

(2) Suppose $\pi_1(B)$ acts trivially on $H_\bullet(F)$. If two of $\widetilde{H}_\bullet(F), \widetilde{H}_\bullet(E), \widetilde{H}_\bullet(B)$ are T-local, then so is the third.

Proof. (1) Apply the long exact sequence in homotopy (Corollary 1.4.6) and induct, using the Five Lemma.

(2) Consider the morphism of fibrations

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[r, "q"] \arrow[d, "q"] & B \arrow[d, "\mathrm{id}"] \\ * \arrow[r] & B \arrow[r, "\mathrm{id}"'] & B \end{tikzcd}$$

The Serre spectral sequence implies that $q_* : \widetilde{H}_\bullet(E;\mathbb{Z}/p\mathbb{Z}) \to \widetilde{H}_\bullet(B;\mathbb{Z}/p\mathbb{Z})$ is an isomorphism if and only if $\widetilde{H}_\bullet(F;\mathbb{Z}/p\mathbb{Z}) = 0$. Thus for any prime $p \notin T$, if two of the following hold:

- $\widetilde{H}_\bullet(F;\mathbb{Z}/p\mathbb{Z}) = 0$
- $\widetilde{H}_\bullet(E;\mathbb{Z}/p\mathbb{Z}) = 0$
- $\widetilde{H}_\bullet(B;\mathbb{Z}/p\mathbb{Z}) = 0$

then the third holds as well. Thus Lemma 4.5.4 implies statement (2). □

We call a space *simple* if $\pi_1(X)$ is abelian and acts trivially on $\pi_\bullet(X)$.

Theorem 4.5.6. *Let X be a simple space. Then the following are equivalent for every set of primes T:*

(a) $\pi_\bullet(X)$ is T-local.

(b) $\widetilde{H}_\bullet(X)$ is T-local.

Proof. (Sketch) First consider $X = K(A,1)$ for an abelian group A. Then $\pi_1(X) = A = H_1(X)$ and $\pi_k(X) = 0$ for $k \neq 1$, so (b) $\implies$ (a). Conversely, if $\pi_\bullet(X) = \pi_1(X) = A$ is T-local, we may assume A is the T-localization of a finitely generated abelian group. Using the Künneth formula, we may individually consider the cases $A = \mathbb{Z}_T$ and $A = (\mathbb{Z}/q^k\mathbb{Z})_T$ for q prime. In the latter case, if $q \notin T$ then $A = 0$ so (a) $\implies$ (b) is trivial. If $q \in T$, $A = \mathbb{Z}/q^k\mathbb{Z}$ and we have

$$\widetilde{H}_n(K(\mathbb{Z}/q^k\mathbb{Z},1);\mathbb{Z}) = \begin{cases} \mathbb{Z}/q^k\mathbb{Z}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Therefore $\widetilde{H}_\bullet(X)$ is T-local in all cases. If $A = \mathbb{Z}_T$, we have

$$K(\mathbb{Z}_T,1) = \mathrm{Tel}(S^1 \xrightarrow{p_1} S^1 \xrightarrow{p_2} S^1 \xrightarrow{p_3} \cdots),$$

the mapping telescope (see Section 4.2) of the sequence $S^1 \xrightarrow{p_1} S^1 \xrightarrow{p_2} S^1 \xrightarrow{p_3} \cdots$ where $p_1, p_2, p_3, \ldots$ are the primes outside T, repeated infinitely often. In particular,

$$H_k(K(\mathbb{Z}_T,1);\mathbb{Z}) = \begin{cases} \mathbb{Z}_T, & k = 1 \\ 0, & k \geq 2. \end{cases}$$

Hence (a) $\implies$ (b) for all $X = K(A,1)$.

Next, suppose $X = K(A,n)$ for $n \geq 2$. Then we have a fibration sequence

$$K(A,n-1) \to K(A,n)$$

(since $\Omega K(A,n) = K(A,n-1)$ and $PK(A,n) = *$), so the equivalence of (a) and (b) follows from Proposition 4.5.5 and induction. Finally, suppose X is an arbitrary simple space. Then by Theorem 4.3.1, there is a Postnikov tower

$$\begin{tikzcd}
 & & X \arrow[dl] \arrow[dr] & & \\
\cdots & X_{n-1} \arrow[l] \arrow[d] & & X_n \arrow[ll] \arrow[d] & \cdots \arrow[l] \\
 & K(\pi_n(X),n+1) & & K(\pi_{n+1}(X),n+2) &
\end{tikzcd}$$

Assuming (a) holds for X, Proposition 4.5.5 implies (a) also holds for each $K(\pi_n(X),n+1)$. Thus by the special case above, (b) also holds for each $K(\pi_n(X),n+1)$. By induction and Proposition 4.5.5, (b) also holds for each X_n. Now by the property of Postnikov towers (Theorem 4.3.1), $\pi_k(X) \to \pi_k(X_n)$ is an isomorphism for $k \leq n$ and hence $H_k(X) \to H_k(X_n)$ is an isomorphism for $k \leq n$. This shows that (b) holds for X.

Conversely, suppose (b) holds for X. There exists a simply connected cover $\widetilde{X} \to X \to K(\pi_1(X),1)$ – e.g. take $\widetilde{X}$ to be the homotopy fibre of $X \to K(\pi_1(X),X)$, which is simply connected by Lemma 3.1.2. Then by assumption, $H_1(X)$ is T-local, so $\pi_1(X)$ is also T-local. By the special case above, $H_\bullet(K(\pi_1(X), 1))$ is T-local, which implies by Proposition 4.5.5 that (b) holds for $\widetilde{X}$. Thus by the Hurewicz theorem (3.3.4), $\pi_2(\widetilde{X}) \cong H_2(\widetilde{X})$ is T-local. Now induct, using an $(n+1)$-connected cover $\widetilde{X}^n \to X \to X_{n-1}$ and the same argument. This proves (b) $\implies$ (a) holds for all X. □

Definition. *A space X is T-**local** if $\pi_\bullet(X)$ is T-local, or equivalently, if $\widetilde{H}_\bullet(X)$ is T-local.*

Theorem 4.5.7. *Suppose $\ell : X \to X'$ is a map between simple spaces. Then the following are equivalent:*

(a) $\ell_* : \pi_\bullet(X) \to \pi_\bullet(X')$ *agrees with T-localization.*

(b) $\ell_* : H_\bullet(X) \to H_\bullet(X')$ *agrees with T-localization.*

Further, if these conditions hold then ℓ is the universal map from X to a T-local space. Explicitly, if $f : X \to Y$ is any map to a T-local space Y, then there exists a map $h : X' \to Y$ which is unique up to homotopy and makes the following diagram commute:

$$\begin{tikzcd} X \arrow[rr, "\ell"] \arrow[dr, "f"'] & & X' \arrow[dl, dashed, "h"] \\ & Y & \end{tikzcd}$$

Definition. *Such a map $\ell : X \to X'$ is called a T-**localization** of X.*

Theorem 4.5.8. *T-localizations exist for every simple space X.*

Proof. (Sketch) One starts by verifying that $K(\pi_n(X), n+1) \to K(\pi_n(X)_T, n+1)$ is a T-localization. Then induct using the Postnikov tower for X. □

Corollary 4.5.9 (T-Hurewicz Theorem)**.** *Suppose X is a simply connected space and T is a set of primes. Then the conditions*

(1) $\pi_k(X) \otimes \mathbb{Z}_T = 0$ *for all* $k < n$

(2) $H_k(X; \mathbb{Z}_T) = 0$ *for all* $k < n$

are equivalent, and when either holds, $\pi_n(X) \otimes \mathbb{Z}_T \cong H_n(X; \mathbb{Z}_T)$.

Proof. By Theorem 4.5.8, there exists a space X_T with $\pi_n(X_T) = \pi_n(X) \otimes \mathbb{Z}_T$. Then the result follows from the ordinary Hurewicz theorem (3.3.4). □

Definition. *A map $f : X \to Y$ is called a T-**equivalence** if the induced map $f_* : H_\bullet(X; \mathbb{Z}_T) \to H_\bullet(Y; \mathbb{Z}_T)$ is an isomorphism.*

Let W_T be the collection of all T-equivalences in the category $\mathtt{Top}$. In the same way that we construct the homotopy category $h(\mathtt{Top})$ by inverting all homotopy equivalences in $\mathtt{Top}$, we can form the category $\mathtt{Top}_T = \mathtt{Top}[W_T^{-1}]$ by formally inverting all T-equivalences in $\mathtt{Top}$. We will denote the Hom sets in this category by $\mathrm{Hom}_T(X, Y)$.

Proposition 4.5.10. *For all spaces X and Y with T-localizations X_T and Y_T, respectively, there is a bijection* $\mathrm{Hom}_T(X,Y) \cong [X_T, Y_T]$ *which is natural in each variable.*

This says that the T-localization functor $X \mapsto X_T$ factors through the "T-local category" $\mathtt{Top}_T$. This generalizes in the following way. Given a generalized homology theory $E_\bullet$, call $f : X \to Y$ an E-equivalence if $f_* : E_\bullet(X) \to E_\bullet(Y)$ is an isomorphism.

Theorem 4.5.11 (Bousfield Localization)**.** *For every generalized homology theory $E_\bullet$, there exists a functor $L_E : \mathtt{Top} \to \mathtt{Top}$ and morphisms $\eta_X : X \to L_E X$ for each $X \in \mathtt{Top}$ such that*

(1) $L_E(X)$ is an E-local space.

(2) There is a natural bijection $\mathrm{Hom}_E(X,Y) \cong [L_E X, L_E Y]$.

Example 4.5.12. If $E_\bullet(-) = H_\bullet(-;\mathbb{Z}/p\mathbb{Z})$, then L_E is called the *p-completion functor.*

Example 4.5.13. If $E_\bullet(X) = K_\bullet(X)$ is the complex K-theory for all X, then L_K is more exotic. For instance, since $\widetilde{K}_\bullet(K(\mathbb{Z}/2\mathbb{Z},2)) = 0$, it follows that $L_K(K(\mathbb{Z}/2\mathbb{Z},2)) = *$.

4.6 Rational Localization

In this section, we study the localization of spaces at the set $T = \varnothing$. In this case, $\mathbb{Z}_T = \mathbb{Q}$ and we call T-localization *rational localization*. Analogously, T-local abelian groups are *rational abelian groups*, T-local spaces are *rational spaces* and the canonical map $\ell : X \to X_0 := X_T$ is called the *rationalization* of X. A T-equivalence $f : X \to Y$ will be called a *rational equivalence*. By Theorem 4.5.8, if $\ell : X \to X_0$ is the rationalization of X, we have

$$\pi_n(X_0) \cong \pi_n(X) \otimes \mathbb{Q} \quad \text{and} \quad H_n(X_0;\mathbb{Z}) \cong H_n(X;\mathbb{Q})$$

for all n. Thus rationalization can be understood as the removal of torsion in homotopy and homology groups. The remaining features of homotopy theory are still of immense interest. For example, let $S^n_{\mathbb{Q}} := (S^n)_0$ be the rationalization of the n-sphere, called the nth *rational homotopy sphere.*

Proposition 4.6.1. *For each $n \geq 0$,*

$$\widetilde{H}_k(S^n;\mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{if } k = n, \\ 0, & \text{if } k \neq n. \end{cases}$$

Proof. Immediate from the universal coefficient theorem. $\square$

Proposition 4.6.2. *For every n, $H^n(K(\mathbb{Z},n);\mathbb{Q}) \cong \mathbb{Q}$. Moreover, if $x \in H^n(K(\mathbb{Z},n);\mathbb{Q})$ is a generator, then*

(1) If n is odd, then $H^\bullet(K(\mathbb{Z},n);\mathbb{Q})$ is an exterior algebra $\bigwedge[x]$.

(2) If n is even, then $H^\bullet(K(\mathbb{Z},n);\mathbb{Q})$ is a polynomial algebra $\mathbb{Q}[x]$.

Proof. (Sketch) By Proposition 3.5.1, we have isomorphisms

$$\pi_n(K(\mathbb{Z},n)) \cong [S^n, K(\mathbb{Z},n)] \cong \widetilde{H}^n(S^n) \cong \mathbb{Z}$$

so take a map $f : S^n \to K(\mathbb{Z},n)$ which generates $\pi_n(K(\mathbb{Z},n))$. By Corollary 4.5.9, the induced map

$$f^* : H^n(K(\mathbb{Z},n);\mathbb{Q}) \longrightarrow H^n(S^n;\mathbb{Q})$$

is an isomorphism, and by Proposition 4.6.1, $H^n(S^n;\mathbb{Z}) \cong \mathbb{Q}$.

We now prove (1) and (2) by induction. For $n = 1$, $K(\mathbb{Z},1) = S^1$ by Example 3.5.5 and $H^\bullet(S^1;\mathbb{Q})$ is an exterior algebra on $x \in H^1(S^1;\mathbb{Q})$ by Proposition 4.6.1. For $n = 2$, we have $K(\mathbb{Z},2) = \mathbb{C}P^\infty$ by Example 3.5.5 and it is known that $H^\bullet(\mathbb{C}P^\infty;\mathbb{Q}) \cong \mathbb{Q}[x]$. One then inducts using the Serre spectral sequence. $\square$

Theorem 4.6.3. *Let S^n be the n-sphere. Then*

(1) If n is odd, then $\pi_k(S^n)$ is finite for all $k \neq n$.

(2) If n is even, then $\pi_k(S^n)$ is finite for all $k \neq n, 2n-1$ and $\pi_{2n-1}(S^n) \cong \mathbb{Z} \oplus T$ for T a finite abelian group.

Proof. (1) By (1) of Proposition 4.6.2, $H^\bullet(K(\mathbb{Z},n);\mathbb{Q})$ is an exterior algebra on $x \in H^n(K(\mathbb{Z},n);\mathbb{Q})$, but so is $H^n(S^n;\mathbb{Q})$ by Proposition 4.6.1. Moreover, as in the proof of Proposition 4.6.2, a map $f : S^n \to K(\mathbb{Z},n)$ generating $\pi_n(K(\mathbb{Z},n))$ induces an isomorphism $f^* : H^n(K(\mathbb{Z},n);\mathbb{Q}) \to H^n(S^n;\mathbb{Q})$, so it follows that $f^* : H^\bullet(K(\mathbb{Z},n);\mathbb{Q}) \to H^\bullet(S^n;\mathbb{Q})$ is an isomorphism of graded algebras. In particular, f is a rational homotopy equivalence in the sense of the previous section, so combining Whitehead's second theorem (Corollary 3.3.6) with the base change functor $- \otimes \mathbb{Q}$ (which is exact since $\mathbb{Q}$ is flat), we get that

$$f_* : \pi_\bullet(S^n) \otimes \mathbb{Q} \longrightarrow \pi_\bullet(K(\mathbb{Z},n)) \otimes \mathbb{Q}$$

is an isomorphism. Consequently, $\pi_n(S^n) \otimes \mathbb{Q} \cong \mathbb{Q}$ and $\pi_k(S^n) \otimes \mathbb{Q} = 0$ for all $k \neq n$. Since each $\pi_k(S^n)$ is finitely generated, this shows $\pi_k(S^n)$ is finite for all $k \neq n$.

(2) Again let $x \in H^n(K(\mathbb{Z},n);Q)$ be a generator so that by (2) of Proposition 4.6.2, $H^\bullet(K(\mathbb{Z},n);\mathbb{Q}) \cong \mathbb{Q}[x]$. Consider $x^2 \in H^{2n}(K(\mathbb{Z},n)) = [K(\mathbb{Z},n), K(\mathbb{Z},2n)]$. Then x^2 determines a map $K(\mathbb{Z},n) \to K(\mathbb{Z},2n)$ with homotopy fibre F. Let $f : S^n \to K(\mathbb{Z},n)$ generate $\pi_n(K(\mathbb{Z},n))$ as above. By Proposition 3.5.1, $\pi_n(K(\mathbb{Z},2n)) = 0$ so f factors through a map $S^n \to F$:

$$\begin{tikzcd}
F \arrow[r] & K(\mathbb{Z},n) \arrow[r, "x^2"] & K(\mathbb{Z},2n) \\
& S^n \arrow[ul, dashed, "g"] \arrow[u, "f"'] &
\end{tikzcd}$$

Applying the long exact sequence in homotopy (Corollary 1.4.6) to the fibration sequence $F \to K(\mathbb{Z},n) \to K(\mathbb{Z},2n)$, we get $\pi_k(F) = 0$ for $k \neq n, 2n-1$ and $\pi_n(F) \cong \pi_{2n-1}(F) \cong \mathbb{Z}$.

Thus $g_* : \pi_n(S^n) \to \pi_n(F)$ is an isomorphism so by the local Hurewicz theorem (Corollary 4.5.9), $g^* : H^n(F;\mathbb{Q}) \to H^n(S^n;\mathbb{Q})$ is also an isomorphism. But $H^\bullet(K(\mathbb{Z},n);\mathbb{Q}) \cong \mathbb{Q}[x]$ and $H^\bullet(K(\mathbb{Z},2n);\mathbb{Q}) \cong \mathbb{Q}[x^2]$, so it follows that $H^\bullet(F;\mathbb{Q}) \cong \mathbb{Q}[x]/(x^2)$, i.e. exactly the same as $H^\bullet(S^n;\mathbb{Q})$. Thus g is a rational homotopy equivalence and the proof finishes as before. □

Corollary 4.6.4. *The natural map $S^n_{\mathbb{Q}} \to K(\mathbb{Q},n)$ is a rational homotopy equivalence if n is odd and has homotopy fibre $K(\mathbb{Q},2n-1)$ if n is even.*

A rational space X is of finite type if $H_\bullet(X;\mathbb{Q})$ is a finite dimensional $\mathbb{Q}$-vector space. We have the following general structure theorem for rational H-spaces.

Theorem 4.6.5. *Every rational H-space X of finite type is a product of rational Eilenberg-Maclane spaces $\prod_{n\geq 1} K(\pi_n(X),n)$.*

5 Stable Homotopy Theory

Recall that Freudenthal's suspension theorem (Corollary 3.2.5) says that if X is a d-dimensional CW-complex and Y is $(n-1)$-connected, then the suspension map

$$\Sigma : [X, Y] \longrightarrow [\Sigma X, \Sigma Y]$$

is a bijection for $d < 2n-1$ and a surjection for $d = 2n-1$. This implies that the sequence $[X, Y], [\Sigma X, \Sigma Y], [\Sigma^2 X, \Sigma^2 Y], \ldots$ eventually stabilizes. This is the jumping off point for a theory of "stable" homotopy theory, in which topological spaces are replaced with more general objects called *spectra* and the homotopy category $h(\mathtt{Top})$ is upgraded to a *stable homotopy category*. There are various approaches throughout the history of homotopy theory to the problem of building a useful stable category. We describe one of the first approaches in Section 5.1 before introducing the modern version in Section 5.2.

5.1 The Spanier-Whitehead Category

In this section we describe the *Spanier-Whitehead category* $\mathtt{SW}$. The objects of $\mathtt{SW}$ are all finite CW-complexes X and we define the morphisms by

$$\mathrm{Hom}_{\mathtt{SW}}(X, Y) = \varinjlim [\Sigma^n X, \Sigma^n Y].$$

Freudenthal's suspension theorem implies – as in the introduction – that the sequence $[\Sigma^n X, \Sigma^n Y]$ stabilizes, so it's equivalent to write $\mathrm{Hom}_{\mathtt{SW}}(X, Y) = [\Sigma^N X, \Sigma^N Y]$ for some large enough $N \in \mathbb{N}$.

Lemma 5.1.1. *Let X and Y be finite CW-complexes. Then*

(a) The suspension functor induces a bijection $\mathrm{Hom}_{\mathtt{SW}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathtt{SW}}(\Sigma X, \Sigma Y)$*.*

(b) $\mathrm{Hom}_{\mathtt{SW}}(X, Y)$ *is an abelian group.*

(c) For any morphisms f, g, h in $\mathtt{SW}$,

$$(f+g) \circ h = f \circ h + g \circ h \qquad \text{and} \qquad h \circ (f+g) = h \circ f + h \circ g.$$

In particular, $\mathtt{SW}$ is an additive category.

Proof. (a) and (c) are obvious. For (b), we may take $N \geq 2$ in the alternate description $\mathrm{Hom}_{\mathtt{SW}}(X, Y) = [\Sigma^N X, \Sigma^N Y]$, so that $[\Sigma^N X, \Sigma^N Y] = [\Sigma^{N-1} X, \Omega\Sigma^N Y]$, $\Sigma^{N-1} X$ is a co-H-space, $\Omega\Sigma^N Y$ is an H-space and the result is implied by Theorem 0.3.7. $\square$

Next, we introduce some new objects to our stable category. For each finite CW-complex X and each $n \geq 1$, we let $\Sigma^{-n} X$ denote the nth formal 'desuspension' of X, with

$$\mathrm{Hom}_{\mathtt{SW}}(\Sigma^{-n} X, \Sigma^{-m} Y) = [\Sigma^{N-n} X, \Sigma^{N-m} Y]$$

for large enough $N \in \mathbb{N}$. Now for every finite CW-complex X we have suspensions $\Sigma^n X$ for every *integer* $n \in \mathbb{Z}$. This defines a desuspension functor $\Sigma^{-1} : \mathtt{SW} \to \mathtt{SW}, X \mapsto \Sigma^{-1} X$.

Example 5.1.2. In $\mathtt{SW}$, there are now spheres in 'negative dimensions': $S^{-n} = \Sigma^{-n} S^0$.

Proposition 5.1.3. $\Sigma : \mathtt{SW} \to \mathtt{SW}, X \mapsto \Sigma X$ *is an equivalence of categories with inverse* Σ^{-1}.

Take a map $f : X \to Y$ and form the cofibration sequence

$$X \xrightarrow{f} Y \xrightarrow{g} C(f) \xrightarrow{h} \Sigma X \to \cdots$$

Then by Theorem 1.2.5, for each W there is a long exact sequence

$$[X, W] \xleftarrow{f^*} [Y, W] \xleftarrow{g^*} [C(f), W] \xleftarrow{h^*} [\Sigma X, W] \leftarrow \cdots$$

In the Spanier-Whitehead category, this becomes a long exact sequence of *abelian groups*:

$$\mathrm{Hom}_{\mathtt{SW}}(X, W) \leftarrow \mathrm{Hom}_{\mathtt{SW}}(Y, W) \leftarrow \mathrm{Hom}_{\mathtt{SW}}(C(f), W) \leftarrow \mathrm{Hom}_{\mathtt{SW}}(\Sigma X, W) \leftarrow \cdots$$

What's amazing is that cofibration sequences also induce *covariant* exact sequences in $\mathtt{SW}$.

Lemma 5.1.4. *For a map $f : X \to Y$ of finite CW-complexes and any finite CW-complex W,*

$$\mathrm{Hom}_{\mathtt{SW}}(W, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathtt{SW}}(W, Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathtt{SW}}(W, C(f))$$

is an exact sequence.

Proof. It's clear that $g_* \circ f_* = 0$. Suppose $g_*(\alpha) = 0$ for $\alpha \in \mathrm{Hom}_{\mathtt{SW}}(W, Y)$. Viewing this as the following diagram, we want to find some $\beta : W \to X$ such that $f_*(\beta) = \alpha$:

$$\begin{tikzcd}
& W \arrow[r] \arrow[dl, dashed, "\beta"'] \arrow[d, "\alpha"] & * \arrow[r] \arrow[d] & \Sigma W \arrow[r] \arrow[d, dashed, "\beta'"] & \Sigma W \arrow[d, "\Sigma\alpha"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & C(f) \arrow[r, "h"'] & \Sigma X \arrow[r, "\Sigma f"'] & \Sigma Y
\end{tikzcd}$$

By Proposition 1.2.7, there exists a map $\beta' : \Sigma W \to \Sigma X$ making the two right squares in the diagram commute, but by Freudenthal's suspension theorem, $[W, X] \to [\Sigma W, \Sigma X]$ is a bijection so $\beta' = \Sigma \beta$ for some $\beta : W \to X$. By commutativity, $f_*(\beta) = \alpha$ so we are done. $\square$

As a result, homotopy cofibre sequences in $\mathtt{SW}$ are the same as homotopy fibre sequences. In technical language, this says that $\mathtt{SW}$ is a *triangulated category*, i.e. an additive category $\mathcal{C}$ with an isomorphism $\Sigma : \mathcal{C} \to \mathcal{C}$ and *distinguished triangles*

$$\begin{tikzcd}
X \arrow[rr] & & Y \arrow[dl] \\
& Z \arrow[ul] &
\end{tikzcd}$$

inducing long exact sequences in $\mathrm{Hom}_{\mathcal{C}}(W,-)$ and $\mathrm{Hom}_{\mathcal{C}}(-,W)$ for any $W \in \mathcal{C}$ (plus a few other other axioms).

Fix $X \in \mathtt{SW}$ and for each $Z \in \mathtt{SW}$ and $n \in \mathbb{Z}$, write $X_n(Z) = \mathrm{Hom}_{\mathtt{SW}}(\Sigma^n, Z)$. This defines a collection of covariant functors $X_n(-) : \mathtt{SW} \to \mathtt{AbGps}$. Similarly, fixing $Y \in \mathtt{SW}$ and defining $Y^n(Z) = \mathrm{Hom}_{\mathtt{SW}}(Z, \Sigma^n Y)$ gives a collection of contravariant functors $Y^n(-) : \mathtt{SW} \to \mathtt{AbGps}$.

Proposition 5.1.5. *For each $X, Y \in \mathtt{SW}$, $X_\bullet(-)$ is a homology theory and $Y^\bullet(-)$ is a generalized cohomology theory.*

It turns out that $\mathtt{SW}$ may be embedded as a full subcategory of a category $\mathcal{S}$ in which these functors $X_\bullet(-)$ and $Y^\bullet(-)$ are representable. We will describe this category in Section 5.2.

Let $X, Y \in \mathtt{SW}$, $n, m \in \mathbb{Z}$ and define a smash product in $\mathtt{SW}$ by

$$\Sigma^m X \wedge \Sigma^n Y := \Sigma^{m+n}(X \wedge Y).$$

We let $\mathbf{S}$ denote the 0-sphere S^0 as an object of $\mathtt{SW}$. An additive category $\mathcal{C}$ with a product $\wedge$ is *symmetric monoidal* (or a *tensor category*) if $\wedge$ is associative, commutative and unital.

Proposition 5.1.6. *The Spanier-Whitehead category $\mathtt{SW}$ is a symmetric monoidal category with respect to the smash product $\wedge$ and the unit $\mathbf{S}$.*

Proposition 5.1.7. *Given $Y, Z \in \mathtt{SW}$, there is a unique element $F(Y,Z) \in \mathtt{SW}$ satisfying the following properties:*

(a) There is an evaluation map $\mu_{Y,Z} : F(Y,Z) \wedge Y \to Z$.

(b) $F(Y,-)$ and $- \wedge Y$ are adjoint in the sense that for all $X \in \mathtt{SW}$,

$$\mathrm{Hom}_{\mathtt{SW}}(X, F(Y,Z)) \xrightarrow{-\wedge Y} \mathrm{Hom}_{\mathtt{SW}}(X \wedge Y, F(Y,Z) \wedge Y) \xrightarrow{\mu_{Y,Z}} \mathrm{Hom}_{\mathtt{SW}}(X \wedge Y, Z)$$

is an isomorphism.

Definition. *The* **Spanier-Whitehead dual** *of an object $X \in \mathtt{SW}$ is $D(X) := F(X, \mathbf{S})$. This comes equipped with an evaluation map $\mu_X : D(X) \wedge X \to \mathbf{S}$.*

Suppose we are given $D(X) \in \mathtt{SW}$ for each $X \in \mathtt{SW}$ and natural morphisms $\mu_X : D(X) \wedge X \to \mathbf{S}$. Then for any $Y, Z \in \mathtt{SW}$, the isomorphism

$$Z \wedge D(Y) \wedge Y \xrightarrow{1_Z \wedge \mu_Y} Z \wedge \mathbf{S}$$

is adjoint to

$$Z \wedge D(Y) \longrightarrow F(Y,Z).$$

Hence we can take $F(Y,Z) := Z \wedge D(Y)$ as the definition of F.

Example 5.1.8. For any $n \in \mathbb{N}$, $S^n \wedge S^{-n} \to S^0$ is an isomorphism, showing that $D(S^n) = S^{-n}$.

Lemma 5.1.9. *For all $X \in \mathtt{SW}$, the natural map $X \to D(D(X))$ is a homotopy equivalence.*

Theorem 5.1.10 (Atiyah)**.** *Let M be a smooth, compact n-manifold with no boundary and suppose $M \subseteq \mathbb{R}^N$. Let $\nu(M) \subseteq \mathbb{R}^N$ be its normal bundle and let $t(M)$ be the one-point compactification of $\nu(M)$, called the* **Thom space***. Then*

$$t(M) \cong \Sigma^N D(M \wedge S^0) \cong D(M) \wedge S^N$$

in the Spanier-Whitehead category.

Remark. When $\nu(M) = 0$, we say M is a *framed manifold* (this happens e.g. if M is parallelizable). If M is framed, $t(M) = \Sigma^N(M \wedge S^0)$ so M is in some sense self-dual in $\mathtt{SW}$.

5.2 The Homotopy Category of Spectra

In this section we define a notion of 'topological spectra' with which we describe stable homotopy theory.

Example 5.2.1. The object $\mathbf{S} \in \mathtt{SW}$ is called the **sphere spectrum**.

Definition. *The abelian group $\pi_n^S = \mathrm{Hom}_{\mathtt{SW}}(S^n, \mathbf{S})$ is called the* n**th stable homotopy group** *of the sphere spectrum. More generally, $\pi_n^S(X) = \mathrm{Hom}_{\mathtt{SW}}(S^n, X)$ is the nth stable homotopy group of X.*

Lemma 5.2.2. *For any maps $f : S^m \to S^0$ and $g : S^n \to S^0$, the maps*

$$S^{m+n} \xrightarrow{f \wedge g} S^0 \qquad \text{and} \qquad S^{m+n} \xrightarrow{\Sigma^n f} S^n \xrightarrow{g} S^0$$

are the same.

Proposition 5.2.3. *$\pi_\bullet^S = \bigoplus_{n\in\mathbb{Z}} \pi_n^S$ is a graded commutative ring. Moreover, for any $X \in$ $\mathtt{SW}$, the group $\pi_\bullet^S(X) = \bigoplus_{n\in\mathbb{Z}} \pi_n^S(X)$ is a π_n^S-module.*

This defines a functor $\pi_\bullet^S : \mathtt{SW} \to \pi_\bullet^S\text{-}\mathtt{Mod}$. There are many interesting open questions in stable homotopy theory related to the stable homotopy group $\pi_\bullet^S$ and the modules $\pi_\bullet^S(X)$.

Conjecture 5.2.4 (Freyd)**.** *Suppose $f : X \to Y$ is a morphism in $\mathtt{SW}$ such that $f_* : \pi_\bullet^S(X) \to \pi_\bullet^S(Y)$ is the zero map. Then f is nullhomotopic.*

We now define the *homotopy category of spectra $\mathcal{S}$*, a generalization of the Spanier-Whitehead category. The objects of $\mathcal{S}$ are sequences of based topological spaces $\mathbf{X} = (X_0, X_1, X_2, \ldots)$ together with maps $\sigma_n : X_n \to \Omega X_{n+1}$, or equivalently by adjointness, maps $\hat{\sigma}_n : \Sigma X_n \to X_{n+1}$. The morphisms are a little more delicate to define. A *strict map* in $\mathcal{S}$ is a map $f : \mathbf{X} \to \mathbf{Y}$ consisting of based maps $f_n : X_n \to Y_n$ for each n such that the diagrams

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "f_n"'] & \Omega X_{n+1} \arrow[d, "\Omega f_{n+1}"] \\ Y_n \arrow[r] & \Omega Y_{n+1} \end{tikzcd}$$

commute. It turns out that these strict maps are not enough to fully study the homotopy theory of spectra. We will add more morphisms in a moment.

Definition. *For a spectrum* $\mathbf{X} \in \mathcal{S}$*, define the* **homotopy group**

$$\pi_n(\mathbf{X}) = \varinjlim \pi_{n+k}(X_k)$$

where the maps $\pi_{n+k}(X_k) \to \pi_{n+k+1}(X_{k+1})$ *are induced by suspension (see Corollary 0.2.5).*

Proposition 5.2.5. $\pi_n : \mathcal{S} \to \texttt{AbGps}$ *is a functor.*

Definition. *A strict map* $f : \mathbf{X} \to \mathbf{Y}$ *is a* **weak equivalence** *if the induced map* $f_* : \pi_n(\mathbf{X}) \to \pi_n(\mathbf{Y})$ *is an isomorphism for all* n*.*

It turns out that $\pi_n(\mathbf{X}) = \pi_n(\mathrm{Tel}(X_0 \xrightarrow{\sigma_0} \Omega X_1 \xrightarrow{\Omega\sigma_1} \Omega^2 X_2 \to \cdots))$ where Tel denotes the mapping telescope of the given loop sequence (see Section 4.2). The advantage of this perspective is that $\mathrm{Tel}(X_0 \to \Omega X_1 \to \Omega^2 X_2 \to \cdots)$ is an actual topological space. This suggests the following modification to our collection of morphisms in $\mathcal{S}$.

Definition. *For a spectrum* $\mathbf{X} \in \mathcal{S}$*, the* **$\Omega$-spectrum** *associated to* $\mathbf{X}$ *is the spectrum* $\mathbf{X}^f = (X_0^f, X_1^f, X_2^f, \ldots)$ *where*

$$X_n^f := \mathrm{Tel}(X_n \xrightarrow{\sigma_n} \Omega X_{n+1} \xrightarrow{\Omega\sigma_{n+1}} \Omega^2 X_{n+2} \to \cdots).$$

Now define the set of morphisms between two spectra $\mathbf{X}, \mathbf{Y} \in \mathcal{S}$ by:

$$[\mathbf{X}, \mathbf{Y}]_{\mathcal{S}} := \{\text{strict maps } g : \mathbf{X} \to \mathbf{Y}^f\}/\sim$$

where $\sim$ is the homotopy equivalence relation. Thus we have fully defined the category $\mathcal{S}$.

Lemma 5.2.6. *For every* $\mathbf{X} \in \mathcal{S}$*, there is a weak equivalence* $\mathbf{X} \to \mathbf{X}^f$ *in* $\mathcal{S}$*.*

Example 5.2.7. [Suspension spectra] For each based space Z, we define an associated spectrum $\Sigma^\infty Z$ by

$$(\Sigma^\infty Z)_n = \Sigma^n Z \qquad \text{and} \qquad \hat{\sigma}_n : \Sigma(\Sigma^n Z) \to \Sigma^{n+1} Z.$$

This defines a functor $\Sigma^\infty : h(\texttt{Top}_*) \to \mathcal{S}$ such that the image of the subcategory $\texttt{CW} \subseteq h(\texttt{Top}_*)$ of finite CW-complexes lands in $\texttt{SW}_*$, the category of (based) formal suspensions $\Sigma^n X$, itself a subcategory of $\mathcal{S}$ via $Z \mapsto \Sigma^\infty Z$. On the other hand, we can define a 'looping functor'

$$\begin{aligned} \Omega^\infty : \mathcal{S} &\longrightarrow h(\texttt{Top}_*) \\ \mathbf{X} &\longmapsto \Omega^\infty \mathbf{X} := X_0^f. \end{aligned}$$

Then $(\Sigma^\infty, \Omega^\infty)$ are an adjoint pair. An interesting object in stable homotopy theory is $\Omega^\infty\Sigma^\infty Z = \varinjlim \Omega^n \Sigma^n Z$. Note that for any based spaces Z, W,

$$[\Sigma^\infty Z, \Sigma^\infty W]_{\mathcal{S}} = [Z, \Omega^\infty \Sigma^\infty W]_{\mathcal{S}} = \varinjlim [Z, \Omega^n \Sigma^n W].$$

Example 5.2.8. For each spectrum $\mathbf{Y} = (Y_n \xrightarrow{\sigma_n} \Omega Y_{n+1})$, define the functors

$$\begin{aligned} \mathbf{Y}^n : \mathtt{Top}_* &\longrightarrow \mathtt{AbGps} \\ Z &\longmapsto [Z, Y_n]_{\mathtt{Top}_*} = [\Sigma^\infty Z, \Sigma^n Y]_{\mathcal{S}}. \end{aligned}$$

Then $\mathbf{Y}^\bullet$ is a generalized cohomology theory that captures the functors $Y^\bullet$ from Section 5.1 when $\mathbf{Y} = \Sigma^\infty Z$ is an Ω-spectrum. In the category $\mathcal{S}$, these cohomology theories $\mathbf{Y}^\bullet$ are now represented by spectra $\mathbf{Y}$.

Theorem 5.2.9 (Brown Representability for Spectra)**.** *For any (reduced) generalized cohomology theory $E^\bullet : \mathtt{Top}_* \to \mathtt{AbGps}$, there exist spaces $E_n \in \mathtt{Top}_*$ such that $E^n(Z) \cong [Z, E_n]_*$ for all based spaces Z and there are isomorphisms $E^n(Z) \xrightarrow{\sim} E^{n+1}(\Sigma Z)$ induced by a homotopy equivalence $E_n \to \Omega E_{n+1}$.*

In particular, these E_n form a spectrum and we have:

Corollary 5.2.10. *There is a bijective correspondence between (reduced) generalized cohomology theories and Ω-spectra given by sending $E^\bullet$ to $(E_0 \xrightarrow{\sigma_0} \Omega E_1 \xrightarrow{\Omega\sigma_1} \Omega^2 E_2 \to \cdots)$.*