Fibre Bundles

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0 Introduction

This document follows two courses in fibre bundles taught respectively by Dr. Slava Krushkal in Spring 2017 and Dr. Thomas Mark in Spring 2018 at the University of Virginia. The main concepts covered are:

- Locally trivial fibre bundles
- Principal bundles and classifying spaces
- Vector bundles and their important properties
- Characteristic classes, including Stiefel-Whitney, Chern and Pontryagin classes
- Orientability
- Euler class
- Elements of $K$-theory
- Connections and Chern-Weil theory
- The Atiyah-Singer index theorem.

Standard references include: Milnor-Stasheff’s *Characteristic Classes*, Hatcher’s *Vector Bundles and K-Theory* and Husemöller’s *Fibre Bundles.*
1 Fibre Bundles

1.1 Locally Trivial Fibre Bundles

Definition. A bundle is a triple $(E, B, p)$ consisting of topological spaces $E, B$ and a continuous map between them, $p : E \to B$. We call $E$ the total space of the bundle and $B$ the base space.

Definition. A morphism of bundles $(E_1, B_1, p_1) \to (E_2, B_2, p_2)$ is a pair of maps $f : B_1 \to B_2, \tilde{f} : E_1 \to E_2$, making the diagram

\[
\begin{array}{ccc}
E_1 & \xrightarrow{\tilde{f}} & E_2 \\
p_1 & & \downarrow{p_2} \\
B_1 & \xrightarrow{f} & B_2
\end{array}
\]

commute and preserving fibres, that is, for any $x \in B_1$ and $y \in p_1^{-1}(x)$, $\tilde{f}(y) \in p_2^{-1}(f(x))$.

Example 1.1.1. When $f$ is the identity $B \to B$, we say the bundles and morphisms of bundles are over $B$.

Example 1.1.2. Let $E \xrightarrow{p} B$ be a bundle and $A \subseteq B$ be a subspace. Then the restriction $p|_A : E|_A := p^{-1}(A) \to A$ is a bundle over $A$ and the natural inclusion $E|_A \hookrightarrow E$ induces a bundle morphism

\[
\begin{array}{ccc}
E|_A & \hookrightarrow & E \\
p|_A & & \downarrow{p} \\
A & \hookrightarrow & B
\end{array}
\]

Example 1.1.3. Let $F$ be any space. The trivial bundle over $B$ with fibre $F$ is the canonical projection $p : B \times F \to B$.

Definition. A locally trivial fibre bundle over $B$ with fibre $F$, hereafter shortened to a fibre bundle, is a bundle $E \xrightarrow{p} B$ that is locally isomorphic to a trivial bundle with fibre $F$. That is, there exists a collection of open sets $\{U_i\}$ together with bundle isomorphisms $\varphi_i : E|_{U_i} \cong U_i \times F$, called local trivializations.

Given a fibre bundle $E \xrightarrow{p} B$ with locally trivial open sets $\{U_i\}$, there is a choice of isomorphism $\varphi_i : E|_{U_i} \to U_i$ for each $U_i$. For each $x \in U_i$, $p^{-1}(x)$ is homeomorphic to $F$ via $\varphi_i$, so when $x \in U_i \cap U_j$, there are two different ways to identify $p^{-1}(x)$ with $F$. In particular, $\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$ is a homeomorphism and induces a homeomorphism $F \to F$. In other words, for each pair of open sets $U_i, U_j$, there is a function

\[
g_{ij} : U_i \cap U_j \to \text{Homeo}(F)
\]
called a \textit{transition map}. One can show that the $g_{ij}$ are continuous. Note that the $g_{ij}$ satisfy a few nice conditions, namely that $g_{ii} = 1$, $g_{ij}g_{ji} = 1$ and $g_{ij}g_{jk}g_{ki} = 1$ pointwise on $F$. In fact, the data of a covering $\{U_i\}$ and continuous transition maps $\{g_{ij}\}$ satisfying these conditions are enough to determine a fibre bundle.

**Proposition 1.1.4.** Let $B$ and $F$ be spaces and suppose there is an open cover $\{U_i\}$ of $B$ and continuous maps $g_{ij} : U_i \cap U_j \to \text{Homeo}(F)$ satisfying:

(a) For all $x \in U_i$, $g_{ii}(x) = 1$, where 1 is the identity map $F \to F$.

(b) For all $x \in U_i \cap U_j$, $g_{ij}(x)g_{ji}(x) = 1$.

(c) For all $x \in U_i \cap U_j \cap U_k$, $g_{ij}(x)g_{jk}(x)g_{ki}(x) = 1$.

Then there exists a fibre bundle over $B$ with fibre $F$ which is trivial over the $U_i$ and has the $g_{ij}$ as its transition maps. Moreover, such a bundle is unique up to isomorphism of bundles over $B$.

**Proof.** (Sketch) Define the total space of the bundle by

$$E := \prod_i (U_i \times F) / \sim$$

where $\sim$ is the equivalence relation identifying $(x, y) \in U_i \times F$ and $(x, g_{ij}(x)(y)) \in U_j \times F$ for any $x \in U_i \cap U_j$ and $y \in F$. The bundle map $p : E \to B$ is given by the natural projection on each disjoint piece, which is well-defined on all of $E$ by (a) – (c) and is continuous since the $g_{ij}$ are continuous. It is straightforward to check that $E \xrightarrow{p} B$ is a locally trivial bundle with fibre $F$ having transition maps $g_{ij}$.

For uniqueness, suppose $\tilde{E} \xrightarrow{\tilde{p}} B$ is another bundle with $g_{ij}$ as its transition maps. For each $U_i$, there is a fibre-preserving homeomorphism $\tilde{p}^{-1}(U_i) \to U_i \times F$. Define a bundle map $\tilde{E} \to E$ by sending a point $\tilde{x} \in \tilde{p}^{-1}(U_i)$ to the corresponding point in $U_i \times F \subseteq E$. One now checks that this is a bundle isomorphism.

Given two bundles $E_1 \xrightarrow{p_1} B$ and $E_2 \xrightarrow{p_2} B$ with fibre $F$ and local trivializations $\{(U_i, \varphi_i)\}$ and $\{(V_j, \psi_j)\}$, respectively, when can we tell if they are isomorphic? First, we may refine the covers $\{U_i\}$ and $\{V_j\}$ by taking intersections until each bundle is trivial over both collections. Thus we may assume both bundles are trivial over $\{U_i\}$. Suppose $f : E_1 \to E_2$ is an isomorphism of bundles over $B$. Then each $U_i$ corresponds to a local isomorphism of bundles

$$f_i := \varphi_i \circ f \circ \varphi_i^{-1} : U_i \times F \to E_1|_{U_i} \to E_2|_{U_i} \to U_i \times F.$$

In fact, this gives us a necessary and sufficient condition for $f$ to induce a bundle isomorphism.

**Lemma 1.1.5.** The map $f : E_1 \to E_2$ is an isomorphism of bundles over $B$ if and only if there exist local bundle isomorphisms $f_i : U_i \times F \to U_i \times F$ that are compatible with the transition maps $\varphi_j \circ \varphi_i^{-1}$ and $\psi_j \circ \psi_i^{-1}$, i.e. the following diagram commutes:
1.1 Locally Trivial Fibre Bundles

Fibre Bundles

\[(U_i \cap U_j) \times F \xrightarrow{f_i} (U_i \cap U_j) \times F \]
\[\varphi_j \circ \varphi_i^{-1} \quad \text{and} \quad \psi_j \circ \psi_i^{-1}\]
\[(U_i \cap U_j) \times F \xrightarrow{f_j} (U_i \cap U_j) \times F\]

Example 1.1.6. Let \(M \subseteq \mathbb{R}^N\) be a smooth manifold and define the tangent bundle to \(M\) by

\[TM = \{(x, v) \in \mathbb{R}^N \times \mathbb{R}^N \mid x \in M, v \in T_xM\}\].

The projection \(p : TM \to M, (x, v) \mapsto x\) gives a fibre bundle structure on \(TM\) with fibre \(p^{-1}(x) = T_xM \cong \mathbb{R}^n\) if \(M\) is an \(n\)-manifold. If \(\{(U_i, \varphi_i)\}\) is an atlas of coordinate charts for the manifold structure on \(M\), then the derivatives of their transition maps,

\[g_{ij} : x \mapsto d_{\varphi_i(x)}(\varphi_j \circ \varphi_i^{-1}),\]

define the transition maps \(g_{ij} : U_i \cap U_j \to \text{Homeo}(\mathbb{R}^n)\) of the tangent bundle. Since these derivatives are linear, the \(g_{ij}\) in fact have target \(GL_n(\mathbb{R}) \subset \text{Homeo}(\mathbb{R}^n)\). This reflects the fact that \(TM\) is a vector bundle (see Chapter 2).

For example, let \(M = S^2\) be the 2-sphere. We may view \(S^2 = \mathbb{C}P^1\), the complex projective line, which has a covering by two coordinate charts

\[U_0 = \{[z_0, z_1] \mid z_0 \neq 0\} \quad \text{and} \quad U_1 = \{[z_0, z_1] \mid z_1 \neq 0\}\]

with homeomorphisms \(\varphi_0 : U_0 \to \mathbb{C}, [z_0, z_1] \mapsto \frac{z_0}{z_1}\) and \(\varphi_1 : U_1 \to \mathbb{C}, [z_0, z_1] \mapsto \frac{z_0}{z_1}\). Then the transition maps \(\varphi_j \circ \varphi_i^{-1} : \mathbb{C} \to \mathbb{C}\) are both given by \(\lambda \mapsto \frac{1}{\lambda}\). Thus the transition maps \(g_{01}, g_{10}\) for the tangent bundle \(TS^2\) are each given by \(\lambda \mapsto -\frac{1}{\lambda^2}\), viewed as a \(2 \times 2\) matrix with real entries in \(GL_2(\mathbb{R})\), or a scalar element of \(GL_1(\mathbb{C}) = \mathbb{C}^\times\). Notice that \(g_{01} : \mathbb{C}^\times \to \mathbb{C}^\times\) is homotopic to a map \(S^1 \to S^1\) of degree \(-2\). It is no accident that \(S^2\) has Euler characteristic \(2\)  we will see this connection explicitly borne out by the Euler class (see Section 3.5).

Let \(G\) be a topological group and \(F\) a space on which \(G\) acts by homeomorphisms, i.e. such that there is a continuous homomorphism \(\rho : G \to \text{Homeo}(F)\).

Definition. A fibre bundle with fibre \(F\) has structure group \(G\) if it has transition functions \(g_{ij} : U_i \cap U_j \to G\).

Example 1.1.7. Fibre bundles with fibre \(F = \mathbb{R}^n\) (resp. \(F = \mathbb{C}^n\)) and structure group \(G = GL_n(\mathbb{R})\) (resp. \(GL_n(\mathbb{C})\)) are called real (resp. complex) vector bundles (see Chapter 2).

Example 1.1.8. A bundle with fibre \(F = S^n\) and structure group \(G = O(n+1)\) is called an (orthogonal) sphere bundle.

Example 1.1.9. The trivial structure group \(G = \{1\}\) corresponds to the trivial bundle over any base (with any fibre).

Given a bundle \(E \xrightarrow{p} B\) with fibre \(F\) and structure group \(G\), there may be other spaces on which \(G\) acts, so that the same transition functions \(g_{ij} : U_i \cap U_j \to G\) produce bundles with those other spaces as fibres.
Example 1.1.10. Let $E \xrightarrow{p} B$ be a sphere bundle with fibre $S^n$ and structure group $O(n+1)$. Then $O(n+1) \subseteq GL_{n+1}(\mathbb{R})$ naturally acts on all of $\mathbb{R}^{n+1}$, so it determines a bundle $E' \xrightarrow{p'} B$ with fibre $\mathbb{R}^{n+1}$ and the same transition maps (and $E'$ is in fact a vector bundle over $B$). Conversely, a bundle $E'$ with fibres $\mathbb{R}^{n+1}$ and structure group $O(n+1)$ determines a “unit sphere bundle” $E$ inside $E'$, with fibres $S^n$.

Example 1.1.11. For a bundle with fibre $\mathbb{R}^n$ and structure group $GL_n(\mathbb{R})$, the natural action of $GL_n(\mathbb{R})$ on $\mathbb{R}^n$ preserves a vector space structure on each fibre. In this way, any bundle with this data becomes a vector bundle (see Chapter 2).

The takeaway from these examples is that if $G$ acts on a space $F$ and preserves a certain structure on $F$, then bundles with structure group $G$ inherit this structure.

1.2 Principal Bundles

Continuing from the discussion in the previous section, any group $G$ has a natural action on itself by left multiplication. Therefore, for any base $B$ with covering $\{U_i\}$ and transition maps $g_{ij} : U_i \cap U_j \to G$, there is a canonically determined bundle $P \to B$ with fibre $G$, structure group $G$ and transition maps $g_{ij}$. The bundle $P \to B$ is called a $G$-principal bundle over $B$.

Definition. A principal $G$-bundle is a space $P$ with a free right action of $G$ such that the quotient map $p : P \to P/G$ defines a fibre bundle over the base $B = P/G$ with $G$-equivariant local trivializations $\varphi_i : p^{-1}(U_i) \to U_i \times G$.

Lemma 1.2.1. A principal $G$-bundle $P \to B = P/G$ is equivalent to the data of a base $B$, a covering $\{U_i\}$ and transition maps $g_{ij} : U_i \cap U_j \to G$.

Remark. Note that the fibres $G$ in a $G$-principal bundle $P \to B$ need not have a group structure (compatible with the bundle structure); they are in general only $G$-sets. A principal bundle whose fibres have a compatible group structure is sometimes called a group bundle.

Let $p : P \to B$ be a principal $G$-bundle and $F$ a left $G$-space, via a continuous homomorphism $\rho : G \to \text{Homeo}(F)$. Then $p$ defines a fibre bundle $E_{(P,\rho)} \to B$ with fibre $F$, called an associated bundle.

Definition. Let $P \to B$ be a principal $G$-bundle and $\rho : G \to \text{Homeo}(F)$ a continuous homomorphism. The associated bundle for this data is the fibre bundle $E_{(P,\rho)} \to B$ with total space

$$E_{(P,\rho)} := P \times_G F = \{(y, t) \in P \times F \mid (y \cdot g, t) \sim (y, \rho(g) \cdot t)\}$$

and local trivializations

$$E_{(P,\rho)}|_{U_i} = (U_i \times G) \times_G F \to U_i \times F$$

$$[(x, h), t] \mapsto (x, \rho(h^{-1}) \cdot t).$$
**Example 1.2.2.** For $G = GL_n(\mathbb{R})$, a principal $G$-bundle is the same thing as a vector bundle with fibre $\mathbb{R}^n$ via the associated bundle operation

\[
\{\text{principal } GL_n(\mathbb{R})\text{-bundles}\} \rightarrow \{n\text{-dimensional vector bundles}\}
\]

\[(P \rightarrow B) \mapsto (P \times_{GL_n(\mathbb{R})} \mathbb{R}^n \rightarrow B).
\]

Conversely, let $V$ be an $n$-dimensional real vector space and let $F(V)$ be the set of ordered bases, or $n$-frames, of $V$. Then $GL_n(\mathbb{R})$ acts freely and transitively on $F(V)$. Given a vector bundle $E \rightarrow B$ with fibre $V \cong \mathbb{R}^n$, let $F(E)$ denote the frame bundle associated to $E$:

\[
F(E) := \{(x, e_1, \ldots, e_n) \mid x \in B, (e_1, \ldots, e_n) \in F(V)\}.
\]

Then by construction there is a free right action of $GL_n(\mathbb{R})$ on $F(E)$, so $F(E)$ is a principal $GL_n(\mathbb{R})$-bundle. Moreover, one can check that the assignments $P \mapsto P \times_{GL_n(\mathbb{R})} \mathbb{R}^n$ and $E \mapsto F(E)$ are inverses.

**Example 1.2.3.** A similar proof to that in Example 1.2.2 using orthogonal frames shows that there is a bijective correspondence between principal $O(n)$-bundles and $n$-dimensional vector bundles with inner product.

**Definition.** A section of a fibre bundle $E \overset{p}{\rightarrow} B$ is a map $s : B \rightarrow E$ such that $p \circ s = id_B$.

**Lemma 1.2.4.** A principal $G$-bundle $P \overset{p}{\rightarrow} B$ admits a section $s : B \rightarrow P$ if and only if $P$ is a trivial $G$-bundle.

**Proof.** ($\Longleftrightarrow$) If $P \cong B \times G$ then any function $f : B \rightarrow G$, for example the constant function $b \mapsto e \in G$, defines a section $x \mapsto (x, f(x))$.

($\Longrightarrow$) If $s : B \rightarrow P$ is a section, define

\[
\varphi : P \rightarrow B \times G
\]

\[
y \mapsto (p(y), g(y))
\]

where $g(y) \in G$ is the unique element such that $y = s(p(y)) \cdot g(y)$. Since $G$ acts freely on $P$, this map is well-defined. Moreover, it is easy to check $\varphi$ is $G$-equivariant. Finally, the continuity of $\varphi$ follows from the lemma below. ☐

**Lemma 1.2.5.** For a principal $G$-bundle $P \rightarrow B$, define

\[
P^* = \{(y_1, y_2) \in P \times P \mid y_2 = y_1 g \text{ for some } g \in G\}.
\]

Then the map $\tau_P : P^* \rightarrow G, (y_1, y_1 g) \mapsto g$ is continuous.

**Proof.** Locally, $P \cong U \times G$ so because continuity is a local question, we may assume $P = U \times G$. If $y_1 = (x, g_1)$ and $y_2 = (x, g_2)$, then $\tau_P$ is given by $\tau_P(y_1, y_2) = g_1^{-1}g_2$. This is evidently the composition

\[
(U \times G) \times (U \times G) \rightarrow G \times G \overset{q}{\rightarrow} G
\]

where the first map is projection in each coordinate and $q : (g_1, g_2) \mapsto g_1^{-1}g_2$. Since $G$ is a topological group, each of these is continuous so $\tau_P$ is continuous. ☐
Definition. Let $P$ be any free right $G$-space. We say $P$ admits local sections if there is an open cover $\{U_i\}$ of $B = P/G$ such that the quotient map $p : P \to P/G$ has the property that each $p|_{p^{-1}(U_i)} : P|_{U_i} \to U_i$ has a section, called a slice. The set of $G$-orbits of a slice is called a tube.

Corollary 1.2.6. A free right $G$-space $P$ is a principal $G$-bundle over $B = P/G$ if and only if $P$ admits local sections and $\tau_P : P^* \to G$ is continuous.

Example 1.2.7. Let $G$ be a topological group and $H$ a subgroup (which is a topological group with the subspace topology). Then $H$ acts on $G$ by right multiplication, giving a fibre bundle $G \to G/H$ with fibres $H$. By Corollary 1.2.6, this is a principal $H$-bundle if and only if there exists a slice for the $H$-action through the identity $e \in G$. (The map $\tau_P$ is guaranteed to be continuous in this case. Moreover, by homogeneity, it is enough to have a slice through the identity.)

Proposition 1.2.8. Let $G$ be a topological group, $H \subseteq G$ a subgroup and suppose there exist subsets $T \subseteq G$ and $W \subseteq H$, each containing $e$ and with $W$ open, such that multiplication $m : T \times W \to TW \subseteq G$ is a homeomorphism onto its image. Then $G \to G/H$ is a principal $H$-bundle.

Proof. By Example 1.2.7, it’s enough to find a slice through $e \in G$. Write $W = H \cap V$ for some open set $V \subseteq G$. Since $e \in V$ and $G$ is a topological group, there is an open set $U \subseteq V$ containing $e$ and satisfying $U^{-1}U \subseteq V$. Set $S = T \cap U$ and consider the multiplication map $m : S \times H \to SH \subseteq G$. By hypothesis, restriction of $m$ to $S \times W$ is a homeomorphism, but since $H$ is a topological group, this means $m$ is a local homeomorphism on all of $S \times H$. Moreover, if $(s_1, h_1)$ and $(s_2, h_2)$ have the same image under $m$, then $s_1 h_1 = s_2 h_2$, so $s_2^{-1}s_1 = h_2 h_1^{-1}$. But $s_2^{-1}s_1 \in U^{-1}U \subseteq V$ and $h_2 h_1^{-1} \in H$, so $s_2^{-1}s_1 = h_2 h_1^{-1} \in V \cap H = W$. Since $m$ is a homeomorphism on $S \times W$, it follows that $(s_1, h_1) = (s_2, h_2)$. Thus $m$ is a one-to-one local homeomorphism, hence a global homeomorphism. This gives us a slice $SH \to S \times H = p^{-1}(SH)$ through $e$ as required.

Corollary 1.2.9. Let $G$ be a Lie group and $H \subseteq G$ a closed subgroup. Then $G \to G/H$ is a principal $H$-bundle.

Proof. A closed subgroup of a Lie group is itself a Lie group, so the tangent space at the identity $T_eH$ is a vector subspace of $T_eG$. Let $V$ be the orthogonal complement of $T_eH \subseteq T_eG$ and define a map

$$f : V \times H \to G$$

$$(v, h) \mapsto \exp(v)h,$$

where $\exp : V \to G$ is the restriction of the exponential map to $V$. Then the derivative of $f$ is equal to the identity $V \times T_eH \to V \times T_eG \cong G$, so $f$ is a local diffeomorphism. Thus there exist open sets $\tilde{T} \subseteq V$ and $W \subseteq H$, with $e \in W$, such that $f|_{\tilde{T} \times W}$ is a diffeomorphism. Taking $T = \exp(\tilde{T}) \subseteq G$, Proposition 1.2.8 implies $G \to G/H$ is a principal $H$-bundle.

Proposition 1.2.10. Let $H \subseteq G$ be any subgroup of a topological group for which $G \to G/H$ is a principal $H$-bundle. Then for every subgroup $K \subseteq H$, the natural map $G/K \to G/H$ is a fibre bundle with fibre $H/K$.  

7
Example 1.2.11. Let \(k < n\) be integers and consider the inclusion of orthogonal groups \(O(n - k) \hookrightarrow O(n)\) given by the “lower diagonal inclusion”

\[
A \mapsto \begin{pmatrix} I_k & 0 \\ 0 & A \end{pmatrix}.
\]

The Lie group \(O(n)\) acts (on the right) on the set of orthonormal \(k\)-tuples of vectors in any \(n\)-dimensional inner product space with stabilizers \(O(n - k)\). Consider the Stiefel manifold

\[V_{n,k} := \{\text{orthonormal } k\text{-tuples in } \mathbb{R}^n\}.\]

Then \(V_{n,k} \cong O(n)/O(n - k)\) and by Corollary 1.2.9, the quotient map \(O(n) \to V_{n,k}\) is a principal bundle with fibre \(O(n - k)\). Similarly, the Grassmannian manifold \(\text{Gr}_k(\mathbb{R}^n) = V_{k,n}/O(k)\) has a canonical principal given as follows. Take \(G = O(n), H = O(n - k) \times O(k)\) and \(K = O(n - k)\). Then \(H/K = O(k)\) so by Proposition 1.2.10 and Corollary 1.2.9, the quotient map \(V_{n,k} \to \text{Gr}_k(\mathbb{R}^n)\) is a principal bundle with fibre \(O(k)\). (See Section 3.3 for further details.)

Lemma 1.2.12. Suppose \(P_1 \to B\) and \(P_2 \to B\) are principal \(G\)-bundles and \(f : P_1 \to P_2\) is a morphism of \(G\)-bundles over \(B\). Then \(f\) is an isomorphism.

Proof. Locally, \(f\) looks like \(U \times G \to U \times G, (x, g) \mapsto (x, f_0(x)g)\) where \(f_0\) is the function \(U \to G\) defined uniquely by \(f(x, e) = (x, f_0(x))\) (and extended by \(G\)-equivariance). The inverse on \(U \times G\) is then \(f^{-1} : (x, h) \mapsto (x, f_0(x)^{-1}h)\), so \(f\) is a local homeomorphism. Since a bundle morphism preserves fibres, \(f\) is also a global homeomorphism, hence a bundle isomorphism.

Definition. Let \(P \to B\) be a principal \(G\)-bundle and \(f : B' \to B\) be a continuous map. The pullback bundle of \(p\) along \(f\) (or the induced bundle) is the bundle \(f^*P : f^*P \to B'\) with total space

\[f^*P := \{(b', y) \in B' \times P \mid f(b') = p(y)\}.
\]

In other words, \(f^*P\) is the pullback of \(f\) and \(p\) in the category of principle \(G\)-bundles over \(B\):

```
\begin{array}{ccc}
f^*P & \to & P \\
\downarrow f^*p & & \downarrow p \\
B' & \to & B
\end{array}
```

Example 1.2.13. If \(i : A \hookrightarrow B\) is a subspace, the pullback \(i^*P\) of any bundle \(P \to B\) coincides with the restriction \(P|_A \to A\) of Example 1.1.2.

Example 1.2.14. If \(f : B' \to B\) is a constant map, then every pullback bundle along \(f\) is trivial.
Example 1.2.15. Suppose $Q \xrightarrow{q} B'$ and $P \xrightarrow{\ell} B$ are principal $G$-bundles and $F : Q \to P$ is a morphism of principal $G$-bundles, lying over $f : B' \to B$. Then it follows from Lemma 1.2.12 that $Q \cong f^*P$.

From now on, we will assume $B$ is paracompact, so that it admits partitions of unity. Let $I = [0,1]$ and $r : B \times I \to B \times \{1\}$ the natural projection.

Lemma 1.2.16. Let $a, b \in \mathbb{R}$, $A$ be a space and $P \to A \times [a,b]$ be a principal $G$-bundle. If for $a \leq c \leq b$, $P|_{A \times [a,c]}$ and $P|_{A \times [c,b]}$ are both trivial, then so is $P$.

Proof. Write $B_1 = A \times [a,c]$, $B_2 = A \times [c,b]$ and let $\varphi_i : P|_{B_i} \to B_i \times G$ be local trivializations for $i = 1, 2$. Over $A \times \{c\}$, the composition $H := \varphi_2 \circ \varphi_1^{-1} : A \times \{c\} \times G \to A \times \{c\} \times G$ is of the form $H(a,c,g) = (a,c,h(a)g)$ for some $h : A \to G$. Extend $H$ across all of $B_1$ by $H(a,t,g) = (a,t,h(a)g)$ for $a \leq t \leq c$. Then $H \circ \varphi_1$ and $\varphi_2$ agree on $P|_{B_1 \cap B_2} = P|_{A \times \{c\}}$. Hence these give a well-defined isomorphism of bundles $P \to (B_1 \cup B_2) \times G = A \times [a,b] \times G$. \qed

Lemma 1.2.17. Suppose $P \to B \times I$ is a principal $G$-bundle. Then there exists a cover $\{U_i\}$ of $B$ such that each $P|_{U_i \times I}$ is trivial.

Proof. Since $P$ is locally trivial and $I$ is compact, for each $b \in B$ there is a sequence $0 = t_0 < t_1 < \cdots < t_n = 1$ and neighborhoods $U_i$ of $b$ such that each $P|_{U_i \times [t_j,t_{j+1}]}$ is trivial. Set $U(b) = \bigcap U_i$. Then by Lemma 1.2.16, $P|_{U(b) \times I}$ is trivial. \qed

Theorem 1.2.18. If $P \to B \times I$ is a principal $G$-bundle, then $P \cong r^*(P|_{B \times \{1\}})$.

Proof. Set $P_i = P|_{B \times \{t\}}$ for each $0 \leq t \leq 1$. By Lemma 1.2.17, there exists a cover $\{U_i\}$ of $B$ with each $P|_{U_i \times I} \to U_i \times I$ all trivial. Take a partition of unity subordinate to the $U_i$, i.e. a collection of functions $\eta_i : B \to I$ with $\eta_i$ supported on $U_i$ and $\max \eta_i \equiv 1$ on $B$. Let $h_i : (U_i \times I) \times G \to P|_{U_i \times I}$ be local trivializations. By paracompactness, for each $b \in B$, there is a neighborhood $U(b)$ meeting finitely many of the $U_i$.

Let $S$ be the index set of the $U_i$ equipped with some ordering and for each $i \in S$, define a function

$$r_i : B \times I \to B \times I$$

$$(b,t) \mapsto (b,\max\{\eta_i(b),t\}).$$

Let $f_i : P \to P$ be the function that is the identity outside $U_i \times I$ and is given by

$$f_i(h_i(x,t,g)) = h_i(r_i(x,t),g)$$

on $U_i \times I$ (which makes sense because every element of $U_i$ is of the form $h_i(x,t,g)$ for some $x \in U_i$, $t \in I$, $g \in G$). These define a map $f : P \to P$ which is the “composition over $S$” of all the $f_i$ in the order given by the ordering on $S$, which is well-defined since for each $b \in B$, $U(b)$ meets only finitely many of the $U_i$ and therefore only finitely many of the $f_i$ are not the identity on $U(b)$. By the same token, $r : B \times I \to B \times \{1\}$ is the composition over $S$ of the $r_i$. Since $\max \eta_i(b) = 1$ for all $b \in B$, the map $f$ determines a bundle morphism $P \to P_1$. Finally, Lemma 1.2.12 implies that $P \cong r^*P_1$ as desired. \qed
Corollary 1.2.19. If \( i_0, i_1 : B \to B \times I \) are the inclusions of \( B \times \{0\} \) and \( B \times \{1\} \), respectively, and \( P \to B \times I \) is a principal \( G \)-bundle, then \( i_0^*P \cong i_1^*P \) as bundles over \( B \).

Corollary 1.2.20. If \( f_0, f_1 : B' \to B \) are homotopic and \( P \to B \) is a principal \( G \)-bundle, then \( f_0^*P \cong f_1^*P \) as bundles over \( B' \).

Proof. Suppose \( F : B' \times I \to B \) is a homotopy between \( f_0 \) and \( f_1 \). Then \( P_0 \times I = F \circ i_0 \) and \( P_1 \times I = F \circ i_1 \), so by Corollary 1.2.19,

\[
f_0^*P = i_0^*F^*P = i_1^*F^*P = f_1^*P.
\]

\( \square \)

Corollary 1.2.21. If \( B \) is a contractible space, then any principal \( G \)-bundle over \( B \) is trivial.

Corollary 1.2.22. Suppose \( E \to B \) is an arbitrary fibre bundle with fibre \( F \) and structure group \( G \). Then the previous results hold for \( E \), namely:

1. If \( \widetilde{E} \to B \times I \) is a fibre bundle with structure group \( G \), then \( i_0^*\widetilde{E} \cong i_1^*\widetilde{E} \) as bundles over \( B \).

2. If \( f_0, f_1 : B' \to B \) are homotopic, then \( f_0^*E \cong f_1^*E \) as bundles over \( B' \).

3. If \( B \) is contractible, then \( E \cong B \times F \).

Proof. Apply the previous results to the associated bundle construction on \( E \). \( \square \)

For a space \( X \) and a group \( G \), let \( \text{Bun}_G(X) \) denote the set of isomorphism classes of principal \( G \)-bundles over \( X \).

Corollary 1.2.23. The assignment \( X \mapsto \text{Bun}_G(X) \) is a contravariant homotopy functor \( \text{Top} \to \text{Set} \).

We may ask if the functor \( \text{Bun}_G(\cdot) \) is representable. This leads to the notion of a classifying space for principal \( G \)-bundles.

1.3 Classifying Spaces and Universal Bundles

Definition. A principal \( G \)-bundle \( P \xrightarrow{p} B \) is a universal bundle for \( G \)-bundles if for each space \( X \), the map

\[
[X, B] \mapsto \text{Bun}_G(X)
\]

\( (f : X \to B) \mapsto (f^*P \to X) \)

is a bijection. For a bundle \( Q \to X \), a map \( f : X \to B \) such that \( f^*P \cong Q \) is called a classifying map for \( Q \).

Remark. Any arbitrary \( G \)-bundle \( P \to B \) induces a natural transformation \( \alpha_P : [\cdot, B] \to \text{Bun}_G(\cdot) \) since pullback is natural, so saying that \( P \) is a universal \( G \)-bundle is equivalent to saying that \( \alpha_P \) is a natural isomorphism.
**Lemma 1.3.1.** If a universal bundle exists for a group $G$, then it is unique up to homotopy equivalence.

**Proposition 1.3.2.** Suppose $p : E \to B$ is a fibre bundle with contractible fibre $F$. Then $p$ is a homotopy equivalence. Moreover, $p$ admits a section and any two sections of $p$ are homotopic through sections.

**Theorem 1.3.3.** A principal $G$-bundle $P \to B$ is a universal bundle for $G$-bundles if and only if $P$ is contractible.

*Proof.* $(\iff)$ Suppose $P \xrightarrow{p} B$ is a $G$-bundle with $P$ contractible. Let $Q \xrightarrow{q} X$ be any principal $G$-bundle. By definition, $P$ is a right $G$-space, but we may view it as a left $G$-space via $\rho(p) := pg^{-1}$ for any $p \in P, g \in G$. This data determines an associated bundle $E = E_{(p,\rho)} = Q \times_G P$, so that $E \to X$ is a fibre bundle with fibre $P$. In particular, since the fibre is contractible, Corollary 1.2.22 and Lemma 1.2.4 say that $E$ admits a section, but a section of an associated bundle corresponds to a map $\tilde{s} : Q \to P$ satisfying $\tilde{s}(yg) = \rho(g)^{-1}\tilde{s}(y) = \tilde{s}(y)g$ for all $y \in Q, g \in G$. This shows that $\tilde{s}$ is a $G$-equivariant map $Q \to P$ taking fibres to fibres, so it also induces a map $f : X \to B$. By Lemma 1.2.12, $Q \cong f^*P$, which shows that $[X,B] \to \text{Bun}_G(X)$ is surjective. On the other hand, any map $f : X \to B$ such that $f^*P \cong Q$ comes from a section $X \to E$ and any two such sections are homotopic through sections by Proposition 1.3.2. Thus any two maps $f, f' : X \to B$ inducing isomorphisms $f^*P \cong Q, f'^*P \cong Q$ are homotopic, so $[X,B] \to \text{Bun}_G(X)$ is injective. Hence $P \to B$ is a universal bundle.

$(\implies)$ We will construct universal bundles $EG \to BG$ for all $G$ and show that their total spaces are contractible. Once that is done, the proof will be complete. 

**Definition.** Let $EG \to BG$ be a universal bundle for a group $G$. The space $BG$ is called a classifying space for $G$.

For $G = O(k)$, the $k$th orthogonal group, recall the principal bundle $V_k(\mathbb{R}^n) \to \text{Gr}_k(\mathbb{R}^n)$ with fibre $O(k)$.

**Proposition 1.3.4.** For any $n \geq 1$, the inclusion $O(n) \hookrightarrow O(n+1)$ (resp. $SO(n) \hookrightarrow SO(n+1)$) given by

$$A \mapsto \begin{pmatrix} I_n & 0 \\ 0 & A \end{pmatrix}$$

induces maps $\pi_i(O(n)) \to \pi_i(O(n+1))$ (resp. $\pi_i(SO(n)) \to \pi_i(SO(n+1))$) which are isomorphisms for $i \leq n - 2$ and surjective for $i = n - 1$. Similarly, the inclusion $U(n) \hookrightarrow U(n+1)$ (resp. $SU(n) \hookrightarrow SU(n+1)$) induce $\pi_i(U(n)) \to \pi_i(U(n+1))$ (resp. $\pi_i(SU(n)) \to \pi_i(SU(n+1))$) which are isomorphisms for $i \leq 2n - 1$ and surjective for $i = 2n$.

*Proof.* The map $O(n+1) \to S^n \subseteq \mathbb{R}^{n+1}$ is a fibre bundle with fibre $O(n)$ and $\pi_i(S^n) = 0$ for all $i < n$, so the result follows from the long exact sequence in homotopy groups

$$\cdots \to \pi_{i+1}(S^n) \to \pi_i(O(n)) \to \pi_i(O(n+1)) \to \pi_i(S^n) \to \cdots$$

\[\square\]
Corollary 1.3.5. For all $k \leq n$, $O(n-k) \hookrightarrow O(n)$ (resp. $SO(n-k) \hookrightarrow SO(n)$) induces maps $\pi_i(O(n-k)) \to \pi_i(O(n))$ (resp. $\pi_i(SO(n-k)) \to \pi_i(SO(n))$) which are isomorphisms for all $i \leq n-k-2$ and surjective for $i = n-k-1$. A similar statement holds for $U(n)$ and $SU(n)$.

Corollary 1.3.6. The Stiefel manifold $V_k(\mathbb{R}^n)$ has $\pi_i(V_k(\mathbb{R}^n)) = 0$ for $i \leq n-k-1$. Similarly, the complex Stiefel manifold $V_k(\mathbb{C}^n)$ (defined using unitary $k$-frames) has $\pi_i(V_k(\mathbb{C}^n)) = 0$ for $i \leq 2(n-k)$.

Corollary 1.3.7. $V_k(\mathbb{R}^n) \to \text{Gr}_k(\mathbb{R}^n)$ is universal for $O(k)$-bundles over CW-complexes of dimension at most $n-k-1$.

Let $V_k = V_k(\mathbb{R}^\infty) := \bigcup_{n=1}^{\infty} V_k(\mathbb{R}^n)$ be the infinite Stiefel manifold with the direct limit topology, using the standard inclusions $V_k(\mathbb{R}^n) \hookrightarrow V_k(\mathbb{R}^{n+1})$. Similarly, let $\text{Gr}_k = \text{Gr}_k(\mathbb{R}^\infty) := \bigcup_{n=1}^{\infty} \text{Gr}_k(\mathbb{R}^n)$ be the infinite Grassmannian manifold.

Theorem 1.3.8. $V_k \to \text{Gr}_k$ is a universal bundle for principal $O(k)$-bundles.

Proof. Note that each $V_k$ admits a CW-structure that is compatible with the inclusions $V_k(\mathbb{R}^n) \hookrightarrow V_k(\mathbb{R}^{n+1})$ (see the Schubert cell description in the proof of Theorem 3.3.9), so this induces an infinite CW-structure on $V_k$. Further, by Corollary 1.3.6, $\pi_i(V_k) = 0$ for all $i$ so by Whitehead’s theorem in homotopy theory, $V_k$ is contractible. Hence by Theorem 1.3.3, $V_k \to \text{Gr}_k$ is universal for $O(k)$-bundles.

Here’s an alternate proof that $V_k$ is contractible. Define the ‘even’ and ‘odd’ embeddings of a Euclidean space into the space with twice its dimension:

$$i_1 : \mathbb{R}^n \hookrightarrow \mathbb{R}^{2n}, \ (x_1, \ldots, x_n) \mapsto (x_1, 0, x_2, 0, \ldots, x_n, 0)$$

$$i_2 : \mathbb{R}^n \hookrightarrow \mathbb{R}^{2n}, \ (x_1, \ldots, x_n) \mapsto (0, x_1, 0, x_2, \ldots, 0, x_n).$$

These induce inclusions $V_k(\mathbb{R}^n) \hookrightarrow V_k(\mathbb{R}^{2n})$. Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$ and define a homotopy

$$F : V_k(\mathbb{R}^n) \times I \to V_k(\mathbb{R}^{2n})$$

$$(v_1, \ldots, v_n, t) \mapsto ((1-t)i_1(v_1, \ldots, v_n) + ti_2(v_1, \ldots, v_n).$$

One can check that $F$ commutes with the inclusions $V_k(\mathbb{R}^n) \hookrightarrow V_k(\mathbb{R}^{n+1})$, and so induces $\overline{F} : V_k \times I \to V_k$, a homotopy from the constant map induced by $i_2$ to the identity map induced by $i_1$. □

Corollary 1.3.9. For every compact Lie group $G$, there exists a universal bundle $EG \to BG$ with $EG$ contractible.

Proof. Any compact Lie group $G$ embeds as a closed subgroup of an orthogonal group $O(k)$ (this is equivalent to saying every such $G$ has a finite dimensional faithful representation with an inner product). In particular, there is a right $G$-action on $V_k = V_k(\mathbb{R}^\infty)$, so $V_k \to V_k/G$ is a principal bundle with fibre $B$. By Theorem 1.3.8, $V_k$ is contractible so $V_k \to V_k/G$ is universal. □
Notice that for any compact Lie group \( G \), an embedding \( G \hookrightarrow O(k) \) induces a map \( BG = V_k/G \to V_k/O(k) = \text{Gr}_k \). This suggests the following construction of classifying spaces due to Milnor.

For arbitrary spaces \( X \) and \( Y \), define their *join* \( X \ast Y \) by

\[
X \ast Y := \{ t_0x + t_1y \mid t_i \in [0, 1], x \in X, y \in Y \} = X \times Y / \sim
\]

where \((x, 0, y) \sim (x', 0, y')\) and \((x, 1, y) \sim (x', 1, y')\) for all \( x, x' \in X, y, y' \in Y \). More generally, the join of a collection of spaces \( X_0, \ldots, X_n \) is given by

\[
X_0 \ast \cdots \ast X_n := \left\{ \sum_{i=0}^n t_i x_i : x_i \in X_i, t_i \in [0, 1], \sum_{i=0}^n t_i = 1 \right\} = X_0 \times \cdots \times X_n \times \Delta_n / \sim
\]

where \( \Delta_n \) is the standard \( n \)-simplex and \( \sim \) is analogous to the above. Then the inclusions \( X_i \hookrightarrow X_0 \ast \cdots \ast X_n \) as the ‘faces’ of the simplex are nullhomotopic. Let \( X_1 = X \) and inductively define \( X_n = X_{n-1} \ast X \). Then the sequence \( X_1, X_2, X_3, \ldots \) becomes increasingly ‘connected’ in the sense that their homotopy groups eventually vanish.

**Proposition 1.3.10.** For any space \( X \), let \( X_n = X \ast \cdots \ast X \) be the \( n \)-fold join of \( X \). Then the infinite join \( X_\infty = \bigcup_{n=1}^\infty X_n \) is contractible.

Given a topological group \( G \), let \( G_\infty \) be the infinite join of \( G \) as above. By Proposition 1.3.10, \( G_\infty \). Moreover, we can define a \( G \)-action on \( G_\infty \) by:

\[
\left( \sum_{i=0}^\infty t_i g_i \right) \cdot g = \sum_{i=0}^\infty t_i g_i g.
\]

(Note that the formal sums have finitely many \( t_i \) nonzero, so they are in reality honest sums.) It’s easy to see that this action is free.

**Lemma 1.3.11.** \( G_\infty \to G_\infty / G \) is a principal \( G \)-bundle.

**Proof.** Consider the ‘coordinate functions’

\[
T_i : G_\infty \to [0, 1]
\]

\[
\sum_{j=0}^\infty t_j g_j \mapsto t_i.
\]

Then by construction, the \( T_i \) descend to \( G_\infty / G \). We claim that \( G_\infty \to G_\infty / G \) is trivial over each \( U_i = T_i^1((0, 1]) \). Define

\[
\varphi_i : U_i \times G \to G_\infty
\]

\[
\left( \sum_{j=0}^\infty t_j g_j, g \right) \mapsto \sum_{j=0}^\infty t_j g_j g_i^{-1} g.
\]

Then \( \varphi_i \) is independent of the choice of representative of \( \left[ \sum_{j=0}^\infty t_j g_j \right] \) and it’s easy to check that \( \varphi_i \) is a bundle isomorphism. \( \Box \)
There are some technicalities present with our construction:

1. Is the $G$-action on $G_{\infty}$ continuous? Note that we have projections

$$\pi_i : p^{-1}(U_i) \to G$$

and the map $m : G_{\infty} \times G \to G_{\infty}$ representing the $G$-action composed with any of the $T_i$ or $\pi_i$ is continuous, but the topology on $G_{\infty}$ is different from the one induced by the $T_i, \pi_i$. So instead, we may take $G_{\infty}$ to have this topology, i.e. the coarsest so that all $T_i$ and $\pi_i$ are continuous. But then one still has to check that $G_{\infty}$ is contractible with respect to this topology – see Dold for the proof of this.

2. Is $G_{\infty}/G$ paracompact? Milnor showed that the answer is no in general, but the cover $\{U_i\}$ of $G_{\infty}/G$ constructed above is numerable so the rest of the results in Section 1.2 still apply.

Setting these issues aside, we have proven:

**Corollary 1.3.12.** For all groups $G$, the bundle $G_{\infty} \to G_{\infty}/G$ is a universal $G$-bundle.

**Corollary 1.3.13.** Let $G \to H$ be a continuous homomorphism of topological groups. Then there is a morphism of principal bundles

$$
\begin{array}{ccc}
EG & \to & EH \\
\downarrow & & \downarrow \\
BG & \to & BH
\end{array}
$$

That is, $G \mapsto BG$ is a functor $\text{Top} \to \text{Group}$.

**Theorem 1.3.14.** A continuous homomorphism $\alpha : G \to H$ induces a natural transformation $\text{Bun}_G(-) \to \text{Bun}_H(-)$. Moreover, this is a natural isomorphism if and only if $\alpha$ is a homotopy equivalence.

**Proof.** By definition, $\text{Bun}_G(-)$ is represented by $BG$ so Corollary 1.3.13 implies that $\alpha$ induces a map $B\alpha : BG \to BH$, and hence a function

$$\text{Bun}_G(-) = [-, BG] \xrightarrow{(B\alpha)_*} [-, BH] = \text{Bun}_H(-).$$

For any map $f : X \to Y$, the diagram

$$
\begin{array}{ccc}
[X, BG] & \xrightarrow{(B\alpha)_*} & [Y, BG] \\
\downarrow f^* & & \downarrow f^* \\
[Y, BG] & \xrightarrow{(B\alpha)_*} & [Y, BH]
\end{array}
$$

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commutes by associativity of composition. Thus \((B\alpha)_*\) is a natural transformation. Now \((B\alpha)_*\) is a bijection \([X, BG] \to [X, BH]\) for all spaces \(X\) if and only if \(B\alpha\) is a homotopy equivalence, so it suffices to show \(B\alpha\) is a homotopy equivalence precisely when \(\alpha\) is one. Consider the pullback \(\overline{EH} := (B\alpha)^*EH:\)

\[
\begin{array}{ccc}
\overline{EH} & \longrightarrow & EH \\
\downarrow & & \downarrow \\
BG & \longrightarrow & BH \\
\downarrow_{B\alpha} & & \\
\end{array}
\]

Then \(B\alpha\) is a homotopy equivalence if and only if \(\overline{EH}\) is universal for \(H\)-bundles (or equivalently by Theorem 1.3.3, \(\overline{EH}\) is contractible). On the other hand, let \(E\alpha : EG \to EH\) be the map induced on total spaces by \(\alpha : G \to H\). Then \(\overline{EH}\) is a universal \(H\)-bundle if and only if there is a \(G\)-bundle morphism

\[
\begin{array}{ccc}
EG & \longrightarrow & \overline{EH} \\
\downarrow & & \downarrow \\
BG & \longmapsto & \\
\end{array}
\]

If \(E\alpha\) is a homotopy equivalence \(EG \to \overline{EH}\), it induces a homotopy equivalence on each fibre, but this is just \(\alpha\) on each fibre so \(\alpha\) is itself a homotopy equivalence. Conversely, if \(\alpha\) is a homotopy equivalence, then over each locally trivial open set \(U \subseteq BG\), \(E\alpha\) has the form \(id \times \alpha : U \times G \to U \times H\). So \(E\alpha\) is locally a fibrewise homotopy equivalence and one can prove that this implies \(E\alpha\) is globally a homotopy equivalence. Hence \(E\alpha\) is a homotopy equivalence if and only if \(\alpha\) is one, so we are done.

**Remark.** For any group \(G\), there is a fibration \(EG \to BG\) with fibre \(G\), giving rise (by homotopy theory) to a fibration sequence

\[
\cdots \to \Omega G \to \Omega EG \to \Omega BG \to G \to EG \to BG.
\]

Then for any topological space \(X\), there is an exact sequence of sets

\[
\cdots [X, \Omega G] \to [X, \Omega EG] \to [X, \Omega BG] \to [X, G] \to [X, EG] \to [X, BG] = \text{Bun}_G(X).
\]

Since \(EG\) is contractible, \([X, \Omega^k BG] = 0\) for all \(k \geq 0\) so there is a bijection \([X, \Omega^k BG] \leftrightarrow [X, \Omega^{k-1}G]\). In particular, taking \(X = S^n\) gives isomorphisms of groups \(\pi_n(\Omega BG) \cong \pi_n(G)\) for all \(n\), so by Whitehead’s theorem, \(\Omega BG \to G\) is a homotopy equivalence. Here is another perspective.

**Lemma 1.3.15.** For a topological group \(G\), let \(PBG\) be the path space of \(BG\). Then there exists a bundle morphism
where $\text{PBG} \to \text{BG}, \gamma \mapsto \gamma(1)$ is the endpoint fibration.

Proof. Since $\text{EG}$ is contractible (Theorem 1.3.3), $p : \text{EG} \to \text{BG}$ extends to a map $\tilde{p} : C\text{EG} \to \text{BG}$ where $C\text{EG} = \text{EG} \wedge I$ is the cone on $\text{EG}$. By adjointness, $\tilde{p}$ determines a map $\tilde{p} : \text{EG} \to \text{PBG}$ given explicitly by $\tilde{p}(y)(t) = \tilde{p}(t, y)$. Since $\tilde{p}$ extends $p$, for $t = 1$ we have $\tilde{p}(y)(1) = \tilde{p}(1, y) = p(y)$ so indeed the diagram above commutes. Moreover, since $\text{EG}$ is contractible, $\tilde{p}$ is a homotopy equivalence on fibres and thus $\text{PBG} \to \text{BG}$ is, too. \qed

Corollary 1.3.16. There is a homotopy equivalence $G \to \Omega \text{BG}$.

Proof. Suppose $P \to X$ is a principal $G$-bundle classified by a map $f : X \to \text{BG}$. Then $P$ is the homotopy fibre of $f$, and $f$ is the pullback of $\text{PBG} \to \text{BG}$. Thus we have two bundle morphisms

\[
\begin{array}{ccc}
f^*\text{PBG} & \to & \text{PBG} \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & \text{BG}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
P & \to & \text{EG} \\
\downarrow & & \downarrow \text{p} \\
X & \xrightarrow{f} & \text{BG}
\end{array}
\]

Since $f$ and $p$ are both homotopy equivalences, we get $f^*\text{PBG} \simeq P = f^*\text{EG}$. \qed

1.4 Reduction of the Structure Group

Let $\alpha : G \to H$ be a continuous homomorphism of topological groups and suppose $P_G \to X$ is a principal $G$-bundle. Then by the results in Section 1.3, $\alpha$ induces a principal $H$-bundle over $X$ in two ways:

1. Composition of a classifying map $f_G : X \to \text{BG}$ for $P_G$ with the induced map $B\alpha : \text{BG} \to \text{BH}$ gives an $H$-bundle $P_H = (B\alpha \circ f_G)^*\text{EH}$.

2. The map $\alpha$ determines a $G$-space structure on $H$ via $\rho(g)h = \alpha(g)h$, so there is an associated bundle $P_G \times_G H \to X$ with fibre $H$.

We may ask the converse of this question: when is an $H$-bundle $P_H \to X$ induced by a $G$-bundle over $X$? There are two perspectives on this question, mirroring the perspectives above:

1. A classifying map $f_H : X \to \text{BH}$ for $P_H$ may by induced by composition of some $f_G$ with $B\alpha$: 

\[
\begin{array}{ccc}
f_H & \to & \text{BH} \\
\downarrow & & \downarrow \text{p} \\
X & \xrightarrow{f} & \text{BG}
\end{array}
\]
Example 1.4.1. If $\alpha : G \to H$ is the inclusion of a closed subgroup, the universal $G$-bundle is given by $EH \to EH/G$ ($EH$ is contractible and $EH \to EH/G$ has fibre $G$ by Proposition 1.2.8). Thus composition with $B\alpha$ gives a fibre bundle $EH/G \to BH$ with fibre $H/G$, but not necessarily a principal bundle.

Example 1.4.2. If $\alpha : G \to H$ is a quotient map, then $H \cong G/K$ where $K = \ker\alpha$, and $EG \to EG/K = EK$ is a principal $K$-bundle, thus universal since $EG$ is contractible. Also, $BK \to BG$ is a principal bundle with fibre $G/K = H$ classified by the map $B\alpha : BG \to BH$. This is because $E\alpha : EG \to EH$ is given by $G \to G/K$ on fibres, so it factors through $EG/K$ and thus $EG/K = (B\alpha)^*EH$.

Lemma 1.4.3. For a quotient map $\alpha : G \to H = G/K$, $BK$ is the homotopy fibre of the associated map $B\alpha : BG \to BH$.

These examples suggest answers to the reduction question in the form posed in (1) above: if $\alpha : G \to H$ is a closed inclusion, we have a fibration sequence $B(H/G) \to BG \to BH$, whereas if $\alpha$ is a quotient map, $BK \to BG \to BH$ is the corresponding fibration sequence. In the former case, for every $X$ there is an exact sequence

$$[X, H/G] \to [X, G] \to [X, H] \to [X, B(H/G)] \to [X, BG] \to [X, BH].$$

This proves:

Proposition 1.4.4. Let $\alpha : G \to H$ be the inclusion of a closed subgroup. Then a principal $H$-bundle $P_H \to X$ classified by $f : X \to BH$ is induced by a principal $G$-bundle over $X$ if and only if the map $B\alpha \circ f : X \to B(H/G)$ is nullhomotopic.

Example 1.4.5. Let $G = SO(n)$, $H = O(n)$ and $\alpha : SO(n) \hookrightarrow O(n)$ the standard inclusion. Here, principal $O(n)$-bundles identify canonically with real vector bundles of dimension $n$ (Example 1.1.10) and $SO(n)$-bundles identify with real oriented vector bundles of dimension $n$. Reduction of the structure group in this case just amounts to determining if a bundle is orientable or not. By Proposition 1.4.4, an $O(n)$-bundle $E \to X$ is orientable (that is, reduces to an $SO(n)$-bundle) if and only if

$$X \to BO(n) \to B(O(n)/SO(n)) = B(\mathbb{Z}/2\mathbb{Z})$$

is nullhomotopic. Notice that $E(\mathbb{Z}/2\mathbb{Z})$ may be taken to be any contractible space with a free $\mathbb{Z}/2\mathbb{Z}$-action, so in particular we may take $E(\mathbb{Z}/2\mathbb{Z}) = S^\infty$, so that $B(\mathbb{Z}/2\mathbb{Z}) = S^\infty/\mathbb{Z}/2\mathbb{Z} =$
1.4 Reduction of the Structure Group

$\mathbb{R}P^\infty$. Usefully, $\mathbb{R}P^\infty = K(\mathbb{Z}/2\mathbb{Z}, 1)$, an Eilenberg-Maclane space, which by definition means for any space $X$,

$$[X, B(\mathbb{Z}/2\mathbb{Z})] = [X, K(\mathbb{Z}/2\mathbb{Z}, 1)] = H^1(X; \mathbb{Z}/2\mathbb{Z}).$$

Consequently, $X \to B(\mathbb{Z}/2\mathbb{Z})$ is nullhomotopic if and only if $(B\alpha \circ f)^*\omega = 0$ in $H^1(X; \mathbb{Z}/2\mathbb{Z})$, where $\omega$ is a generator of the cohomology ring $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\omega]$. This class $\omega_1(E) := (B\alpha \circ f)^*\omega \in H^1(X; \mathbb{Z}/2\mathbb{Z})$ is our first example of a characteristic class, called the first Stiefel-Whitney class, which we develop in Section 3.2.

Let’s return to perspective (2) on the structure group reduction question, which asks if $P \to X$ is the associated bundle for some $P \to X$. If $\alpha : G \to H$ is the inclusion of a closed subgroup, then $P \to X$ reduces to a $G$-bundle if and only if

$$P_H = P_G \times_G H$$

for some space $P_G$ with a free left $G$-action.

**Proposition 1.4.6.** Let $\alpha : G \to H$ be the inclusion of a closed subgroup. Then a principal $H$-bundle $P_H \to X$ reduces to a $G$-bundle if and only if $P_H/G \to X$ has a section.

**Proof.** $(\implies)$ If $P_H = P_G \times_G H$ for some free left $G$-space $P_G$, then $P_H/G = (P_G \times_G H)/G = P_G \times_G (H/G)$, so

$$X := P_H/H = (P_G \times_G H)/H$$

$$= P_G \times_G (H/H)$$

$$= P_G \times_G (G/G) = P_G,$$

so let $s : X \to P_H/G$ be the map corresponding to $P_G \to P_G \times_G (H/G)$. This defines a section by construction.

$(\impliedby)$ Suppose $s$ is a section of $P_H/G \to X = P_H/H$. Let $Q = s^*(P_H \to P_H/G)$ be the induced bundle:

$$\begin{array}{ccc}
Q & \longrightarrow & P_H \\
\downarrow & & \downarrow \\
X & \longrightarrow & P_H/G
\end{array}$$

This determines a map $\gamma : Q \times_G H \to P_H$, $\gamma([q, h]) = \tilde{s}(q)(h)$, where $\tilde{s}$ is the lift of $s$ as in the proof of Theorem 1.3.3. It’s easy to check that $\gamma$ is $G$-equivariant, so by Lemma 1.2.12, $Q \times_G H \cong P_H$. \hfill $\Box$

This gives another proof of Lemma 1.2.4: a principal $G$-bundle $P \to B$ is trivial if and only if $P = P/\{1\}$ admits a section.

**Corollary 1.4.7.** For any closed subgroup $G \leq H$, the quotient $H/G$ is contractible if and only if every $H$-bundle reduces to a $G$-bundle.
1.5 Gauge Groups

This section follows a talk given by Peter Johnson for Dr. Mark’s class.

Definition. The **gauge group** of a principal $G$-bundle $p : P \to B$, denoted $\text{Aut}_B(P)$, is the group of principal $G$-bundle isomorphisms

\[
P \xrightarrow{p} B \xrightarrow{\sigma} P
\]

We endow $\text{Aut}_B(P)$ with the subspace topology inherited from the compact-open topology on $\text{Map}_G(P, P)$. One can check that this makes $\text{Aut}_B(P)$ into a topological group. This means $\text{Map}_G(P, EG)$ is nonempty. Moreover, $\text{Aut}_B(P)$ acts freely on the right on $\text{Map}_G(P, EG)$ via $\sigma f = \sigma \circ f$.

Question. What is the classifying space for $\text{Aut}_B(P)$-bundles?

Let $\text{Cl}_p(B, BG)$ be the set of classifying maps $B \to BG$ for $p : P \to B$. By Corollary 1.2.20, all elements of $\text{Cl}_p(B, BG)$ are homotopic. It follows that $\text{Map}_G(P, EG)$ is path-connected (it is also nonempty by the existence of classifying maps).

Lemma 1.5.1. For any principal $G$-bundle $p : P \to B$, $\text{Map}_G(P, EG)/ \text{Aut}_B(P) \cong \text{Cl}_p(B, BP)$ as topological spaces.

**Proof.** Define a map

\[
S : \text{Map}_G(P, EG) \longrightarrow \text{Cl}_p(B, BG)
\]

\[
\sigma \mapsto \bar{\sigma}
\]

where $\bar{\sigma}$ is the map induced by the $G$-equivariant $\sigma$ under the identification $B = P/G$ and $BG = EG/G$. Suppose $f \in \text{Aut}_B(P)$ and $\sigma \in \text{Map}_G(P, EG)$. Then there is a $G$-equivariant commutative diagram

\[
P \xrightarrow{f} P \xrightarrow{\sigma} EG
\]

\[
B \xrightarrow{id} B \xrightarrow{\bar{\sigma}} BG
\]

It follows that $\bar{\sigma} f = \bar{\sigma}$, i.e. the map $S$ descends to the quotient $\text{Map}_G(P, EG)/ \text{Aut}_B(P)$. Now suppose $\sigma_1, \sigma_2 \in \text{Map}_G(P, EG)$ such that $\bar{\sigma}_1 = \bar{\sigma}_2$. Then there exists some map $\alpha : P \to G$ such that for all $x \in P$,

\[
\sigma_1(x) = \sigma_2(x)\alpha(x) = \sigma_2(x\alpha(x)).
\]

Then the map $f : P \to P, x \mapsto x\alpha(x)$ lies in $\text{Aut}_B(P)$. One can also prove that $S$ is a homeomorphism, so $\text{Map}_G(P, EG)/ \text{Aut}_B(P) \cong \text{Cl}_p(B, BP)$ as spaces. \qed
**Theorem 1.5.2.** For all principal $G$-bundles $p : P \to B$, $\text{Map}_G(P, EG) \to \text{Cl}_p(B, BG)$ is universal for $\text{Aut}_B(P)$-bundles.

**Proof.** It is straightforward to prove that $\text{Map}_G(P, EG) \to \text{Cl}_p(B, BG)$ is a locally trivial fibre bundle with fibre $\text{Aut}_B(P)$. Moreover, the gauge group acts freely on $\text{Map}_G(P, EG)$, so we indeed have a principal $\text{Aut}_B(P)$-bundle and we need only prove $\text{Map}_G(P, EG)$ is contractible (by Theorem 1.3.3).

Consider the product bundle

$$p \times \text{id} : P \times \text{Map}_G(P, EG) \to B \times \text{Map}_G(P, EG).$$

By our observations above, $\text{Map}_G(Q, EG)$ is nonempty and path-connected for any bundle $Q$, so in particular this is true for

$$\text{Map}_G(P \times \text{Map}_G(P, EG), EG) \cong \text{Map}(\text{Map}_G(P, EG), \text{Map}_G(P, EG))$$

(this identification is by the adjoint property of $\times$ and Map). Therefore there is a path in $\text{Map}(\text{Map}_G(P, EG), \text{Map}_G(P, EG))$ from the identity map to a constant map, which proves $\text{Map}_G(P, EG)$ is contractible. \hfill $\square$

Now assume the topological group $G$ is abelian, so that the multiplication map $m : G \times G \to G$ is a continuous homomorphism. This determines a map $Bm : BG \times BG \to BG$ and similarly, inversion $i : G \to G$ and the identity $e : * \to G$ determine maps $Bi : BG \to BG$ and $Be : * \to BG$. Together, these give $BG$ the structure of an abelian topological group. Repeating this procedure gives a sequence of abelian topological groups

$$B, BG, B^2G, B^3G, \ldots$$

where $B^nG = B(B^{n-1}G)$. By Corollary 1.3.16, $G \cong \Omega BG$ so for $G$ abelian, we get $G \cong \Omega^nB^nG$ for all $n \geq 0$.

**Lemma 1.5.3.** If $G$ is any abelian group with the discrete topology, then for all $n \geq 0$, $B^nG = K(G, n)$.

**Proof.** Let $m \geq 0$. Then

$$\pi_m(B^nG) = [S^m, B^nG] = [S^0, \Omega^mB^nG] \text{ by adjointness}$$

$$= [S^0, \Omega^{m-n}\Omega^nB^nG] = [S^0, \Omega^{m-n}G]$$

$$= [S^{m-n}, G] = \pi_{m-n}(G) = \begin{cases} G, & m = n \\ 0, & m \neq n. \end{cases}$$

\hfill $\square$

This proves:

**Theorem 1.5.4.** For all abelian $G$ and $n \geq 0$, $K(G, n)$ is an abelian topological group and $BK(G, n-1) = K(G, n)$. 

20
Lemma 1.5.5. If $G$ is abelian, then for any principal $G$-bundle $p : P \to B$, $\text{Aut}_B(P) \cong \text{Map}(B, G)$. In particular, the gauge group does not depend on the total space $B$.

Thus the gauge group $\text{Aut}_B(K(G, n))$ is well-defined for all abelian $G$ and $n \geq 0$. This group can be viewed as:

$$\text{Aut}_B(K(G, n)) \cong \text{Map}(B, K(G, n)) = [B, K(G, n)] \cong H^n(B; G).$$

**Theorem 1.5.6** (Thom). For all spaces $B$ and $K(G, n)$, $\text{Map}(B, K(G, n))$ is homotopy equivalent to

$$\prod_{q=0}^{n} K(H^{n-q}(B; G), q).$$

**Proof.** We induct on $n$. When $n = 0$, $K(G, 0) = G$ and $K(H^0(B; G), 0) \cong G$ so they are in fact homeomorphic. For $n > 0$,

$$\text{Map}(B, K(G, n)) = \prod_{[B, K(G, n)]} B(\text{Map}(B, K(G, n - 1)))$$

where the product is indexed over the connected components of $\text{Map}(B, K(G, n))$, represented by elements of $[B, K(G, n)]$. By induction,

$$\text{Map}(B, K(G, n)) \simeq K(H^n(B; G), 0) \times B \left( \prod_{q=0}^{n-1} K(H^{n-q-1}(B; G), q) \right)$$

$$= \prod_{q=0}^{n} K(H^{n-q}(B; G), q).$$

Hence the statement holds by induction. □

As a beautiful consequence, we derive a proof of the Künneth formula from algebraic topology.

**Corollary 1.5.7** (Künneth Formula). For any CW-complexes $X$ and $Y$ and any abelian group $G$, there isomorphisms

$$H^n(X \times Y; G) \cong \bigoplus_{q=0}^{n} H^q(X; H^{n-q}(Y; G)).$$

**Proof.** By definition of $K(G, n)$,

$$H^n(X \times Y; G) = [X \times Y, K(G, n)]$$

$$= [X, \text{Map}(Y, K(G, n))] \text{ by adjointness}$$

$$= \prod_{q=0}^{n} [X, K(H^{n-q}(Y; G), q)] \text{ by Theorem 1.5.6}$$

$$= \bigoplus_{q=0}^{n} H^q(X; H^{n-q}(Y; G)).$$

□
2 Vector Bundles

2.1 Vector and Tangent Bundles

A vector bundle is a special type of fibre bundle in which the fibre has the structure of a (Euclidean) vector space. The definition of a vector bundle is also quite similar to that of a manifold, in that there are certain “local triviality” and compatibility conditions placed on the space. Recall:

Definition. A topological (resp. smooth) $n$-dimensional manifold is a space $M$ such that for any point $p \in M$, there is a neighborhood $U \subset M$ containing $p$ which is homeomorphic (resp. diffeomorphic) to an open subset of $\mathbb{R}^n$.

We introduce the definition of a vector bundle in a parallel fashion.

Definition. A topological (resp. smooth) $n$-dimensional vector bundle is an object $\xi$ consisting of the following data: a total space $E$, a base $B$ and a continuous (resp. smooth) map $\pi : E \to B$, where $E$ and $B$ are topological (resp. smooth) manifolds, such that for each $p \in B$, the fibre $F_p := \pi^{-1}(p)$ is a vector space of dimension $n$. Moreover, for each $p \in B$, there must exist a neighborhood $U \subseteq B$ containing $p$ and a homeomorphism (resp. diffeomorphism) $h : U \times \mathbb{R}^n \to \pi^{-1}(U)$ making the following diagram commute:

$$
\begin{array}{ccc}
U \times \mathbb{R}^n & \xrightarrow{h} & \pi^{-1}(U) \\
\downarrow{\text{proj}} & & \downarrow{\pi} \\
U & & \\
\end{array}
$$

and for all $x \in U$, $h|_{\{x\} \times \mathbb{R}^n} : \{x\} \times \mathbb{R}^n \to \pi^{-1}(x)$ is a vector space isomorphism.

Alternatively, one can define a manifold from a topological space $M$ using transition maps (which are either continuous or smooth, depending on if the manifold is topological or smooth). This has an analagous interpretation for vector bundles. Given two overlapping neighborhoods $U, V \subseteq B$, consider the local trivializations

$$
\begin{array}{ccc}
(U \cap V) \times \mathbb{R}^n & \xrightarrow{h_U} & \pi^{-1}(U \cap V) \\
\downarrow{\text{proj}} & & \downarrow{\pi} \\
U \cap V & & \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
(U \cap V) \times \mathbb{R}^n & \xleftarrow{h_V} & \pi^{-1}(U \cap V) \\
\downarrow{\text{proj}} & & \downarrow{\pi} \\
U \cap V & & \\
\end{array}
$$

The vector space isomorphism condition then guarantees that for any $x \in U \cap V$, the composition $\{x\} \times \mathbb{R}^n \xrightarrow{h_U} \pi^{-1}(x) \xrightarrow{h_V^{-1}} \{x\} \times \mathbb{R}^n$ is a vector space isomorphism, i.e. an element of $GL_n(\mathbb{R})$. Such a map is called a clutching map, or a transition map. What’s more, this defines a continous (resp. smooth) map $U \cap V \to GL_n(\mathbb{R})$. We will see in Section 2.4 that this is enough data to reconstruct the original vector bundle.
2.1 Vector and Tangent Bundles

Examples.

1. The **trivial bundle** of dimension $n$ over any manifold $B$ consists of $E = B \times \mathbb{R}^n$ and the projection map $\pi : B \times \mathbb{R}^n \to B, (p, v) \mapsto p$. For each $b \in B$, $\pi^{-1}(p) = \{b\} \times \mathbb{R}^n \cong \mathbb{R}^n$. The local triviality condition in the definition of a vector bundle intuitively says that every vector bundle locally looks like the trivial bundle. Notice that there is a natural projection onto the fibre: $B \times \mathbb{R}^n \to \mathbb{R}^n, (p, v) \mapsto v$. In fact, the existence of such a projection characterizes the trivial bundle; that is, any nontrivial bundle (in the sense of bundle equivalence to be defined in a moment) does not have a projection onto the fibre.

2. Perhaps the most important vector bundle is the tangent bundle $p : TM \to M$ of a smooth manifold $M$ (see Example 1.1.6).

3. The **canonical or tautological bundle** is an important example of a vector bundle over $\mathbb{R}P^n$, defined as follows. The total space is given by

$$E = \{(x, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} : v = \lambda x \text{ for some scalar } \lambda \in \mathbb{R}\}.$$  

In other words, viewing $\mathbb{R}P^n$ as the quotient of the unit sphere $S^n \subset \mathbb{R}^{n+1}$ by the antipodal action, a point $(x, v) \in E$ is determined by a pair of points $x, -x$ on the $n$-sphere and a vector $v$ lying on the line between them. Here, the projection map $\pi : E \to \mathbb{R}P^n$ is given by $\pi(x, v) = x$.

Let us show that this information determines the structure of a vector bundle over $\mathbb{R}P^n$, denoted by $\gamma^1_n$. Take a chart $U \subset S^n$ that is small enough so that $U \cap -U = \emptyset$. If $p : S^n \to \mathbb{R}P^n$ is the double cover, then $p|_U : U \to U' = p(U)$ and $p|_{-U} : -U \to U'$ are diffeomorphisms. We construct a trivialization of $U'$ by defining

$$h : U' \times \mathbb{R} \to \pi^{-1}(U')$$

$$(\bar{x}, t) \mapsto (\bar{x}, tx)$$

where $x$ is the point in $p^{-1}(\bar{x}) = \{x, -x\}$ lying in $U$. (The subtlety with this definition is that one may choose $-x \in U$ instead, still have $p(-x) = \bar{x}$, but obtain a different trivialization.) Since $U$ and $-U$ don’t overlap, $h$ is a diffeomorphism. Further, $h$ clearly commutes with $\pi$ and is an isomorphism on fibres, so $h$ is a local trivialization.

**Proposition 2.1.1.** For any smooth manifold $M$ of dimension $n$, the tangent bundle $TM$ is a smooth vector bundle of dimension $n$.

**Proof.** We need only check the local triviality condition, since the projection $\pi : TM \to M$ is already defined. Fix $p \in M$ and choose a coordinate chart $U \subseteq M$ of $p$; this comes equipped with a diffeomorphism $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$. Define $h : U \times \mathbb{R}^n \to \pi^{-1}(U)$ by $h(x, v) = (x, D\varphi(x)\varphi^{-1}(v))$, where $D\varphi^{-1}$ is the differential of the inverse diffeomorphism $\varphi^{-1} : \varphi(U) \to U$ at a point $u \in \varphi(U)$. By definition, $T_xM = \text{im}(D\varphi(x)\varphi^{-1})$ for any $x \in U$, so $h$ is well-defined. Moreover, for each $x$, $h|_{\{x\} \times \mathbb{R}^n} : \{x\} \times \mathbb{R}^n \to \pi^{-1}(x)$ is an isomorphism since $\varphi$ and $\varphi^{-1}$ are diffeomorphisms. This completes the proof. \qed
Vector bundles are used in modern differential topology to distinguish between smooth manifolds. We next define the notion of equivalence of vector bundles.

**Definition.** Given two vector bundles \( \pi_1 : E_1 \rightarrow B \) and \( \pi_2 : E_2 \rightarrow B \) over the same base \( B \), we say \((E_1, B, \pi_1)\) and \((E_2, B, \pi_2)\) are **isomorphic vector bundles** if there exists a diffeomorphism \( \varphi : E_1 \rightarrow E_2 \) such that for all \( b \in B \), \( \varphi|_{\pi_1^{-1}(b)} : \pi_1^{-1}(b) \rightarrow \pi_2^{-1}(b) \) is a vector space isomorphism. More succinctly, \((E_1, B, \pi_1) \cong (E_2, B, \pi_2)\) if and only if the following diagram commutes:

\[
\begin{array}{ccc}
E_1 & \xrightarrow{\varphi} & E_2 \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
B & & B
\end{array}
\]

**Example 2.1.2.** We claim that the tangent bundle \( TS^1 \) to the unit circle is isomorphic to the trivial bundle over \( S^1 \). To show this, we must construct a diffeomorphism \( \varphi : S^1 \times \mathbb{R} \rightarrow TS^1 \) making the diagram commute:

\[
\begin{array}{ccc}
S^1 \times \mathbb{R} & \xrightarrow{\varphi} & TS^1 \\
\downarrow{\text{proj}} & & \downarrow{\pi} \\
S^1 & & 
\end{array}
\]

Given any point \( p = (x, y) \in S^1 \), \( T_pS^1 \) is spanned by \(( -y, x) \in T_pS^1 \). Define the map

\[
\varphi : S^1 \times \mathbb{R} \rightarrow TS^1 \subset \mathbb{R}^2 \times \mathbb{R}^2
\]

\[
((x, y), t) \mapsto ((x, y), t(-y, x)).
\]

This is clearly smooth and one-to-one, and restricts on each \( \{p\} \times \mathbb{R} \) to a diffeomorphism with \( T_pS^1 \). Therefore \( TS^1 \) is the trivial bundle.

**Example 2.1.3.** One may be tempted to think that all vector bundles are isomorphic to the trivial bundle. Consider the tautological bundle over \( \mathbb{R}P^n \):

\[
E = \{(x, tx) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \mid x \in S^n, t \in \mathbb{R}\}.
\]

Then each point \( x \in \mathbb{R}P^n \) may be viewed as a line in \( \mathbb{R}^{n+1} \) through the origin, so \( \pi^{-1}(x) \) consists of all vectors in that line defined by \( x \). In the case \( n = 1 \), \( \mathbb{R}P^1 \cong S^1 \) and \( E \) looks like a quotient of an infinite rectangle:
Then $E$ is homeomorphic to the Möbius band. The line bundle $E \to S^1$ cannot be the trivial bundle over $S^1$ because $E$ is not diffeomorphic to $S^1 \times \mathbb{R}$ (e.g. $S^1 \times \mathbb{R}$ is orientable and the Möbius band is not). Hence we have exhibited a nontrivial vector bundle over $\mathbb{R}P^1 \cong S^1$. In general, the tautological bundle $\pi : E \to \mathbb{R}P^n$ is nontrivial, as we show in Section 2.2.

**Definition.** A manifold $M$ whose tangent bundle $TM$ is isomorphic to the trivial bundle is called parallelizable.

In Example 2.1.2 we showed that $S^1$ is parallelizable. It turns out that for $n \geq 2$ except $n = 3, 7$, $S^n$ is not parallelizable. We will prove this in the next section.

### 2.2 Sections

**Definition.** Given a vector bundle $\xi : E \xrightarrow{\pi} B$, a section of the bundle is a smooth map $s : B \to E$ such that $\pi \circ s = id_B$. The set of all sections of $\xi$ is denoted $\Gamma(\xi)$, or $\Gamma(B, E)$ when the map $\pi$ is understood.

**Definition.** When $E = TM$ is the tangent bundle over a manifold $M$, a section $s : M \to TM$ is instead called a smooth tangent vector field on $M$.

**Example 2.2.1.** The trivial section $s_0 : B \to E, p \mapsto 0 \in \pi^{-1}(p)$ is smooth for any vector bundle $\pi : E \to B$. In particular, sections exist for every vector bundle over $B$. One can show that the space of sections for any bundle of nonzero dimension is uncountable. The existence of a zero section is an important characteristic of vector bundles which does not generalize to fibre bundles. What’s more, any smooth vector bundle $E \xrightarrow{\pi} B$ is a homotopy equivalence between $E$ and $B$ – the image of $B$ under the 0 section is a strong deformation retract of $E$.

**Example 2.2.2.** The sections of the trivial bundle $B \times \mathbb{R}^n \to B$ are precisely all of the smooth functions $B \to \mathbb{R}^n$, that is, $\Gamma(B, B \times \mathbb{R}^n) = C^\infty(B, \mathbb{R}^n)$.

An interesting question is: given a bundle $\xi$, does there exist a nonvanishing section, i.e. a section $s : B \to E$ for which $s(x) \neq 0$ for all $x \in B$? More generally:

**Definition.** Given an $n$-bundle $\xi$ and a number $1 \leq k \leq n$, we say sections $s_1, \ldots, s_k \in \Gamma(\xi)$ are nowhere linearly dependent, or simply linearly independent sections, if for every $x \in B$, $s_1(x), \ldots, s_k(x)$ are linearly independent in $\pi^{-1}(x)$.

Then nowhere vanishing is the same as nowhere linearly dependent for $k = 1$. The existence of $k$ linearly independent sections of a vector bundle is an important one, and one that can be answered with the information encoded in Stiefel-Whitney classes, to be introduced later.

**Example 2.2.3.** There is always a nonvanishing section on any trivial bundle $B \times \mathbb{R}^n \to B$, for $n \geq 1$, since for a fixed nonzero vector $v \in \mathbb{R}^n$, $s(b) = (b, v)$ is smooth.

We have the following analogue of Lemma 1.2.12.
Lemma 2.2.4. Suppose \( f : E_1 \to E_2 \) is a smooth map of vector bundles over \( B \) that makes the following diagram commute:

\[
\begin{array}{ccc}
E_1 & \xrightarrow{f} & E_2 \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
B
\end{array}
\]

Then if \( f \) is an isomorphism on each fibre, it is a bundle isomorphism.

Proof. Fix \( p \in B \) and take local trivializations \( U_1, U_2 \subseteq B \) of \( E_1, E_2 \) about \( p \). Then both bundles are trivial over \( U_1 \cap U_2 \), i.e. there exist diffeomorphisms \( h_1 : (U_1 \cap U_2) \times \mathbb{R}^n \to \pi_1^{-1}(U_1 \cap U_2) \) and \( h_2 : (U_1 \cap U_2) \times \mathbb{R}^n \to \pi_2^{-1}(U_1 \cap U_2) \) commuting with the projections. Consider the diagram:

\[
\begin{array}{ccc}
(U_1 \cap U_2) \times \mathbb{R}^n & \xrightarrow{h_1} & \pi_1^{-1}(U_1 \cap U_2) \\
\downarrow{\pi_1} & \xrightarrow{f} & \pi_2^{-1}(U_1 \cap U_2) \xleftarrow{h_2} & (U_1 \cap U_2) \times \mathbb{R}^n \\
U_1 \cap U_2 \subseteq B
\end{array}
\]

Define the composition \( h = h_2^{-1} \circ f \circ h_1 : (U_1 \cap U_2) \times \mathbb{R}^n \to (U_1 \cap U_2) \times \mathbb{R}^n \). Then by hypothesis, \( h \) is smooth, commutes with the projection map and is a vector space isomorphism on each fibre \( \{x\} \times \mathbb{R}^n \) for \( x \in U_1 \cap U_2 \). Fixing such an \( x \), this map is given by \( h(x, v) = (x, M(x) \cdot v) \) for some matrix \( M(x) \in M_n(\mathbb{R}) \), whose entries vary smoothly in terms of \( x \). Moreover, since \( h \) is an isomorphism on fibres, \( \det M(x) \neq 0 \) and there is even a formula for \( M(x)^{-1} \) which is also smooth in each entry in terms of \( x \). Hence \( h \) is a diffeomorphism, so \( f \) is as well. \( \square \)

We have the following analogue of Lemma 1.2.4.

Theorem 2.2.5. An \( n \)-dimensional vector bundle is trivial if and only if it admits \( n \) linearly independent sections.

Proof. ( \( \Rightarrow \) ) The trivial bundle \( B \times \mathbb{R}^n \to B \) admits sections \( s_i : B \to B \times \mathbb{R}^n, x \mapsto (x, e_i) \), where \( e_i \) is the \( i \)th standard basis vector in \( \mathbb{R}^n \). These are clearly linearly independent at every point in the base. For any trivial bundle \( E \xrightarrow{\pi} B \) with bundle isomorphism \( \varphi : B \times \mathbb{R}^n \to E \), set \( t_i = s_i \circ \varphi : B \to E \). These are \( n \) linearly independent sections, since

\[
\begin{array}{ccc}
B \times \mathbb{R}^n & \xrightarrow{\varphi} & E \\
\downarrow{\pi} & & \\
B
\end{array}
\]

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commutes and \( \varphi \) is an isomorphism on each fibre.

(\( \iff \)) Conversely, suppose \( E \xrightarrow{\pi} B \) is an \( n \)-dimensional bundle with linearly independent sections \( s_1, \ldots, s_n \). Define a map

\[
\psi : B \times \mathbb{R}^n \rightarrow E
\]

\[
(x, v_1, \ldots, v_n) \mapsto \sum_{i=1}^{n} v_i s_i(x).
\]

Clearly \( \psi \) is a vector space isomorphism on each fibre of the bundle, since the \( s_i(x) \) are linearly independent, and the following diagram commutes:

\[
\begin{array}{ccc}
B \times \mathbb{R}^n & \xrightarrow{\psi} & E \\
\downarrow \rho & & \downarrow \pi \\
B & & B
\end{array}
\]

Thus by Lemma 2.2.4, \( \psi \) is a bundle isomorphism, so \( E \) is trivial.

**Example 2.2.6.** (Parallelizable spheres) Consider \( S^m \) for \( m \geq 1 \). It turns out that only a select few spheres are parallelizable. For instance, \( S^1 \) has a nonvanishing vector field:

\[
F(x, y) = (-y, x)
\]

Thus by Theorem 2.2.5, \( S^1 \) is parallelizable (see to Example 2.1.2). However, the hairy ball theorem states that \( S^2 \) has no nonvanishing tangent vector fields, so \( S^2 \) is not parallelizable. This is related to the fact that \( \chi(S^2) = 2 \) and the Euler class of the tangent bundle \( TS^2 \rightarrow S^2 \) is nontrivial. A similar proof shows that all even-dimensional spheres are not parallelizable.

This suggests only look at odd-dimensional spheres to find nonvanishing sections, since any such sphere has Euler characteristic 0. For example, consider \( S^3 \). As a subset of \( \mathbb{R}^4 \), this is given by: \( S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\} \). To show \( S^3 \) is parallelizable, we exhibit three linearly independent sections:

\[
\begin{align*}
s_1(x_1, x_2, x_3, x_4) &= (-x_2, x_1, -x_4, x_3) \\
s_2(x_1, x_2, x_3, x_4) &= (-x_3, -x_4, x_1, x_2) \\
s_3(x_1, x_2, x_3, x_4) &= (-x_4, -x_3, x_2, x_1).
\end{align*}
\]
One easily checks that these are smooth and linearly independent. Therefore by Theorem 2.2.5, $S^3$ is parallelizable. Unfortunately, this technique does not work for odd higher dimensional spheres – in fact, $S^7$ is the only other parallelizable sphere!

**Example 2.2.7.** Here we prove that the tautological line bundle $\gamma^n_1$ over $\mathbb{R}P^n$ is nontrivial for all $n$. We have shown this already for $\mathbb{R}P^1$ (see Example 2.1.3), but the general case is straightforward. By Theorem 2.2.5, it suffices to show there is no nonvanishing section of $\gamma^n_1$. Suppose $s : \mathbb{R}P^n \to E$ is such a section. Composing with the double cover $p : S^n \to \mathbb{R}P^n$ gives a smooth map $s \circ p : S^n \to \mathbb{R}P^n \times \mathbb{R}^{n+1}$. Now a map is smooth if and only if each component is smooth, so we have

$$s \circ p(x) = (\bar{x}, t(x))$$

for some smooth function $t(x)$. If $s$ is nonvanishing, $t(x) \neq 0$ for all $x \in S^n$. However, since $p(-x) = p(x) = \bar{x}$, we have $(\bar{x}, -tx) = s \circ p(-x) = s \circ p(x) = (\bar{x}, t(x))$, and so $t(x) = -t(x)$ which is only possible if $t(x) = 0$. Therefore no nonvanishing section exists, so $\gamma^n_1$ is nontrivial.

### 2.3 The Induced Bundle

**Definition.** A bundle map (or a morphism of (vector) bundles) is a pair of smooth maps between vector bundles of the same dimension

\[
\begin{array}{ccc}
E_1 & \xrightarrow{f} & E_2 \\
\pi_1 & \downarrow & \pi_2 \\
B_1 & \xrightarrow{\bar{f}} & B_2
\end{array}
\]

making the diagram commute, and such that $f$ maps the fibres of $\pi_1$ isomorphically to the fibres of $\pi_2$.

**Example 2.3.1.** If $\bar{f} : B \to B$ is the identity, then by Lemma 2.2.4, any bundle map $f : E_1 \to E_2$ over $B$ must be a bundle isomorphism.

**Definition.** Let $\xi : E \xrightarrow{\pi} B$ be a vector bundle and suppose $f : B' \to B$ is a smooth map. Then the induced bundle of $\xi$ by $f$ is a bundle $f^*\xi : E' \xrightarrow{\pi'} B'$ consisting of total space

$$E' = \{(x, y) \in B' \times E \mid f(x) = \pi(y)\}$$

and projection $\pi'(x, y) = x$.

**Proposition 2.3.2.** For any $\xi : E \xrightarrow{\pi} B$ and $f : B' \to B$, the induced bundle $f^*\xi$ is a vector bundle over $B'$.

**Proof.** Fix $x \in B'$. Then

$$\pi'^{-1}(x) = \{(x, y) \in E' \mid \pi(y) = f(x)\} \cong \{y \in E \mid \pi(y) = f(x)\} = \pi^{-1}(f(x))$$
which is a vector space by hypothesis. Hence each fibre of $f^*\xi$ is a vector space. To show $f^*\xi$ is a bundle, we must find a neighborhood $U' \subseteq B'$ over which $f^*\xi$ is trivial. For $f(x) \in B$, we know there is a neighborhood $U \subseteq B$ and a diffeomorphism $h : \pi^{-1}(U) \to U \times \mathbb{R}^n$ which is a bundle isomorphism over $U$. Set $U' = f^{-1}(U)$, which is an open neighborhood of $x \in B'$. Then $(\pi')^{-1}(U') = \{(x, y) \in E' \mid x \in U', y \in \pi^{-1}(U)\}$ so the map

$$h' : (\pi')^{-1}(U') \to U' \times \mathbb{R}^n$$

$$(x, y) \mapsto (x, p(h^{-1}(y)))$$

is well-defined, where $p : U \times \mathbb{R}^n \to \mathbb{R}^n$ is projection. By construction, $h'$ is a bundle isomorphism over $U'$ (there is even an easy formula for the inverse), so $U'$ is a trivialization of $f^*\xi$ over $x \in B'$.

**Example 2.3.3.** If $\xi : E \to B$ is a trivial bundle, then any pullback $f^*\xi$ along $f : B' \to B$ is a trivial bundle over $B'$. Indeed, the triviality of $\xi$ means we can pick $U = B$ in the proof of Proposition 2.3.2, and then $U' = B'$, so the entire bundle is trivial.

**Example 2.3.4.** Every trivial bundle is a pullback:

$$\begin{array}{ccc}
B \times \mathbb{R}^n & \to & \{x\} \times \mathbb{R}^n \\
\xi & \downarrow & \xi_0 \\
B & \xrightarrow{f} & \{x\}
\end{array}$$

Let $f : B \to \{x\}$ be the unique map from $B$ to a point space and $\xi_0$ the $n$-bundle over a point. Then $f^*\xi_0 = \xi$.

**Lemma 2.3.5.** Suppose $\xi : E \xrightarrow{\pi} B$ and $\eta : E' \xrightarrow{\pi'} B'$ are vector bundles and

$$\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\pi_1 & \downarrow & \pi_2 \\
B & \xrightarrow{g} & B'
\end{array}$$

is a bundle map. Then $\eta$ is isomorphic to the induced bundle $g^*\xi$.

**Proof.** By Lemma 2.2.4, it suffices to construct a bundle map $\eta \to g^*\xi$. Let $g^*\xi : E'' \xrightarrow{\pi''} B'$ be the induced bundle and define a map $\varphi : E' \to E''$ by $\varphi(y) = (\pi'(y), f(y))$. By definition of the induced total space, $\varphi(y) \in E''$. It is immediate that $\varphi$ makes the diagram

$$\begin{array}{ccc}
E' & \xrightarrow{\varphi} & E'' \\
\pi' & \downarrow \varphi & \pi'' \\
B' & \xrightarrow{\pi''} & B''
\end{array}$$

commute and is an isomorphism on fibres. Therefore $\varphi$ is a bundle map. \qed
2.4 Bundle Operations

Definition. Given two bundles $\xi_1 : E_1 \xrightarrow{\pi_1} B_1$ and $\xi_2 : E_2 \xrightarrow{\pi_2} B_2$, one forms the (Cartesian) product bundle $\xi_1 \times \xi_2 : E_1 \times E_2 \xrightarrow{\pi_1 \times \pi_2} B_1 \times B_2$. Here, for a point $p = (p_1, p_2)$, the fibre is $(\pi_1 \times \pi_2)^{-1}(p) = \pi_1^{-1}(p_1) \times \pi_2^{-1}(p_2) \cong \pi_1^{-1}(p_1) \oplus \pi_2^{-1}(p_2)$.

Lemma 2.4.1. Given two manifolds $M$ and $N$, the tangent bundle $T(M \times N)$ is isomorphic to $TM \times TN$ as vector bundles.

Example 2.4.2. Consider the torus $X \cong S^1 \times S^1$. The tangent bundle to the torus is a product bundle $TX = TS^1 \times TS^1$. Since $S^1$ is parallelizable, $TS^1$ is trivial and therefore so is $TS^1 \times TS^1$, so the torus is also parallelizable.

We would like to define a product of bundles over a common base $B$ that yields a bundle over $B$. For this, we utilize the induced bundle construction from Section 2.3.

Definition. Given two bundles $\xi_1 : E_1 \xrightarrow{\pi_1} B$ and $\xi_2 : E_2 \xrightarrow{\pi_2} B$ over the same base $B$, the direct sum, or Whitney sum of $\xi_1$ and $\xi_2$, denoted $\xi_1 \oplus \xi_2$, is the induced bundle of the product bundle over the diagonal map $d : B \to B \times B$, that is,

$$E_1 \oplus E_2 = d^*(E_1 \times E_2) \xrightarrow{\pi_1 \oplus \pi_2} E_1 \times E_2$$

Categorically, there are many ways of obtaining new vector spaces from old ones using functors. Examples of functors on vector spaces include: direct sum ($\oplus$), tensor product ($\otimes$), exterior power ($\wedge^k$) and the vector space dual ($V^*$), itself a special case ($V^* = \text{Hom}(V, \mathbb{R})$) of the Hom functor ($\text{Hom}(V, W)$). It turns out that these examples are smooth functors. It is a general principle that any smooth functor of vector spaces determines an analogous construction of vector bundles.

It is useful to expand upon our definition of vector bundles in terms of transition maps in order to construct further bundle operations. The basic idea is that transition maps arise from overlapping charts. Specifically, let $\{U_i\}$ be a locally trivial covering of $B$ with respect to some bundle $\xi : E \xrightarrow{\pi} B$ and let $h_i : (U_i \times \mathbb{R}^n) \to \pi^{-1}(U_i)$ be the local trivializations. Then for each pair of overlapping charts $U_i \cap U_j \neq \emptyset$, there is a homeomorphism $(U \cap V) \times \mathbb{R}^n \to (U \cap V) \times \mathbb{R}^n$, which reduces to a vector space isomorphism on each fibre $\{x\} \times \mathbb{R}^n$, induced by the diagram

$$
\begin{align*}
(U_i \cap U_j) \times \mathbb{R}^n &\xrightarrow{h_i} \pi^{-1}(U_i \cap U_j) \\
&\xrightarrow{h_j} (U_i \cap U_j) \times \mathbb{R}^n
\end{align*}
$$
This defines a smooth map \( h_{ij} : U_i \cap U_j \to GL_n(\mathbb{R}) \) with inverse \( h_{ji} \), by which we mean for each \( x \in U_i \cap U_j \), \( h_{ji}(x) = h_{ij}(x)^{-1} \).

Conversely, given a collection of charts \( \{U_i\} \) and smooth maps on the intersections \( h_{ij} : U_i \cap U_j \to GL_n(\mathbb{R}) \), we can reconstruct the original bundle \( E \xrightarrow{\pi} B \). Explicitly, set
\[
E = \coprod_i (U_i \times \mathbb{R}^n) / \sim
\]
where \( \sim \) is the equivalence relation \((p, \vec{x}) \sim (p, h_{ij}(\vec{x}))\) for \((p, \vec{x}) \in (U_i \cap U_j) \times \mathbb{R}^n \subseteq U_i \times \mathbb{R}^n\).

The projection \( \pi \) is induced by \( U_i \times \mathbb{R}^n \to U_i \) for each chart \( U_i \). Local triviality is guaranteed by construction, so \( E \xrightarrow{\pi} B \) is the desired bundle.

There is a subtlety in this construction. For three locally trivial charts \( U_1, U_2 \) and \( U_3 \) of a vector bundle with \( U_1 \cap U_2 \cap U_3 \neq \emptyset \), we must have \( h_{31}h_{23}h_{12} = 1 \) in \( GL_n(\mathbb{R}) \). This comes from the fact that on \( U_1 \cap U_2 \cap U_3 \), \((h_3^{-1} \circ h_2) \circ (h_1^{-1} \circ h_2) = \text{id} \). However, such a condition is not necessarily guaranteed when constructing a bundle from abstract transition maps. We will implicitly assume the condition is always met.

**Example 2.4.3.** Consider the tautological line bundle \( \gamma_1 : E \to \mathbb{R}P^1 = S^1 \). Let’s instead view this bundle as a family of maps into \( GL_1(\mathbb{R}) = \mathbb{R} \setminus \{0\} = \mathbb{R}^* \). Take two overlapping charts \( U \) and \( V \):

Then \( \pi^{-1}(U) \cong U \times \mathbb{R} \) and \( \pi^{-1}(V) \cong V \times \mathbb{R} \), and \( U \cap V = A \cup B \), the union of two disjoint open intervals on the circle. We can define the map \( h_{UV} : U \cap V \to \mathbb{R}^* \) by our choices of sending \( A \) and \( B \) each to either \( \mathbb{R}_{>0} \) or \( \mathbb{R}_{<0} \). Thus the equivalence classes of line bundles over \( S^1 \) are determined by whether the maps \( h_{UV} \) send the disjoint pieces of \( U \cap V \) into the same connected component of \( \mathbb{R}^* \) or the opposite components. Hence the only line bundles over the circle are, up to isomorphism, the tautological and trivial bundles.

**Definition.** A **structure group** for a vector bundle \( \xi \) with fibres \( V \) is a subgroup \( G \) of \( GL(V) \) such that all transition maps of \( \xi \) take values in \( G \).

**Example 2.4.4.** Let \( E \xrightarrow{\pi} S^2 \) be a two-dimensional bundle over the 2-sphere. Consider an open covering \( U \cup V = S^2 \) with \( U \) as the northern hemisphere and \( V \) as the southern hemisphere.
Then $U \cong D^2, V \cong D^2$ and $U \cap V$ is homotopy equivalent to the circle $S^1$. The total space of any bundle may be constructed by $E = (U \times \mathbb{R}^2) \bigsqcup (V \times \mathbb{R}^2)/ \sim$ where the gluing is induced by the single transition map $h_{UV} : S^1 \to GL_2(\mathbb{R})$. In fact, the special orthogonal group $SO(2) \subset GL_2(\mathbb{R})$ is a structure group for any 2-bundle over $S^2$. Identifying $SO(2)$ with $S^1 \subset \mathbb{R}^2$, any transition map may be viewed as a map $h : S^1 \to S^1$. Of course, up to homotopy, any such map can be identified with an integer $n$, its winding number. For each $n \in \mathbb{Z}$, let

$$h_n(x) = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$$

be the corresponding transition map $h_n : S^1 \to S^1 = SO(2)$. We will show that these $h_n$ give non-isomorphic 2-bundles over $S^2$.

To construct new bundles from old ones, suppose $\xi$ and $\eta$ are vector bundles over $B$, of respective dimensions $m$ and $n$, with some specified transition maps. By refining the charts of each bundle if necessary, we may assume there is a collection of charts $\{U\}$ over which both bundles are trivial. Denote the transition maps for $\xi$ by $g_{UV} : U \cap V \to GL_m(\mathbb{R})$ and for $\eta$ by $h_{UV} : U \cap V \to GL_n(\mathbb{R})$.

Now any functorial operation on vector spaces defines an operation on vector bundles. Explicitly, if $\mathcal{F}$ is a functor on the category of pairs of vector spaces $(W_1, W_2)$, then for any bundles $\xi$ with fibres $W_1$ and $\eta$ with fibres $W_2$, there is a vector bundle $\mathcal{F}(\xi, \eta)$ induced by the transition maps

$$\mathcal{F}(g_{UV}, h_{UV}) : U \cap V \to GL(\mathcal{F}(W_1, W_2)).$$

**Example 2.4.5.** The Whitney sum $\xi \oplus \eta$ over a base $B$ may be constructed from the direct sum functor, with transition maps

$$(g \oplus h)_{UV} := g_{UV} \oplus h_{UV} : U \cap V \to GL_{m+n}(\mathbb{R}).$$

It is immediate that these maps are smooth, so they indeed define a bundle. One can show that this corresponds with the Whitney sum construction defined previously.

**Definition.** The tensor bundle of $\xi$ and $\eta$ over $B$ is the bundle $\xi \otimes \eta$ induced by the tensor product functor, given by transition maps

$$(g \otimes h)_{UV} := g_{UV} \otimes h_{UV} : U \cap V \to GL_{mn}(\mathbb{R}).$$

**Definition.** The dual bundle of a vector bundle $\xi$ is the bundle $\xi^*$ induced by the dual functor $(-)^* = \text{Hom}(-, \mathbb{R})$, using the matrix transpose as transition maps:

$$g_{UV}^* := g_{UV}^t : U \cap V \to GL_n(\mathbb{R}).$$
More generally, we define:

**Definition.** The Hom bundle $\text{Hom}_B(\xi, \eta)$ between two vector bundles $\xi$ and $\eta$ over $B$ is the bundle induced by the Hom functor $\text{Hom}(W_1, W_2) \cong W_1^* \otimes W_2$, defined explicitly by the transition maps

$$\text{Hom}(g, h)_{UV} := g_{UV}^t \otimes h_{UV} : U \cap V \rightarrow GL_{mn}(\mathbb{R}).$$

In other words, $\text{Hom}_B(\xi, \eta) = \xi^* \otimes \eta$, where $\xi^*$ is the dual bundle.

**Definition.** Let $\xi$ be an $n$-dimensional vector bundle over $B$ and $k$ any nonnegative integer. Then the exterior power functor $\wedge^k$ defines the $k$th exterior power bundle $\wedge^k \xi$ which is given by transition maps

$$\wedge^k g_{UV} : U \cap V \rightarrow GL(\frac{n}{k})(\mathbb{R}).$$

**Example 2.4.6.** If $\xi$ and $\eta$ are line bundles over $B$, then $\xi \otimes \eta$ is again a line bundle over $B$. This in fact defines a group structure on the set of equivalence classes of line bundles over $B$.

Take for example the two line bundles over $S^1$: the trivial bundle $\xi_0$ and the tautological bundle $\gamma_1^1$. Let $S^1$ be covered by $\{U, V\}$ as in Example 2.4.3. Then the tensor product on transition maps $g_{UV} \otimes h_{UV} : U \cap V \rightarrow GL_1(\mathbb{R}) = \mathbb{R}^*$ is just given by scalar multiplication:

$$(g_{UV} \otimes h_{UV})(x) = g_{UV}(x)h_{UV}(x).$$

It follows that

$$\xi_0 \otimes \xi_0 \cong \xi_0, \quad \xi_0 \otimes \gamma_1^1 \cong \gamma_1^1, \quad \gamma_1^1 \otimes \xi_0 \cong \gamma_1^1 \quad \text{and} \quad \gamma_1^1 \otimes \gamma_1^1 \cong \xi_0.$$

Hence the group of line bundles over $S^1$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. 

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3 Characteristic Classes

3.1 Metrics on Vector Bundles

There is a natural Euclidean metric on any finite dimensional vector space \( V \) given by fixing an isomorphism \( \varphi : V \to \mathbb{R}^n \) and setting \( \langle v, w \rangle = \sum_{i=1}^n \varphi(v)_i \varphi(w)_i \), where \( x_i \) denotes the \( i \)th component of a vector \( x \in \mathbb{R}^n \). To extend the Euclidean metric over the fibres of a vector bundle, we need a slightly different perspective.

**Proposition 3.1.1.** For a finite dimensional vector space \( V \), there is a bijective correspondence between inner products \( \langle \cdot, \cdot \rangle \) and (positive definite) quadratic forms \( \mu : V \to \mathbb{R} \).

**Proof.** For an inner product \( \langle \cdot, \cdot \rangle \) on \( V \), a quadratic form is given by \( \mu(v) := \langle v, v \rangle \). Conversely, a quadratic form \( \mu \) determines an inner product \( \langle v, w \rangle := \frac{1}{2} (\mu(v+w) - \mu(v) - \mu(w)) \). One now checks that these assignments are inverses. \( \square \)

**Definition.** A Euclidean metric on a vector bundle \( \xi : E \to B \) is a smooth map \( \mu : E \to \mathbb{R} \) such that \( \mu \) gives a positive definite quadratic form on each fibre \( \pi^{-1}(x) \).

**Corollary 3.1.2.** A Euclidean metric on a vector bundle determines an inner product on every fibre.

**Example 3.1.3.** A metric on the tangent bundle \( TM \) to a manifold \( M \) is equivalent to a choice of a Riemannian metric on \( M \).

**Proposition 3.1.4.** Every vector bundle has a Euclidean metric.

**Proof.** Let \( \xi : E \to B \) be a vector bundle bundle. Fix a locally trivial covering \( \{ U_i \} \) of \( B \) such that there is an isomorphism \( h_i \) making

\[
\begin{array}{ccc}
U_i \times \mathbb{R}^n & \xrightarrow{h_i} & \pi^{-1}(U_i) \\
\downarrow & & \downarrow \\
U_i & \xrightarrow{\pi} & B
\end{array}
\]

commute for each \( U_i \). This determines a partition of unity \( \{ \rho_i \} \) of \( B \), i.e. a collection of maps \( \rho_i : B \to \mathbb{R}_{>0} \) such that \( \rho_i \) is supported on \( U_i \) and \( \sum_i \rho_i \equiv 1 \) on \( B \). For each \( U_i \), put the product quadratic form on \( U_i \times \mathbb{R}^n \); that is, set

\[
\mu_i'(x, \bar{v}) = \sum_{j=1}^n v_j^2
\]

for any \((x, \bar{v}) \in U_i \times \mathbb{R}^n\). This determines a quadratic form on each fibre given by

\[
\mu_i = h_i \circ \mu'_i : \pi^{-1}(x) \to \mathbb{R}_{>0}
\]
3.2 Stiefel-Whitney Classes

Characteristic Classes

when \( x \in U_i \). If \( x \not\in U_i \), set \( \mu_i(x) = 0 \). We extend this to a map on the entire total space by

\[
\mu : E \longrightarrow \mathbb{R}_{>0}
\]

\[
x \mapsto \sum_i \rho_i(\pi(x))\mu_i(x)
\]

It is immediate that \( \mu \) is a positive definite quadratic form on each fibre \( \pi^{-1}(x) \), and so defines a Euclidean metric on \( \xi \).

**Definition.** A subbundle of a vector bundle \( \eta : E_1 \xrightarrow{\pi_1} B \) is a bundle \( \xi : E_2 \xrightarrow{\pi_2} B \) such that each fibre \( \pi_2^{-1}(x) \subseteq \pi_1^{-1}(x) \) is a vector subspace, and the induced diagram commutes:

\[
\begin{array}{ccc}
E_1 & \xleftarrow{\pi_1} & E_2 \\
\downarrow & & \downarrow \\
B & \xrightarrow{\pi_2} & B
\end{array}
\]

This is denoted \( \xi \subseteq \eta \).

**Definition.** Let \( \xi : E_2 \xrightarrow{\pi_2} B \) be a subbundle of \( \eta : E_1 \xrightarrow{\pi_1} B \). Then the orthogonal complement of \( \xi \) in \( \eta \) with respect to a Euclidean metric \( \mu \) on \( \eta \) is the bundle \( \xi^\perp \) whose fibres are \( \pi_2^{-1}(x)^\perp \), the orthogonal complement of \( \pi_1^{-1}(x) \) in \( \pi_2^{-1}(x) \), for each \( x \in B \).

**Theorem 3.1.5.** Given a subbundle \( \xi \subseteq \eta \), the orthogonal complement \( \xi^\perp \) is a subbundle of \( \eta \). Further, \( \xi \oplus \xi^\perp \cong \eta \) as bundles.

**Definition.** Let \( M^m \) be a submanifold of \( N^n \). Then the normal bundle of \( M \) in \( N \) is the subbundle orthogonal to \( TM \) in \( TN|_M \):

\[
\mathcal{V}_M^n := (TM)^\perp \subseteq TN|_M.
\]

**Corollary 3.1.6.** Let \( M \) be a submanifold of \( \mathbb{R}^n \). Then \( TM \oplus \mathcal{V}_M^\mathbb{R} \cong (\mathbb{R}^n \times \mathbb{R}^n)|_M \), the trivial \( n \)-bundle over \( \mathbb{R}^n \) restricted to \( M \).

3.2 Stiefel-Whitney Classes

Given a vector bundle \( \xi \) over \( B \), there is an assignment of cohomology classes in each \( H^i(B; \mathbb{Z}/2\mathbb{Z}) \) satisfying a set of axioms which uniquely characterize these so-called Stiefel-Whitney classes.

**Definition.** A set of Stiefel-Whitney classes for a vector bundle \( \xi \) over \( B \) is a choice of \( \omega_i(\xi) \in H^i(B; \mathbb{Z}/2\mathbb{Z}) \) for each \( i \geq 0 \) and a total Stiefel-Whitney class \( \omega(\xi) = \omega_0 + \omega_1 + \omega_2 + \ldots \) which satisfy:

1. If the dimension of the fibres of \( \xi \) is \( n \), then \( \omega_i(\xi) = 0 \) for all \( i > n \).
2. \( \omega_0(\xi) = 1 \) in \( H^0(B; \mathbb{Z}/2\mathbb{Z}) \).
(2) The \( \omega_i \) are natural with respect to pullbacks; that is, for any map \( f : B' \to B \), \( \omega_i(f^*\xi) = f^*\omega_i(\xi) \).

(3) For any other bundle \( \eta \) over \( B \),

\[
\omega(\xi \oplus \eta) = \omega(\xi) \cdot \omega(\eta) = \sum_{i=0}^{\infty} \prod_{j+k=i} \omega_j(\xi)\omega_k(\eta).
\]

(4) For the tautological bundle \( \gamma_1^1 \) over \( \mathbb{R}P^1 \), \( \omega_1(\gamma_1^1) \neq 0 \) in \( H^1(\mathbb{R}P^1; \mathbb{Z}/2\mathbb{Z}) \).

Axiom 4 is necessary to rule out the assignment \( \omega_i = 0 \) for each \( i \), as we want to study a nontrivial set of cohomology classes. In Sections 3.3 and 3.4, we will construct the Stiefel-Whitney classes for any vector bundle \( \xi \) explicitly and prove that the axioms uniquely determine these.

**Remark.** By Axiom 3, \( \omega(\xi_1 \oplus \xi_2) = \omega(\xi_1)\omega(\xi_2) \) for any bundles \( \xi_1, \xi_2 \) but by definition of the Whitney sum, \( \xi_1 \oplus \xi_2 = d^*(\xi_1 \times \xi_2) \) where \( \xi_1 \times \xi_2 \) is the product bundle. Thus \( \omega(\xi_1)\omega(\xi_2) = d^*\omega(\xi_1 \times \xi_2) \). Consider the canonical projections

\[
\begin{array}{ccc}
B_1 & \xrightarrow{p_1} & B_1 \\
\downarrow & & \downarrow \\
B_1 \times B_2 & \xrightarrow{p_2} & B_2
\end{array}
\]

Then \( p_1^*\xi_1 \oplus p_2^*\xi_2 = d^*(p_1^*\xi_1 \times p_2^*\xi_2) \cong \xi_1 \times \xi_2 \) and thus

\[
\omega(\xi_1 \times \xi_2) = p_1^*\omega(\xi_1)p_2^*\omega(\xi_2) = \omega(\xi_1) \times \omega(\xi_2),
\]

where \( \times \) is the algebraic cross product.

**Example 3.2.1.** Let \( \xi_0 \) be the trivial bundle \( M \times \mathbb{R}^n \to M \) over any manifold \( M \). By Example 2.3.4, \( \xi_0 \) is a pullback of a bundle over a point space, so it follows from Axiom 2 that \( \omega(\xi_0) = 1 \).

For any manifold \( M \), we will write \( \omega(M) = \omega(TM) \).

**Proposition 3.2.2.** For any submanifold \( M \subseteq \mathbb{R}^n \), \( \omega(M)\omega(V_M^\mathbb{R}^n) = 1 \).

**Proof.** By Corollary 3.1.6 and Axiom 3, \( \omega(M)\omega(V_M^\mathbb{R}^n) = \omega(M \times \mathbb{R}^n) = 1 \).

**Example 3.2.3.** View \( S^n \subset \mathbb{R}^{n+1} \) and consider the normal bundle \( V_M^\mathbb{R}^{n+1} \). Then the function \( x \mapsto \bar{x} \) is a nonvanishing section of \( V_M^\mathbb{R}^{n+1} \), so by Theorem 2.2.5, the normal bundle over \( S^n \) is trivial. Thus by Axiom 3 and Proposition 3.2.2,

\[
1 = \omega(TS^n)\omega(V_M^\mathbb{R}^{n+1}) = \omega(TS^n) \cdot 1 = \omega(TS^n).
\]

Therefore we have proven that the total Stiefel-Whitney class of any sphere is trivial.
Let \( \omega = \omega(\xi) \) be the total Stiefel-Whitney class of a bundle \( \xi \) over \( B \). By Axiom 1, \( \omega_0 = 1 \) so we can write
\[
\omega = 1 + \omega_1 + \omega_2 + \ldots
\]
We can view this sum as a formal power series \( \omega = 1 + x \), where \( x = \omega_1 + \omega_2 + \ldots \), so that there exists a formal inverse
\[
\bar{\omega} := \omega^{-1} = 1 - x + x^2 - \ldots = 1 - (\omega_1 + \omega_2 + \ldots) + (\omega_1 + \omega_2 + \ldots)^2 + \ldots
\]
Alternatively, there is a recursive formula for the \( \bar{\omega}_i \) in \( \bar{\omega} = \bar{\omega}_0 + \bar{\omega}_1 + \bar{\omega}_2 + \ldots \) coming from the relation \( \omega \cdot \bar{\omega} = 1 \):
\[
\begin{align*}
\bar{\omega}_0 &= 1 \\
\bar{\omega}_i &= \sum_{j, k > 0} \omega_j \bar{\omega}_k \quad \text{for } i \geq 1.
\end{align*}
\]
(Here, we are using the fact that the \( \omega_i \) are elements of \( \mathbb{Z}/2\mathbb{Z} \) to eliminate signs everywhere.)

**Example 3.2.4.** For any manifold \( M \subseteq \mathbb{R}^n \), \( \omega(M) = \omega(\mathcal{V}_M^{\mathbb{R}^n}) \).

**Example 3.2.5.** We will compute the Stiefel-Whitney classes for \( \mathbb{R}P^n \) using our knowledge of \( \omega(TS^n) \). Recall that the cohomology ring of \( \mathbb{R}P^n \) with \( \mathbb{Z}/2\mathbb{Z} \) coefficients is a truncated polynomial ring
\[
H^\bullet(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{F}_2[a]/(a^{n+1})
\]
where \( a \) is an indeterminate. By definition, the tautological bundle \( \gamma^1_n \) is a subbundle of the trivial bundle \( \xi_0^{n+1} \) on \( \mathbb{R}P^n \). Let \( \gamma^\perp \) denote the orthogonal complement of \( \gamma^1_n \) in the trivial bundle. We are interested in computing the Stiefel-Whitney class \( \omega = \omega(\gamma^1_n) \). By Axiom 0, \( \omega_i = 0 \) for all \( i > \dim \gamma^1_n = 1 \), so we have \( \omega = 1 + \omega_1 \). Viewing \( \mathbb{R}P^1 \subseteq \mathbb{R}P^n \), we actually have a bundle map
\[
E(\gamma^1_n) \longrightarrow E(\gamma^1_n)
\]
\[
\mathbb{R}P^1 \xleftarrow{i} \mathbb{R}P^n
\]
which is an isomorphism on each fibre (i.e. a bundle morphism). By Lemma 2.3.5, \( \gamma^1_n \cong i^*(\gamma^1_n) \), the induced bundle. Then by Axiom 2, \( i^*\omega_1(\gamma^1_n) = \omega_1(\gamma^1_1) \) but by Axiom 4, \( \omega_1(\gamma^1_1) \neq 0 \). Thus \( i^*\omega_1(\gamma^1_n) \neq 0 \), so \( \omega_1(\gamma^1_n) \neq 0 \). Hence we deduce that \( \omega(\gamma^1_n) = 1 + a \) where \( a \) is a generator of \( H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) \). Next, by the formulas for inverses of Stiefel-Whitney classes, we have
\[
\omega(\gamma^\perp) = \overline{\omega(\gamma^1_n)} = (1 + a)^{-1}
\]
\[
= 1 - a + a^2 - \ldots \pm a^n
\]
\[
= 1 + a + a^2 + \ldots + a^n \quad \text{using } \mathbb{Z}/2\mathbb{Z}-\text{coefficients.}
\]

**Lemma 3.2.6.** There is a bundle isomorphism \( \mathbb{T}\mathbb{R}P^n \cong \text{Hom}(\gamma^1_n, \gamma^\perp) \).
Proof. View $\mathbb{R}P^n$ as the quotient of $S^n$ by the antipodal action $\sim$. For any $x \in S^n$, a tangent vector $\vec{v} \in T_xS^n$ is of the form $\vec{v} = \alpha'(0)$ for some curve $\alpha(t)$ in $S^n$ with $\alpha(0) = x$. Passing to the quotient $S^n/\sim$, $\vec{v} \in T_xS^n$ is identified with $-\vec{v} \in T_xS^n$. So we see that there is a correspondence $T\mathbb{R}P^n \cong \{(x, -x), (\vec{v}, -\vec{v}) \mid x \in S^n, \vec{v} \in T_xS^n\}$. On the other hand, $T_xS^n = L_x^\perp$, where $L_x$ is the line in $\mathbb{R}^{n+1}$ through $x$ and $-x$. Thus $T\mathbb{R}P^n \cong \text{Hom}(L_x, L_x^\perp)$.

This defines the isomorphism on fibres, since $L_x$ is the fibre in $\mathbb{R}P^n$ to the quotient $T\mathbb{R}P^n$. So by Lemma 2.2.4, where $t_{\vec{v}}$ is the transformation $L_x \to L_x^\perp, \vec{x} \mapsto \vec{v}$. It is routine to verify that $\varphi$ is a smooth map, so by Lemma 2.2.4, $\varphi$ is a bundle isomorphism.

Now that we have $T\mathbb{R}P^n \cong \text{Hom}(\gamma_n^1, \gamma_n^1)$, take the Whitney sum with the trivial line bundle $\text{Hom}(\gamma_n^1, \gamma_n^1) \cong \xi_0^1$ — here, triviality follows from the existence of the nonzero section $id: \gamma_n^1 \to \gamma_n^1$ — on both sides:

$$T\mathbb{R}P^n \oplus \xi_0^1 \cong \text{Hom}(\gamma_n^1, \gamma_n^1) \oplus \text{Hom}(\gamma_n^1, \gamma_n^1)$$

$$\implies T\mathbb{R}P^n \cong \text{Hom}(\gamma_n^1, \gamma_n^1 \oplus \gamma_n^1) \quad \text{since Hom commutes with direct sums}$$

$$\implies T\mathbb{R}P^n \cong \text{Hom}(\gamma_n^1, \xi_0^{n+1}) \quad \text{by definition of the orthogonal complement}$$

$$\cong \bigoplus_{i=1}^{n+1} \text{Hom}(\gamma_n^1, \xi_0^1)$$

$$\cong \bigoplus_{i=1}^{n+1} (\gamma_n^1)^* \quad \text{by definition of the dual bundle}$$

$$\cong \bigoplus_{i=1}^{n+1} \gamma_n^1,$$

where in the last step, we use the fact that each fibre is an inner product space and thus isomorphic to its dual. Applying Axiom 3 to the result $T\mathbb{R}P^n \cong \bigoplus_{i=1}^{n+1} \gamma_n^1$ gives us

$$\omega(\mathbb{R}P^n) = \omega(\gamma_n^{1\oplus 1}) = (1 + a)^{n+1} = 1 + (n + 1)a + \ldots + (n + 1)a^n.$$

(At the end, we use Axiom 0: $a^{n+1} = 0$.)

Example 3.2.7. For the projective plane, the above calculations show that

$$\omega(\mathbb{R}P^2) = (1 + a)^3 = 1 + 3a + 3a^2 \equiv 1 + a + a^2 \quad (\text{mod } 2).$$

In particular, $\mathbb{R}P^2$ has nontrivial Stiefel-Whitney classes in every dimension.

There is a generalization of Proposition 3.2.2 to immersed manifolds.

**Theorem 3.2.8.** Let $M$ be an $m$-manifold and $f: M \to \mathbb{R}^n$ an immersion. Then $TM$ is a subbundle of the pullback $f^*\xi_0^n$ of the trivial $n$-bundle $\xi_0^n$ over $\mathbb{R}^n$, and $\omega(M)\omega(V_f) = 1$, where $V_f = (TM)^\perp$ as a subbundle of $f^*\xi_0^n$. 3 Characteristic Classes
Proof. For any map \( f : M \to \mathbb{R}^n \), we have a commutative diagram

\[
\begin{array}{c}
TM \xrightarrow{df} \mathbb{R}^n \times \mathbb{R}^n \\
\downarrow & \downarrow \\
M \xrightarrow{f} \mathbb{R}^n
\end{array}
\]

which is \( d_x f : T_x M \to \{ f(x) \} \times \mathbb{R}^n \) on each tangent space. So if \( f \) is an immersion, \( TM \) naturally embeds as a subbundle of \( \xi_0^n|_{f(M)} \), the trivial \( n \)-bundle restricted to \( f(M) \). Under the pullback

\[
\begin{array}{c}
f^* E(\xi_0^n) \to E(\xi_0^n) \\
\downarrow & \downarrow \\
M \xrightarrow{f} \mathbb{R}^n
\end{array}
\]

we also have \( TM \subseteq f^* \xi_0^n \) as a subbundle. Hence by Theorem 3.1.5 and the Stiefel-Whitney axioms,

\[\omega(M) \cdot \omega(\mathcal{V}_f) = \omega(f^* \xi_0^n) = f^* \omega(\xi_0^n) = f^*(1) = 1.\]

\[\square\]

Corollary 3.2.9. If an \( n \)-dimensional manifold \( M \) can be immersed into \( \mathbb{R}^{n+1} \), then for each \( i \), \( \omega_i(M) \) is the \( i \)-fold cup product \((\omega_1(M))^i\).

Proof. Suppose \( i : M \to \mathbb{R}^{n+1} \) is an immersion. Then by Theorem 3.2.8, \( \mathbb{R}^{n+1}|_M \cong TM \oplus \mathcal{V}_f \) as bundles and consequently \( \omega(M) \cdot \omega(\mathcal{V}_f) = 1 \). Here, \( \mathcal{V}_f = (TM)\perp \) is a line bundle, so \( \omega(\mathcal{V}_f) = 1 + \omega_1(\mathcal{V}_f) \). Hence we have

\[
1 = \omega(M) \cdot \omega(\mathcal{V}_f) = \omega(M)(1 + \omega_1(\mathcal{V}_f))
\]

\[
\implies \omega(M) = 1 - \omega_1(\mathcal{V}_f) + \omega_1(\mathcal{V}_f)^2 - \ldots + (-1)^n \omega_1(\mathcal{V}_f)^n \quad \text{by “geometric series”}
\]

\[
= 1 + \omega_1(\mathcal{V}_f)^2 + \ldots + \omega_n(\mathcal{V}_f)^n \quad \text{(mod 2)}.
\]

Therefore \( \omega_1(M) = \omega_1(\mathcal{V}_f) \) and for each \( i > 1 \), \( \omega_i(M) = \omega_1(\mathcal{V}_f)^i = \omega_1(M)^i \). \[\square\]

Example 3.2.10. We will show that there does not exist an immersion \( \mathbb{R}P^4 \to \mathbb{R}^6 \). By Example 3.2.5, the total Stiefel-Whitney class of \( \mathbb{R}P^4 \) is

\[
\omega(\mathbb{R}P^4) = (1 + a)^5 = 1 + 5a + 10a^2 + 10a^3 + 5a^4 + a^5
\]

\[\equiv 1 + a + a^4 \quad \text{(mod 2)},\]

where \( a \) is a generator of \( H^1(\mathbb{R}P^4; \mathbb{Z}/2\mathbb{Z}) \). Suppose \( f : \mathbb{R}P^4 \to \mathbb{R}^6 \) is an immersion. By Theorem 3.2.8, \( \mathbb{R}^6|_{\mathbb{R}P^4} \cong T\mathbb{R}P^4 \oplus \mathcal{V}_f \) but \( \mathcal{V}_f \) is a 2-bundle so \( \omega(\mathcal{V}_f) = 1 + \omega_1 + \omega_2 \) for Stiefel-Whitney classes \( \omega_1, \omega_2 \in H^*(\mathbb{R}P^4; \mathbb{Z}/2\mathbb{Z}) \) for the bundle \( \mathcal{V}_f \). Then \( \omega(\mathbb{R}P^4) \cdot \omega(\mathcal{V}_f) = 1 \) which
can be written
\[1 = \omega(\mathbb{R}P^4) \cdot \omega(V_f) = (1 + a + a^4)(1 + \omega_1 + \omega_2) = 1 + (a + \omega_1) + (a\omega_1 + \omega_2) + a\omega_2 + a^4 + a^4\omega_1 + a^4\omega_2.\]

This implies \(a^4 = 0\) in \(H^4(\mathbb{R}P^4; \mathbb{Z}/2\mathbb{Z})\) which is false. Therefore no such immersion exists.

**Example 3.2.11.** By a similar argument, we can show that \(\mathbb{R}P^9\) does not admit an immersion into \(\mathbb{R}^{15}\). Using Example 3.2.5, we have
\[1 = \omega(\mathbb{R}P^9)\omega(V_f) = (1 + a^2 + a^8)(1 + \omega_1 + \omega_2 + \ldots + \omega_6),\]
so it follows from an elementary calculation that
\[\omega(V_f) = 1 + a^2 + a^4 + a^6.\]

As before, this implies \(\mathbb{R}P^9\) does not immerse into \(\mathbb{R}^N\) for \(N < 16\). By Whitney’s embedding theorem, \(\mathbb{R}P^9\) does immerse into \(\mathbb{R}^{17}\), so it is an interesting question whether \(\mathbb{R}P^9\) immerse into \(\mathbb{R}^{16}\).

**Theorem 3.2.12.** Let \(\xi\) be an \(n\)-dimensional bundle over \(B\). If \(\xi\) has a nonvanishing section, then \(\omega_n(\xi) = 0\) in \(H^n(B; \mathbb{Z}/2\mathbb{Z})\).

**Proof.** Let \(s : B \to E\) be a nonvanishing section. Then for each \(x \in B\), \(s(x) \neq 0\) in \(\pi^{-1}(x)\) so this spans a linear subspace \(L_x \subseteq \pi^{-1}(x)\). Since \(s\) is smooth, it follows that the \(L_x \subseteq \pi^{-1}(x)\) define a 1-dimensional subbundle \(\eta\):

\[
\begin{array}{ccc}
E(\eta) & \hookrightarrow & E \\
\downarrow & & \downarrow \pi \\
& & B
\end{array}
\]

Further, \(s\) also defines a nonvanishing section of \(\eta\), so \(\eta\) is a trivial line bundle and thus \(\omega(\eta) = 1\). Fix a Euclidean metric on \(\xi\), so that \(\eta^\perp\) is defined as a subbundle of \(\xi\). Then by Theorem 3.1.5, \(\eta \oplus \eta^\perp \cong \xi\). By the Stiefel-Whitney axioms,
\[\omega(\xi) = \omega(\eta)\omega(\eta^\perp) = 1 \cdot \omega(\eta^\perp) = \omega(\eta^\perp),\]
but since the dimension of \(\eta^\perp\) is \(n - 1\), \(\omega_n(\eta^\perp) = 0\). Hence \(\omega_n(\xi) = 0\).

**Corollary 3.2.13.** If \(n\) is even, then \(\mathbb{R}P^n\) is not parallelizable.

**Proof.** By our calculations in Example 3.2.5, \(\omega_n(\mathbb{R}P^n) = n + 1 \neq 0 \pmod{2}\), so Theorem 3.2.12 implies \(T\mathbb{R}P^n\) has no nonvanishing section. Hence by Theorem 2.2.5, \(T\mathbb{R}P^n\) is nontrivial.
Example 3.2.14. Notice that for even \( n \), \( \omega(S^n) = 1 \) and therefore \( \omega_n(S^n) = 0 \). Thus we cannot use the same argument to show that spheres are not parallelizable. We will introduce the Euler class to study \( TS^n \) more effectively.

Example 3.2.15. Notice that for \( \mathbb{R}P^2 \), the fact that \( \omega(\mathbb{R}P^2) \neq 1 \) does not preclude the existence of an immersion into \( \mathbb{R}^3 \). In fact, there does exist an immersion \( \mathbb{R}P^2 \to \mathbb{R}^3 \) called Boy’s surface (discovered in 1901). This is a special case of a more general result:

Theorem 3.2.16 (Immersion Conjecture, proved by R. Cohen). Let \( M \) be a smooth, compact \( n \)-manifold. Consider the dyadic expansion of \( n \),

\[
n = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_k} \quad \text{for} \quad i_1 < i_2 < \ldots < i_k.
\]

Let \( \alpha(n) = k \), the number of 1’s in this expansion of \( n \). Then there exists an immersion \( M \to \mathbb{R}^{2n-\alpha(n)} \).

Massey also showed that this \( 2n - \alpha(n) \) bound is sharp.

Assume \( M \) is a smooth, closed, compact \( n \)-manifold. Then from algebraic topology we know that \( H^n(M; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \). The cup product gives a pairing

\[
H^n(M; \mathbb{Z}/2\mathbb{Z}) \times H^n(M; \mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z}
\]

\[
(\alpha, c) \longmapsto \alpha \cap c
\]

and in particular, Poincaré duality can be realized as a map

\[
H^n(M; \mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z}
\]

\[
\alpha \longmapsto \alpha \cap [M],
\]

where \([M] \in H_n(M; \mathbb{Z}/2\mathbb{Z})\) is the fundamental class of \( M \).

Definition. The \( i \)-th Stiefel-Whitney number of \( M \) is the element \( \omega_i(M) \cap [M] \) in \( \mathbb{Z}/2\mathbb{Z} \).

More generally, let \( \varepsilon_1, \ldots, \varepsilon_n \geq 0 \) be integers and consider the cohomology class \( \omega_1^{\varepsilon_1} \cdot \cdots \cdot \omega_n^{\varepsilon_n} \). When \( \varepsilon_1 + 2\varepsilon_2 + \ldots + n\varepsilon_n = n \), this belongs to the top cohomology \( H^n(M; \mathbb{Z}/2\mathbb{Z}) \) and so the integer

\[
\omega_1^{\varepsilon_1} \cdot \cdots \cdot \omega_n^{\varepsilon_n} \cap [M] \in \mathbb{Z}/2\mathbb{Z}
\]

is defined. We call this integer the Stiefel-Whitney number for the sequence \( \varepsilon_1, \ldots, \varepsilon_n \).

Theorem 3.2.17. For any smooth, closed, compact \( n \)-manifold \( M \), the Stiefel-Whitney numbers \( \omega_1^{\varepsilon_1} \cdot \cdots \cdot \omega_n^{\varepsilon_n} \cap [M] \) are 0 for all sequences \( \varepsilon_1, \ldots, \varepsilon_n \) satisfying \( \varepsilon_1 + \ldots + n\varepsilon_n = n \) if and only if \( M = \partial W \) for some \((n + 1)\)-manifold \( W \).

Proof. Assume \( M \) is the boundary of a manifold \( W \). Then \( TM \) is a subbundle of the restriction \( TW|_M \) and \( V^W_M = (TM)^\perp \) is a line bundle over \( M \). In fact, since \( M = \partial W \), there is a consistent choice of an outward unit normal vector at every point of \( M \subset W \), and thus a nonvanishing section of \( V^W_M \). So \( V^W_M \) is a trivial line bundle by Theorem 2.2.5. Therefore Theorem 3.1.5 implies that \( \omega(M) = \omega(TW|_M) \).
Now take the fundamental class \([W] \in H_{n+1}(W, M; \mathbb{Z}/2\mathbb{Z})\). Recall that \(\partial_*[W] = [\partial W] = [M]\) in \(H_n(M; \mathbb{Z}/2\mathbb{Z})\). Then by duality for manifolds with boundary, we have

\[
\omega_1^{e_1} \cdots \omega_n^{e_n} \cap [M] = (\omega_1^{e_1} \cdots \omega_n^{e_n}) \cap \partial_*[W] = \delta(\omega_1^{e_1} \cdots \omega_n^{e_n}) \cap [W].
\]

To prove the Stiefel-Whitney number is 0, it suffices to show \(\omega_1^{e_1} \cdots \omega_n^{e_n} \in \text{im } i^*\), since we have an exact sequence

\[
H^n(W; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{i^*} H^n(M; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\delta} H^{n+1}(W, M; \mathbb{Z}/2\mathbb{Z}).
\]

However, notice that if \(i : M \hookrightarrow W\) is the natural embedding, then we have \(TW|_M = i^*(TW)\), the pullback of the tangent bundle of \(W\). Thus by naturality of Stiefel-Whitney classes, for each \(j \geq 0\),

\[
\omega_j(TM) = \omega_j(TW|_M) = \omega_j(i^*TW) = i^*\omega_j(TM).
\]

Hence each \(\omega_j \in \text{im } i^*\) so the proof is complete. (The converse follows from Thom’s theorem.)

**Corollary 3.2.18.** For \(n\) even, \(\mathbb{R}P^n\) is not the boundary of any \((n + 1)\)-manifold.

**Proof.** If \(n\) is even, \(\omega(\mathbb{R}P^n) = 1 + (n + 1)a + \ldots + (n + 1)a^n\) by Example 3.2.5, so in particular \(\omega_n = (n + 1)a^n = a^n \pmod{2}\). Thus \(\omega_n\) is a generator of \(H^n(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})\). It follows that \(\omega_n \cap [\mathbb{R}P^n] \neq 0\) so by Theorem 3.2.17, \(\mathbb{R}P^n\) does not bound a manifold.

**Definition.** Two \(n\)-manifolds \(M_1\) and \(M_2\) are called **cobordant** if there exists a compact \((n + 1)\)-manifold \(W\) such that \(\partial W = M_1 \coprod M_2\).

**Corollary 3.2.19.** Two compact, closed \(n\)-manifolds \(M_1\) and \(M_2\) have the same Stiefel-Whitney numbers for all sequences \(\varepsilon_1, \ldots, \varepsilon_n\) such that \(\varepsilon_1 + \ldots + n\varepsilon_n = n\) if and only if they are cobordant.

**Proof.** This follows directly from Theorem 3.2.17 with \(M = M_1 \coprod M_2\), using the fact that \(\omega_i(M) \cap [M] = \omega_i(M_1) \cap [M_1] - \omega_i(M_2) \cap [M_2]\) for all \(i\).

Let \(\mathcal{M}_n\) be the set of cobordism classes of smooth closed \(n\)-dimensional manifolds.

**Lemma 3.2.20.** For any \(n \geq 1\), \(\mathcal{M}_n\) is a group under disjoint union.

**Proof.** Let \([M]\) denote the cobordism class of a smooth closed manifold \(M\) and \(\alpha_M \in H_n(M; \mathbb{Z}/2\mathbb{Z})\) the (unoriented) fundamental class of \(M\). Also \(\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{N}_0^n\) will denote a sequence which is assumed to satisfy \(\varepsilon_1 + 2\varepsilon_2 + \ldots + n\varepsilon_n = n\). For such a sequence, we let \(s_M(\varepsilon) = (\omega_1(M)^{\varepsilon_1} \cup \cdots \cup \omega_n(M)^{\varepsilon_n}) \cap \alpha_M \in \mathbb{Z}/2\mathbb{Z}\) denote the Stiefel-Whitney number of the sequence \(\varepsilon\).

First, for \([M], [N], [P] \in \mathcal{M}_n\), we have a cobordism
which shows associativity. The fact that $[\emptyset]$ is the zero class is obvious from the definition of addition using disjoint union. Finally, every class $[M]$ is its own inverse since $W = M \times [0, 1]$ gives a cobordism between $M \times \{0\} \bigcup M \times \{1\}$ and the empty set:

Therefore $\mathcal{M}_n$ is a group, and even a 2-group since $[M] + [M] = 0$ for all $M$.

**Theorem 3.2.21.** $\mathcal{M}_n$ is a finite group.

**Proof.** Corollary 3.2.19 says that $M$ and $N$ are cobordant if and only if $s_M(\varepsilon) = s_N(\varepsilon)$ for all sequences $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{N}_0^n$ satisfying $\varepsilon_1 + 2\varepsilon_2 + \ldots + n\varepsilon_n = n$. Of course, there are only a finite number of these sequences satisfying the given identity to begin with, and for each sequence $\varepsilon$ we may specify a 0 or a 1 for its Stiefel-Whitney number. Therefore there are only a finite number of choices $s(\varepsilon) \in \mathbb{Z}/2\mathbb{Z}$ for all sequences $\varepsilon$, so by Corollary 3.2.19 there are a finite number of distinct cobordism classes of $n$-manifolds.

**Example 3.2.22.** For $n = 4$, we exhibit 4 distinct cobordism classes of 4-manifolds. The trivial class corresponds to Stiefel-Whitney numbers $s_0(\varepsilon) = 0$ for all admissible $\varepsilon$ – this is Theorem 3.2.17 – so to show any other cobordism classes are nontrivial, it will suffice to compute nonzero Stiefel-Whitney numbers on any of the following sequences:

$$(4, 0, 0, 0), \quad (2, 1, 0, 0), \quad (1, 0, 1, 0), \quad (0, 2, 0, 0) \quad \text{and} \quad (0, 0, 0, 1).$$

Let’s begin with the class $[\mathbb{R}P^4] \in \mathcal{M}_4$. By Example 3.2.5, $\omega(\mathbb{R}P^4) = 1 + a + a^4$, where $a$ is a generator of $H^1(\mathbb{R}P^4; \mathbb{Z}/2\mathbb{Z})$. In particular, we have Stiefel-Whitney numbers:

$$s_{\mathbb{R}P^4}(0, 0, 0, 1) = \omega_4(\mathbb{R}P^4) \cap a_{\mathbb{R}P^4} = 1 \quad \text{since} \quad \omega_4 = a^4 \quad \text{generates} \quad H^4(\mathbb{R}P^4; \mathbb{Z}/2\mathbb{Z})$$

$$s_{\mathbb{R}P^4}(0, 2, 0, 0) = \omega_2(\mathbb{R}P^4)^2 \cap a_{\mathbb{R}P^4} = 0 \quad \text{since} \quad \omega_2 = 0.$$
Since \( s(0,0,1) = 1 \), \([ \mathbb{R}P^4 ]\) is a nontrivial cobordism class. Next, consider \([ \mathbb{R}P^2 \times \mathbb{R}P^2 ]\). By the (external) product formula for Stiefel-Whitney classes,

\[
\omega(\mathbb{R}P^2 \times \mathbb{R}P^2) = \omega(\mathbb{R}P^2) \times \omega(\mathbb{R}P^2) = (1 + a) \times (1 + b),
\]

but by definition, \((1 + a) \times (1 + b) = p_1^*(1 + a) \cup p_2^*(1 + b)\), where \(p_1, p_2 : \mathbb{R}P^2 \times \mathbb{R}P^2 \to \mathbb{R}P^2\) are the canonical projections and \(a\) (resp. \(b\)) generates \(H^1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})\) for one of the copies of \(\mathbb{Z}/2\mathbb{Z}\). Set \(c = p_1^*a + p_2^*b \in H^1(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})\). Now we really have \(p_1^*(1 + a) \cup p_2^*(1 + b) = 1 + c + c + c^2 \equiv 1 + c^2 \pmod{2}\). Thus we can compute the same Stiefel-Whitney numbers again for this cobordism class:

\[
\begin{align*}
\sigma_{[\mathbb{R}P^2]^2}(0,0,0,1) &= \omega_4((\mathbb{R}P^2)^2) \cap \alpha_{([\mathbb{R}P^2]^2)} = 0 \cap \alpha_{([\mathbb{R}P^2]^2)} = 0 \\
\sigma_{[\mathbb{R}P^2]^2}(0,2,0,0) &= \omega_2((\mathbb{R}P^2)^2) \cap \alpha_{([\mathbb{R}P^2]^2)} = c^2 \cap \alpha_{([\mathbb{R}P^2]^2)} = 1.
\end{align*}
\]

The latter tells us that \([ \mathbb{R}P^2 \times \mathbb{R}P^2 ]\) is nontrivial in the cobordism group, while the former says that \([ \mathbb{R}P^2 \times \mathbb{R}P^2 ] \neq [ \mathbb{R}P^4 ]\). Finally, since every class in \(\mathcal{M}_4\) is its own inverse and \([ \mathbb{R}P^4 ]\) and \([ \mathbb{R}P^2 \times \mathbb{R}P^2 ]\) are nontrivial and distinct, we get a fourth nontrivial class by adding these together: \([ \mathbb{R}P^4 ] + [ \mathbb{R}P^2 \times \mathbb{R}P^2 ] = [ \mathbb{R}P^4 \coprod (\mathbb{R}P^2 \times \mathbb{R}P^2) ]\).

### 3.3 The Universal Bundle

In this section we construct a “universal bundle” \(\gamma^n\) of dimension \(n\) such that every \(n\)-dimensional vector bundle \(\xi\) over \(B\) is induced by a map from \(B\) into the base of \(\gamma^n\), with certain loose conditions assumed for \(B\). Then by the Stiefel-Whitney axioms, \(\omega_i(\xi) = f^*\omega_i(\gamma^n)\) for any such map \(f\), so to explicitly construct the Stiefel-Whitney classes, we construct a set of generators for the cohomology ring of the base of \(\gamma^n\) and show that they satisfy the axioms.

The tautological line bundle \(\gamma_1^n\) over \(\mathbb{R}P^n\) generalizes to a “canonical bundle” \(\gamma_k^n\) of the Grassmannian manifold of the appropriate dimensions. We recall the construction of this manifold here.

**Definition.** For \(n \geq 1\) and any vector space \(V\), the Grassmannian manifold \(\text{Gr}_n(V)\) is the set of all \(n\)-dimensional vector subspaces of \(V\).

We will particularly be focused on the Grassmannian \(\text{Gr}_n(\mathbb{R}^{n+k})\) for \(k \geq 0\).

**Example 3.3.1.** For any \(n\), \(\text{Gr}_1(\mathbb{R}^{n+1}) = \mathbb{R}P^n\), projective \(n\)-space.

**Definition.** For \(n \geq 1, k \geq 0\), the Stiefel manifold \(V_n(\mathbb{R}^{n+k})\) is defined as the set of all \(n\)-frames in \(\mathbb{R}^{n+k}\), i.e. the set of all collections of \(n\) linearly independent vectors in \(\mathbb{R}^{n+k}\).

There is a natural surjection \(p : V_n(\mathbb{R}^{n+k}) \to \text{Gr}_n(\mathbb{R}^{n+k})\) sending an \(n\)-frame to the subspace of \(\mathbb{R}^{n+k}\) it spans.

**Proposition 3.3.2.** \(V_n(\mathbb{R}^{n+k})\) is an open submanifold of \(\mathbb{R}^{n(n+k)}\) for any \(n, k\).

**Definition.** The determinant function \(\det : \mathbb{R}^{n(n+k)} = \mathbb{R}^n \times \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}\) is a continuous function, and clearly \(V_n(\mathbb{R}^{n+k}) = \det^{-1}(\mathbb{R}^{n+k} \setminus \{0\})\).
Now the map \( p : V_n(\mathbb{R}^{n+k}) \to \text{Gr}_n(\mathbb{R}^{n+k}) \) defines the quotient topology on \( \text{Gr}_n(\mathbb{R}^{n+k}) \): a set \( U \subseteq \text{Gr}_n(\mathbb{R}^{n+k}) \) is open if and only if \( p^{-1}(U) \) is open in \( V_n(\mathbb{R}^{n+k}) \).

**Lemma 3.3.3.** \( \text{Gr}_n(\mathbb{R}^{n+k}) \) is a compact manifold of dimension \( nk \).

**Proof.** Given a point \( V \in \text{Gr}_n(\mathbb{R}^{n+k}) \), we need to define an open neighborhood \( U \) of \( V \) and a diffeomorphism \( \varphi : U \to \varphi(U) \subseteq \mathbb{R}^k \). By definition, \( V \) is an \( n \)-dimensional subspace of \( \mathbb{R}^{n+k} \), so it has a \( k \)-dimensional orthogonal complement \( V^\perp \) such that \( \mathbb{R}^{n+k} = V \oplus V^\perp \). Define the open neighborhood

\[
U = \{ X \subseteq \mathbb{R}^{n+k} \mid X \text{ is an } n\text{-dimensional subspace and } X \cap V^\perp = \emptyset \}.
\]

Note that there is a one-to-one correspondence

\[
U \leftrightarrow \text{Hom}(V, V^\perp) \cong \mathbb{R}^{nk}
\]

\[
X \leftrightarrow (L : V \to V^\perp)
\]

where \( L \) is the linear map between \( V \) and its complement such that \( X \) is the graph of \( L \). To show that the correspondence is a diffeomorphism, we show that the composition \( V_n(\mathbb{R}^{n+k}) \to U \to \text{Hom}(V, V^\perp) \) is smooth. For \( X \in U \), choose a basis \( \{w_1, \ldots, w_n\} \) of \( X \) which is an \( n \)-frame in \( V_n(\mathbb{R}^{n+k}) \). Then for each \( 1 \leq i \leq n \), we may write \( w_i = v_i + L(v_i) \) where \( \{v_1, \ldots, v_n\} \) is a basis for \( V \). Notice that \( L \) depends smoothly on the entire \( n \)-frame \( \{w_1, \ldots, w_n\} \) because the matrix entries of \( L \) can be expressed in terms of the two bases \( \{v_i\} \) and \( \{w_j\} \). Conversely, the \( n \)-frames \( \{w_1, \ldots, w_n\} \) also depend smoothly on such \( L \), so the bijection \( U \leftrightarrow \text{Hom}(V, V^\perp) \) is in fact smooth. One now shows that the transition maps on overlaps between these charts \( U \) are smooth, and this completes the proof.

**Remark.** An alternative proof that \( \text{Gr}_n(\mathbb{R}^{n+k}) \) is a smooth manifold comes from Lie theory. Since \( \text{Gr}_n(\mathbb{R}^{n+k}) \) is the set of \( n \)-planes in \( \mathbb{R}^{n+k} \), the Lie group \( GL_{n+k}(\mathbb{R}) \) acts transitively on the Grassmannian as a set. For an \( n \)-plane \( X \in \text{Gr}_n(\mathbb{R}^{n+k}) \), the isotropy subgroup of this action is:

\[
\text{Stab}(X) = \{ M \in GL_{n+k}(\mathbb{R}) \mid M \cdot X = X \}.
\]

With respect to the decomposition \( \mathbb{R}^{n+k} = X \oplus X^\perp \), we can write any \( M \in \text{Stab}(X) \) as a block diagonal matrix

\[
M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.
\]

In the topology on \( GL_{n+k}(\mathbb{R}) \), these matrices form a closed subgroup. Therefore \( \text{Gr}_n(\mathbb{R}^{n+k}) \) is a homogeneous space for \( GL_{n+k}(\mathbb{R}) \), so it follows from the theory of Lie groups that there is a unique topology and a unique smooth structure on \( \text{Gr}_n(\mathbb{R}^{n+k}) \) with respect to which the action of \( GL_{n+k}(\mathbb{R}) \) is smooth.

**Definition.** The canonical \( n \)-bundle over \( \text{Gr}_n(\mathbb{R}^{n+k}) \) is the bundle \( \gamma_n^{n+k} \) with total space

\[
E(\gamma_n^{n+k}) = \{(X, \bar{v}) : X \in \text{Gr}_n(\mathbb{R}^{n+k}), \bar{v} \in X \} \subseteq \text{Gr}_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}
\]

and projection \( \pi : (X, \bar{v}) \to X \). The topology on \( E(\gamma_n^{n+k}) \) is the subspace topology inherited from \( \text{Gr}_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k} \).
Lemma 3.3.4. \( \gamma_{n+k}^n \) is a vector bundle.

Proof. We construct a local trivialization over each chart \( U \) defined in the previous proof. Fix \( V \in \text{Gr}_n(\mathbb{R}^{n+k}) \) and \( U = \{ X : X \cap V^\perp = 0 \} \). Then \( V \cong \mathbb{R}^n \), so fixing an isomorphism once and for all, we may define

\[
h : U \times V \longrightarrow \pi^{-1}(U) \\
(x, \bar{v}) \longmapsto (x, \bar{v} + L(\bar{v})),
\]

where \( L \in \text{Hom}(V, V^\perp) \) is the linear map having graph \( X \). On a fibre, \( \{ X \} \times V \rightarrow \pi^{-1}(X) \) is given by \( \bar{v} \mapsto \bar{v} + L(\bar{v}) \) which is linear and whose inverse is given by orthogonal projection onto \( V \). Further, \( h \) is defined in terms of \( L \in \text{Hom}(V, V^\perp) \) which was shown to be smooth in the previous proof. Therefore \( h \) is a local trivialization of \( \gamma_{n+k}^n \).

Many of the next results were proven in greater generality in Section 1.2; however we give the proofs here in terms of vector bundles since they are integral to understanding the structure of the universal bundle for vector bundles.

Lemma 3.3.5. Let \( \xi \) be an \( n \)-dimensional bundle over a compact \( m \)-manifold \( B \). Then \( \xi \cong f^*\gamma_N^n \) for some map \( f : B \rightarrow \text{Gr}_n(\mathbb{R}^N) \).

Proof. We aim to construct a map \( F : E(\xi) \rightarrow \mathbb{R}^N \) for \( N \) large enough so that each restriction \( F|_{\pi^{-1}(x)} \) is an injective linear map. Given such an \( F \), we then define \( f : B \rightarrow \text{Gr}_n(\mathbb{R}^N) \) by \( f(x) = F(\pi^{-1}(x)) \). This yields a bundle map

\[
\begin{array}{ccc}
E(\xi) & \longrightarrow & E(\gamma_N^n) \\
\downarrow & & \downarrow \\
B & \longrightarrow & \text{Gr}_n(\mathbb{R}^N)
\end{array}
\]

Then Lemma 2.3.5 implies that \( \xi \cong f^*\gamma_N^n \) as desired.

Since \( B \) is compact, there is a finite cover \( \{ U_1, \ldots, U_k \} \) of \( B \) such that \( \xi \) is trivial over each \( U_i \). Let \( h_i : U_i \times \mathbb{R}^n \rightarrow \pi^{-1}(U_i) \) be the local trivialization of \( \xi \) over \( U_i \) and let \( p : U_i \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) be the natural projection map. This determines a map \( g_i = p \circ h_i^{-1} : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^n \rightarrow \mathbb{R}^n \). Fix a partition of unity \( \{ \lambda_i \} \) subordinate to \( \{ U_1, \ldots, U_k \} \), with \( \lambda_i \equiv 1 \) on \( B \setminus U_i \) and \( \sum \lambda_i \equiv 1 \). Then we extend the \( g_i \) maps to the total space by setting \( \tilde{g}_i(x, \bar{v}) = \lambda_i(x)g_i(x, \bar{v}) \) for any \( (x, \bar{v}) \in E(\xi) \). Finally, define

\[
f : E \longrightarrow \mathbb{R}^n \oplus \cdots \oplus \mathbb{R}^n = \mathbb{R}^{nk} \\
(x, \bar{v}) \longmapsto (\tilde{g}_1(x, \bar{v}), \ldots, \tilde{g}_k(x, \bar{v})).
\]

For each point \( (x, \bar{v}) \in E \), \( x \) is in the support of some \( \lambda_i, 1 \leq i \leq k \), so \( \lambda_i(x) \neq 0 \), and then \( \tilde{g}_i \) maps \( \pi^{-1}(x) \subset E(\xi) \) isomorphically to a copy of \( \mathbb{R}^n \subset \mathbb{R}^{nk} \). Hence \( f \) is as required. \( \square \)

We want to consider infinite Euclidean space

\[
\mathbb{R}^\infty = \{(x_1, x_2, \ldots) : x_i \in \mathbb{R}, \text{ finitely many } x_i \text{ are nonzero}\}
\]

with the direct limit topology given by the direct system \( \mathbb{R} \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow \cdots \). That is, \( U \subseteq \mathbb{R}^\infty \) is open (resp. closed) if and only if \( U \cap \mathbb{R}^m \) is open (resp. closed) for all \( m \geq 1 \).
Definition. The \( n \)th infinite Grassmannian is the direct limit of the \( \text{Gr}_n(\mathbb{R}^{n+k}) \) directed by inclusion:

\[
\text{Gr}_n(\mathbb{R}^\infty) := \bigcup_{k=0}^{\infty} \text{Gr}_n(\mathbb{R}^{n+k}).
\]

Definition. The canonical \( n \)-bundle over \( \text{Gr}_n(\mathbb{R}^\infty) \) is called the universal \( n \)-bundle, written \( \gamma^n \). The topology is taken to be the direct limit topology induced by the direct system \( \text{Gr}_n(\mathbb{R}^{n+k}) \hookrightarrow \text{Gr}_n(\mathbb{R}^{n+1+k}) \hookrightarrow \cdots \). Since each \( \text{Gr}_n(\mathbb{R}^{n+k}) \) embeds into \( \text{Gr}_n(\mathbb{R}^\infty) \), we have:

Proposition 3.3.6. Every \( n \)-bundle \( \xi \) over \( B \) is isomorphic to the pullback \( f^*\gamma^n \) for some map \( f : B \to \text{Gr}_n(\mathbb{R}^\infty) \).

Note that there is a map \( E(\gamma^n) \to \mathbb{R}^\infty \), given by \((X, \vec{v}) \mapsto \vec{v}\), which is injective on each fibre.

Definition. Two bundle maps \( f_0, f_1 : \xi \to \eta \) are said to be bundle homotopic if there exists a smooth map \( f_t : E(\xi) \times [0, 1] \to E(\eta) \) extending \( f_0 \) and \( f_1 \) such that for each \( t \in [0, 1] \), \( f_t : E(\xi) \times \{t\} \to E(\eta) \) is a bundle map. Such a map \( f_t \) is called a bundle homotopy.

Lemma 3.3.7. Any two bundle maps \( f_0, f_1 : \xi \to \gamma^n \) are bundle homotopic.

Proof. Recall in the proof of Lemma 3.3.5 we constructed a map \( F : E(\xi) \to \mathbb{R}^N \subseteq \mathbb{R}^\infty \) which is injective on each fibre and, from this, constructed \( f : B \to \text{Gr}_n(\mathbb{R}^N) \subseteq \text{Gr}_n(\mathbb{R}^\infty) \) by sending \( x \) to \( f(x) = F(\pi^{-1}(x)) \). Conversely, given a bundle map

\[
\begin{array}{ccc}
E(\xi) & \longrightarrow & E(\gamma^n) \\
\downarrow & & \downarrow \\
B & \longrightarrow & \text{Gr}_n(\mathbb{R}^\infty)
\end{array}
\]

we get a map \( E(\xi) \to \mathbb{R}^\infty \) by composing with the map \( E(\gamma^n) \to \mathbb{R}^\infty \) discussed above. It is immediate that such a map is injective on fibres.

Now given two bundle maps

\[
\begin{array}{ccc}
E(\xi) & \longrightarrow & E(\gamma^n) \\
\downarrow & & \downarrow \\
f_0, f_1 & \quad & \text{Gr}_n(\mathbb{R}^\infty)
\end{array}
\]

let \( F_0, F_1 : E(\xi) \to \mathbb{R}^\infty \) be the corresponding maps from the previous paragraph. By the above, it suffices to construct \( F_t : E(\xi) \times [0, 1] \to \mathbb{R}^\infty \) such that each \( F_t : E(\xi) \times \{t\} \to \mathbb{R}^\infty \) is linear and injective on fibres. If we define \( F_t = (1-t)F_0 + tF_1 \) then it is immediate that this map is smooth and linear. The only way \( F_t \) fails to be injective on fibres is if there is some nonzero vector \( \vec{v} \in E(\xi) \) such that \( F_0(\vec{v}) = -F_1(\vec{v}) \). However, consider the two
linear maps $L_0, L_1 : \mathbb{R}^\infty \to \mathbb{R}^\infty$ given by $L_0(e_i) = e_{2i}$ and $L_1(e_i) = e_{2i-1}$, where $e_i$ is the $i$th standard basis vector in $\mathbb{R}^\infty$. Geometrically, $L_0$ and $L_1$ “separate” the components of $\mathbb{R}^\infty$ into subspaces with trivial intersection. It is clear that $F_0$ is homotopic to $L_0 \circ F_0$ and moreover, $F_0(\vec{v}) \neq -L_0(F_0(\vec{v}))$ for any nonzero $\vec{v}$ since there is a largest index of $F_0(\vec{v}) \in \mathbb{R}^\infty$ with a nonzero component. Hence $F_0 \simeq L_0 \circ F_0$ through a family of maps that are linear and injective on fibres. Similarly, $F_1 \simeq L_1 \circ F_1$ through such a family of maps. It follows that $L_0 \circ F_0$ and $L_1 \circ F_1$ are homotopic through a family of maps that are injective on each fibre, but clearly $L_0 \circ F_0(\vec{v}) \neq -L_1 \circ F_1(\vec{v})$ for any $\vec{v} \neq 0$, showing we can reduce the general case to this case up to homotopy.

\[ \square \]

**Corollary 3.3.8.** For any $n \geq 1$, the bundle $V_n(\mathbb{R}^\infty) \to \text{Gr}_n(\mathbb{R}^\infty)$ is universal for $O(n)$-bundles.

Thus we have shown that every $n$-bundle $\xi$ over a paracompact base $B$ determines a well-defined homotopy class of maps $f : B \to \text{Gr}_n = \text{Gr}_n(\mathbb{R}^\infty)$ such that $\xi \cong f^*\gamma^n$. So the existence of Stiefel-Whitney classes for $B$ rests on their construction in the cohomology ring of $\text{Gr}_n$.

**Theorem 3.3.9.** For any $n \geq 1$, $H^\bullet(\text{Gr}_n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\omega_1, \omega_2, \ldots, \omega_n]$, the polynomial ring in $\omega_i \in H^1(\text{Gr}_n; \mathbb{Z}/2\mathbb{Z})$. Moreover, assuming the existence of Stiefel-Whitney classes, there is a unique choice of $\omega_i$ satisfying the Stiefel-Whitney axioms for the universal bundle, that is, $\omega_i = \omega_i(\gamma^n)$.

**Proof.** We first describe a CW-structure on $\text{Gr}_n(\mathbb{R}^N)$ and pass to the direct limit to obtain a structure on $\text{Gr}_n$. Group $n$-planes $X \in \text{Gr}_n(\mathbb{R}^N)$ into cells as follows. For each $1 \leq i \leq N$, consider $\mathbb{R}^i \hookrightarrow \mathbb{R}^N$ viewed as $(x_1, \ldots, x_i, 0, \ldots, 0)$ and the sequence of spaces $X \cap \mathbb{R}, X \cap \mathbb{R}^2, \ldots, X \cap \mathbb{R}^N = X$. Then

$$\text{dim}(X \cap \mathbb{R}), \text{dim}(X \cap \mathbb{R}^2), \ldots, \text{dim}(X \cap \mathbb{R}^N) = n$$

is a nondecreasing sequence of $N$ integers between 0 and $n$. Since $N \geq n$ and the dimensions increase by at most one in each step, we must have $k$ places in the sequence where the dimension doesn’t change. On the other hand, given a sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ of $n$ integers $1 \leq \sigma_i \leq N$, we define the **Schubert symbol** for $\sigma$:

$$e(\sigma) = \{ X \in \text{Gr}_n(\mathbb{R}^N) : \text{dim}(X \cap \mathbb{R}^{\sigma_i}) = i, \text{dim}(X \cap \mathbb{R}^{\sigma_i - 1}) = i - 1 \}.$$ 

In other words, $\sigma$ encodes the places in the sequence $\text{dim}(X \cap \mathbb{R}), \ldots, \text{dim}(X \cap \mathbb{R}^N)$ where the dimension jumps, for each $X \in e(\sigma)$. Each $e(\sigma)$ is an open CW-cell. To discuss closed cells, it is convenient to reformulate them in terms of (orthogonal) frames. Given an $n$-plane $X \subset \mathbb{R}^N$, there is a unique orthonormal basis $\{e_1, \ldots, e_n\}$ for $X$ which lies in $H^{\sigma_1} \times H^{\sigma_2} \times \cdots \times H^{\sigma_n}$, where $H^j = \{(x_1, \ldots, x_k, 0, \ldots, 0) \in \mathbb{R}^N : x_k > 0\}$ is the $k$th open half-plane in $\mathbb{R}^N$. Conversely, any such frame determines $X \in e(\sigma)$. Let $V_n^o(\mathbb{R}^N)$ be the set of orthonormal $n$-frames in $\mathbb{R}^N$ and define

$$e'(\sigma) = V_n^o(\mathbb{R}^N) \cap (H^{\sigma_1} \times H^{\sigma_2} \times \cdots \times H^{\sigma_n})$$

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called the Schubert cell for \( \sigma \). By the above description, \( e'(\sigma) \cong e(\sigma) \). One can show that \( e'(\sigma) \) is a closed CW-cell of dimension \((\sigma_1 - 1) + (\sigma_2 - 2) + \ldots + (\sigma_n - n)\) and that the set \( \{e'(\sigma) \mid \sigma = (\sigma_1, \ldots, \sigma_n)\} \) forms a CW-structure on \( \text{Gr}_n \).

To prove \( H^\bullet(\text{Gr}_n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\omega_1, \ldots, \omega_n] \), we first prove \( \mathbb{Z}/2\mathbb{Z}[\omega_1, \ldots, \omega_n] \subseteq H^\bullet(\text{Gr}_n; \mathbb{Z}/2\mathbb{Z}) \). Consider the bundle map

\[
\begin{array}{ccc}
E(\gamma_1) & \longrightarrow & E(\gamma) \\
\mathbb{R}P^1 & \longleftarrow & \mathbb{R}P^\infty \\
\end{array}
\]

By the naturality axiom, \( \omega(\gamma_1) = i^*\omega(\gamma) \) so assuming the Stiefel-Whitney classes exist, we are forced to have \( \omega(\gamma_1) = 1 + a \) for the generator \( a \in H^1(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \). Define the product bundle \( \xi = \gamma^1 \times \cdots \times \gamma^1 \) over \( \mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty \). Then \( \dim \xi = n \) so by Proposition 3.3.6, there is a map \( f : \mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty \to \text{Gr}_n \) inducing \( \xi \) from \( \gamma^n \). Thus \( \omega(\gamma_1 \times \cdots \times \gamma_1) = f^*\omega(\gamma^n) \). Now by the K"unneth formula,

\[
\begin{align*}
H^\bullet(\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) & \cong H^\bullet(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \otimes \cdots \otimes H^\bullet(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \\
& \cong \mathbb{Z}/2\mathbb{Z}[a_1] \otimes \cdots \otimes \mathbb{Z}/2\mathbb{Z}[a_n] \\
& \cong \mathbb{Z}/2\mathbb{Z}[a_1, \ldots, a_n].
\end{align*}
\]

By the remark following the definition of the Stiefel-Whitney axioms in Section 3.2, the total Stiefel-Whitney class of \( \gamma_1 \times \cdots \times \gamma_1 \) can be computed as an algebraic cross product:

\[
\omega(\gamma_1 \times \cdots \times \gamma_1) = \omega(\gamma_1) \times \cdots \times \omega(\gamma_1) = p_1^*\omega(\gamma_1) \times \cdots \times p_n^*\omega(\gamma_1) \quad \text{where} \quad p_i : (\mathbb{R}P^\infty)^n \to \mathbb{R}P^\infty
\]

\[
= (1 + a_1) \cdots (1 + a_n) = 1 + (a_1 + \ldots + a_n) + (a_1a_2 + a_1a_3 + \ldots + a_{n-1}a_n) + \ldots + a_1 \cdots a_n.
\]

Notice that these are just the sum of the elementary symmetric functions of degree 0 through degree \( n \) in the \( a_i \). By linear algebra, the elementary symmetric functions are algebraically independent, and thus there are no polynomial relations among the Stiefel-Whitney classes of \( \xi = \gamma_1 \times \cdots \times \gamma_1 \). Further, \( \xi = f^*\gamma^n \) implies that there are no polynomial relations among the \( \omega_i(\gamma^n) \) either. This proves that \( \mathbb{Z}/2\mathbb{Z}[\omega_1, \ldots, \omega_n] \subseteq H^\bullet(\text{Gr}_n; \mathbb{Z}/2\mathbb{Z}) \).

Next, observe that any \( d \)-cell \( e(\sigma) \) in the CW-structure we have defined on \( \text{Gr}_n \) satisfies \( d = (\sigma_1 - 1) + \ldots + (\sigma_n - n) \), and thus every \( d \)-cell determines a partition of \( d \), but clearly this correspondence is one-to-one. Hence the number of \( d \)-cells, and hence the rank of \( H^d(\text{Gr}_n; \mathbb{Z}/2\mathbb{Z}) \), is equal to the number of partitions \( d = k_1 + \ldots + k_s \). On the other hand, the degree \( d \) part of \( \mathbb{Z}/2\mathbb{Z}[\omega_1, \ldots, \omega_n] \) consists of elements of the form \( \omega_1^{k_1} \cdots \omega_s^{k_s} \), where \( k_1 + 2k_2 + \ldots + sk_s = d \). So every such element determines a partition of \( d \), but since every partition can be written in nondecreasing order \( d = k_s + (k_s + k_{s-1}) + \ldots + (k_s + \ldots + k_1) \), we see that this correspondence is also one-to-one. Hence the \( d \)-part of \( \mathbb{Z}/2\mathbb{Z}[\omega_1, \ldots, \omega_n] \) has rank equal to that of \( H^\bullet(\text{Gr}_n; \mathbb{Z}/2\mathbb{Z}) \), so we conclude that \( H^\bullet(\text{Gr}_n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[\omega_1, \ldots, \omega_n] \).
Using this, we show that there is a unique choice of such generators of the cohomology ring of $\text{Gr}_n$ satisfying the Stiefel-Whitney axioms. Suppose $\omega'_i$ and $\omega''_i$ are two choices of Stiefel-Whitney classes generating this cohomology ring. If $n = 1$, we have a bundle map

$$E(\gamma^1) \longrightarrow E(\gamma^1)$$

$$\mathbb{R}P^1 \overset{i}{\longrightarrow} \mathbb{R}P^\infty$$

Then $H^1(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \cong H^1(\mathbb{R}P^1; \mathbb{Z}/2\mathbb{Z})$. Moreover, $\omega'_1(\gamma^1)$ and $\omega''_1(\gamma^1)$ are both nonzero in $H^1(\mathbb{R}P^1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, so they must be equal. Hence by naturality, $\omega'_1(\gamma_1) = \omega''_1(\gamma_1)$. This proves $\omega'$ and $\omega''$ agree on the universal line bundle $\gamma^1$. For $\xi = \gamma^1 \times \cdots \gamma^1$, we get $\omega'(\xi) = \omega''(\xi)$ from the description of these using the algebraic cross product. Finally, for the universal $n$-bundle $\gamma^n$, we have $f^*\omega'(\gamma^n) = \omega'(\xi)$ and $f^*\omega''(\gamma^n) = \omega''(\xi)$ as above, but since $f^*$ is injective on cohomology, we must have $\omega'(\gamma^n) = \omega''(\gamma^n)$. \(\square\)

**Corollary 3.3.10.** For any bundle $\xi$ over $B$, if Stiefel-Whitney classes $\omega_i(\xi) \in H^i(B; \mathbb{Z}/2\mathbb{Z})$ exist, then they are unique.

In light of the theory of principal bundles in Chapter 1, a vector bundle $p : E \to B$ may be viewed from one of the following equivalent perspectives:

- A locally trivial fibre bundle with fibre $V \cong \mathbb{R}^n$ and structure group $GL(V)$. (For complex bundles, replace $\mathbb{R}$ with $\mathbb{C}$.)

- A principal $O(n)$-bundle, by Example 1.1.10. (For complex bundles, replace $O(n)$ with $U(n)$.)

- A vector bundle with a smooth inner product, via the identification of principal $O(n)$-bundles with such vector bundles. (For complex bundles, replace smooth with Hermitian.)

- Up to isomorphism, real vector bundles over $B$ are classified by homotopy classes of maps $B \to \text{Gr}_n(\mathbb{R}^\infty)$. Moreover, this is compatible with the associated bundle construction, since if $F(E)$ is the frame bundle associated to $E \to B$ (see Example 1.2.2) and $f : B \to \text{Gr}_n(\mathbb{R}^\infty)$ is a classifying map for $F(E)$, we have

$$E = F(E) \times_{\text{GL}_n(\mathbb{R})} \mathbb{R}^n$$

$$= f^*V_n(\mathbb{R}^\infty) \times_{O(n)} \mathbb{R}^n$$

$$= f^*(V_n(\mathbb{R}^\infty)) \times_{O(n)} \mathbb{R}^n$$

$$= f^*\gamma_n$$

where $\gamma_n$ is the tautological bundle over $\text{Gr}_n(\mathbb{R}^\infty)$. 

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3.4 Construction of Stiefel-Whitney Classes

To prove that the classes $\omega_i = \omega_i(\xi)$ exist at all, we need Steenrod squares and the Thom isomorphism theorem. Recall the following definition from algebraic topology:

**Definition.** The **Steenrod squares** for a pair of spaces $(X,Y)$ are maps $Sq^i : H^n(X,Y; \mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(X,Y; \mathbb{Z}/2\mathbb{Z})$ satisfying

1. $Sq^i = 0$ for all $i > n$.
2. $Sq^0 : H^n(X,Y; \mathbb{Z}/2\mathbb{Z}) \to H^n(X,Y; \mathbb{Z}/2\mathbb{Z})$ is the identity for each $n \geq 0$.
3. Each $Sq^i$ is an abelian group homomorphism.
4. For any $\alpha, \beta \in H^\bullet(X,Y; \mathbb{Z}/2\mathbb{Z})$, $Sq^i(\alpha \cup \beta) = \sum_{j=0}^{i} Sq^j(\alpha) \cup Sq^{i-j}(\beta)$.
5. The $Sq^i$ are natural, that is, for any map $f : (X',Y') \to (X,Y)$, $f^* Sq^i(\alpha) = Sq^i(f^* \alpha)$ for all $\alpha \in H^\bullet(X,Y; \mathbb{Z}/2\mathbb{Z})$.

**Definition.** The **total Steenrod square** is the map

$$Sq : H^\bullet(X,Y; \mathbb{Z}/2\mathbb{Z}) \to H^\bullet(X,Y; \mathbb{Z}/2\mathbb{Z})$$

$$\alpha \mapsto \sum_{i=0}^{\infty} Sq^i(\alpha).$$

**Remark.** By Axiom 4, $Sq(\alpha \cup \beta) = Sq(\alpha) \cup Sq(\beta)$ for any cohomology classes $\alpha, \beta$.

Now let $\xi : E \overset{\pi}{\to} B$ be any vector bundle and let $E_0$ be the subset of $E$ consisting of all nonzero vectors in each fibre. Fixing a metric on $\xi$, in each fibre $\pi^{-1}(x)$ we can consider all vectors $v$ satisfying $||v|| \leq 1$. This defines a unit disk bundle as a subbundle of $\xi$:

$$\begin{array}{c}
D \\
\downarrow \\
B
\end{array} \quad \begin{array}{c}
\to
\end{array} \quad \begin{array}{c}
E
\end{array}$$

Let $S = \partial D$ be the unit sphere bundle which is a subbundle of $D$. Then there is a deformation retract $(E, E_0) \to (D, S)$ inducing an isomorphism

$$H^i(E, E_0; \mathbb{Z}/2\mathbb{Z}) \cong H^i(D, S; \mathbb{Z}/2\mathbb{Z})$$

for each $i \geq 0$. Further, for each $x \in B$, the fibre $F = \pi^{-1}(x)$ has a set of nonzero vectors $F_0 = E_0 \cap F$ and an isomorphism

$$H^i(F, F_0; \mathbb{Z}/2\mathbb{Z}) \cong H^i(D^n, S^{n-1}; \mathbb{Z}/2\mathbb{Z})$$

$$\cong \begin{cases} 
\mathbb{Z}/2\mathbb{Z}, & i = n \\
0, & i \neq n.
\end{cases}$$
Theorem 3.4.1 (Thom Isomorphism Theorem). There is a unique fundamental class \( u \in H^n(E, E_0; \mathbb{Z}/2\mathbb{Z}) \) which restricts to the nontrivial class in each fibre \( H^n(F, F_0; \mathbb{Z}/2\mathbb{Z}) \). Moreover, for all \( i \geq 0 \), there is an isomorphism

\[
\varphi : H^i(B; \mathbb{Z}/2\mathbb{Z}) \overset{\pi_*}{\longrightarrow} H^i(E; \mathbb{Z}/2\mathbb{Z}) \overset{\varphi}{\longrightarrow} H^{n+i}(E, E_0; \mathbb{Z}/2\mathbb{Z}).
\]

The map \( \varphi : \alpha \mapsto \pi^*(\alpha) \cup u \) is called the Thom isomorphism. Notice that we must have \( u = \varphi(1) \) where 1 is the generator of \( H^0(B; \mathbb{Z}/2\mathbb{Z}) \).

Theorem 3.4.2. For any vector bundle \( \xi : E \overset{\pi}{\rightarrow} B \), the classes

\[
\omega_i(\xi) := \varphi^{-1} Sq^i(u) \in H^i(B; \mathbb{Z}/2\mathbb{Z})
\]

satisfy the Stiefel-Whitney axioms.

Proof. Most of these follow from the axioms for Steenrod squares. For example, axiom 0 for Steenrod squares implies \( \omega_i = 0 \) for \( i > n \). Likewise, naturality of the \( \omega_i \) follows from the naturality of Steenrod squares and naturality of the Thom isomorphism. Thus we have axioms (1), (3) and (4) to check for the Stiefel-Whitney classes.

(1) By definition, \( \omega_0(\xi) = \varphi^{-1} Sq^0(\varphi(1)) = \varphi^{-1} \varphi(1) = 1 \) since \( \varphi \) is an isomorphism.

(3) Consider two bundles \( \xi : E_1 \rightarrow B_1 \) and \( \eta : E_2 \rightarrow B_2 \). Then \( \omega_i(\xi) = \varphi^{-1} Sq^i(u_1) \) where \( u_1 \in H^m(E_1, (E_1)_0) \) is the fundamental class for \( \xi \) and \( \omega_i(\eta) = \varphi^{-1} Sq^i(u_2) \) where \( u_2 \in H^m(E_2, (E_2)_0) \) is the fundamental class for \( \eta \). We have

\[
u_1 \times u_2 \in H^{m+n}(E_1 \times E_2, (E_1)_0 \times E_2 \cup E_1 \times (E_2)_0) = H^{m+n}(E_1 \times E_2, (E_1 \times E_2)_0).
\]

But the fundamental class \( u \in H^{m+n}(E_1 \times E_2, (E_1 \times E_2)_0) \) is the unique class restricting nontrivially to each fibre \((F, F_0)\). On \( F = (\pi_1 \times \pi_2)^{-1}(x) \), we see that \( u_1 \times u_2 \) restricts nontrivially, since \( F = F_1 \times F_2 = \pi_1^{-1}(x) \times \pi_2^{-1}(x) \), so uniqueness implies \( u = u_1 \times u_2 \). Finally, by properties of the Steenrod squares we get \( Sq(u_1 \times u_2) = Sq(u_1) \times Sq(u_2) \), where \( \times \) is the algebraic cross product. The third Stiefel-Whitney axiom now follows from the first remark in Section 3.2.

(4) Consider \( \omega_1(\gamma_1^1) \in H^1(\mathbb{R}P^1; \mathbb{Z}/2\mathbb{Z}) \), where \( \gamma_1^1 \) is the tautological line bundle. For the fundamental class \( u \in H^1(E, E_0; \mathbb{Z}/2\mathbb{Z}) \), we have

\[\omega_1 = \varphi^{-1} Sq^1(u) = \varphi^{-1}(u \cup u).\]

By the remarks above, \( H^i(E, E_0) \cong H^i(M, \partial M) \), where \( M \) is the Möbius band and \( \partial M \cong S^1 \) is its boundary circle. Under this identification, \( u \) corresponds to the core circle \( S^1 \hookrightarrow M \). Using intersection products, \( u \cup u \) corresponds to the intersection of two circles in general position in \( M \), which must consist of exactly one point. Hence \( u \cup u \neq 0 \), so \( \omega_1 \neq 0 \). Another way to see this is through the excision isomorphism:

\[H^i(M, \partial M) \cong H^i(\mathbb{R}P^2, D^2) \cong H^i(\mathbb{R}P^2)\]

(since \( D^2 \) is contractible in the last step). Thus \( u \) must correspond to \( a \neq 0 \) in \( H^1(\mathbb{R}P^2) \), and since \( a^2 \neq 0 \) in \( H^2(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) \), we must have \( u \cup u \neq 0 \) in \( H^2(E, E_0; \mathbb{Z}/2\mathbb{Z}) \). \( \square \)
3.5 Oriented Bundles

Recall the definition of an orientation on a manifold \( M \). First, on \( T_pM \) for \( p \in M \), an orientation is an equivalence class of ordered bases for \( T_pM \), where two bases \( \mathcal{B} \) and \( \mathcal{B}' \) are said to be equivalent if the change-of-basis matrix \( \mathcal{B} \to \mathcal{B}' \) has positive determinant. Since \( \det \) takes values in \( \mathbb{R} \setminus \{0\} \), there are two equivalence classes and thus two orientations on each \( T_pM \). We say \( M \) is orientable if there is an atlas \( M = \bigcup U_i \) with diffeomorphisms \( \varphi_i : U_i \to \varphi_i(U_i) \subseteq \mathbb{R}^n \) such that the transition maps \( \varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j) \) preserve orientation. That is, for all \( x \in \varphi_i(U_i \cap U_j) \), the differential \( d_x \varphi_{ij} \) has positive determinant.

Equivalently, an orientation on \( M \) is a choice of orientation on each tangent space \( T_pM \) so that if \( p \in U_i \), the differential \( d_p \varphi_i : T_pM \to \mathbb{R}^n \) takes the class of bases of \( T_pM \) to the standard basis \( \{e_1, \ldots, e_n\} \) on \( \mathbb{R}^n \).

One perspective is that an orientation on \( M \) is really just an orientation on the tangent bundle. This can be generalized to arbitrary vector bundles as follows. Let \( \xi : E \xrightarrow{\pi} B \) be a vector bundle with local trivializations \( B = \bigcup U_i, \; h_i : U_i \times \mathbb{R}^n \to \pi^{-1}(U_i) \).

**Definition.** An orientation on \( \xi \) is a choice of orientation on each vector space \( \pi^{-1}(x) \) such that if \( x \in U_i \), the restriction \( h_i|_{\{x\} \times \mathbb{R}^n} : \{x\} \times \mathbb{R}^n \to \pi^{-1}(x) \) takes the standard orientation on \( \mathbb{R}^n \) to the chosen orientation on \( \pi^{-1}(x) \). We say \( \xi \) is orientable if such an orientation exists.

An alternative perspective can be phrased in terms of cohomology. Recall that for \( F = \pi^{-1}(x) \), with nonzero vectors \( F_0 = F \setminus \{0\} \), we have isomorphisms
\[
H^n(F, F_0; \mathbb{Z}) \cong H^n(D^n, S^{n-1}; \mathbb{Z}) \cong H^{n-1}(S^{n-1}; \mathbb{Z}) \cong \mathbb{Z}.
\]
Since \( (D^n, S^{n-1}) \) is homotopy equivalent to \( (\Delta^n, \partial \Delta^n) \) for an \( n \)-simplex \( \Delta^n \), we can view \( \Delta^n \subseteq \mathbb{R}^n \) with ordered vertices \( v_0, \ldots, v_n \). This defines an orientation \([v_0 v_1, \ldots, v_n] \) on \( \mathbb{R}^n \) and also a choice of generator of
\[
H_{n-1}(\partial \Delta^n) \cong H_n(\Delta^n, \partial \Delta^n) \cong H^n(\Delta^n, \partial \Delta^n) \cong H^n(D^n, S^{n-1}).
\]

**Proposition 3.5.1.** An orientation on \( \xi \) is a choice of generator for \( H^n(F, F_0; \mathbb{Z}) \) for each fibre \( F = \pi^{-1}(x) \) such that if \( x \in U_i \), the local trivialization \( h_i : U_i \times \mathbb{R}^n \to \pi^{-1}(U_i) \) induces \( h_i^* : H^n(F, F_0; \mathbb{Z}) \to H^n(\{x\} \times \mathbb{R}^n, \{x\} \times \mathbb{R}_0^n; \mathbb{Z}) \) mapping the chosen generator to the standard generator.

There is a version of the Thom isomorphism theorem (3.4.1) for cohomology with \( \mathbb{Z} \)-coefficients.

**Theorem 3.5.2** (Thom Isomorphism Theorem for Oriented Bundles). Let \( \xi \) be an oriented \( n \)-bundle and specify a generator \( u_F \in H^n(F, F_0; \mathbb{Z}) \) for each fibre \( F \). Then there exists a unique class \( u \in H^n(E, E_0; \mathbb{Z}) \) which restricts to \( u_F \) on each fibre and such that for all \( i \geq 0 \), the map
\[
H^i(E; \mathbb{Z}) \xrightarrow{-\cup u} H^{n+i}(E, E_0; \mathbb{Z})
\]
is an isomorphism.

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Notice that if $B$ is connected, Theorem 3.5.2 says that

$$H^n(E, E_0; \mathbb{Z}) \cong H^0(E; \mathbb{Z}) \cong H^0(B; \mathbb{Z}) \cong \mathbb{Z}$$

so the fundamental class $u \in H^n(E, E_0; \mathbb{Z})$ is just one of the two generators.

Let $j : (E, \emptyset) \to (E, E_0)$ be the inclusion and $\pi : E \to B$ be a bundle projection. Then we have isomorphisms $H^n(E, E_0; \mathbb{Z}) \overset{j^*}{\to} H^n(E; \mathbb{Z}) \overset{\pi^*}{\leftarrow} H^n(B; \mathbb{Z})$.

**Definition.** For an oriented bundle $\xi : E \overset{\pi}{\to} B$, the Euler class of $\xi$ is $e(\xi) := (\pi^*)^{-1}j^*(u)$, where $u \in H^n(E, E_0; \mathbb{Z})$ is the fundamental class.

Although Steenrod squares are only defined for $\mathbb{Z}/2\mathbb{Z}$-coefficients, the top Steenrod square is a cup product, $\text{Sq}^n(\alpha) = \alpha \cup \alpha$, which is defined for any coefficients. Thus we have a similar definition of the Euler class as for the top Stiefel-Whitney class.

**Lemma 3.5.3.** For a bundle $\xi$, $e(\xi) = \varphi^{-1}(u \cup u)$ where $u$ is the fundamental class and $\varphi : H^n(B; \mathbb{Z}) \to H^{2n}(E, E_0; \mathbb{Z})$ is the map $\alpha \mapsto \pi^*(\alpha) \cup u$.

**Proof.** Applying $\varphi$ to $\varphi^{-1}(u \cup u)$ of course gives $u \cup u$, while for $(\pi^*)^{-1}j^*(u)$ it gives $\varphi(\pi^*)^{-1}j^*(u) = j^*(u) \cup u$ by definition of $\varphi$. Then by properties of the cup product, $j^*(u) \cup u = u \cup u$ since $j$ is just an inclusion. Since $\varphi$ is an isomorphism, it follows that $\varphi^{-1}(u \cup u) = (\pi^*)^{-1}j^*(u) = e(\xi)$.

**Corollary 3.5.4.** For any $n$-bundle $\xi$, $\omega_n(\xi) = e(\xi) \mod 2$.

**Corollary 3.5.5.** For any bundle $\xi$ with Euler class $e(\xi)$,

1. (Naturality) If $f : B' \to B$ is a map, then $e(f^*\xi) = f^*e(\xi)$.
2. If $\eta$ is another bundle, then $e(\xi \oplus \eta) = e(\xi) \cup e(\eta)$.
3. If $\xi$ has a nonvanishing section then $e(\xi) = 0$.

**Remark.** Suppose $\varphi$ is an oriented $n$-bundle over an oriented $n$-manifold $B$. Let $s_0 : B \to E$ be the zero section and $s : B \to E$ be any other smooth section of $\xi$ whose image in $E$ lies in general position relative to $s_0(B)$. If $B$ is a closed manifold, then $s_0(B) \cup s(B)$ can only be a finite collection of points for transversality to hold. Thus $s_0(B) \bullet s(B) = e(\xi) \cap [B]$, where $\bullet$ is the algebraic intersection number and $[B] \in H_n(B; \mathbb{Z})$ is the fundamental class. A special case of this is (3) of Corollary 3.5.5, since a nonvanishing section has trivial intersection with $s_0(B)$ and thus $e(\xi) = 0$. A related fact is given in the theorem below.

**Theorem 3.5.6.** Let $M$ be an oriented $n$-manifold. Then $e(TM) \cap [M] = \chi(M)$.

**Proof.** A smooth section $s : M \to TM$ is just a smooth tangent vector field on $M$ and the points of $s_0(M) \cap s(M)$ are just the zeroes of $s$. The sign of each zero is called the index of the zero, written $I(x)$. It is well-known that one can compute the Euler characteristic using indices as follows:

$$\chi(M) = \sum_{s(x) = 0} I(x).$$

Thus we can apply the Remark to obtain the desired equality. □
3.6 Chern Classes

Recall (Section 2.4) that a structure group for $\xi$ with fibres $V$ is a subgroup of $GL(V)$ containing the images of the transition maps of $\xi$. We saw in Section 3.1 that, after endowing a bundle with a metric, we may assume the orthogonal group $O_n(\mathbb{R}) \subset GL_n(\mathbb{R})$ is a structure group for $\xi$. This observation is called a “reduction of the structure group”.

In this section we define complex vector bundles. Parallel to the linear algebra, we will see that the structure group of such a bundle may be chosen to be $U_n(\mathbb{C}) \subset GL_n(\mathbb{C})$, the group of unitary matrices.

**Definition.** A complex vector bundle of dimension $n$ is a vector bundle $\omega : E \xrightarrow{\pi} B$ such that each fibre $\pi^{-1}(x)$ has the structure of a complex vector space of dimension $n$, and the local trivializations $U \times \mathbb{C}^n \rightarrow \pi^{-1}(U)$ reduce to a complex vector space isomorphism $\{x\} \times \mathbb{C}^n \rightarrow \pi^{-1}(x)$ on each fibre.

Notice that every complex vector bundle is also a real vector bundle if we “forget” the complex structure on the fibres. In particular, a complex $n$-bundle $\omega$ is a real $2n$-bundle which we denote by $\omega_{\mathbb{R}}$.

**Example 3.6.1.** For a complex $n$-manifold $M$, the complex tangent bundle $TM$ is a complex vector bundle of dimension $n$, and therefore a real vector bundle of dimension $2n$. That is, $TM$ is a real manifold of dimension $4n$.

For a complex vector space $V$, let $J : V \rightarrow V$ be the $\mathbb{C}$-linear operator $J(\vec{v}) = i\vec{v}$. Then $J^2 = -id_V$. In fact, there is an equivalence between complex $n$-vector spaces and real $2n$-vector spaces with a linear operator $J : V \rightarrow V$ such that $J^2 = -id_V$.

Given any complex vector space $V$, there is an induced orientation on the underlying real vector space $V_{\mathbb{R}}$: if $\{x_1, \ldots, x_n\}$ is a $\mathbb{C}$-basis of $V$ then $\{x_1, Jx_1, \ldots, x_n, Jx_n\}$ is an $\mathbb{R}$-basis of $V_{\mathbb{R}}$. The fact that this is well-defined on $V_{\mathbb{R}}$ (i.e. a difference choice of $x_i$ determines the same orientation on $V_{\mathbb{R}}$) is a consequence of $GL_n(\mathbb{C})$ being connected. Similarly, we have:

**Lemma 3.6.2.** For a complex bundle $\omega$, there is an induced orientation on $\omega_{\mathbb{R}}$.

As a consequence, the Euler class $e(\omega_{\mathbb{R}}) \in H^{2n}(B; \mathbb{Z})$ is defined. This coincides with the $n$th Chern class of the bundle $\omega$.

**Definition.** For a complex $n$-bundle $\omega$ over $B$, a set of Chern classes for $\omega$ is a choice of elements $c_i = c_i(\omega) \in H^{2i}(B; \mathbb{Z})$ for each $i \geq 0$ and a total Chern class $c(\omega) = c_0(\omega) + c_1(\omega) + c_2(\omega) + \ldots$ which satisfy:

1. $c_i(\omega) = 0$ for $i > n$.
2. $c_0(\omega) = 1$ in $H^0(B; \mathbb{Z})$.
3. For any map $f : B' \rightarrow B$ and each $i \geq 0$, $c_i(f^*\omega) = f^*c_i(\omega)$.
4. For the tautological line bundle $\gamma_1^1$ over $\mathbb{C}P^1$, $c_1(\gamma_1^1)$ is a generator of $H^2(\mathbb{C}P^1; \mathbb{Z})$.
In analogy to the construction of Stiefel-Whitney classes, we can consider complex Grassmannian manifolds
\[ \text{Gr}_n(\mathbb{C}^{n+k}) \xrightarrow{k \to \infty} \text{Gr}_n(\mathbb{C}^\infty). \]
Each of these has a canonical bundle \( \gamma^n(\mathbb{C}^{n+k}) \) which fit together to define a universal complex bundle \( \gamma^n \) over \( \text{Gr}_n \).

**Theorem 3.6.3.** \( H^\bullet(\text{Gr}_n; \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_n] \) for the complex Grassmannian \( \text{Gr}_n = \text{Gr}_n(\mathbb{C}^\infty) \), for generators \( c_i \in H^{2i}(\text{Gr}_n; \mathbb{Z}) \) satisfying the axioms for Chern classes on \( \text{Gr}_n \).

**Theorem 3.6.4.** Any complex bundle \( \omega \) of dimension \( n \) over \( B \) is induced by a map \( f : B \to \text{Gr}_n(\mathbb{C}^\infty) \) such that \( \omega \cong f^*\gamma^n \), where \( \gamma^n \) is the universal bundle. Moreover, there is a one-to-one correspondence between isomorphism classes of complex bundles over \( B \) and homotopy classes of maps \( B \to \text{Gr}_n(\mathbb{C}^\infty) \).

These theorems imply that Chern classes, if they exist, must be unique. Now the construction of \( c_i(\omega) \) proceeds as in Section 3.4, using the oriented version of the Thom isomorphism theorem (3.5.2).

Alternatively, for a complex bundle \( \omega : E \xrightarrow{\pi} B \), one can define \( c_n(\omega) := e(\omega_R) \in H^{2n}(B; \mathbb{Z}) \) using the existence of the Euler class from Section 3.5.2. Then for the set of nonzero vectors \( E_0 \subset E \), one constructs a bundle \( \omega_0 : E(\omega_0) \xrightarrow{\pi_0} E_0 \) by taking \( \pi_0^{-1}(\vec{v}) := (\text{Span}(\vec{v}))^\perp \), the orthogonal complement of the vector subspace of \( \pi^{-1}(x) \) spanned by \( \vec{v} \neq 0 \), where the orthogonal complement is taken with respect to a Hermitian inner product on \( \omega \). (The existence of such a Hermitian inner product is constructed in exactly the same manner as Euclidean metrics in Proposition 3.1.4.) Then \( \omega_0 \) is a complex \((n-1)\)-bundle. To proceed, recall the Gysin sequence from algebraic topology:

**Theorem 3.6.5 (Gysin Sequence).** For a real, oriented vector bundle \( \xi \) over \( B \) with Euler class \( e = e(\xi) \), there is an exact sequence
\[ \cdots \to H^i(B) \xrightarrow{-\cup e} H^{i+n}(B) \xrightarrow{\rho^*} H^{i+n}(E_0) \to H^{i+1}(B) \to \cdots \]
where \( \rho = \pi|_{E_0} \).

Now by induction, suppose the Chern classes are constructed for any \((n-1)\)-bundle. In particular, \( c_1(\omega_0), \ldots, c_{n-1}(\omega_0) \in H^\bullet(E_0) \) exist for the complex bundle \( \omega_0 \) defined above. Notice that when \( 1 \leq k \leq n-1 \), the Gysin sequence gives us an isomorphism
\[ H^{2k}(B) \xrightarrow{\rho^*} H^{2k}(E_0) \]
so we can define \( c_k(\omega) := (\rho^*)^{-1}c_k(\omega_0) \). Together with \( c_n(\omega) = e(\omega_R) \), this constructs the Chern classes for \( \omega \).

**Corollary 3.6.6.** If \( \omega \) has \( k \) sections which are linearly independent over \( \mathbb{C} \), then \( c_i(\omega) = 0 \) for each \( n-k+1 \leq i \leq n \).
Example 3.6.7. Let $\gamma_1^1$ be the tautological line bundle over $\mathbb{C}P^1 \cong S^2$, the Riemann sphere. Then $H^2(\mathbb{C}P^1; \mathbb{Z})$ has a canonical choice of generator agreeing with the orientation on $S^2$ induced by its complex structure. By the axioms for Chern classes, we must have $c_1(\gamma_1^1) = a \in H^2(\mathbb{C}P^1)$. Likewise, for the canonical line bundle $\gamma_n^1$ over $\mathbb{C}P^n$, we have $c_1(\gamma_n^1) = a \in H^2(\mathbb{C}P^n)$. A similar proof as the one in Example 3.2.5 for $\mathbb{R}P^n$ shows that the total Chern class of the complex projective space is
\[ c(\mathbb{C}P^n) := c(T\mathbb{C}P^n) = (1 + a)^{n+1}. \]

Example 3.6.8. Consider a complex line bundle $\omega$ over an oriented 4-manifold $M$. Let $s_0, s : M \to E = E(\omega)$ be sections intersecting in general position in $E$, with $s_0$ the 0-section. Set $S' = s_0(M) \cap s(M) \subseteq E$. Since $\omega_{\mathbb{R}}$ is a 2-dimensional real bundle, we must have $\dim S' = 2$ by transversality. Thus, since $\pi$ is a local diffeomorphism, $S = \pi(S') \subseteq M$ is a 2-dimensional (oriented) submanifold. Let $[S] \in H_2(S)$ be the fundamental class of $S$. For $i : S \to M$, we get the following duality between the fundamental classes of $M$ and $S$, given by the first Chern class:
\[ c_1(\omega) \cap [M] = i^*[S]. \]
This can be interpreted similarly for oriented bundles of arbitrary dimension.

### 3.7 Pontrjagin Classes

For any real vector bundle $\xi : E \to X$, we obtain an associated complex bundle $\xi_{\mathbb{C}} : E_{\mathbb{C}} \to X$ by setting $(E_{\mathbb{C}})_x = E_x \otimes \mathbb{C}$ for every $x \in X$. If $\xi$ is a bundle of real dimension $n$, then $\xi_{\mathbb{C}}$ is a bundle of complex dimension $n$. Further, every fibre $F_x$ of $E_{\mathbb{C}}$ has a decomposition $F_x = E_x \oplus iE_x$, so as a real vector bundle of dimension $2n$, $\xi_{\mathbb{C}} \cong \xi \oplus \xi$.

**Lemma 3.7.1.** For any real vector bundle, $\xi_{\mathbb{C}}$ is isomorphic to the complex conjugate $\overline{\xi_{\mathbb{C}}}$.

For a real bundle $\xi : E \to X$, consider the total Chern class of the associated complex bundle:
\[ c(\xi_{\mathbb{C}}) = 1 + c_1(\xi_{\mathbb{C}}) + \ldots + c_n(\xi_{\mathbb{C}}). \]
It is clear that for each $1 \leq j \leq n$, $c_j(\overline{\xi_{\mathbb{C}}}) = (-1)^j c_j(\xi_{\mathbb{C}})$, so we have
\[ c(\overline{\xi_{\mathbb{C}}}) = 1 - c_1(\xi_{\mathbb{C}}) + \ldots + (-1)^n c_n(\xi_{\mathbb{C}}). \]
Thus each $c_{2k+1}(\xi_{\mathbb{C}})$ is an element of order two in $H^*(X; \mathbb{Z})$.

**Definition.** For a real vector bundle $\xi : E \to X$, define the $j$th Pontrjagin class of $\xi$ to be
\[ p_j(\xi) := (-1)^j c_{2j}(\xi_{\mathbb{C}}) \in H^{4j}(X; \mathbb{Z}) \]
and the total Pontrjagin class to be
\[ p(\xi) := 1 + p_1(\xi) + \ldots + p_{\left\lfloor \frac{n}{2} \right\rfloor}(\xi). \]

We don’t quite have a Whitney sum formula for Pontrjagin classes, but we do have the following sum formula.
Proposition 3.7.2. Let $\xi$ and $\eta$ be two real vector bundles over $X$. Then

$$2(p(\xi \oplus \eta) - p(\xi)p(\eta)) = 0,$$

or in other words, $p(\xi \oplus \eta) \equiv p(\xi)p(\eta) \mod 2$.

Example 3.7.3. Let $X = S^n$ and $E = TS^n$. Then by Example 3.2.2, $c(TS^n) = c(TS^n \oplus \varepsilon^1) = c(\varepsilon^{n+1}) = 1$, so $p(TS^n) = 1$.

Example 3.7.4. Let $X = \mathbb{C}P^n$ and $E = T\mathbb{C}P^n$, so that by Example 3.6.7, $c(T\mathbb{C}P^n) = (1 + x)^{n+1}$ where $x$ is a generator in $H^2(\mathbb{C}P^n; \mathbb{Z})$. Then by definition of Pontrjagin classes $p_j = p_j(T\mathbb{C}P^n)$, we have

$$(1 - p_1 + \ldots (-1)^n p_n) = (1 - c_1 + \ldots (-1)^n c_n)(1 + c_1 + \ldots + c_n)$$

$$= (1 - x)^{n+1}(1 + x)^{n+1} = (1 - x^2)^{n+1}.$$

Therefore $p(T\mathbb{C}P^n) = (1 + x^2)^{n+1}$.

3.8 General Characteristic Classes

We adopt a more abstract perspective in this section to describe all possible characteristic classes of vector bundles. This affords an algebraic description of the Stiefel-Whitney, Chern and Euler classes we constructed in previous sections and also allows us access to other types of classes.

Let $K$ be a field (usually $K = \mathbb{R}$ or $\mathbb{C}$) and let $\text{Vect}_n(K)$ be the category of vector bundles with fibre $K^n$. For a base $X$, $\text{Vect}_n(K,X)$ will denote the subcategory of $\text{Vect}_n(K)$ consisting of bundles with base $X$.

Definition. For a triple $(n,p,R)$ consisting of $n,p \in \mathbb{N}_0$ and an abelian group $R$, a characteristic class for this data is a natural transformation

$$c : \text{Vect}_n(K,-) \to H^p(-;R).$$

That is, for each bundle $E \to X$, there is a class $c(E) \in H^p(X;R)$ which is natural with respect to pullback.

Example 3.8.1. If $E \to B$ is a trivial vector bundle, then it is the pullback of a vector bundle over a point, so it follows that $c(E) = 0$ for any characteristic class $c$.

Lemma 3.8.2. The set of characteristic classes for $(n,p,R)$ forms an abelian group under addition. Moreover, if $R$ is a ring then this group acquires the structure of a ring.

Proof. Pullback is natural. When $R$ is a ring, the multiplication operation on characteristic classes is given by cup product. \qed

Lemma 3.8.3. For all $n,p \in \mathbb{N}_0$ and abelian groups $R$, there is a bijection between characteristic classes for $(n,p,R)$ and the set $H^p(BGL_n(K);R)$.
3.8 General Characteristic Classes

Proof. If \( c_0 \in H^p(BGL_n(K); R) \), then for any vector \( n \)-bundle \( E \) classified by a map \( f \), \( c(E) := f^*c_0 \) defines a characteristic class for \( (n, p, R) \). Conversely, any characteristic class \( c : Vect_n(K, X) \to H^p(X; R) \) determines a class \( c_0 := c(EGL_n(K)) \in H^p(BGL_n(K); R) \), where \( f^*c(EGL_n(K)) = c(f) \) for \( f \) any classifying map of \( E \to X \). These associations are clearly inverses.

Example 3.8.4. When \( n = 1 \), we have

\[
BO(1) = B(\mathbb{Z}/2\mathbb{Z}) = \mathbb{R}^\infty / (\mathbb{Z}/2\mathbb{Z}) = \mathbb{R}P^\infty \quad \text{and} \quad H^\bullet(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\omega]
\]

\[
BU(1) = BS^1 = S^\infty / S^1 = \mathbb{C}P^\infty \quad \text{and} \quad H^\bullet(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[c]
\]

for generators \( \omega \in H^1(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \) and \( c \in H^2(\mathbb{C}P^\infty; \mathbb{Z}) \). So in some sense, the only “interesting” characteristic class of a line bundle is for \( p = 1 \). By Proposition 3.3.6 and Theorem 3.6.4, these are the first Stiefel-Whitney class \( \omega_1(L) \in H^1(X; \mathbb{Z}/2\mathbb{Z}) \) and the first Chern class \( c_1(L) \in H^2(X; \mathbb{Z}) \). Altogether, the classification of line bundles can be summarized by the isomorphism

\[
\text{Vect}_1(\mathbb{R}, X) = [X, \mathbb{R}P^\infty] = [X, K(\mathbb{Z}/2\mathbb{Z}, 1)] \cong H^1(X; \mathbb{Z}/2\mathbb{Z})
\]

\[
\text{Vect}_1(\mathbb{C}, X) = [X, \mathbb{C}P^\infty] = [X, K(\mathbb{Z}, 2)] \cong H^2(X; \mathbb{Z}).
\]

These are a rather special case, due to the fact that \( \mathbb{R}P^\infty = K(\mathbb{Z}/2\mathbb{Z}, 1) \) and \( \mathbb{C}P^\infty = K(\mathbb{Z}, 2) \).

Definition. Let \( p : E \to X \) be a \( K \)-vector bundle of dimension \( n \). The associated projective bundle is a fibre bundle \( \mathbb{P}E \to X \) with fibre \( KP^{n-1} \), the projective \((n - 1)\)-space over \( K \), and total space

\[ \mathbb{P}E = \{(x, \ell) \in B \times KP^{n-1} : \ell \text{ is a line in } E_x = p^{-1}(x)\}. \]

Equivalently, \( \mathbb{P}E = F(E) \times_{GL_n(K)} KP^{n-1} \), the associated bundle for the projective action of \( GL_n(K) \) on \( KP^{n-1} \).

Each fibre of \( \mathbb{P}E \) admits a tautological line bundle which induces a line bundle \( L \to \mathbb{P}E \) given by

\[ L = \{(x, \ell, v) \in PE \times K^n : v \in \ell\}. \]

Since \( L \) is a line bundle, it has a distinguished characteristic class \( q = \omega_1(L) \in H^1(\mathbb{P}E; \mathbb{Z}/2\mathbb{Z}) \) when \( K = \mathbb{R} \) or \( q = c_1(L) \in H^2(\mathbb{P}E; \mathbb{Z}) \) when \( K = \mathbb{C} \).

Lemma 3.8.5. Let \( R = \mathbb{Z}/2\mathbb{Z} \) when \( K = \mathbb{R} \) and \( R = \mathbb{Z} \) when \( K = \mathbb{C} \). The classes \( 1, q, q^2, \ldots, q^{n-1} \in H^\bullet(\mathbb{P}E; R) \) restrict to a generating set for \( H^\bullet(KP^{n-1}; R) \) under the fibre inclusion \( KP^{n-1} \hookrightarrow \mathbb{P}E \).

Proof. We give the proof for \( K = \mathbb{R} \), but the proof for \( K = \mathbb{C} \) is identical, with appropriate dimension changes.

If \( i : RP^{n-1} \hookrightarrow \mathbb{P}E \) is the inclusion of the fibre over \( x \in X \), then \( i^*(L) = L_0 \) is the the tautological line bundle on \( RP^{n-1} \). Thus by naturality of characteristic classes,

\[ i^*q = i^*\omega_1(L) = \omega_1(L_0) \in H^1(\mathbb{R}P^{n-1}; \mathbb{Z}/2\mathbb{Z}). \]
Notice that the inclusion \( \mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^{\infty} \) is a classifying map for the tautological bundle over \( \mathbb{R}P^{n-1} \) (this follows from the proof of Theorem 3.3.9). Then the induced map on cohomology is

\[
\mathbb{Z}/2\mathbb{Z} = H^*(\mathbb{R}P^{\infty}; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^*(\mathbb{R}P^{n-1}; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[\omega]/(\omega^n),
\]

so \( \omega \) maps to a generator \( \tilde{\omega} \) of \( H^*(\mathbb{R}P^{n-1}; \mathbb{Z}/2\mathbb{Z}) \). This proves \( \tilde{\omega} \) is the Stiefel-Whitney class of the tautological line bundle over \( \mathbb{R}P^{n-1} \) and \( i^*q_j = \tilde{\omega}^j \) for each \( 0 \leq j \leq n-1 \) by construction.

**Theorem 3.8.6 (Leray-Hirsch).** Let \( p : E \to X \) be a general fibre bundle with fibre \( F \), let \( R \) be a commutative ring and suppose

1. \( H^k(F; R) \) is a finitely generated, free \( R \)-module for all \( k \geq 0 \).
2. There exist classes \( q_1, q_2, \ldots, q_n \in H^*(E; R) \) whose restrictions \( i^*q_j \) along the inclusion \( i : F \hookrightarrow E \) form a basis for the cohomology ring \( H^*(F; R) \).

Then \( H^*(E; R) \) is a free module over \( H^*(X; R) \) with basis \( \{q_1, q_2, \ldots, q_n\} \).

**Proof.** It is equivalent to prove that the map

\[
\Phi : H^*(X; R) \otimes H^*(F; R) \longrightarrow H^*(E; R)
\]

\[
x \otimes i^*q_j \longmapsto p^*x \cup q_j
\]

is an isomorphism of \( R \)-modules. First assume \( X \) is a finite dimensional CW-complex. In this case, we induct on its dimension \( d \), with the case \( d = 0 \) being trivial. For \( d > 0 \), let \( X' \) be the complement of the centers of all the \( d \)-cells of \( X \) (viewing them as disks) and let \( E' \) be the pullback of \( E \) along the inclusion \( X' \hookrightarrow X \). We will drop the coefficients in \( H^*(-; R) \) for convenience. Consider the diagram

\[
\begin{array}{cccccc}
\cdots & \longrightarrow & H^k(X, X') \otimes H^*(F) & \longrightarrow & H^k(X) \otimes H^*(F) & \longrightarrow & H^k(X') \otimes H^*(F) & \longrightarrow & \cdots \\
\Phi_{X,X'} \downarrow & & \Phi_X \downarrow & & \Phi_{X'} \downarrow & & \\
\cdots & \longrightarrow & H^k(E, E') & \longrightarrow & H^k(E) & \longrightarrow & H^k(E') & \longrightarrow & \cdots
\end{array}
\]

The top row is obtained by tensoring the long exact sequence in cohomology for the pair \( (X, X') \) with \( H^*(F) \), which remains exact since \( H^*(F) \) is free. The bottom row is the long exact sequence for the pair \( (E, E') \). The vertical arrow \( \Phi_{X,X'} \) can be defined in analogy to the \( \Phi \) in the theorem using relative cup products. Since pullbacks and cup products are natural, the entire diagram commutes so by the Five Lemma from commutative algebra, if two out of the three \( \Phi \) are isomorphisms, so is the third. Now notice that \( \Phi_{X'} \) is an isomorphism because \( X' \) deformation retracts to the \((d-1)\)-skeleton \( X^{(d-1)} \) by construction, meaning \( p^{-1}(X^{(d-1)}) \to E' \) is a weak homotopy equivalence. One can prove that \( \Phi_{X,X'} \) is also an isomorphism by excision, so it follows that \( \Phi_X \) is an isomorphism by induction.

For an arbitrary CW-complex \( X \), the pair \( (X, X^{(d)}) \) is \( d \)-connected by homotopy theory, and one can show this implies \( (E, p^{-1}(X^{(d)})) \) is also \( d \)-connected. As a result, the diagram
Theorem 3.8.6 is a special case of the Leray-Serre spectral sequence, which, for any fibration $E \to X$ with homotopy fibre $F$, has an $E_2$-page $E_2^{p,q} = H^p(X; R) \otimes H^q(F; R)$ and differential $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$. The inclusion $F \hookrightarrow E$ and the corresponding restriction map $H^\bullet(E) \to H^\bullet(F)$ induce a map

$$H^\bullet(E) \to E_\infty^\bullet \to H^0(X) \otimes H^q(F)$$

explicitly given by $c \mapsto 1 \otimes i^*c$, whose image lies in the kernels of all of the $d_r$. When $E \to X$ is a fibre bundle satisfying the conditions of Theorem 3.8.6, restriction $H^\bullet(E) \to H^\bullet(F)$ is surjective, so $H^\bullet(F) \subseteq \ker d_r$ for all $r$ and therefore there are no nonzero differentials on the $E_2$-page of the spectral sequence. Hence $H^\bullet(X) \otimes H^\bullet(F) \to H^\bullet(E)$ is an isomorphism.

Let $E \to X$ be a real, dimension $n$ vector bundle and let $\mathbb{P}E \to X$ be the associated projective bundle with tautological line bundle $L \to \mathbb{P}E$. As we saw, there is a Stiefel-Whitney class $q = \omega_1(L) \in H^1(\mathbb{P}E; \mathbb{Z}/2\mathbb{Z})$ with the property that $1, q, q^2, \ldots, q^{n-1}$ generate $H^\bullet(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$. That is, $\mathbb{P}E$ satisfies the hypotheses of the Leray-Hirsch theorem, so we have proven:

**Corollary 3.8.7.** For all real vector bundles $E \to X$, the cohomology ring $H^\bullet(\mathbb{P}E; \mathbb{Z}/2\mathbb{Z})$ is a free module over $H^\bullet(X; \mathbb{Z}/2\mathbb{Z})$ with basis $\{1, q, q^2, \ldots, q^{n-1}\}$.

**Corollary 3.8.8.** The Stiefel-Whitney classes of the bundle $E \to X$ are the unique classes $\omega_1(E), \ldots, \omega_n(E) \in H^\bullet(X; \mathbb{Z}/2\mathbb{Z})$ such that

$$q^n + \omega_1(E)q^{n-1} + \omega_2(E)q^{n-2} + \ldots + \omega_{n-1}(E)q + \omega_n(E) = 0$$

in $H^\bullet(\mathbb{P}E; \mathbb{Z}/2\mathbb{Z})$.

Similarly, when $E \to X$ is a complex vector bundle with distinguished Chern class $q = c_1(L) \in H^2(\mathbb{P}E; \mathbb{Z})$, the same argument proves the following corollaries.

**Corollary 3.8.9.** For all complex vector bundles $E \to X$, $H^\bullet(\mathbb{P}E; \mathbb{Z})$ is a free module over $H^\bullet(X; \mathbb{Z})$ with basis $\{1, q, q^2, \ldots, q^{n-1}\}$.

**Corollary 3.8.10.** The Chern classes of $E \to X$ are the unique classes $c_1(E), \ldots, c_n(E) \in H^\bullet(X; \mathbb{Z})$ such that

$$q^n + c_1(E)q^{n-1} + c_2(E)q^{n-2} + \ldots + c_{n-1}(E)q + c_n(E) = 0$$

in $H^\bullet(\mathbb{P}E; \mathbb{Z})$. 

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Example 3.8.11. These results give an explicit proof of the statement in Example 3.8.1: $E \to X$ is trivial if and only if $H^\bullet(X) \otimes H^\bullet(F) \cong H^\bullet(E)$ as rings, in which case $q^n = 0$. So $E$ is a trivial bundle if and only if $\omega_i(E) = 0$ (for real bundles) or $c_i(E) = 0$ (for complex bundles) for all $1 \leq i \leq n$.

Theorem 3.8.12. Suppose $p : E \to X$ is a vector bundle. Then there exists a space $Y$ and a map $f : Y \to X$ so that for any coefficient ring $R$,

(1) $f^* : H^\bullet(X; R) \to H^\bullet(Y; R)$ is injective.

(2) The induced bundle $f^*E \to Y$ is isomorphic to a Whitney sum of line bundles.

Proof. Let $\tilde{p} : \mathbb{P}E \to X$ be the projective bundle associated to $E$. Then by the Leray-Hirsch theorem (3.8.6), $\tilde{p}^* : H^\bullet(X; R) \to H^\bullet(\mathbb{P}E; R)$ is injective. Now the induced bundle $\tilde{p}^*E \to \mathbb{P}E$ contains the tautological line bundle $L \to \mathbb{P}E$ as a subbundle, so there is an exact sequence of vector bundles

$$0 \to L \to \tilde{p}^*E \to Q \to 0$$

where $Q$ is the quotient bundle with fibres $Q_x = (\tilde{p}^*E)_x/L_x$. This shows that $\tilde{p}^*E \cong L \otimes Q$. Since $Q$ is then a bundle of strictly smaller dimension than $E$, we may repeat this process on $q : \mathbb{P}Q \to \mathbb{P}E$ to obtain isomorphisms

$$q^*(\tilde{p}^*E) = q^*(L \otimes Q) \cong q^*L \otimes q^*Q \cong q^*L' \otimes q^*Q'$$

for some line bundle $L'$ and quotient bundle $Q'$. Continue this process until $Q'$ is a line bundle. \qed

Remark. Let $V$ be a real $n$-dimensional vector space and recall that a (complete) flag in $V$ is a sequence of subspaces

$$F_\bullet : 0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = V.$$ 

A flag is completely described by a maximal orthogonal collection of 1-dimensional subspaces (once an inner product is chosen) $V_1, \ldots, V_n$ such that $V_1 \oplus \cdots \oplus V_j = F_j$ for each $1 \leq j \leq n$. Let $\mathcal{F}(V)$ be the flag manifold of $V$, i.e. the space of all flags $F_\bullet$ in $V$ with the topology inherited from the identification $\mathcal{F}(V) \cong O(n)/O(1)^\otimes n$. Then there are tautological line bundles $L_1, \ldots, L_n$ over $\mathcal{F}(V)$, explicitly given by

$$L_j = \{((V_1, \ldots, V_n), w) \in \mathcal{F}(V) \times V \mid w \in V_j\}.$$ 

For any vector bundle $p : E \to X$, let $\mathcal{F}(p) : \mathcal{F}(E) \to X$ be the associated flag bundle with fibres $\mathcal{F}(E)_x = \mathcal{F}(E_x)$, the flag manifold on the fibre $E_x$. Let $L_1, \ldots, L_n$ be the tautological line bundles over $\mathcal{F}(E)$ as above. Then there is an isomorphism of bundles

$$L_1 \oplus \cdots \oplus L_n \xrightarrow{\cong} \mathcal{F}(p)^*E \to \mathcal{F}(E)$$
Taking $Y = F(E)$ gives an alternate proof of Theorem 3.8.12.

In the complex case, we can take $F(V) \cong U(n)/U(1)^n$ and the same construction works.

By Example 3.8.4, we know the characteristic class for any line bundle: it is the pullback along a classifying map of the generator $\omega \in H^\bullet(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z})$ in the real case and $c \in H^\bullet(\mathbb{C}P^\infty; \mathbb{Z})$ in the complex case. Now Theorem 3.8.12 says that the characteristic classes of a bundle of any dimension can (up to pullback) be written as products of characteristic classes for line bundles.

Using classifying spaces, we can give another interpretation of orientability (see Section 3.5).

**Proposition 3.8.13.** Let $p : E \to X$ of dimension $n$ be a real vector bundle. Then the following are equivalent:

1. $p$ is orientable.
2. The structure group of $p$ reduces to $SO(n)$, that is, the classifying map $f : X \to BO(n)$ lifts to $BSO(n)$:

$$
\begin{array}{ccc}
BSO(n) & \xrightarrow{\tilde{f}} & X \\
\downarrow{\pi} & & \downarrow{f} \\
BO(n) & & BO(n)
\end{array}
$$

3. $\omega_1(E) = 0$.

**Proof.** (1) $\iff$ (2) follows from Proposition 3.5.1.

(2) $\iff$ (3) We know $O(n)$ has two connected components, with $SO(n)$ identified as the connected component of the identity. Consider the short exact sequence of groups

$$
1 \to SO(n) \to O(n) \to \mathbb{Z}/2\mathbb{Z} \to 1.
$$

By the construction in Corollary 1.3.9, $BSO(n) = EO(n)/SO(n)$ and there exists a principal bundle $BSO(n) \to BO(n)$ with fibre $O(n)/SO(n) \cong \mathbb{Z}/2\mathbb{Z}$. Note that $\pi : BSO(n) \to BO(n)$ is a double covering, so by covering theory a classifying map $f : X \to BO(n)$ lifts to $BSO(n)$ if and only if $f_*(\pi_1(X)) \subseteq \pi_*(\pi_1(BSO(n)))$. Since $SO(n) \cong \Omega BSO(n)$, $\pi_1(SO(n)) \cong \pi_{i+1}(BSO(n))$ for all $i$. In particular,

$$
\pi_1(BO(n)) \cong \pi_0(O(n)) = \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \pi_1(BSO(n)) \cong \pi_0(SO(n)) = 1
$$

so we see that $\pi : BSO(n) \to BO(n)$ is a universal cover. Thus $f : X \to BO(n)$ lifts if and only if $f_* : \pi_1(X) \to \pi_1(BO(n))$ is trivial. This in turn is equivalent to $f_* : H_1(X; \mathbb{Z}) \to \pi_1(BO(n))$ being trivial, since $\pi_1(BO(n)) = \mathbb{Z}/2\mathbb{Z}$ is abelian. Notice that

$$
f_* \in \text{Hom}(H_1(X; \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \cong \text{Hom}(H_1(X; \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})
$$

$$
\cong \text{Hom}(H_1(X; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \quad \text{by universal coefficients}.
$$
So we see that \( f_* = 0 \) in \( \text{Hom}(H_1(X; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \) is equivalent to having \( f^* = 0 \) in \( \text{Hom}(H^1(BO(n); \mathbb{Z}/2\mathbb{Z}), H^1(X; \mathbb{Z}/2\mathbb{Z})) \), which in turn is equivalent to \( f^* \omega_1 = \omega_1(E) \), where \( \omega_1 \) is a generator in \( H^1(BO(n); \mathbb{Z}/2\mathbb{Z}) \).

**Definition.** A relative fibre bundle (or bundle pair) is a pair of spaces \((E, E')\) with \( E' \subseteq E \) and a fibre bundle \( p : E \to X \) such that \( p' := p|_{E'} : E' \to X \) is a fibre bundle and, if \( F \) (resp. \( F' \)) is the fibre of \( p \) (resp. \( p' \)), then the structure group \( G \) of \( p \) preserves \( F' \) as a subset of \( F \).

**Example 3.8.14.** Let \( p : E \to X \) be a vector bundle and \( \sigma : X \to E \) the zero section, i.e. \( \sigma(x) = 0 \in p^{-1}(x) \) for all \( x \in X \). Then \( E' = E \setminus \sigma(X) \) forms a bundle pair \((E, E')\) with fibre pair \((\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})\) which retracts to the pair \((D^n, D^n \setminus \{0\})\).

Given a bundle pair \((E, E')\) \to X with structure group \( G \), given by a homomorphism \( \rho : G \to \text{Homeo}(F, F') \) – the subgroup of \( \text{Homeo}(F) \) preserving \( F' \) – there is an induced action

\[
\rho_\ast : G \to \text{Aut}(H^n(F, F'; R))
\]

for any coefficient ring \( R \) and any \( n \geq 0 \). Let \( \mathcal{H}^n(F, F'; R) \to X \) be the associated bundle

\[
\mathcal{H}^n(F, F'; R) := P_G \times_G H^n(F, F'; R) \to X,
\]

where \( P_G \) is the principal \( G \)-bundle associated to \( E \).

**Definition.** Let \( p : E \to X \) be a real vector bundle of dimension \( n \) and set \( E' = E \setminus \sigma(x) \). The orientation bundle for \( E \) with coefficients in \( R \) is the bundle

\[
\Theta_R(E) := \mathcal{H}^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R) \to X.
\]

We say \( E \to X \) is \( R \)-orientable if \( \Theta_R(E) \) is a trivial bundle.

Since \( H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R) \cong R \), a bundle \( E \to X \) is \( R \)-orientable if and only if \( \Theta_R(E) \) admits a section \( \sigma : X \to \Theta_R(E) \) such that \( \sigma(x) \) is a unit in \( R \) for all \( x \in X \).

**Example 3.8.15.** For \( R = \mathbb{Z}/2\mathbb{Z} \) and any bundle \( E \to X \), \( \Theta_{\mathbb{Z}/2\mathbb{Z}}(E) \) always has a section given by \( x \mapsto \sigma(x) \) where \( \sigma(x) \) is the unique non-zero element of \( H^n(E_x, E'_x; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \). That is, every vector bundle is \( \mathbb{Z}/2\mathbb{Z} \)-orientable.

**Proposition 3.8.16.** A vector bundle \( E \to X \) is orientable if and only if it is \( \mathbb{Z} \)-orientable, that is, if and only if \( \Theta_{\mathbb{Z}}(E) \) is trivial.

**Proof.** We claim that \( \Theta_{\mathbb{Z}}(E) \otimes \mathbb{R} \cong \wedge^n E \), where \( n \) is the dimension of \( E \) as a bundle. Given transition functions

\[
g_{\alpha\beta} : U_\alpha \cap U_\beta \to O(n)
\]

for \( E \), the transition functions for \( \wedge^n E \) are given by \( h_{\alpha\beta} : \det g_{\alpha\beta} : U_{\alpha\beta} \to O(n) \to \{\pm 1\} \). On the other hand, the transition maps for \( \Theta_{\mathbb{Z}}(E) \) are given by the associated bundle construction,

\[
(g_{\alpha\beta})_\ast : H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}) = \mathbb{Z} \to \mathbb{Z}.
\]
That is, \((g_{\alpha \beta})_* \in \text{Aut}(\mathbb{Z}) = \{\pm 1\}\) and \((g_{\alpha \beta})_* = 1\) if and only if \(g_{\alpha \beta}\) is orientation-preserving, i.e. \(\det(g_{\alpha \beta}) = 1\). Tensoring with \(\mathbb{R}\), we see that \(\Theta_{\mathbb{Z}}(E) \otimes \mathbb{R}\) has precisely the same transition maps as \(\wedge^n E\), so \(\Theta_{\mathbb{Z}}(E) \otimes \mathbb{R} \cong \wedge^n E\) as claimed.

Now, it’s easy to see that \(\wedge^n E\) being trivial is equivalent to \(\omega_1(E) = 0\), so by Proposition 3.8.13, \(E \to X\) is orientable if and only if \(\Theta_{\mathbb{Z}}(E)\) is trivial. \(\square\)

**Corollary 3.8.17.** For any vector bundle \(E \to X\), there is a double covering \(q : \tilde{X} \to X\) such that \(q^* E \to \tilde{X}\) is orientable, and \(\tilde{X}\) is connected if and only if \(E \to X\) is not orientable.

By the results above, a bundle \(E \to X\) is orientable if there exists a class \(u \in H^n(E, E'; \mathbb{R})\) with the property that \(u|_{E_x} \in H^n(E_x, E'_x; \mathbb{Z})\) is a generator for all \(x \in X\). This generalizes to any coefficient ring \(R\) as follows.

**Definition.** A class \(u \in H^n(E, E'; R)\) is called an \(R\)-Thom class for the bundle \(E \to X\) if for every \(x \in X\), \(u_x := u|_{E_x}\) is a generator in \(H^n(E_x, E'_x; R)\). When \(R = \mathbb{Z}\), such a \(u\) is simply called a Thom class.

Note that a Thom class \(u \in H^n(E, E'; \mathbb{Z})\) is precisely the class prescribed by the Thom isomorphism theorem (3.4.1). To relate this class to orientability, we need a relative version of Theorem 3.8.6.

**Theorem 3.8.18** (Relative Leray-Hirsch). Suppose \((E, E') \to X\) is a relative fibre bundle such that \(H^\bullet(F, F'; R)\) is a finitely generated, free \(R\)-module and there exist classes \(q_1, \ldots, q_n \in H^\bullet(E, E'; R)\) whose restrictions are free generators of each \(H^\bullet(E_x, E'_x; R), x \in X\). Then \(H^\bullet(E, E'; R)\) is free over \(H^\bullet(X; R)\) with basis \(\{q_1, \ldots, q_n\}\).

**Proof.** Let \(\hat{p} : \hat{E} \to X\) be the relative mapping cylinder of \(p\) defined by \(\hat{E} = E \cup M_{p'}\), where \(p' = p|_E : E' \to X\) and

\[
M_{p'} = (E' \times [0, 1] \cup X)/(e, 1) \sim p'(e).
\]

Then \((\hat{E}, M_{p'})\) is an excisive pair with

\[
H^\bullet(\hat{E}, M_{p'}; R) \cong H^\bullet(E, E'; R).
\]

Observe that the natural inclusion \(i_0 : X \hookrightarrow M_{p'}\) has the property that \(\hat{p} \circ i_0 = \text{id}\), i.e. \(\hat{p}\) is a retraction. Thus \(i_0^* \hat{p}^* = 0\) on \(H^\bullet(X; R)\), so \(i_0^*\) is surjective. This shows that \((\hat{E}, M_{p'})\) and \((\hat{E}, X)\) have the same cohomology, so we get along exact sequence

\[
\cdots \to H^k(\hat{E}; R) \xrightarrow{i_0^*} H^k(X; R) \to H^k(\hat{E}, X; R) \to H^{k+1}(\hat{E}; R) \to H^{k+1}(X; R) \to \cdots
\]

By exactness, \(H^\bullet(\hat{E}; R) \cong H^\bullet(\hat{E}, X; R) \oplus H^\bullet(X; R)\) and result follows from the ordinary Leray-Hirsch theorem (3.8.6) applied to the bundle \(\hat{E}\). \(\square\)

**Theorem 3.8.19.** A dimension \(n\) vector bundle \(E \to X\) is \(R\)-orientable if and only if there exists an \(R\)-Thom class \(u \in H^n(E, E'; R)\).
Proof. ( \( \iff \) ) follows from the definition of \( R \)-orientability.

( \( \implies \) ) Suppose \( E \to X \) is \( R \)-orientable. We will prove the more general statement that, when \( X \) is connected, the restriction
\[
H^k(E, E'; R) \to H^k(E_x, E'_x; R)
\]
is an isomorphism for all \( k \leq n \); the result we are after is the special case \( k = n \). Suppose \( X \) is a CW-complex of finite dimension \( d \) and induct on \( d \). Let \( U \) be the complement in \( X \) of the centers of all the \( d \)-cells in \( X \), so that \( U \) deformation retracts to \( X^{(d-1)} \). Let \( V = X \setminus X^{(d-1)} \). Then \( X = U \cup V \) and \( U \cap V \) deformation retracts to a disjoint union \( \coprod_{\sigma} S^{d-1} \) of a copy of \( S^{d-1} \) for each \( d \)-cell \( \sigma \). Consider the Mayer-Vietoris sequence (with coefficients suppressed) for \((E|_U, E'|_U), (E|_V, E'|_V)\):
\[
\cdots \to H^{n-1}(E|_{U \cap V}, E'|_{U \cap V}) \to H^n(E|_{U \cup V}, E'|_{U \cup V}) \to H^n(E|_U, E'|_U) \oplus H^n(E|_V, E'|_V) \to H^n(E|_{U \cap V}, E'|_{U \cap V}) \to \cdots
\]
By induction, \( H^{n-1}(E|_{U \cap V}, E'|_{U \cap V}) = 0 \) and we also have
\[
H^n(E|_U, E'|_U) \cong R
\]
\[
H^n(E|_V, E'|_V) \cong \bigoplus_{\sigma} R
\]
\[
H^n(E|_{U \cap V}, E'|_{U \cap V}) \cong \bigoplus_{\sigma} R.
\]
For each \( d \)-cell \( \sigma \), \( H^n(E|_{U \cap V \cap \sigma}, E'|_{U \cap V \cap \sigma}) \cong R \) and the maps
\[
R \cong H^n(E|_{U \cap \sigma}, E'|_{U \cap \sigma}) \to H^n(E|_{U \cap V \cap \sigma}, E'|_{U \cap V \cap \sigma}) \cong R
\]
\[
R \cong H^n(V|_{V \cap \sigma}, E'|_{V \cap \sigma})
\]
are both isomorphisms \( R \cong R \) since they factor through the restrictions to \((E_x, E'_x)\) for any \( x \in U \cap V \cap \sigma \). Let \( \eta_x \in H^n(E_x, E'_x) \) be the generator determined by the trivialization of \( \Theta_R(E) \). This determines units \( \eta_U \in H^n(E|_{U \cap \sigma}, E'|_{U \cap \sigma}) \) and \( \eta_V \in H^n(E|_{V \cap \sigma}, E'|_{V \cap \sigma}) \), but both of these map to the same element of \( H^n(E|_{U \cap V \cap \sigma}, E'|_{U \cap V \cap \sigma}) \) by the above maps and they depend continuously on \( x \in U \cap V \cap \sigma \). Thus the \( \eta_U, \eta_V \) are independent of \( x \). This shows that the \( \eta_U, \eta_V \) for all \( d \)-cells \( \sigma \) generate \( \ker \psi \), so \( \ker \psi \cong R \). Thus by exactness of the long sequence above, \( H^n(E|_{U \cap V}, E'|_{U \cap V}) \cong H^n(E, E') \cong \ker \psi \cong R \), and \( \Theta_R(E) \) is therefore trivial.

This proves the theorem when \( X \) is a finite-dimensional CW-complex. The general case follows by a similar argument to that in the proof of the Leray-Hirsch theorem (3.8.6). \( \square \)

Remark. The Thom class is natural: if \( E \to X \) is an oriented bundle, i.e. an orientable bundle with choice of Thom class \( u \in H^n(E, E'; R) \), then for any map \( f : Y \to X \), \( \tilde{f} : f^*E \to E \) is an orientation-preserving isomorphism on fibres, so
\[
\tilde{f}^* : H^n(E, E'; R) \to H^n(f^*E, f^*E'; R)
\]
determines a Thom class \( f^*u := \tilde{f}^*u \) for the induced bundle.
Let \( p : E \to X \) be an oriented bundle (for \( R = \mathbb{Z} \)) and let \( \sigma : X \to E \) be the zero section. Then \( p \circ \sigma = \text{id} \) and \( \sigma \circ p \) is homotopic to the identity. Consider the diagram

\[
\cdots \to H^k(E, E') \xrightarrow{i^*} H^k(E) \xrightarrow{j^*} H^k(E') \xrightarrow{\delta} H^{k+1}(E, E') \xrightarrow{\Phi^{-1}} H^{k-n}(X) \xrightarrow{\psi} H^k(X) \xrightarrow{p^*} H^k(E') \xrightarrow{\Phi} \cdots
\]

where \( \Phi(\alpha) = p^* \alpha \cup u \) is the Thom isomorphism (Theorem 3.4.1), the top row is the long exact sequence for \( (E, E') \) and the bottom row is the Gysin sequence (Theorem 3.6.5). Since \( p \) and \( \sigma \) are homotopy inverses, \( p^* \) is an isomorphism. This diagram shows that the Thom isomorphism theorem and the Gysin sequence are logically equivalent: either \( \Phi \) is an isomorphism by exactness of the bottom row and the Five Lemma, or the bottom row is defined so that the entire diagram commutes, in which case exactness follows immediately.

Note that for a class \( \alpha \in H^\bullet(X) \), commutativity gives us

\[
\psi(\alpha) = \sigma^* i^* \Phi(\alpha) = \sigma^* i^*(p^* \alpha \cup u) = \alpha \cup \sigma^* i^* u.
\]

**Definition.** The **Euler class** of an oriented bundle \( E \to X \) is the class \( e(E) := \sigma^* i^* u \in H^n(X; \mathbb{Z}) \), where \( \sigma : X \to E \) is the zero section, \( i : E \hookrightarrow (E, E') \) is the natural inclusion and \( u \in H^n(E, E'; \mathbb{Z}) \) is a Thom class.

The following properties of the Euler class from Section 3.5 are now immediate from the commutative diagram above:

- \( e(E) = \Phi^{-1}(u^2) \) (Lemma 3.5.3).
- Euler class is natural (Corollary 3.5.5(1)).
- If \( E \to X \) is trivial, then \( e(E) = 0 \).
- \( e(E_1 \oplus E_2) = e(E_1) \cup e(E_2) \) (Corollary 3.5.5(2)).
- If \( E \) admits a nonvanishing section, then \( e(E) = 0 \) (Corollary 3.5.5(3)).

**Corollary 3.8.20.** Euler class is a characteristic class of type \((n, n, \mathbb{Z})\) for oriented vector bundles.

**Remark.** Note that the converse to Corollary 3.5.5(3) does not hold in general, that is, there are vector bundles with trivial Euler class that do not admit nonvanishing global sections. However, the converse does hold when \( n \leq 2 \), or when \( n = d \) and \( X \) is a CW-complex of finite dimension \( d \).

**Example 3.8.21.** Consider an oriented \( n \)-bundle \( E \to S^k \). Then \( E \) is trivial over the open hemispheres \( S^k_+ \) and \( S^k_- \) (they are each contractible) so \( E \) is determined by a the homotopy class of a single transition map

\[
g : S^k_+ \cap S^k_- \cong S^{k-1} \to SO(n),
\]
or equivalently, by an element of \([S^{k-1}, SO(n)] = \pi_{k-1}(SO(n))/\pi_1(SO(n))\). For \(k = 4\) and small values of \(n\) for example, the category \(\text{Vect}_n^{SO}(S^k)\) of oriented \(n\)-bundles over \(S^k\) can be computed:

\[
\begin{align*}
\text{Vect}_1^{SO}(S^4) &\cong \pi_3(SO(1))/\pi_1(SO(1)) = \pi_3(*)/\pi_1(*) = 0 \\
\text{Vect}_2^{SO}(S^4) &\cong \pi_3(SO(2))/\pi_1(SO(2)) = 0 \\
\text{Vect}_3^{SO}(S^4) &\cong \pi_3(SO(3))/\pi_1(SO(3)) = \mathbb{Z}/(\mathbb{Z}/2\mathbb{Z}).
\end{align*}
\]

In particular, there is some nontrivial oriented vector bundle of dimension 3 over \(S^4\). Such a bundle \(E \to S^4\) has \(e(E) \in H^3(S^4) = 0\) but no nonvanishing sections, since otherwise it would be a Whitney sum of lower rank, nontrivial bundles, but by the above computation, none exist.

Recall that for a complex vector bundle \(p : E \to X\) of complex dimension \(n\), with associated tautological line bundle \(L \to \mathbb{CP}E\) (over its complex projective bundle) classified by \(q \in H^2(\mathbb{CP}E; \mathbb{Z})\), the Chern classes \(c_i(E) \in H^{2i}(X; \mathbb{Z})\) are the unique classes satisfying the relation

\[q^n + c_1(E)q^{n-1} + \ldots + c_{n-1}(E)q + c_n(E) = 0.\]

Every complex bundle \(E\) can be regarded as a real vector bundle of dimension \(2n\), say \(E_\mathbb{R}\), so we may consider the Stiefel-Whitney classes \(\omega_i(E_\mathbb{R})\). Let \(\mathbb{RP}E \to X\) be the associated real projective bundle for \(E_\mathbb{R}\). Then \(\mathbb{RP}E \to X\) has fibres \(\mathbb{RP}^{n-1}\) and tautological (real) line bundle \(L \to \mathbb{RP}E\) which is classified by some \(h : \mathbb{RP}E \to \mathbb{RP}^\infty\). Likewise, \(\mathbb{CP}E \to X\) has fibres \(\mathbb{CP}^{n-1}\) and tautological (complex) line bundle \(Q \to \mathbb{CP}E\) classified by \(g : \mathbb{CP}E \to \mathbb{CP}^\infty\). Define a map \(\varphi : \mathbb{RP}E \to \mathbb{CP}E\) by sending a real line \(\ell \subset E\) to the unique complex line \(\varphi(\ell)\) containing \(\ell\). There is a similarly defined map \(\bar{\varphi} : \mathbb{RP}^\infty \to \mathbb{CP}^\infty\).

**Lemma 3.8.22.** For a complex line bundle \(E \to X\) with \(g, h, \varphi, \bar{\varphi}\) as above, the diagram

\[
\begin{array}{ccc}
\mathbb{RP}E & \xrightarrow{\varphi} & \mathbb{CP}E \\
h \downarrow & & \downarrow g \\
\mathbb{RP}^\infty & \xrightarrow{\bar{\varphi}} & \mathbb{CP}^\infty
\end{array}
\]

commutes up to homotopy.

**Proof.** A classifying map for a vector bundle is equivalent to a map from the total space to \(\mathbb{R}^\infty\) (in the real case) or \(\mathbb{C}^\infty\) (in the complex case) that is a linear injection on fibres. For \(g, h\) above, fix such maps \(\tilde{g} : Q \to \mathbb{C}^\infty\) and \(\tilde{h} : L \to \mathbb{R}^\infty\). One can see that \(h\) is equal to the composition of \(L \to Q\) with \(\tilde{g}\), proving the lemma.

This allows us to make an explicit comparison between the Chern classes for \(E\) and the Stiefel-Whitney classes for \(E_\mathbb{R}\).

**Theorem 3.8.23.** For a complex vector bundle \(E \to X\) of dimension \(n\) and for all \(i \geq 0\), \(\omega_{2i+1}(E_\mathbb{R}) = 0\) in \(H^{2i+1}(X; \mathbb{Z}/2\mathbb{Z})\) and \(\omega_{2i}(E_\mathbb{R}) \equiv c_i(E) \mod 2\) in \(H^{2i}(X; \mathbb{Z}/2\mathbb{Z})\).
3.8 General Characteristic Classes

Proof. By Lemma 3.8.22, the diagram

\[
\begin{array}{ccc}
L & \xrightarrow{\phi} & Q \\
\downarrow & & \downarrow \\
\mathbb{R}P^\infty & \xrightarrow{h} & \mathbb{C}P^\infty \\
\downarrow & & \downarrow \\
\tilde{L} & \xrightarrow{\tilde{\phi}} & \tilde{Q} \\
\end{array}
\]

commutes up to homotopy. Observe that \(\tilde{\phi}\) is a fibre bundle over \(\mathbb{C}P^\infty\) with fibre \(\mathbb{R}P^1 = S^1\). The generator \(x \in H^1(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z})\) restricts to the generator of \(H^1(S^1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\), so by Leray-Hirsch (Theorem 3.8.6), \(H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z})\) is a free module over \(H^*(\mathbb{C}P^\infty; \mathbb{Z}/2\mathbb{Z})\) with basis \(\{1, x\}\). Hence \(x^2 = d_1 x + d_2\) for some classes \(d_i \in H^1(\mathbb{C}P^\infty; \mathbb{Z}/2\mathbb{Z})\), but \(x^2\) generates \(H^2(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z})\) and \(d_1 \in H^1(\mathbb{C}P^\infty; \mathbb{Z}/2\mathbb{Z}) = 0\), so we must have \(d_2 \neq 0\) in \(H^2(\mathbb{C}P^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\). In particular, \(d_2\) generates this group, so \(x^2 = d_2 = \tilde{\phi}^* c\) for some class \(c \in H^2(\mathbb{C}P^\infty; \mathbb{Z}/2\mathbb{Z})\). Set \(q = h^* x\) and \(r = q^* c\). Then by commutativity of the diagram,

\[
q^2 = (h^* x)^2 = h^*(x^2) = h^* \tilde{\phi}^* c = \varphi^* g^* c \equiv \varphi^* r \mod 2.
\]

By Corollary 3.8.10, the Chern classes \(c_i(E)\) satisfy

\[
\begin{align*}
& r^n + c_1(E)r^{n-1} + \ldots + c_{n-1}(E)r + c_n(E) = 0 \\
& \implies (\varphi^* r)^n + c_1(E)(\varphi^* r)^{n-1} + \ldots + c_{n-1}(E)(\varphi^* r) + c_n(E) \equiv 0 \pmod{2} \text{ by naturality} \\
& \implies q^n + c_1(E)q^{n-2} + \ldots + c_{n-1}(E)q^2 + c_n(E) \equiv 0 \pmod{2}.
\end{align*}
\]

Therefore the odd Stiefel-Whitney classes of \(E_{\mathbb{R}}\) are trivial and by Corollary 3.8.8, \(\omega_{2i}(E_{\mathbb{R}}) \equiv c_i(E) \mod 2.\)
4 K-Theory

In this chapter we follow Hatcher’s notes on K-theory to study the following main topics:

- Complex K-theory $K(X)$ and reduced K-theory $\tilde{K}(X)$
- Real (ordinary) K-theory $KO(X)$ and its reduced form $\tilde{KO}(X)$
- Bott periodicity for complex K-theory.

For the most part we will assume all vector bundles are complex bundles over a fixed compact base $X$ unless otherwise specified.

To first give some abstract nonsense, we record that any abelian semigroup $(S, \oplus)$ induces a canonical group in one of two (equivalent) ways:

1. Let $F(S)$ be the free abelian group generated by the elements of $S$ and let $E(S)$ be the subgroup of $F(S)$ generated by all elements of the form $(x \oplus y) - (x + y)$. The quotient group $K(S) = K_0(S) = F(S)/E(S)$ is called the Grothendieck group of $S$. Note that $i : S \to K(S), s \mapsto [s]$ is an embedding and that $K(S)$ is universal with respect to all abelian groups in the following sense: for every morphism of semigroups $S \to G$ where $G$ is an abelian group, there is a unique homomorphism of abelian groups $K(S) \to G$ making the diagram commute:

   \[
   \begin{array}{ccc}
   K(S) & \to & G \\
   \downarrow & & \downarrow \\
   S & \overset{i}{\to} & G
   \end{array}
   \]

2. The product $S \times S$ is a semigroup and the image of the diagonal map $\Delta : S \hookrightarrow S \times S$ is a sub-semigroup. We claim the quotient $(S \times S)/\Delta(S)$ is a group. An element of this quotient is $(x, y)\Delta = \{(x, y) + (a, a) \mid a \in S\}$, and addition and inverses are given by

   $$(x, y)\Delta + (x', y')\Delta = (x + x', y + y')\Delta \quad \text{and} \quad -(x, y)\Delta = (y, x)\Delta.$$ 

   The map $i : S \to (S \times S)/\Delta(S), s \mapsto (s, 0)\Delta$ in fact satisfies the same universal property as in (1), so $(S \times S)/\Delta(S) \cong K(S)$ as abelian groups. The notation in construction (2) suggests that we view $(x, y)\Delta \in K(S)$ as a formal difference $x - y$.

   When $S = \text{Vect}(X)$, the semigroup (category) of vector bundles over a compact Hausdorff space $X$ under Whitney sum, $K(X) := K(\text{Vect}(X))$ is called the $K$-theory of $X$. In the next section, we give an explicit construction of this abelian group.
4.1 K-Theory and Reduced K-Theory

Informally, the idea of $K$-theory is to construct an abelian group out of vector bundles over a fixed base, with the addition operation given by Whitney sum $\xi \oplus \eta$. We first make note of the following types of equivalence relations on vector bundles over $X$. Let $\varepsilon^n$ denote the trivial $n$-bundle $X \times \mathbb{C}^n \to X$ (or $X \times \mathbb{R}^n \to X$ in the real case).

- A natural isomorphism of bundles is denoted $\xi \cong \eta$. Isomorphism is of course an equivalence relation.
- We say $\xi$ and $\eta$ are stably equivalent, written $\xi \approx \eta$ if $\xi \oplus \varepsilon^n \cong \eta \oplus \varepsilon^n$ for some $n$. Note that for $\xi \approx \eta$ to make sense, $\xi$ and $\eta$ must have the have fibres of the same dimension. It is easy to check stable equivalence is an equivalence relation on the set of $n$-bundles over $X$ for each $n \geq 1$.
- Define a third equivalence relation by $\xi \sim \eta$ if $\xi \oplus \varepsilon^n \cong \eta \oplus \varepsilon^m$ for some $m, n$. In this case, $\xi$ and $\eta$ may have fibres of different dimensions.
- If $E_1 \xrightarrow{\xi} X$ and $E_2 \xrightarrow{\eta} X$ are the total spaces of the vector bundles over $X$, we will usually abuse notation in each of three cases above by writing $E_1 \cong E_2, E_1 \approx E_2$ or $E_1 \sim E_2$.

**Example 4.1.1.** For any sphere $S^k$, the tangent bundle $TS^k$ is stably equivalent to $\varepsilon^n$ (for any $n$) since $\nu_x^{\mathbb{R}^{k+1}} \cong \varepsilon^1$ and $TS^k \oplus \nu_x^{\mathbb{R}^{k+1}} \cong \varepsilon^{k+1}$ by Corollary 3.1.6.

Let $\xi_1 : E_1 \to X, \xi_2 : E_2 \to X$ be two finite-dimensional complex bundles over $X$. To define an abelian group structure on bundles, we introduce the formal difference $E_1 - E_2$. Two formal differences $E_1 - E_2$ and $E'_1 - E'_2$ are said to be equal if $E_1 \oplus E'_2 \approx E'_1 \oplus E_2$.

**Definition.** The **K-theory** of $X$ is the abelian group

$$K(X) = \{ E_1 - E_2 \mid E_1, E_2 \text{ are bundles over } X \}$$

(with equivalence of formal differences as defined above), equipped with the addition operation

$$(E_1 - E_2) + (E'_1 - E'_2) = E_1 \oplus E'_1 - E_2 \oplus E'_2.$$

**Proposition 4.1.2.** For any compact space $X$, $K(X)$ is an abelian group with identity class $[E - E] = [\varepsilon^0 - \varepsilon^0]$ and inverses given by $-(E_1 - E_2) = E_2 - E_1$.

**Proof.** For any bundle $E$ over $X$, notice that

$$(E_1 - E_2) + (E - E) = E_1 \oplus E - E_2 \oplus E = E_1 - E_2,$$

since $E_1 \oplus E_2 \oplus E \cong E_1 \oplus E \oplus E_2$. Therefore $E - E$ represents the identity class (for any $E$; in particular, $[\varepsilon^0 - \varepsilon^0] = 0$). Similarly, $(E_1 - E_2) + (E_2 - E_1) = 0$. □

**Corollary 4.1.3.** $K(X)$ coincides with the Grothendieck group of the semigroup $\text{Vect}(X)$. 

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Definition. The reduced K-theory of $X$ is the set of $\sim$-equivalence classes of complex vector bundles over $X$,

$$\tilde{K}(X) = \{E \to X\}/\sim$$

with addition given by $[E_1] + [E_2] = [E_1 \oplus E_2]$.

**Proposition 4.1.4.** $\tilde{K}(X)$ is an abelian group with identity class $[\varepsilon^0]$ and inverses given by $-[E] = [E^\perp]$, where $E$ is viewed as a subbundle of $X \times \mathbb{R}^N$ for some large $N$ and $E^\perp$ is the orthogonal complement of $E$ in $X \times \mathbb{R}^N$ with respect to some fixed metric on $E$.

**Proof.** It is clear that $E \oplus \varepsilon^0 \cong E$ and $E \oplus E^\perp \cong \varepsilon^N \sim \varepsilon^0$ for any $E$. \qed

**Lemma 4.1.5.** For any space $X$, there is an isomorphism $K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$.

**Proof.** Note that any class in $K(X)$ may be represented by $E - \varepsilon^n$ for some $n$, since $E_1 - E_2 = E_1 \oplus E_2^\perp - E_2 \oplus E_1^\perp - \varepsilon^n$. Define $\varphi : K(X) \to \tilde{K}(X)$ by $\varphi(E - \varepsilon^n) = [E]$. For $E, E'$, suppose $E - \varepsilon^n = E' - \varepsilon^m$ in $K(X)$. Then $E \oplus \varepsilon^m \sim E' \oplus \varepsilon^n$ but this implies $E \sim E'$. Thus $\varphi$ is well-defined. Clearly it is also surjective and an abelian group homomorphism.

To finish, note that for any bundle $E$,

$$E - \varepsilon^n \in \ker \varphi \iff E \sim \varepsilon^0 \iff E \oplus \varepsilon^k \cong \varepsilon^0 \oplus \varepsilon^\ell$$

for some $k, \ell$

$$\iff E - \varepsilon^n = E \oplus \varepsilon^k - \varepsilon^n \oplus \varepsilon^k \cong \varepsilon^\ell - \varepsilon^{n+k}$$

for some $k, \ell$.

Therefore $\ker \varphi = \{\varepsilon^a - \varepsilon^b \mid a, b \in \mathbb{Z}\}$ and there is an obvious map $\ker \varphi \to \mathbb{Z}$ given by $\varepsilon^a - \varepsilon^b \mapsto a - b$ which is an isomorphism. Define a splitting $K(X) \to \ker \varphi$ by $E \mapsto E|_{x_0} = \varepsilon^n - \varepsilon^m$ for a fixed basepoint $x_0 \in X$. This gives us the desired isomorphism $K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$ which depends on the choice of basepoint. \qed

**Proposition 4.1.6.** $K(X)$ and $\tilde{K}(X)$ are rings with multiplication given by $\otimes$ and unit represented by the trivial line bundle $\varepsilon^1$.

**Proof.** Similar to the proofs that $K(X)$ and $\tilde{K}(X)$ are abelian groups. \qed

We will commonly replace $\varepsilon^n$ with $n$ and write $nE = \underbrace{E + \ldots + E}_{n} = E \otimes \varepsilon^n$.

**Theorem 4.1.7.** $K$ and $\tilde{K}$ are both contravariant functors $\textbf{Top} \to \textbf{Rings}$. In particular, for any $f : X \to Y$ there are induced maps

$$f^* : K(Y) \to K(X)$$

$$E_1 - E_2 \mapsto f^*E_1 - f^*E_2$$

and $$f^* : \tilde{K}(Y) \to \tilde{K}(X)$$

$$E \mapsto f^*E$$

which both satisfy $f^*(E_1 \oplus E_2) \cong f^*E_1 \oplus f^*E_2$ and $f^*(E_1 \otimes E_2) \cong f^*E_1 \otimes f^*E_2$.
Definition. For a pair of spaces \((X, A)\), the relative \(K\)-theory of the pair is
\[
K(X, A) := \tilde{K}(X/A).
\]

By convention \(X/\emptyset = X \coprod \ast\), the disjoint union of \(X\) with a point, so that \(K(X, \emptyset) = \tilde{K}(X \coprod \ast) = K(X)\).

There is an alternative construction that yields relative \(K\)-theory. For a pair \((X, A)\), let \(L_n(X, A)\) be the set of equivalence classes of all sequences
\[
(E_0, E_1, \ldots, E_n; \alpha_1, \ldots, \alpha_n)
\]
where \(E_i \in \text{Vect}(X)\), \(\alpha_i : E_{i-1} \mid_A \to E_i \mid_A\) are bundle morphisms and the sequence of bundles
\[
0 \to E_0 \mid_A \xrightarrow{\alpha_1} E_1 \mid_A \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} E_n \mid_A \to 0
\]
is exact. The equivalence relation on sequences is defined as follows. Define an elementary sequence in \(L_n(X, A)\) to be a sequence of the form
\[
(0, 0, \ldots, E_{i-1}, E_i, 0, \ldots, 0; 0, \ldots, \alpha_i, \ldots, 0)
\]
where \(E_{i-1} = E_i\) and \(\alpha_i = \text{id}\). Then two elements of \(L_n(X, A)\) are called equivalent if they become isomorphic after a direct sum with some finite number of elementary sequences.

Example 4.1.8. For \(n = 1\), \(L_1(X, A)\) consists of triples \((E_0, E_1; \alpha)\) where \(\alpha : E_0 \to E_1\) is a bundle isomorphism over \(A\).

Lemma 4.1.9. For each \(n \geq 1\), the map
\[
L_n(X, A) \longrightarrow L_{n+1}(X, A)
\]
\[
(E_0, \ldots, E_n; \alpha_1, \ldots, \alpha_n) \longmapsto (E_0, \ldots, E_n, 0; \alpha_1, \ldots, \alpha_n, 0)
\]
is an isomorphism. Therefore,
\[
L(X, A) = \lim_{\to} L_n(X, A)
\]
is defined and equals \(L_n(X, A)\) for any \(n\).

Proposition 4.1.10. There is a unique natural isomorphism of functors
\[
\chi : L(X, A) \longrightarrow K(X, A)
\]
such that when \(A = \emptyset\), \(\chi(E_0, E_1, \ldots, E_n) = E_0 - E_1 + \ldots + (-1)^n E_n\) in \(K(X)\).

4.2 External Products in \(K\)-theory

Recall the algebraic cross product on cohomology, \(H^\bullet(X) \otimes H^\bullet(Y) \to H^\bullet(X \times Y)\). There is an analogous external product in \(K\)-theory, defined by:
\[
K(X) \otimes K(Y) \longrightarrow K(X \times Y)
\]
\[
\alpha \otimes \beta \longmapsto (\pi_1^* \alpha) \otimes (\pi_2^* \beta)
\]
where \(\pi_1 : X \times Y \to X\) and \(\pi_2 : X \times Y \to Y\) are the natural projections.
Proposition 4.2.1. The external product on $K$-groups is a ring homomorphism.

Definition. A functor $H : \text{Top} \times \text{Top} \rightarrow \text{GradedAbGps}$ is called a generalized (or extraordinary) cohomology theory if it satisfies the following four out of five Eilenberg-Steenrod Axioms:

1. (Homotopy) If $f \simeq g$ then $f^* = g^*$.
2. (Excision) For any pair $(X, A)$ and any open set $U \subseteq X$ such that $\overline{U} \subseteq A$, the inclusion $i : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism $i^*$.
3. (Additivity) If $P = \{x\}$ is a point space, then $H^n(P) = 0$ unless $n = 0$.
4. (Exactness) For each pair $(X, A)$, there is an exact sequence
   \[ \cdots \rightarrow H^{n-1}(A) \rightarrow H^n(X, A) \rightarrow H^n(X) \rightarrow H^n(A) \rightarrow \cdots \]

Theorem 4.2.2. $K$ and $\tilde{K}$ are generalized cohomology theories.

There is an external product for reduced $K$-theory,
\[ \tilde{K}(X) \otimes \tilde{K}(Y) \rightarrow \tilde{K}(X \wedge Y) \]

where $X \wedge Y = (X \times Y)/X \vee Y$ is the smash product of $X$ and $Y$.

Proposition 4.2.3. For a compact, Hausdorff space $X$ and a closed subset $A \subseteq X$, the sequence $A \xrightarrow{i} X \xrightarrow{p} X/A$ induces an exact sequence
\[ \tilde{K}(X/A) \xrightarrow{p^*} \tilde{K}(X) \xrightarrow{i^*} \tilde{K}(A). \]

Proof. The sequence $A \rightarrow X \rightarrow X/A$ factors through a point $A/A = *$:
\[ \begin{array}{ccc}
A & \rightarrow & X \\
& \downarrow & \downarrow \\
& * & \rightarrow \\
\end{array} \]

Therefore the induced sequence of reduced $K$-groups factors through 0:
\[ \tilde{K}(X/A) \rightarrow \tilde{K}(X) \rightarrow \tilde{K}(A) \]

so $i^*p^* = 0$. In other words im $p^* \subseteq \ker i^*$.

On the other hand, an element of $\ker i^*$ is a bundle $E \rightarrow X$ whose restriction $E|_A$ is stably trivial, i.e. $E|_A \oplus \varepsilon^n_A \cong \varepsilon^n_A$ for some trivial bundles $\varepsilon^n_A, \varepsilon^n_A$ over $A$. Consider the trivial bundle $\varepsilon^n_X$ over $X$. Then $(E \oplus \varepsilon^n_X)|_A$ is trivial since $[E \oplus \varepsilon^n_X] = [E]$. Thus there is some trivialization $h : \pi^{-1}(a) \times \mathbb{C}^n \rightarrow E|_A$. We can use this to identify all the fibres $\pi^{-1}(a)$ for $a \in A$. Define
the bundle $E/h \to X/A$ by $E/h = E/\pi^{-1}(a_1) \sim \pi^{-1}(a_2)$ for any $a_1, a_2 \in A$. One now checks that $E/h$ is locally trivial – this reduces to checking local triviality near $A/A = *$ which follows from Tietze’s extension theorem: if $E$ is trivial over $A$, there exists an open set $A \subseteq U \subseteq X$ such that $E$ is trivial over $U$. Alternatively, we can use a deformation retract $r : U \to A$ in $X$ and pull back the trivial bundle $E|_A$ to $U$. This proves $\ker i^* \subseteq \im p^*$.

**Theorem 4.2.4** (Long Exact Sequence in Reduced $K$-theory). For a closed subset $A \subseteq X$, there is a long exact sequence of abelian groups

$$\cdots \to \tilde{K}(\Sigma^2 A) \to \tilde{K}(\Sigma(X/A)) \to \tilde{K}(\Sigma X) \to \tilde{K}(\Sigma A) \to \tilde{K}(X/A) \to \tilde{K}(X) \to \tilde{K}(A).$$

**Definition.** For a space $X$, the nth reduced $K$-group is defined to be $\tilde{K}^n(X) := \tilde{K}(\Sigma^n X)$.

**Theorem 4.2.5.** For any spaces $X$ and $Y$, there is an isomorphism

$$\tilde{K}(X \times Y) \cong \tilde{K}(X \wedge Y) \oplus \tilde{K}(X) \oplus \tilde{K}(Y).$$

**Proof.** In the exact sequence

$$\tilde{K}(\Sigma(X \times Y)) \to \tilde{K}(\Sigma(X \vee Y)) \to \tilde{K}(X \wedge Y) \to \tilde{K}(X \times Y) \to \tilde{K}(X \vee Y)$$

there are splittings $\tilde{K}(\Sigma(X \vee Y)) \to \tilde{K}(\Sigma(X \times Y))$ and $\tilde{K}(X \vee Y) \to \tilde{K}(X \times Y)$. Then use the fact that $\tilde{K}(X \vee Y) \cong \tilde{K}(X) \oplus \tilde{K}(Y)$. \qed

Now fix a point $(x_0, y_0) \in X \times Y$ and suppose $E \in \ker(K(X) \to K(\{x_0\})) = \tilde{K}(X)$ and $E' \in \ker(K(Y) \to K(\{y_0\})) = \tilde{K}(Y)$. Then $p_1^*E$ is trivial over $X \times \{y_0\}$, $p_2^*E'$ is trivial over $\{x_0\} \times Y$ and therefore $p_1^*E \otimes p_2^*E'$ is trivial over $X \vee Y$. This defines the reduced external product $\tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X \wedge Y)$.

### 4.3 Bott Periodicity

One version of Bott periodicity says that for any space $X$, there is an isomorphism $\tilde{K}(X) \cong \tilde{K}(\Sigma^2 X)$, where $\Sigma$ is the reduced suspension functor. In the course of proving this, we will establish that the external product

$$K(X) \otimes K(S^2) \longrightarrow K(X \times S^2)$$

is an isomorphism. We will also compute the $K$-theory of the 2-sphere to be the polynomial ring $K(S^2) = \mathbb{Z}[H]/(H - 1)^2$.

To this end, let $H = [\gamma_1^1]$ be the equivalence class in $K(S^2)$ of the (complex) canonical line bundle over $S^2 \cong \mathbb{C}P^1$. Concretely, we will view the complex projective line as the space of homogeneous coordinates

$$\mathbb{C}P^1 = \{[z_0, z_1] : z_0, z_1 \in \mathbb{C}, [\lambda z_0, \lambda z_1] = [z_0, z_1] \text{ for all } \lambda \in \mathbb{C}^*\}.$$ 

The fibres of $\gamma_1^1$ are given by $\pi^{-1}([z_0, z_1]) = \{\lambda(z_0, z_1) \mid \lambda \in \mathbb{C}\}$. Identifying $[z_0, z_1]$ with $\frac{z_0}{z_1}$, we get a bijection between $\mathbb{C}P^1$ and the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. We may cover
the sphere with two charts, $D_0^2$ containing $0 \in \mathbb{C}$ and $D_\infty^2$ containing the point at infinity. Then explicit trivializations of the bundle $\gamma_1^1$ over $D_0^2$ and $D_\infty^2$ are given by the following nonvanishing sections:

$$
D_0^2 \to E(\gamma_1^1)
$$

$$
[z_0, z_1] \mapsto \left( \frac{z_0}{z_1}, 1 \right)
$$

and

$$
D_\infty^2 \to E(\gamma_1^1)
$$

$$
[z_0, z_1] \mapsto \left( 1, \frac{z_1}{z_0} \right).
$$

The map $\varphi : \pi^{-1}(D_0^2) \to \pi^{-1}(D_\infty^2)$ is defined on the boundary by

$$
S^1 \times \mathbb{C} \to S^1 \times \mathbb{C}
$$

$$
(z, 1) \mapsto (1, z^{-1}),
$$

so that the transition map $S^1 \to GL_1(\mathbb{C}) = \mathbb{C}^\ast$ is given by $z \mapsto$ multiplication by $z$. Let $1 = [\varepsilon^1] \in K(S^2)$ be the unit class.

**Lemma 4.3.1.** $(H - 1)^2 = 0$ in $K(S^2)$.

**Proof.** Note that $(H - 1)^2 = 0$ is equivalent to $H^2 + 1 = 2H$, so it’s enough to show $(H \otimes H) \oplus \varepsilon^1 \cong H \oplus H$. We prove this equivalence by analyzing both sides as a 2-bundle over $S^2$. The two transition maps $S^2 \to GL_2(\mathbb{C})$ are given by matrices

$$
\begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}.
$$

It’s enough to show these are homotopic, since then the two bundles are isomorphic. Using the fact that $GL_2(\mathbb{C})$ is connected, we may choose a path $\gamma : [0, 1] \to GL_2(\mathbb{C})$ with $\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then the homotopy

$$
\Phi_t = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \gamma_t \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \gamma_t
$$

has $\Phi_0 = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$ and $\Phi_1 = \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}$ so the maps are homotopic. \qed

The lemma shows there exists an injective map $\mathbb{Z}[H]/(H - 1)^2 \hookrightarrow K(S^2)$. We claim the composition

$$
K(X) \otimes \mathbb{Z}[H]/(H - 1)^2 \to K(X) \otimes K(S^2) \to K(X \times S^2)
$$

is an isomorphism. To show this, we must further analyze bundles over $X \times S^2$. Consider a bundle $E \xrightarrow{\pi} X$. Taking the Cartesian product bundle (Section 2.4) with the trivial bundle $D^2 \to D^2$ gives a bundle

$$
E \times D^2 \xrightarrow{\pi'} X \times D^2
$$

where $(\pi')^{-1}(x, z) = \pi^{-1}(x)$. This gives two bundles $E \times D_0^2 \to X \times D_0^2$ and $E \times D_\infty^2 \to X \times D_\infty^2$ which glue together along $S^1$ by an automorphism.
called a generalized clutching function. Given such a clutching function \( f \), we get a bundle over \( X \times S^2 \) by gluing \( E \times D_0^2 \coprod f_* E \times D_\infty^2 \). Denote this bundle by \([E, f]\). We will show that any bundle over \( X \times S^2 \) is of this form.

**Example 4.3.2.** The identity map is a clutching function, and for any bundle \( E \to X \), the resulting bundle \([E, id] \to X \times S^2 \) is equal to \( E \times S^2 \). On the level of \( K \)-groups, \([E, id] = E \times 1 \in K(X \times S^2)\).

**Example 4.3.3.** Define a clutching function \( f : E \times S^1 \to E \times S^1 \) by \((\vec{v}, z) \mapsto (z\vec{v}, z)\). On a fibre \( \pi^{-1}(x) \), this acts by

\[
\pi^{-1}(x) \otimes \mathbb{C} \longrightarrow \pi^{-1}(x) \times \mathbb{C} \\
\vec{v} \otimes 1 \mapsto \vec{v} \otimes z.
\]

Here, we get \([E, f] = E \times H \in K(X \times S^2)\). More generally, for each \( n \) define

\[
f_n : E \times S^1 \longrightarrow E \times S^1 \\
(\vec{v}, z) \longmapsto (z^n \vec{v}, z).
\]

Then \([E, f_n] = E \times H^n \in K(X \times S^2)\), where \( H^n = H^\otimes n \) is the \( n \)-fold tensor power of the canonical class in \( K(S^2)\).

**Lemma 4.3.4.** For any generalized clutching function \( f : E \times S^1 \to E \times S^1 \), \([E, z^n f] = [E, f] \otimes p_2^* H^n\), where \( p_2 : X \times S^2 \to S^2 \) is the canonical projection.

**Proof.** Set \( E' = [E, f] \) and \( \tilde{H} = p_2^* H \). Then \([E, z^n f] \cong [[E, f], z^n] = [E', z^n]\) so it’s enough to prove the statement when \( f = id\). In this case we have

\[
[E, z^n] = E \times D_0^2 \cup z^n E \times D_\infty^2 \\
\cong (E \times D_0^2) \otimes \tilde{H} \cup z^n (E \times D_\infty^2) \otimes \tilde{H} \\
\cong (E \times D_0^2) \otimes \tilde{H} \cup z^n (E \times D_\infty^2) \otimes \tilde{H} \\
\cong (E \times D_0^2 \cup z^n E \times D_\infty^2) \otimes \tilde{H} \\
= [E, 1] \otimes \tilde{H}^n.
\]

\(\square\)

**Lemma 4.3.5.** Any bundle over \( X \times S^2 \) is isomorphic to \([E, f]\) for some bundle \( E \to X \) and some generalized clutching function \( f \).

**Proof.** Given a bundle \( E_1 \to X \times S^2 \), consider the restrictions \( E_0 \to X \times D_0^2 \) and \( E_\infty \to X \times D_\infty^2 \). Set \( E = E_1|_{E_1 \times \{1\}} \to X \times \{1\} \) where we view \( 1 \in S^1 \subset S^2 \). Since \( D_0^2 \) is contractible, the maps \( id \) and \( i \circ p \) in the diagram
4.3 Bott Periodicity

\[ X \times D_0^2 \xrightarrow{id} X \times D_0^2 \]
\[ \begin{array}{c}
\downarrow p \\
\scriptstyle i \\
X \times \{1\}
\end{array} \]

are homotopic. Since homotopic maps induce isomorphic bundles, we get
\[ E_0 = \text{id}^*E_0 \cong (i \circ p)^*E_0 = p^*i^*E_0 = p^*E = E \times D_0^2. \]

Similarly, \( E_{\infty} \cong E \times D_\infty^2 \), so a clutching function can be defined by restricting to \( X \times S^1 \) on each piece. \( \square \)

The hardest piece of the proof that \( K(X) \otimes K(S^2) \to K(X \times S^2) \) is an isomorphism is contained in the following lemma.

**Lemma 4.3.6.** \( \mu : K(X) \otimes K(S^2) \to K(X \times S^2) \) is surjective.

**Proof.** By Lemma 4.3.5, any bundle over \( X \times S^2 \) is isomorphic to \([E, f]\) for some bundle \( E \to X \) and clutching map \( f : E \times S^1 \to E \times S^1 \). We now change \( f \) by homotopy through generalized clutching maps to simplify the scenario. Ultimately, we will demonstrate that \([E, f]\) is the image of some linear combination of elements of \( K(X) \otimes K(S^2) \). The following steps will lead towards this goal:

1. Given a generalized clutching map \( f : E \times S^1 \to E \times S^1 \), there is a Laurent polynomial generalized clutching map
   \[ q(x, z) = \sum_{n=-N}^{N} b_n(x)z^n, \]
   with \( b_n(x) \) continuous functions on \( X \), to which \( f \) is homotopic through generalized clutching maps.

2. \([E, q] = [E, p]\) for a polynomial clutching map \( p(x, z) \).

3. For any polynomial clutching map \( p(x, z) = \sum_{n=0}^{k} a_n(x)z^n \), where \( a_n(x) \) are continuous on \( X \), there exists a linear clutching function \( L(x, z) \) such that
   \[ [E, p] \otimes [kE, 1] \cong [(k + 1)E, L]. \]

4. For any bundle \( \widetilde{E} \) over \( X \) and any linear clutching function \( L \), \([\widetilde{E}, L]\) is in the image of \( \mu : K(X) \otimes K(S^2) \to K(X \times S^2) \).
Once we have (1) – (4), surjectivity of $\mu$ will follow from:

$$[E, f] = [E, q] = [E, z^{-N}p] \text{ by (1) – (2)}$$

$$= [E, p] \otimes p^*_2 H^{-N} \text{ where } H^{-1} = H^* \text{ is the dual bundle to } H$$

$$= ((k + 1)E, L) - [kE, 1] \otimes p^*_2 H^{-N} \text{ by (3)}$$

$$= [(k + 1)E, L] \otimes p^*_2 H^{-N} - [kE, 1] \otimes p^*_2 H^{-N}$$

$$= [(k + 1)E, L] \otimes p^*_2 H^{-N} - \mu(kE \otimes H^{-N})$$

$$= [%E_1, z] \otimes p^*_2 H^{-N} - \mu(kE \otimes H^{-N}) \text{ for } (k + 1)E = \tilde{E}_+ \oplus \tilde{E}_-$$

(1) In local trivializations of $E \to X$, the clutching function $f(x, z) : E \times S^1 \to E \times S^1$ is just a matrix with entries depending on $x$ and $z$. Each entry is of the form $h : X \times S^1 \to \mathbb{C}$. By Fourier analysis, as long as $X$ is compact, any such function $X \times S^1 \to \mathbb{C}$ can be approximated uniformly by Laurent polynomial functions in $z$:

$$h(x, z) = \sum_{n=-\infty}^{\infty} \left( \frac{1}{2\pi} \int_0^{2\pi} f(x, e^{i\theta}) e^{-inz} d\theta \right) z^n.$$

Next, any endomorphism $E \times S^1 \to E \times S^1$ can be approximated by Laurent polynomial endomorphisms by defining the approximation on each local trivialization $h_i : U_i \times \mathbb{C}^n \to \pi^{-1}(U_i)$ and using a partition of unity to define on the entire space $E \times S^1$. Finally, since $\text{Aut}(E \times S^1)$ is an open subset of $\text{End}(E \times S^1)$ (with respect to the operator norm), this implies that any $f \in \text{Aut}(E \times S^1)$ can be approximated by Laurent polynomial clutching functions. Let $q(x, z)$ be the Laurent polynomial clutching function approximating $f(x, z)$ within some $\varepsilon$-neighborhood of $f(x, z)$ in $\text{Aut}(E \times S^1)$. Then the linear homotopy $tq + (1 - t)f$ induces a bundle isomorphism $[E, f] \cong [E, q]$ since each level of the homotopy is also a clutching function.

(2) Define the polynomial clutching map $p(x, z) = z^N q(x, z)$. Then by Lemma 4.3.4,

$$[E, p] = [E, z^N q] = [E, q] \otimes p^*_2 H^N = [E, f] \otimes p^*_2 H^N.$$

(3) Consider two $(k + 1) \times (k + 1)$ matrices whose entries are endomorphisms of bundles:

$$A = \begin{pmatrix}
1 & -z & 0 & 0 & \cdots & 0 \\
0 & 1 & -z & 0 & \cdots & 0 \\
0 & 0 & 1 & -z & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 1 & -z \\
(a_k(x) & a_{k-1}(x) & \cdots & a_1(x) & a_0(x))
\end{pmatrix} \text{ and } B = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & p(x, z)
\end{pmatrix}$$

Then $A, B \in \text{End}((k + 1)E)$ and in fact $B : (k + 1)E \to (k + 1)E$ is an automorphism, that is, a generalized clutching map. Notice that $A$ is similar to $B$ (by elementary row/column operations), so $A \in \text{Aut}((k + 1)E)$ as well. Even better, $A$ and $B$ are homotopic through clutching functions, and $B$ represents $[E, p] \otimes [kE, 1]$ while $A$ represents $[(k + 1)E, L]$ for some linear $L$, since all the entries of $A$ are linear in $z$. 

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(4) Let $L(x, z) = a_1(x)z + a_0(x)$ be a linear clutching function. Define $H_t(x, z) = a_1(x)\frac{z + t}{1 + tz} + a_0(x)$ and consider the homotopy

$$H_t(x, z)(1 + tz) = a_1(x)(z + t) + a_0(x)(1 + tz) = (a_1(x) + ta_0(x))z + (ta_1(x) + a_0(x)).$$

Note that if $z \in S^1 \subset \mathbb{C}$, then $\frac{z + t}{1 + tz} \in S^1$ as well:

$$\left\| \frac{z + t}{1 + tz} \right\| = \left\| \frac{z + t\bar{z}}{1 + t\bar{z}} \right\| = \left\| \frac{1 + t\bar{z}}{1 + t\bar{z}} \right\| = \left\| \frac{1 + t\bar{z}}{1 + t\bar{z}} \right\| = 1.$$

So this implies

$$L\left( x, \frac{z + t}{1 + tz} \right) = a_1(x)\frac{z + t}{1 + tz} + a_0(x)$$

is a clutching function as well (i.e. is an automorphism on fibres). Further, multiplication by the nonzero $1 + tz \in \mathbb{C}$ is also an automorphism, so $[a_1(x)\frac{z + t}{1 + tz} + a_0(x)](1 + tz)$ is an automorphism on each fibre for any $0 \leq t < 1$. At $t = 0$, we get $a_1(x)z + a_0(x) = L(x, z)$. Plugging in $t = 1$, we do get an automorphism $a_1(x) + a_0(x)$ of the fibre $\pi^{-1}(x)$, but since $X$ is compact, this means $a_1(x) + ta_0(x)$ is still an automorphism of $\pi^{-1}(x)$ for any $0 \leq t \leq 1$. Therefore we can invert this coefficient to obtain

$$(a_1(x) + ta_0(x))^{-1}H_t(x, z) = z + \frac{ta_1(x) + a_0(x)}{a_1(x) + ta_0(x)}, \quad 0 \leq t \leq 1.$$
Theorem 4.3.7. \( K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X) \otimes K(S^2) \to K(X \times S^2) \) is an isomorphism.

Proof. To check the composition is one-to-one, we need only return to each step of the previous proof and check that there is a unique choice of generalized clutching functions up to homotopy. This is done in Hatcher’s notes. \(\square\)

Corollary 4.3.8. \( K(S^2) \cong \mathbb{Z}[H]/(H-1)^2 \).

Proof. Apply Theorem 4.3.7 to \( X = \{x\} \). \(\square\)

Corollary 4.3.9. The external product map \( K(X) \otimes K(S^2) \to K(X \times S^2) \) is an isomorphism.

As abelian groups, \( \mathbb{Z}[H]/(H-1)^2 = \mathbb{Z} \oplus \mathbb{Z}[H] \). Since \( \widetilde{K}(S^2) = \ker(K(S^2) \to K(\ast)) \), this implies \( \widetilde{K}(S^2) = \mathbb{Z}[H-1] \) is infinite cyclic. We will use Theorem 4.2.5 to analyze the reduced \( K \)-theory external product. Note that when \( Y = S^2 \), the smash product \( X \wedge Y \) becomes the reduced suspension \( \Sigma^2 X \).

Theorem 4.3.10 (Bott Periodicity). Let \( X \) be a compact space and \( \Sigma \) the reduced suspension functor. Then there is an isomorphism \( \widetilde{K}(X) \cong \widetilde{K}(\Sigma^2 X) \).

Proof. Consider the diagram

\[
\begin{array}{c}
K(X) \otimes K(S^2) \\
\cong \quad \cong \\
\cong \\
(\widetilde{K}(X) \oplus \mathbb{Z}) \otimes (\widetilde{K}(S^2) \oplus \mathbb{Z}) \\
\longrightarrow \quad \longrightarrow \\
\longrightarrow \quad \longrightarrow \\
\widetilde{K}(X \times S^2) \oplus \mathbb{Z}
\end{array}
\]

The top row is an isomorphism by Corollary 4.3.9 and the vertical arrows are isomorphisms by Lemma 4.1.5. Moreover, the diagram commutes so the bottom row is also an isomorphism. Now

\[
(\widetilde{K}(X) \oplus \mathbb{Z}) \otimes (\widetilde{K}(S^2) \oplus \mathbb{Z}) = (\widetilde{K}(X) \otimes \widetilde{K}(S^2)) \oplus \widetilde{K}(X) \oplus \widetilde{K}(S^2) \oplus \mathbb{Z}
\cong (\widetilde{K}(X) \otimes \mathbb{Z}[H-1]) \oplus \widetilde{K}(X) \oplus \widetilde{K}(S^2) \oplus \mathbb{Z}
\cong \widetilde{K}(X) \oplus \widetilde{K}(X) \oplus \widetilde{K}(S^2) \oplus \mathbb{Z}.
\]

On the other hand, Theorem 4.2.5 implies that

\[
\widetilde{K}(X \times S^2) \oplus \mathbb{Z} \cong \widetilde{K}(X \wedge S^2) \oplus \widetilde{K}(X) \oplus \widetilde{K}(S^2) \oplus \mathbb{Z}
\cong \widetilde{K}(\Sigma^2 X) \oplus \widetilde{K}(X) \oplus \widetilde{K}(S^2) \oplus \mathbb{Z}.
\]

Therefore the isomorphism on the bottom row of the diagram implies \( \widetilde{K}(X) \cong \widetilde{K}(\Sigma^2 X) \). \(\square\)

Corollary 4.3.11. For any \( n \geq 1 \), \( \widetilde{K}(S^n) \cong \widetilde{K}(S^{n+2}) \).
Bott’s original incarnation of periodicity went as follows. Let
\[ \mathcal{U} = \lim_{\to} U_n(\mathbb{R}) \]
be the infinite unitary group, the direct limit of the finite (real) unitary groups \( U_n(\mathbb{R}) \) for \( n \geq 1 \). Then \( \pi_{k+2}(\mathcal{U}) = \pi_k(\mathcal{U}) \) for all \( k \geq 0 \). It is a fact that the infinite Grassmannian
\[ \text{Gr}_\infty = \lim_{\to} \text{Gr}_n(\mathbb{R}^\infty) \]
(the limit of the Grassmannians constructed in Section 3.3) is the classifying space of \( \mathcal{U} \), i.e. \( B\mathcal{U} = \text{Gr}_\infty \). By the results in Section 3.3, this means \( B\mathcal{U} \) is the base of the universal bundle \( \gamma \). The version of periodicity in Theorem 4.3.10 then follows from the original version of periodicity by comparing \( \pi_n(\mathcal{U}) \) to \( \tilde{K}^n(B\mathcal{U}) \).

Some vital consequences of Bott periodicity include:

- For any CW-complex \( X \) and any fixed \( k \geq 0 \), the groups \( \pi_{n+k}(\Sigma^n X) \) are eventually constant as \( n \) gets large. This is the beginning of stable homotopy theory.
- The above allows one to define the \( k \)th stable homotopy group, \( \pi^S_k = \lim_{\to} \pi_{n+k}(S^n) \), and the \( J \)-homomorphism \( J : \pi_k(SO(n)) \to \pi_{n+k}(S^n) \) for each \( n \geq 1 \).
- Real Bott periodicity, i.e. for real vector bundles, says that real reduced \( K \)-theory \( \widetilde{KO}(X) \) is 8-periodic:
\[ \widetilde{KO}(X) \cong \widetilde{KO}(\Sigma^8 X). \]
- This further implies the image of the \( J \)-homomorphism in \( \pi_{n+k}(S^n) \) is 8-periodic.

### 4.4 Characteristic Classes in \( K \)-Theory

Let \( X \) be any topological space and consider the total Chern class \( c : \text{Vect}(X) \to H^\bullet(X; \mathbb{Z}) \).

**Lemma 4.4.1.** The total Chern class extends naturally to a functor
\[ c : K(X) \to H^\bullet(X; \mathbb{Z}) \]
\[ E - F \leftrightarrow c(E)c(F)^{-1}. \]

More generally, if \( R \) is a commutative ring and \( f(t) \in R[[t]] \) is a formal power series over \( R \) which is a unit (the constant term of \( f \) is 1), then \( f \) induces a characteristic class in \( K \)-theory as follows. First, if \( L \to X \) is a complex line bundle, set \( c_f(L) = f(c_1(L)) \in H^\bullet(X; \mathbb{R}) \).

**Example 4.4.2.** For the polynomial \( f(t) = 1 + t \), \( c_f(L) = 1 + c_1(L) \) is the ordinary Chern class of \( L \).

If \( E \to X \) is a bundle which splits as a Whitney sum of line bundles \( E = L_1 \oplus \cdots \oplus L_n \), then define
\[ c_f(E) = c_f(L_1) \cdots c_f(L_n) = f(c_1(L_1)) \cdots f(c_1(L_n)). \]
In general, by Theorem 3.8.12 any \( E \to X \) pulls back along some \( q : Y \to X \) to a sum of line bundles over \( Y \), say \( q^*E \cong L_1 \oplus \cdots \oplus L_n \).
Definition. The elements $x_1, \ldots, x_n \in H^\bullet(X; \mathbb{Z})$ such that $q^*x_i = c_1(L_i)$ for each $i$ are called the Chern roots of $E$.

Therefore for an arbitrary vector bundle $E \to X$, we can define a characteristic class

$$c_f(E) = f(x_1) \cdots f(x_n)$$

where $x_i$ are the Chern roots of $E$. Since $c_f$ is symmetric in the $x_i$, it can be written as a polynomial in $\sigma_1, \ldots, \sigma_n$, the elementary symmetric polynomials in the $x_i$, but since $\sigma_j = c_j(E)$ for all $1 \leq j \leq n$, $c_f$ gives a value in $H^\bullet(X; \mathbb{Z})$ which is a sum of Chern classes of $X$.

To make this construction compatible with the tensor product of bundles, i.e. the ring structure of $K(X)$, we define:

Definition. For a space $X$, the Chern character of $X$ is the function

$$\text{ch} : K(X) \to H^\bullet(X; \mathbb{Q})$$

defined uniquely by the following:

(1) For a line bundle $L \to X$, $\text{ch}(L) = e^{c_1(L)}$, where $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$ is the formal exponential.

(2) For a Whitney sum of line bundles $E = L_1 \oplus \cdots \oplus L_n$,

$$\text{ch}(E) = \sum_{i=1}^{n} e^{c_1(L_i)}.$$

(3) For an arbitrary vector bundle $E \to X$,

$$\text{ch}(E) = \sum_{i=1}^{n} e^{x_i}$$

where $x_1, \ldots, x_n$ are the Chern roots of $E$.

Example 4.4.3. If $E = L_1 \oplus L_2$ is a sum of line bundles, then

$$\text{ch}(E) = \left(1 + x_1 + \frac{x_1^2}{2} + \frac{x_1^3}{6} + \ldots\right) \left(1 + x_2 + \frac{x_2^2}{2} + \frac{x_2^3}{6} + \ldots\right)$$

$$= 2 + \sigma_1(x_1, x_2) + \frac{1}{2}(\sigma_1^2 - 2\sigma_2)(x_1, x_2) + \ldots$$

$$= \text{rank}(E) + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \ldots$$

Such formulas are easily obtained for vector bundles of arbitrary rank.

Lemma 4.4.4. The Chern character $\text{ch} : K(X) \to H^\bullet(X; \mathbb{Q})$ is a ring homomorphism. Moreover, there is an induced ring homomorphism $\text{ch} : K(X, A) \to H^\bullet(X, A; \mathbb{Q})$ for any pair $(X, A)$. 

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Example 4.4.5. Let $\pi : E \to X$ be a complex vector bundle of rank $n$ and write $\widetilde{E} = \pi^* E \to E$. Then there is a sequence of vector bundles

$$0 \to \wedge^0 \widetilde{E} \xrightarrow{\alpha_1} \wedge^1 \widetilde{E} \xrightarrow{\alpha_2} \wedge^2 \widetilde{E} \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_n} \wedge^n \widetilde{E} \to 0$$

where over $v \in E_x$, $\alpha_j(v)(\omega) = (v, v \wedge \omega)$ for $(v, \omega) \in \wedge^{j-1} \widetilde{E}$. In fact, this sequence is exact when restricted to $E_0 = E \setminus s_0(X)$, so it defines an element $\lambda_E \in K(E, E_0)$ by Proposition 4.1.10. Taking the pullback along the zero section gives

$$s_0^* \lambda_E = \sum_{j=0}^n (-1)^j \wedge^j E \in K(X).$$

This allows us to calculate the Chern character of $E$ as follows. Assume $E = L_1 \oplus \cdots \oplus L_n$, or pullback to $\mathbb{P}E$ to be in this situation. Then $\wedge^\bullet E = \bigotimes_{i=1}^n (\wedge^\bullet L_i) = \bigotimes_{i=1}^n (\mathbb{C} \oplus L_i)$ so we have

$$\text{ch} \left( \wedge^\bullet E \right) = \text{ch} \left( \bigotimes_{i=1}^n (1 + L_i) \right) = \prod_{i=1}^n \text{ch}(1 + L_i) = \prod_{i=1}^n (1 + e^{x_i})$$

where $x_i = c_1(L_i)$. On the other hand, notice that

$$\text{ch}(\lambda_E) = \text{ch} \left( \prod_{i=1}^n (1 - L_i) \right) = \prod_{i=1}^n (1 - e^{x_i}).$$

Theorem 4.4.6 (Thom Isomorphism for K-Theory). Let $E \to X$ be a complex vector bundle. Then

1. The ring $\widetilde{K}(E) = K(E, E_0)$ is a module over $K(X)$ via pullback and tensor product.
2. $K(E, E_0)$ is a free $K(X)$-module generated by $\lambda_E$. In particular, the map

$$\Phi_K : K(X) \longrightarrow K(E, E_0), \; \alpha \longmapsto \alpha \lambda_E$$

is an isomorphism.

Definition. $\lambda_E \in K(E, E_0)$ is called the Thom class in K-theory of $E \to X$. 
5 Chern-Weil Theory

Let $M$ be a smooth manifold and $E \to M$ a smooth vector bundle. Denote by $\Gamma(M, E)$ the space of smooth sections of $E \to M$.

**Definition.** A differential $k$-form on a smooth manifold $M$ is a smooth section $\omega \in \Omega^k(M) := \Gamma(M, \wedge^k T^* M)$, where $T^* M = (T M)^*$ is the cotangent bundle of $M$.

**Example 5.0.1.** A differential 0-form is just a smooth function $f : M \to \mathbb{R}$. A differential 1-form $\omega \in \Omega^1(M)$ is a smooth assignment $x \mapsto \omega_x \in \hom(TM, \mathbb{R})$.

If $x_1, \ldots, x_n$ are local coordinates in a chart $U$ of a manifold $M$, there is a corresponding basis $\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\}$ of $T^* M|_U$. Let $\{dx_1, \ldots, dx_n\}$ denote the corresponding dual basis of $T M|_U$. Then any differential form $\omega \in \Omega^k(M)$ can be written locally on $U$ as

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \ldots, i_k} \; dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

for a smooth function $\omega_{i_1, \ldots, i_k}$ on $U$ defined by

$$\omega_{i_1, \ldots, i_k} = \omega \left( \frac{\partial}{\partial x_{i_1}}, \ldots, \frac{\partial}{\partial x_{i_k}} \right) .$$

Recall that for any smooth manifold $M$, there is a unique operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ called the **exterior derivative** which satisfies

(a) For a smooth function $f \in \Omega^0(M)$ and a vector field $X \in \Gamma(M, TM)$, $df(X) = X(f)$.

(b) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$ then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.

(c) For all $\omega \in \Omega^k(M)$, $d^2 \omega = 0$.

(d) For a smooth function $f : M \to N$, there is a unique linear map $f^* : \Omega^k(N) \to \Omega^k(M)$ satisfying:

(i) If $g \in \Omega^0(N)$, then $f^* g = g \circ f$.

(ii) $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$ for all $\omega, \eta \in \Omega^*(N)$.

(iii) $f^*(d\omega) = d(f^*\omega)$ for all $\omega \in \Omega^*(N)$.

Since $d^2 = 0$, the sequence $\cdots \to \Omega^k(M) \to \Omega^{k+1}(M) \to \cdots$ forms a chain complex with differential $d$. The cohomology of this complex is called the **de Rham cohomology** of $M$, written $H^*_{dR}(M)$. De Rham cohomology has the following properties:

- $\wedge$ descends to a signed-commutative operation on $H^*_{dR}(M)$, also denoted $\wedge$, and this gives the de Rham cohomology the structure of a ring.

- Every smooth function $f : M \to N$ induces linear maps $f^* : H^k_{dR}(N) \to H^k_{dR}(M)$ for all $k$. In fact, $f^*$ commutes with $\wedge$ and so preserves the ring structure of the de Rham cohomology.
5.1 Connections on a Vector Bundle

Our goal is to generalize de Rham theory on a smooth manifold by considering “vector bundle-valued differential forms”, i.e. sections of $\bigwedge^k T^* M \otimes E$ for a vector bundle $E \to M$. Given a (real or complex) vector bundle $E \to M$, any section of $\bigwedge^k T^* M \otimes E$ may be thought of as an alternating linear map $\bigwedge^k T^* M \to E$.

Proposition 5.1.2. If $\nabla^1$ and $\nabla^2$ are connections on a bundle $E \to M$, then $\nabla^1 - \nabla^2$ satisfies

$$ (\nabla^1 - \nabla^2)(f\sigma) = f(\nabla^1 - \nabla^2)\sigma $$

for all $f \in C^\infty(M)$ and $\sigma \in \Omega^p(E)$. That is, $\nabla^1 - \nabla^2$ is a section of $\text{End}(E) = E^* \otimes E$.

Remark. For two real bundles $E_1, E_2 \to M$, an $\mathbb{R}$-linear $F : \Gamma(E_1) \to \Gamma(E_2)$ corresponds to a section of $\text{Hom}(E_1, E_2) = E_1^* \otimes E_2$ if and only if $F(f\sigma) = fF(\sigma)$ for all $f \in C^\infty(M)$ and $\sigma \in \Gamma(E_1)$. Such an $F$ is called tensorial.
Example 5.1.3. Proposition 5.1.2 shows that every connection on \( M \times \mathbb{R} \) is of the form \( d^a \) for some \( a \in \Omega^1(M) \). More generally, let \( E \cong M \times \mathbb{R}^k \) be a trivial \( k \)-bundle. Then any \( \sigma \in \Gamma(E) \) is of the form \( \sigma = (\sigma_1, \ldots, \sigma_k) \) for \( \sigma_i \in \Omega^0(M) \). There is a trivial connection \( d \) on \( M \times \mathbb{R}^k \) given by the componentwise differential operator:

\[
d : \Omega^0(M; M \times \mathbb{R}^k) \to \Omega^1(M; M \times \mathbb{R}^k) \\
(\sigma_1, \ldots, \sigma_k) \mapsto (d\sigma_1, \ldots, d\sigma_k).
\]

Any connection on \( M \times \mathbb{R}^k \) is then of the form \( \nabla = d + A \) for some \( A \in \Omega^1(M; \text{End}(M \times \mathbb{R}^k)) = \Omega^1(M; \mathfrak{gl}_k(\mathbb{R})) \)

where \( \mathfrak{gl}_k(\mathbb{R}) = M_k(\mathbb{R}) \) is the space of \( k \times k \) matrices with real entries (the Lie algebra of \( GL_k(\mathbb{R}) \) in fact). One can regard \( A \) as a matrix of 1-forms on \( M \), and for a section \( \sigma \in \Omega^0(M; M \times \mathbb{R}^k) \), \( \nabla \sigma \) is explicitly given by

\[
\nabla \sigma = d\sigma + \sum_{j=1}^k A_{ij} \sigma_j.
\]

Proposition 5.1.4. Let \( E \to M \) be a vector bundle. Then

1. \( E \to M \) admits a connection.
2. Any convex combination of connections is again a connection.
3. The space \( \text{Conn}(E) \) of all connections on \( E \) is an affine space modelled on \( \Omega^1(M; \text{End}(E)) \).

Proposition 5.1.5. Let \( E \to M \) be a vector bundle with connection \( \nabla \). Then there is a unique linear differential operator \( d^\nabla : \Omega^k(M; E) \to \Omega^{k+1}(M; E) \) satisfying:

1. \( d^\nabla \sigma = \nabla \sigma \) for all \( \sigma \in \Omega^0(M; E) \).
2. If \( \alpha \in \Omega^p(M) \) and \( \omega \in \Omega^q(M; E) \), then \( d^\nabla (\alpha \wedge \omega) = d\alpha \wedge \omega + (-1)^p \alpha \wedge d^\nabla \omega \).

Proof. Locally, \( \omega \) can be written \( \omega = \sum_{i=1}^r \omega_i \otimes \sigma_i \) for some \( \omega_i \in \Omega^p(M) \) and \( \sigma \in \Omega^0(M; E) \). Then if \( d^\nabla \) exists, it must satisfy

\[
d^\nabla \omega = \sum_{i=1}^r (d\omega_i \otimes \sigma_i + (-1)^p \omega_i \otimes d^\nabla \sigma_i) = \sum_{i=1}^r (d\omega_i \otimes \sigma_i + (-1)^p \omega_i \otimes \nabla \sigma_i).
\]

So we may take this as the definition of \( d^\nabla \), or we may define \( d^\nabla \) globally by

\[
(d^\nabla \omega)(X_0, \ldots, X_p) = \sum_{i=1}^p (-1)^i \nabla X_i \omega(X_0, \ldots, \hat{X}_i, \ldots, X_p)
+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p),
\]

where \( \hat{X}_\ell \) denotes the deletion of \( X_\ell \) from the given tuple. Now one can check that either definition gives an operator satisfying (1) and (2).
Definition. For a connection $\nabla$, the operator $d^\nabla : \Omega^\ast(M; E) \to \Omega^\ast(M; E)$ is called the
\textbf{covariant exterior derivative} with respect to $\nabla$.

Compare the properties of $d^\nabla$ in Proposition 5.1.5 to those of $d$ in the previous section. In particular, notice that $d^2 = 0$, but this is not required for $d^\nabla$. In general, $(d^\nabla)^2 \neq 0$ but we can measure how far a given connection is from defining a de Rham cohomology theory by studying the value of $(d^\nabla)^2$. First, note that the composition

$$
\left(\bigwedge^p T^* M \otimes \text{End}(E)\right) \otimes \left(\bigwedge^q T^* M \otimes E\right) \cong \left(\bigwedge^p T^* M \otimes \bigwedge^q T^* M\right) \otimes (\text{End}(E) \otimes E)
$$

$$
\downarrow \wedge \otimes \text{ev}
$$

$$
\bigwedge^{p+q} T^* M \otimes E
$$

defines a wedge operation $\wedge : \Omega^p(M; \text{End}(E)) \otimes \Omega^q(M; E) \to \Omega^{p+q}(M; E)$.

Proposition 5.1.6. For a connection $\nabla$ on a vector bundle $E \to M$, there is a unique
2-form $R^\nabla \in \Omega^2(M; \text{End}(E))$ satisfying $(d^\nabla)^2 \omega = R^\nabla \wedge \omega$ for all $\omega \in \Omega^p(M; E)$.

Proof. For $p = 0$, note that the assignment

$$
\Omega^0(M; E) \to \Omega^2(M; E)
$$

$$
\sigma \mapsto (d^\nabla)^2 \sigma
$$

is tensorial (in the sense of the remark above) since for all smooth functions $f \in C^\infty(M)$,

$$
d^\nabla (d^\nabla (f \sigma)) = d^\nabla (df \wedge \sigma + f \wedge d^\nabla \sigma)
$$

$$
= d^2 f \wedge \sigma - df \wedge d^\nabla \sigma + df \wedge d^\nabla \sigma + f \wedge (d^\nabla)^2 \sigma
$$

$$
= f \wedge (d^\nabla)^2 \sigma
$$

since $d^2 f = 0$. Therefore by that earlier remark, $\sigma \mapsto (d^\nabla)^2 \sigma$ corresponds to a section $R^\nabla \in \Omega^2(M; \text{End}(E))$. So we have $(d^\nabla)^2 \sigma = R^\nabla \wedge \sigma$ for all 0-forms $\sigma$. Now we show that this extends to all $p$-forms. Any $\omega \in \Omega^p(M; E)$ may be written locally as $\omega = \sum_{i=1}^r \alpha_i \wedge \sigma_i$ for some $\alpha_i \in \Omega^p(M)$ and $\sigma_i \in \Omega^0(M; E)$, so it suffices to prove the case when $\omega = \alpha \wedge \sigma$ identically. In this case, we have

$$
d^\nabla (d^\nabla \omega) = d^\nabla (d^\nabla (\alpha \wedge \sigma))
$$

$$
= d^\nabla (d\alpha \wedge \sigma + (-1)^p \alpha \wedge d^\nabla \sigma)
$$

$$
= d^2 \alpha \wedge \sigma + (-1)^p d\alpha \wedge d^\nabla \sigma + \alpha \wedge (d^\nabla)^2 \sigma
$$

$$
= \alpha \wedge (d^\nabla)^2 \sigma.
$$

\hfill \Box

Definition. The 2-form $R^\nabla \in \Omega^2(M; \text{End}(E))$ is called the \textbf{curvature} of the connection $\nabla$.

Example 5.1.7. When $E = TM$ and $\nabla$ is a particular connection called the \textit{Levi-Civita connection} (or \textit{Riemannian connection}; see Section 5.3), $R^\nabla$ recovers the Riemannian curvature of the manifold $M$. (In the case of a surface, this is the Gaussian curvature.)
Example 5.1.8. Take $E = M \times \mathbb{R}^k$ to be the trivial bundle and $\nabla = d$ to be the trivial connection. Then a section of $E$ may be viewed as a column vector

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_k \end{pmatrix}: M \rightarrow \mathbb{R}^k$$

and $d\sigma$ is just the corresponding column vector of 1-forms

$$d\sigma = \begin{pmatrix} d\sigma_1 \\ \vdots \\ d\sigma_k \end{pmatrix}.$$ 

Thus $d\nabla \sigma = d\sigma$ and $d^2 \sigma$ is just the zero vector, so $R_\nabla = 0$. In general, $R_\nabla$ may be regarded as an obstruction in $\Omega^2(M; \text{End}(E))$ to the connection $\nabla$ being locally trivial on $E$.

Locally, any bundle looks like $U \times \mathbb{R}^k$ for $U \subseteq M$ open, so any connection $\nabla$ on the bundle looks like $\nabla = d + A$ on $U$, where $d$ is the trivial connection on $U \times \mathbb{R}^k$ and $A \in \Omega^1(U; \mathfrak{gl}_k(\mathbb{R}))$ is a matrix of 1-forms. If $\{e_1, \ldots, e_k\}$ is a basis of sections of $E$, then for each $1 \leq i \leq k$, $de_i = 0$ so

$$\nabla e_i = \sum_{j=1}^k A_{ji} e_j.$$ 

Here, $A$ is called the connection matrix for $\nabla$ on $U$. More generally, for $\sigma = \sum_{i=1}^k \sigma_i \wedge e_i$, we have

$$\nabla \sigma = (d + A)\sigma = \sum_{i=1}^k (d(\sigma_i \wedge e_i) + A(\sigma_i \wedge e_i))$$

$$= \sum_{i=1}^k \left( d\sigma_i \wedge e_i + \sum_{j=1}^k A_{ji}(\sigma_i \wedge e_j) \right)$$

$$= \sum_{i=1}^k \left( d\sigma_i + \sum_{j=1}^k A_{ji} \sigma_j \right) \wedge e_i.$$ 

For curvature, $R_\nabla$ locally looks like a matrix of 2-forms satisfying $R_\nabla \wedge e_i = \sum_{j=1}^k (R_\nabla)_{ji} \wedge e_j$. Thus for $\sigma \in \Omega^0(M; E)$,

$$R_\nabla \wedge \sigma = (d + A)(d + A)\sigma = (d + A)(d\sigma + A \wedge \sigma)$$

$$= d^2 \sigma + d(A \wedge \sigma) + A \wedge d\sigma + A \wedge (A \wedge \sigma)$$

$$= dA \wedge \sigma - A \wedge d\sigma + A \wedge d\sigma + (A \wedge A) \wedge \sigma$$

$$= (dA + A \wedge A)\sigma$$

where $A \wedge A$ is the ‘wedge product of matrices’ done by standard matrix multiplication using wedge product for each product of entries. This shows that $R_\nabla = dA + A \wedge A$. 

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Example 5.1.9. Let \( M = \mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \) be the upper half-plane and \( E = T\mathbb{H}^2 \) its tangent bundle. There is a nontrivial connection \( \nabla = d + A \) on \( T\mathbb{H}^2 \) given by the matrix

\[
A = \begin{pmatrix}
-\frac{dy}{y} & -\frac{dx}{y} \\
\frac{dx}{y} & -\frac{dy}{y}
\end{pmatrix}.
\]

Consider the vector fields \( \partial_x \) and \( \partial_y \), the partial differential operators in the \( x \) and \( y \) directions, respectively, which can be viewed as the column vectors \( \partial_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \partial_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). Then \( \sigma = \partial_x \) is a section whose value under the connection \( \nabla \) is:

\[
\nabla \partial_x = (d + A) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = d \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{dy}{y} & -\frac{dx}{y} \\
\frac{dx}{y} & -\frac{dy}{y}
\end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{dy}{y} \\ \frac{dx}{y} \end{pmatrix}.
\]

Applying this to the vector fields \( \partial_x, \partial_y \) gives:

\[
\nabla \partial_x \sigma = \begin{pmatrix} -\frac{dy}{y} \\ \frac{dx}{y} \end{pmatrix} \partial_x = \begin{pmatrix} 0 \\ \frac{1}{y} \end{pmatrix}
\]

and

\[
\nabla \partial_y \sigma = \begin{pmatrix} -\frac{dy}{y} \\ \frac{dx}{y} \end{pmatrix} \partial_y = \begin{pmatrix} 0 \\ -\frac{1}{y} \end{pmatrix}.
\]

To compute curvature, it’s easy to show that \( A \wedge A = 0 \) and

\[
dA = \begin{pmatrix} 0 & -\frac{dx\wedge dy}{y^2} \\
\frac{dx\wedge dy}{y^2} & 0
\end{pmatrix}.
\]

It turns out that \( \frac{dx\wedge dy}{y^2} \) is the standard volume form on \( \mathbb{H}^2 \), so the curvature of the connection \( \nabla \) encodes some geometry of \( \mathbb{H}^2 \).

Globally, we defined \( d\nabla \) explicitly by

\[
(d\nabla \omega)(X_0, \ldots, X_p) = \sum_{i=1}^{p} (-1)^i \nabla_{X_i} \omega(X_0, \ldots, \hat{X}_i, \ldots, X_p)
+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p)
\]

for \( \omega \in \Omega^p(M; E) \). As a special case, if \( \sigma \in \Omega^0(M; E) = \Gamma(E) \), then \( (d\nabla \sigma)(X) = \nabla_X \sigma \) for all vector fields \( X \). Applying \( d\nabla \) twice gives

\[
d\nabla(d\nabla \sigma)(X, Y) = \nabla_X (d\nabla \sigma)(Y) - \nabla_Y (d\nabla \sigma)(X) - (d\nabla \sigma)([X, Y])
= \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X, Y]} \sigma.
\]

This proves:
Proposition 5.1.10. For any connection $\nabla$ on $E \to M$ and any two vector fields $X, Y \in \Gamma(TM)$, the curvature of $\nabla$ along $(X, Y)$ equals

$$R(\nabla)(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}.$$ 

Remark. The proof that $d^2 = 0$ really relies on the fact that mixed partial derivatives commute. Therefore, in some sense $R(\nabla)$ is a measurement of the failure of $\nabla_X$ and $\nabla_Y$ to commute for general vector fields $X$ and $Y$. This can be seen by the formula $R(\nabla) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$.

Above, we showed that locally, $R(\nabla) = dA + A \wedge A$ where $A$ is the connection matrix on a locally trivial coordinate chart $U \subseteq M$. We now investigate what happens under change of coordinates. Suppose $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ are two local bases of $E \to M$. Then there is a matrix of smooth functions $g = (g_{ij}) : U \to GL_n(\mathbb{R})$ such that each $f_i$ can be written

$$f_i = \sum_{j=1}^{n} g_{ji} e_j \quad \text{or equivalently} \quad e_j = \sum_{k=1}^{n} (g^{-1})_{kj} f_k.$$ 

If $\nabla$ is a connection on $E$, let $A$ and $B$ be the two connection matrices with respect to these local bases, i.e. the matrices satisfying

$$\nabla e_i = \sum_{j=1}^{n} A_{ji} e_j \quad \text{and} \quad \nabla f_i = \sum_{j=1}^{n} B_{ji} f_j$$

for each $1 \leq i \leq n$. Then we have

$$\sum_{j=1}^{n} B_{ji} f_j = \nabla f_i = \nabla \left( \sum_{j=1}^{n} g_{ji} e_j \right)$$

$$= \sum_{j=1}^{n} ((dg_{ji}) e_j + g_{ji} \nabla e_j) \quad \text{by axiom (2) for connections}$$

$$= \sum_{j=1}^{n} (dg_{ji}) e_j + \sum_{j,k=1}^{n} g_{ji} A_{kj} e_k$$

$$= \sum_{j,k} (dg_{ji}) (g^{-1})_{kj} f_k + \sum_{j,k,\ell} g_{ji} A_{ki} (g^{-1})_{\ell k} f_\ell$$

$$= \sum_{j,k} (dg_{ki}) (g^{-1})_{jk} f_j + \sum_{j,k,\ell} g_{ki} A_{\ell k} (g^{-1})_{j,\ell} f_j \quad \text{after reindexing}$$

$$= \sum_{j,k,\ell} ((g^{-1})_{jk} dg_{ki} + (g^{-1})_{j,\ell} A_{\ell k} g_{ki}) f_j.$$ 

More succinctly, $B = g^{-1} d g + g^{-1} A g$. For curvature, we have the following proposition:

Proposition 5.1.11. Let $R^A_\nabla$ and $R^B_\nabla$ be the curvature forms for $\nabla$ associated to each local basis and let $g : M \to GL_n(\mathbb{R})$ be the change of basis function between the local bases. Then

$$R^B_\nabla = g^{-1} R^A_\nabla g.$$
Proof. By the calculations above, $R^A = dA + A \wedge A$, $R^B = dB + B \wedge B$ and $B = g^{-1}dg + g^{-1}Ag$. Putting these together,

$$R^B = dB + B \wedge B = d(g^{-1}dg + g^{-1}Ag) + (g^{-1}dg + g^{-1}Ag) \wedge (g^{-1}dg + g^{-1}Ag)$$

$$= (-g^{-1}dg g^{-1} \wedge dg + g^{-1} \wedge d^2g - g^{-1}dg g^{-1}Ag + g^{-1}dA \wedge g - g^{-1}A \wedge dg)$$

$$+ (g^{-1}dg \wedge g^{-1}dg + g^{-1}dg \wedge g^{-1}Ag + g^{-1}Ag \wedge g^{-1}dg + g^{-1}Ag \wedge g^{-1}Ag)$$

$$= g^{-1}dAg + g^{-1}A \wedge Ag$$

$$= g^{-1}(dA + A \wedge A)g = g^{-1}R^A g.$$

\[\square \]

Remark. Globally, any vector bundle $E \to M$ is identified with the principal $GL_n(\mathbb{R})$-bundle $P_E = F(E) \to M$ (the frame bundle of Example 1.2.2). Further, the Lie group $GL_n(\mathbb{R})$ acts on its Lie algebra $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$ by the adjoint representation

$$GL_n(\mathbb{R}) \times \mathfrak{gl}_n(\mathbb{R}) \longrightarrow \mathfrak{gl}_n(\mathbb{R})$$

$$(g, A) \longmapsto \text{Ad}_g(A) := gAg^{-1}.$$  

For each pair of vector fields $X, Y$ on $M$, the curvature $R_{\nabla}(X, Y)$ is a matrix, so $R_{\nabla}(\cdot, \cdot)$ may really be thought of as a $\mathfrak{gl}_n(\mathbb{R})$-valued function $R_{\nabla} : P_E \to \mathfrak{gl}_n(\mathbb{R})$. Moreover, Proposition 5.1.11 shows that $R_{\nabla}$ satisfies

$$R_{\nabla}(X, Y)(p \cdot g) = \text{Ad}_{g^{-1}}(R_{\nabla}(X, Y)(p))$$

for all $X, Y \in \Gamma(TM), p \in P_E, g \in GL_n(\mathbb{R})$. Therefore $R_{\nabla}$ is a section of the associated bundle $P_E \times \mathfrak{gl}_n(\mathbb{R})$.

### 5.2 Characteristic Classes from Curvature

In this section, we use the curvature of a connection on a vector bundle to produce some de Rham cohomology classes that encode the obstruction to $d^\nabla$ being a differential (i.e. squaring to zero). Suppose $f : \mathfrak{gl}_n(\mathbb{R}) \to \mathbb{R}$ is a function satisfying $f(g^{-1}Ag) = f(A)$ for all $A \in \mathfrak{gl}_n(\mathbb{R})$ and $g \in GL_n(\mathbb{R})$. Let $I_n(\mathbb{R})$ be the $\mathbb{R}$-algebra of all such functions $f$ which are polynomials.

Example 5.2.1. Examples of functions in $I_n(\mathbb{R})$ are $\text{tr}(A), \det(A)$ and in general, any coefficient of the characteristic polynomial $\chi(A) = \det(A - tI)$.

Fix a connection $\nabla$ on a vector bundle $E \to M$. For each $f \in I_n(\mathbb{R})$, $f(R_{\nabla})$ is a differential form on $M$ which is independent of the choice of local bases by Proposition 5.1.11. Notice that if $f$ is homogeneous of degree $k$, then $f(R_{\nabla}) \in \Omega^{2k}(M) = \Omega^{2k}(M; \mathbb{C})$. We can similarly construct $f(R_{\nabla}) \in \Omega^{2k}(M; \mathbb{C})$ for a connection on any complex vector bundle.

Recall the elementary symmetric functions from linear algebra,

$$\sigma_k(x_1, \ldots, x_n) = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}.$$
Lemma 5.2.2. $I_n(\mathbb{R})$ is a polynomial algebra generated by the functions $\sigma_k : \mathfrak{gl}_n(\mathbb{R}) \to \mathbb{R},$ where $\sigma_k(A)$ is the $k$th elementary symmetric function on the eigenvalues of $A$. The same things holds for $I_n(\mathbb{C}).$

Proof. The proof is identical for $\mathbb{R}$ and $\mathbb{C},$ so we will prove the real case only. Let $S_n(\mathbb{R}) \subset \mathbb{R}[x_1, \ldots, x_n]$ be the subalgebra of symmetric polynomials, i.e. polynomials invariant under permutation of the variables $x_1, \ldots, x_n.$ To define a map $I_n(\mathbb{R}) \to S_n(\mathbb{R}),$ take $f \in I_n(\mathbb{R})$ and consider its restriction to the subspace $T_n \subset \mathfrak{gl}_n(\mathbb{R})$ of diagonal matrices. Then $f|_{T_n}$ may be thought of as a polynomial in the $n$ entries along the diagonal of such a matrix. Moreover, $f$ is invariant under conjugation, so $f|_{T_n}$ is a symmetric polynomial in $x_1, \ldots, x_n,$ the variables representing diagonal entries in $T_n.$ Thus we have a map $\alpha : I_n \to S_n(\mathbb{R}), f \mapsto f|_{T_n}.$ From linear algebra, $S_n(\mathbb{R}) \cong \mathbb{R}[\sigma_1, \ldots, \sigma_n],$ so because the coefficients of the characteristic polynomial $\chi = \det((x_{ij} - tf)$ are elementary symmetric functions, it follows that $\alpha$ is surjective.

On the other hand, the same proof shows that $\alpha : I_n(\mathbb{C}) \to S_n(\mathbb{C})$ is well-defined and surjective. If $f$ is in the kernel of $\alpha,$ it vanishes on all matrices similar to a diagonal matrix. In particular, $f$ vanishes on all upper triangular matrices with distinct, nonzero diagonal entries, and by continuity, all upper triangular matrices. But since $f$ is invariant under the symmetric group action, $f$ vanishes on all matrices in $GL_n(\mathbb{R}),$ so $f = 0.$ This shows $I_n(\mathbb{C}) \cong S_n(\mathbb{C}).$ Finally, since $I_n(\mathbb{C}) = I_n(\mathbb{R}) \otimes \mathbb{C}$ and $S_n(\mathbb{C}) = S_n(\mathbb{R}) \otimes \mathbb{C},$ we get the same isomorphism in the real case.

Lemma 5.2.3. For each $j \geq 1,$ let $s_j \in I_n(\mathbb{R})$ be defined by $s_j(A) = \text{tr}(A^j).$ Then $s_j(A) = \alpha_1^j + \ldots + \alpha_n^j$ where $\alpha_i$ are the eigenvalues of $A,$ and $I_n(\mathbb{R}) = \mathbb{R}[s_1, \ldots, s_n].$ The same thing holds for $I_n(\mathbb{C}).$

Proof. Standard linear algebra. 

Let $\nabla$ be a connection on a vector bundle $E \to M.$ Then the connection $R_\nabla,$ viewed as a matrix of differential 2-forms, can be turned into a single differential form by applying any function in $I_n(\mathbb{R})$ (or $I_n(\mathbb{C}).$

Example 5.2.4. Suppose $L \to M$ is a complex line bundle with connection $\nabla.$ Then $R_\nabla \in \Omega^2(M; \text{End}(L)) = \Omega^2(M)$ since $\text{End}(L)$ is a trivial line bundle. Thus $R_\nabla$ is a $1 \times 1$ matrix, i.e. a single 2-form $R_\nabla = (\omega).$ Locally, $R_\nabla = dA + A \wedge A = dA$ since $A$ is a local 1-form, so we see that $dR_\nabla = d^2A = 0.$ So here, $R_\nabla = (\omega)$ is a closed 2-form. The de Rham class $[R_\nabla] = [\omega] \in H^2_{dR}(M)$ corresponds to the trace (or the determinant, since they are the same thing in the one-dimensional case). Moreover, any other connection $\nabla'$ on $L$ is of the form $\nabla' = \nabla + a$ for some $a \in \Omega^1(M; \text{End}(L)) = \Omega^1(M).$ If $A'$ is its local connection matrix, then $A' = A + a,$ so

$$R_{\nabla'} = dA' = dA + da = R_\nabla + da$$

and as a consequence $[R_\nabla] = [R_{\nabla'}].$ This generalizes as follows.

Lemma 5.2.5 (Bianchi’s Identity). Let $\nabla$ be a connection with local connection matrix $A = (A_{ij})$ and curvature matrix $R = R^A_\nabla = (R_{ij}).$ Then

$$dR = [R, A] = R \wedge A - A \wedge R.$$
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Proof. Write $R_{ij} = dA_{ij} + \sum_{k=1}^{n} A_{ik} \wedge A_{kj}$. Then

$$dR_{ij} = d^2 A_{ij} + \sum_{k=1}^{n} (dA_{ik} \wedge A_{kj} - A_{ik} \wedge dA_{kj})$$

$$= \sum_{k,\ell=1}^{n} [(R_{ik} - A_{i\ell} \wedge A_{k\ell}) \wedge A_{kj} - A_{ik} \wedge (R_{kj} - A_{k\ell} \wedge A_{ij})]$$

$$= \sum_{k=1}^{n} \left[ (R_{ik} \wedge A_{kj} - A_{ik} \wedge R_{kj}) - \sum_{\ell=1}^{n} (A_{i\ell} \wedge A_{k\ell} \wedge A_{kj} - A_{ik} \wedge A_{k\ell} \wedge A_{ij}) \right]$$

$$= [R, A]_{ij} - ((A^3)_{ij} - (A^3)_{ij}) = [R, A]_{ij}.$$ 

Since they agree in the $ij$ component for all $1 \leq i, j \leq n$, $dR = [R, A]$. \hfill \Box

**Theorem 5.2.6.** Let $E \to M$ be a vector bundle of rank $n$ and $\nabla$ a connection on $E$. Then for any $f \in I_n(\mathbb{R})$ of degree $k$,

1. $f(R_\nabla) \in \Omega^{2k}(M)$ is a closed $2k$-form, i.e. $d(f(R_\nabla)) = 0$.

2. The de Rham cohomology class $[f(R_\nabla)] \in H^{2k}_{dR}(M)$ is independent of $\nabla$.

**Proof.** (1) By Lemma 5.2.3, it suffices to show the case when $f = s_j = tr(A^j)$ is the $j$th elementary symmetric power function. In this case, we have

$$d(s_j(R_\nabla)) = d(tr(R^j_\nabla)) = tr(d(R^j_\nabla)) \text{ by linearity of } d$$

$$= tr(dR_\nabla \wedge R^{j-1}_\nabla + R_\nabla \wedge dR_\nabla \wedge R^{j-2}_\nabla + \ldots + R^{j-1}_\nabla \wedge dR_\nabla)$$

$$= tr((R_\nabla, A) \wedge R^{j-1}_\nabla + R_\nabla \wedge [R_\nabla, A] \wedge R^{j-2}_\nabla + \ldots + R^{j-1}_\nabla \wedge [R_\nabla, A]) \text{ by Lemma 5.2.5}$$

$$= tr((R_\nabla \wedge A - A \wedge R_\nabla) \wedge R^{j-1}_\nabla + \ldots + R^{j-1}_\nabla \wedge (R_\nabla \wedge A - A \wedge R_\nabla))$$

$$= tr([R^j_\nabla, A] - 0) \text{ after cancelling terms}$$

$$= tr([R^j_\nabla, A]) = 0$$

by linearity again. Therefore $[s_j(R_\nabla)]$ is closed.

(2) If $\nabla, \nabla'$ are two connections on $E$ with curvatures $R, R'$, respectively, then consider the product bundle $\tilde{E} = E \times [0, 1] \to M \times [0, 1]$. A section $\tilde{s} \in \Gamma(M \times [0, 1], \tilde{E})$ is of the form $(x, t) \mapsto \tilde{s}(x, t) \in E_x \times \{t\} = E_x$, so in fact $\tilde{s}(\cdot, t)$ is a section of $E$ for each fixed $t \in [0, 1]$. Write $T(M \times [0, 1]) = TM \oplus \mathbb{R}(\partial_t)$. One can prove that there is a unique connection $\tilde{\nabla}$ on $\tilde{E}$ satisfying the following conditions:

- If $\tilde{s} \in \Gamma(M \times [0, 1], \tilde{E})$ is independent of $t$, then $\tilde{\nabla}_{\partial_t}\tilde{s} = 0$.

- If $X \in TM \subseteq T(M \times [0, 1])$, then $\tilde{\nabla}_X\tilde{s} = (1 - t)\nabla_X(\tilde{s}|_{M \times \{t\}}) + t\nabla'_X(\tilde{s}|_{M \times \{t\}})$.

Observe that if $f \in I_n(\mathbb{R})$, then

$$f(R_\nabla)|_{M \times \{0\}} = f(R_\nabla) \quad \text{and} \quad f(R_\nabla)|_{M \times \{1\}} = f(R_\nabla).$$

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Let \( i_0, i_1 : M \hookrightarrow M \times [0,1] \) be the inclusions of \( M \times \{0\} \) and \( M \times \{1\} \), respectively. Then
\[
[f(R_{\nabla})] = i_0^*[f(R_{\tilde{\nabla}})|_{M \times \{0\}}] = i_0^*[f(R_{\tilde{\nabla}})] \\
= i_1^*[f(R_{\tilde{\nabla}})] \quad \text{since } i_0, i_1 \text{ are homotopic} \\
= i_1^*[f(R_{\tilde{\nabla}})|_{M \times \{1\}}] = [f(R_{\tilde{\nabla}})].
\]
Therefore \([f(R_{\nabla})]\) is independent of \( \nabla \).

\[\textbf{Corollary 5.2.7.} \quad \text{For any } f \in I_n(\mathbb{R}) \text{ (or } I_n(\mathbb{C}) \text{ in the complex case), the assignment}
\]
\[
\text{Vect}_n(M) \longrightarrow H^\bullet_{dR}(M) \\
E \mapsto [f(R_{\nabla})],
\]
\[\text{for any connection } \nabla \text{ on } E, \text{ is a characteristic class for the category of smooth real (or complex) manifolds.}\]

\[\textbf{Proof.} \quad \text{We need only check } [f(R_{\nabla})] \text{ is natural. Suppose } h : M \rightarrow N \text{ is a smooth map,}
\]
\[E \rightarrow N \text{ is a vector bundle and } \tilde{E} = h^*E \text{ is the pullback bundle, so that we have a diagram}
\]
\[
\begin{array}{ccc}
\tilde{E} & \xrightarrow{\tilde{h}} & E \\
\downarrow & & \downarrow \\
M & \xrightarrow{h} & N
\end{array}
\]
\[\text{If } \{e_1, \ldots, e_n\} \text{ is a local basis of } E \text{ on an open neighborhood } U \subseteq N, \text{ let } \{\tilde{e}_1, \ldots, \tilde{e}_n\} \text{ be the}
\]
\[\text{induced local basis of } \tilde{E} \text{ on } h^{-1}(U) \subseteq M. \text{ Then for a connection } \nabla \text{ on } E, \text{ define the induced}
\]
\[\text{connection } \tilde{\nabla} = h^*\nabla \text{ locally on } h^{-1}(U) \text{ by}
\]
\[\tilde{\nabla}_X \tilde{e}_j = \nabla_{h_*(X)} e_j
\]
\[\text{for any } X \in \Gamma(h^{-1}(U), TM), \text{ where } h_* \text{ denotes the pushforward: } (h_*(X))(v) = (d_\ast h(X))_{h(x)}(v). \]
\[\text{One now checks that } \tilde{\nabla} \text{ is a connection on } \tilde{E}, \text{ and it clear that for any } f \in I_n(\mathbb{R}) \text{ or } I_n(\mathbb{C}),
\]
\[\text{we have } h^*[f(R_{\nabla})] = [h^*f(R_{\nabla})] = [f(R_{\tilde{\nabla}})]. \quad \square
\]

\[\textbf{Remark.} \quad \text{There is a more general notion of a connection on a principal } G\text{-bundle, with the}
\]
\[\text{special cases } G = O(n) \text{ and } G = U(n) \text{ recovering the theory described in this chapter. The}
\]
\[\text{replacement for } I_n(\mathbb{R}) \text{ and } I_n(\mathbb{C}) \text{ in that setting is the space of polynomial functions on the}
\]
\[\text{Lie algebra } g = \text{Lie}(G) \text{ which are invariant under Ad}(G). \]

\[\textbf{Example 5.2.8.} \quad \text{Let } M = S^2 \cong \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} \text{ and let } E = TS^2 \text{ be the tangent bundle,}
\]
\[\text{which is a complex line bundle. Viewing } S^2 \text{ in stereographic coordinates, we can write down}
\]
\[\text{a connection } \nabla \text{ on } TS^2 \text{ by specifying a } 1 \times 1 \text{ connection matrix } A = (\omega), \text{ where}
\]
\[
\omega = \frac{-2}{1 + x^2 + y^2} (x \, dx + y \, dy + i(x \, dy - y \, dx)).
\]
This defines $\nabla$ locally on the coordinate chart $S^2 \setminus \{\infty\} \cong \mathbb{C}$, but under the natural transition function from this chart to $S^2 \setminus \{0\} \cong \mathbb{C}$ given by $(x, y) \mapsto \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$, the definition of $A$ is preserved. Hence $A$ defines a global connection $\nabla$ on $TS^2$. By Example 5.2.4, $R_\nabla = dA = (d\omega)$, and we have

$$d\omega = d\left(\frac{-2}{1 + x^2 + y^2}\right) \wedge (x \, dx + y \, dy + i(x \, dy - y \, dx))$$

$$+ \left(\frac{-2}{1 + x^2 + y^2}\right) \wedge d(x \, dx + y \, dy + i(x \, dy - y \, dx))$$

$$= \left(\frac{4x}{(1 + x^2 + y^2)^2} \, dx + \frac{4y}{(1 + x^2 + y^2)^2} \, dy\right) \wedge (x \, dx + y \, dy + i(x \, dy - y \, dx))$$

$$+ \left(\frac{-2}{1 + x^2 + y^2}\right) \wedge (dx \wedge dx + dy \wedge dy + i(dx \wedge dy - dy \wedge dx))$$

$$= \frac{4}{(1 + x^2 + y^2)^2} (x^2 \, dx \wedge dx + xy \, dx \wedge dy + i(x^2 \, dx \wedge dy - xy \, dx \wedge dx))$$

$$+ \frac{4}{(1 + x^2 + y^2)^2} (xy \, dy \wedge dx + y^2 \, dy \wedge dy + i(xy \, dy \wedge dy - y^2 \, dy \wedge dx))$$

$$- \frac{2i}{1 + x^2 + y^2} (dx \wedge dy - dy \wedge dx) \quad \text{since } dx \wedge dx = dy \wedge dy = 0$$

$$= \frac{4}{(1 + x^2 + y^2)^2} (xy + i(x^2 - xy + iy^2) \, dx \wedge dy - \frac{4i(1 + x^2 + y^2)}{(1 + x^2 + y^2)^2} dx \wedge dy)$$

$$= \frac{-4i}{(1 + x^2 + y^2)^2} \, dx \wedge dy.$$

Now to compute the cohomology class $[R_\nabla] \in H^2_{dR}(S^2; \mathbb{C})$, recall that by de Rham’s theorem, $H^2_{dR}(S^2; \mathbb{C}) \cong \mathbb{C}$ via integration over the fundamental class: $\alpha \mapsto \langle \alpha, [S^2] \rangle = \int_{S^2} \alpha$. Therefore for $\alpha = d\omega$, we have

$$\int_{S^2} d\omega = \int_{\mathbb{CP}^1 \setminus \{\infty\}} d\omega = \int_{\mathbb{C}} \frac{-4i}{(1 + x^2 + y^2)^2} \, dx \, dy$$

$$= \int_0^{2\pi} \int_0^\infty \frac{-4ir}{(1 + r^2)^2} \, dr \, d\theta = \int_0^{2\pi} \left[\frac{-2i}{1 + r^2}\right]_r=0^{r=\infty} \, d\theta$$

$$= \int_0^{2\pi} -2i \, d\theta = -4\pi i.$$

So $[R_\nabla] = -4\pi i$. Compare this to the Chern class (which is equal to the Euler class in this case) using Example 3.6.7:

$$c(\mathbb{CP}^1) = c(T\mathbb{CP}^1) = (1 + x)^2 = 1 + 2x + x^2,$$

so $c_1(\mathbb{CP}^1) = 2$. This shows that $[\frac{i}{2\pi} R_\nabla] = c_1(\mathbb{CP}^1)$.

**Definition.** For a complex vector bundle $E \to M$ over a complex manifold $M$ of dimension $n$, the de Rham-Chern classes of $E$ are the classes $c_j^{dR}(E) \in H^j_{dR}(M; \mathbb{C})$ appearing as the
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jth coefficient of the total de Rham-Chern class

\[ c^{dR}(E) := \det \left( 1 + \frac{i}{2\pi} R_\nabla \right) \]

where \( \nabla \) is any connection on \( E \). Explicitly, for \( 0 \leq j \leq n \),

\[ c^{dR}_j(E) = \left[ \left( \frac{i}{2\pi} \right)^j \sigma_j(R_\nabla) \right] \]

where \( \sigma_j \in I_n(\mathbb{C}) \) is the \( j \)th elementary symmetric function.

**Example 5.2.9.** Example 5.2.8 shows that \( c^{dR}_1(T\mathbb{C}P^1) = c_1(T\mathbb{C}P^1) \), viewed in \( H^{2}_{dR}(\mathbb{C}P^1; \mathbb{C}) \cong H^2(\mathbb{C}P^1; \mathbb{Z}) \otimes \mathbb{C} \).

To extend this identification of Chern classes to any manifold and any vector bundle, it will suffice to check the axioms for Chern classes hold for \( c^{dR}(E) \). Note that naturality is guaranteed by Corollary 5.2.7.

Given any complex vector bundle \( E \rightarrow M \) with connection \( \nabla \), we can define associated connections on various related bundles such as \( E^*, E^\otimes n, \bigwedge^k E \), etc. For example, a connection \( \nabla^* \) on the dual bundle \( E^* \) should satisfy the following product rule for all \( \sigma \in \Gamma(M, E), \alpha \in \Gamma(M, E^*) \):

\[ d(\alpha(\sigma)) = (\nabla^*\alpha)(\sigma) + \alpha(\nabla\sigma). \]

In fact, we can take this as a definition of \( \nabla^* \), so that for \( X \in \Gamma(TM) \),

\[ (\nabla^*_X\alpha)(\sigma) = d(\alpha(\sigma))(X) - \alpha(\nabla_X\sigma). \]

Similarly, for \( E, F \rightarrow M \) with connections \( \nabla^E, \nabla^F \), we can define \( \nabla \) on \( E \otimes F \) as follows, for \( \sigma_1 \in \Gamma(M, E), \sigma_2 \in \Gamma(M, F) \):

\[ \nabla(\sigma_1 \otimes \sigma_2) = \nabla\sigma_1 \otimes \sigma_2 + \sigma_1 \otimes \nabla\sigma_2. \]

In the case of two complex line bundles \( L_1, L_2 \rightarrow M \) with connections \( \nabla^1, \nabla^2 \), let \( e_1 \) be a local parameter for \( L_1 \), \( e_2 \) a local parameter for \( L_2 \) and let

\[ \nabla^1 e_1 = A_1 e_1 \quad \text{and} \quad \nabla^2 e_2 = A_2 e_2 \]

define the local connection matrices. Then the connection \( \nabla \) on \( L_1 \otimes L_2 \) satisfies

\[ \nabla(e_1 \otimes e_2) = \nabla^1 e_1 \otimes e_2 + e_1 \otimes \nabla^2 e_2 = A_1 e_1 \otimes e_2 + e_1 \otimes A_2 e_2 = (A_1 + A_2)(e_1 \otimes e_2). \]

That is, \( \nabla \) has local connection matrix \( A = A_1 + A_2 \). For curvature, we compute:

\[ R_\nabla = dA + A \wedge A = dA = d(A_1 + A_2) = dA_1 + dA_2 = R_{\nabla^1} + R_{\nabla^2}. \]

This shows \([R_\nabla] = [R_{\nabla^1}] + [R_{\nabla^2}]\), which implies:

\[ c^{dR}_1(L_1 \otimes L_2) = \left[ \frac{i}{2\pi} R_{\nabla} \right] = \left[ \frac{i}{2\pi} (R_{\nabla^1} + R_{\nabla^2}) \right] \]

\[ = \left[ \frac{i}{2\pi} R_{\nabla^1} \right] + \left[ \frac{i}{2\pi} R_{\nabla^2} \right] = c^{dR}_1(L_1) + c^{dR}_1(L_2). \]

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Proposition 5.2.10. For the tautological line bundle $\gamma_1 = \gamma_1^1$ on $\mathbb{CP}^1$, $c_1^{dR}(\gamma_1) = \pm c_1(\gamma_1)$ in $H^2_{dR}(\mathbb{CP}^1; \mathbb{C})$.

Proof. By Lemma 3.8.3, complex line bundles over $\mathbb{CP}^1$ are classified by $H^2(\mathbb{CP}^1; \mathbb{Z})$ via $L \mapsto c_1(L)$, and this assignment is additive with respect to $\otimes$. We know from Example 3.6.7 that $x = c_1(\gamma_1)$ is a generator in $H^2_{dR}(\mathbb{CP}^1; \mathbb{C})$ and from Example 5.2.8, $c_1(T\mathbb{CP}^1) = \pm 2x$. Thus $T\mathbb{CP}^1 \cong \pm(\gamma_1 \otimes \gamma_1)$, where for a line bundle $L$, $-L = L^*$. Moreover, by the work above,

$$2c_1^{dR}(\gamma_1) = c_1^{dR}(\gamma_1 \otimes \gamma_1) = \pm c_1^{dR}(T\mathbb{CP}^1) = \pm 2x$$

so $c_1^{dR}(\gamma_1) = \pm x$.

By convention, we choose the “positive generator” in $H^2_{dR}(\mathbb{CP}^1; \mathbb{C})$ to be the $x$ such that $c_1(T\mathbb{CP}^1) = 2x$. In this case, $T\mathbb{CP}^1 \cong (\gamma_1 \otimes \gamma_1)^*$, so $c_1(\gamma_1) = -x$.

It remains to establish the Whitney formula for de Rham-Chern classes.

Proposition 5.2.11. Let $E, F \to M$ be complex vector bundles. Then $c^{dR}(E \oplus F) = c^{dR}(E)c^{dR}(F)$.

Proof. Let $\nabla^E$ and $\nabla^F$ be connections on $E$ and $F$, respectively. For $\sigma_1 \in \Gamma(M, E)$ and $\sigma_2 \in \Gamma(M, F)$, the formula

$$\nabla(\sigma_1, \sigma_2) = (\nabla^E \sigma_1, \nabla^F \sigma_2)$$

defines a connection $\nabla$ on $E \oplus F$, and the corresponding covariant exterior derivative $d^\nabla$ is of the form $d^\nabla(\omega_1, \omega_2) = (d^\nabla^E \omega_1, d^\nabla^F \omega_2)$ for $\omega_1 \in \Omega^p(M; E), \omega_2 \in \Omega^q(M; F)$. In particular, the curvature of $\nabla$ is given by $R^\nabla(\sigma_1, \sigma_2) = (R^\nabla^E \sigma_1, R^\nabla^F \sigma_2)$. On the level of matrices, $R^\nabla = R^\nabla^E \oplus R^\nabla^F,$ so

$$c^{dR}(E \oplus F) = \det \left(1 + \frac{i}{2\pi} R^\nabla\right) = \det \left(1 + \frac{i}{2\pi} R^\nabla^E\right) \det \left(1 + \frac{i}{2\pi} R^\nabla^F\right) = c^{dR}(E)c^{dR}(F).$$

Theorem 5.2.12. If $E \to M$ is a complex vector bundle, then $c^{dR}(E)$ is equal to the image of $c(E)$ in $H^*_{dR}(M; \mathbb{C})$ via the de Rham isomorphism $H^*(M; \mathbb{Z}) \otimes \mathbb{C} \cong H^*_{dR}(M; \mathbb{C})$.

Recall that any real vector bundle $E \to M$ has an associated complex bundle $E_{\mathbb{C}} \to M$ given by tensoring every fibre with $\mathbb{C}$. The Pontrjagin class of $E$ was defined to be

$$p(E) = 1 - c_2(E_{\mathbb{C}}) + \ldots + (-1)^n c_{2n}(E_{\mathbb{C}}).$$

Corollary 5.2.13. If $E \to M$ is a real vector bundle, then the total Pontrjagin class of $E$ is

$$p(E) = \det \left(1 + \frac{1}{2\pi} R\nabla\right) \in H^*_{dR}(M)$$

where $\nabla$ is any connection on $E$, and the $j$th Pontrjagin class $p_j(E)$ is the $j$th homogeneous component of this determinant.
Example 5.2.14. Viewing $M = S^2$ in Example 5.2.8 as a real 2-manifold, one can compute

$$R^\text{real}_\nabla = \frac{-4}{(1 + x^2 + y^2)^2} \begin{pmatrix} 0 & -dx \wedge dy \\ dx \wedge dy & 0 \end{pmatrix}.$$ 

Therefore $p(TS^2) = 1$ so the top Pontrjagin class is trivial.

Let $E \to M$ be a real vector bundle equipped with a metric. Recall that this is equivalent to $E$ having structure group $O(n)$, where $n$ is the dimension of the fibre, and since $GL_n(\mathbb{R})$ retracts to $O(n)$, every real bundle admits a metric.

**Definition.** A connection $\nabla$ on $E \to M$ is **compatible with the metric** $\langle \cdot, \cdot \rangle$ on $E$ if for any sections $v, w \in \Gamma(M, E)$,

$$d\langle v, w \rangle = \langle \nabla v, w \rangle + \langle v, \nabla w \rangle.$$ 

**Remark.** A choice of metric on a bundle $E \to M$ is equivalent to a choice of section $\sigma \in \Gamma(M, E^* \otimes E^*)$ satisfying

- $h(v, w) = h(w, v)$ for all $v, w \in \Gamma(M, E)$;
- $h(v, v) \geq 0$ for all $v \in \Gamma(M, E)$, with $h(v, v) = 0$ if and only if $v = 0$.

Then a connection $\nabla$ is compatible with the metric if and only if $\nabla^* h = 0$, where $\nabla^*$ is the connection induced by $\nabla$ on $E^* \otimes E^*$.

**Lemma 5.2.15.** Every real vector bundle with a metric admits a compatible connection.

**Proposition 5.2.16.** Suppose $E \to M$ is a vector bundle with a metric and a compatible connection $\nabla$ and let $\{e_1, \ldots, e_n\}$ be a local orthonormal basis for $E$. Then the connection matrix $A$ for $\nabla$ with respect to the $e_j$ is skew-symmetric, i.e. $A_{ij} = -A_{ji}$.

**Proof.** An orthonormal basis satisfies $\langle e_i, e_j \rangle = \delta_{ij}$ for all $i, j$, so the compatibility condition gives us

$$0 = d\delta_{ij} = d\langle e_i, e_j \rangle = \langle \nabla e_i, e_j \rangle + \langle e_i, \nabla e_j \rangle$$

$$= \sum_{k=1}^{n} (\langle A_{ik} e_k, e_j \rangle + \langle e_i, A_{jk} e_j \rangle)$$

$$= \sum_{k=1}^{n} (A_{ik} \delta_{kj} + A_{jk} \delta_{ik})$$

$$= A_{ij} + A_{ji}.$$ 

Therefore $A$ is skew. 

**Remark.** The set $\mathfrak{o}(n)$ of skew-symmetric $n \times n$ skew-symmetric matrices is precisely the Lie algebra of $O(n)$. A compatible connection is sometimes called an $O(n)$-connection, and there is a more general theory of $G$-connections as hinted at earlier.
Corollary 5.2.17. The curvature of a compatible connection $\nabla$ on a vector bundle $E \to M$ is an element $R_\nabla$ of $\Omega^2(M; \text{End}_{\text{skew}}(E))$, where $\text{End}_{\text{skew}}(E)$ denotes the subbundle of $\text{End}(E)$ given by skew-symmetric operators on each fibre.

Proof. Locally, $R_\nabla = dA + A \wedge A$ by Example 5.1.8, so for each $1 \leq i, j \leq n$,

$$(R_\nabla)_{ij} = dA_{ij} + \sum_{k=1}^{n} A_{ki} \wedge A_{jk} = -dA_{ji} - \sum_{k=1}^{n} A_{kj} \wedge A_{ik} = -(R_\nabla)_{ji}. \quad \square$$

Corollary 5.2.18. For any symmetric polynomial $f \in I_n(\mathbb{R})$ of odd degree $k$ and any connection $\nabla$ on $E \to M$, $[f(R_\nabla)] = 0$ in $H^{2k}_{dR}(M; \mathbb{R})$.

Proof. By Theorem 5.2.6, $[f(R_\nabla)]$ is independent of $\nabla$ so we may choose $\nabla$ compatible with the metric, so that by Corollary 5.2.17, $R_\nabla$ is skew. In the case $f = s_k$, $s_k(A) = \text{tr}(A^k)$ and if $A$ is skew, so is $A^k$. Therefore $s_k(R_\nabla) = 0$ and by Lemma 5.2.3, this proves it for any $f \in I_n(\mathbb{R})$. \quad \square

Corollary 5.2.19. The Pontrjagin classes $p_j(E) = (-\frac{1}{2\pi})^{2j} \sigma_{2j}(R_\nabla) \in H^{4j}_{dR}(M; \mathbb{R})$ are trivial for $j$ odd.

Remark. The Euler class also has an interpretation in terms of curvature. Let $\mathfrak{so}(n)$ be the space of $n \times n$ skew-symmetric matrices, which is equivalently the Lie algebra of $SO(n)$. (In fact, since $SO(n)$ is a connected component of $O(n)$, $\mathfrak{so}(n) = \mathfrak{o}(n)$.) The Pfaffian is a skew-symmetric polynomial $PF \in I_{2n}(\mathfrak{so}(2n))$ defined by

$$PF(x_{ij}) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} x_{(1)} \sigma(2) x_{(3)} \sigma(4) \cdots x_{(2n-1)} \sigma(2n).$$

Then for any compatible connection $\nabla$, the matrix $R_\nabla$ is skew, so $PF(R_\nabla)$ defines a de Rham cohomology class which is independent of $\nabla$. One can prove that:

$$e(E) = \frac{1}{(2\pi)^n} [PF(R_\nabla)] \in H^{2n}_{dR}(M; \mathbb{R}).$$

It’s easy to see that this class satisfies $e(E) = c_n^{dR}(E)$ and $e(E)^2 = p_n(E_{\mathbb{R}})$ as expected.

5.3 The Levi-Civita Connection

Let $(M, g)$ be a Riemannian manifold.

Lemma 5.3.1. If $\{e_1, \ldots, e_n\}$ is a local orthonormal basis for $TM$ with respect to $g$ and $\{\theta_1, \ldots, \theta_n\}$ is the dual basis for $T^*M$, then there is a unique matrix $A = (A_{ij})$ of differential 1-forms satisfying:

(i) $A_{ij} = -A_{ji}$ (that is, $A$ is skew-symmetric);
(ii) \( d\theta_i = -\sum_{j=1}^{n} A_{ji} \wedge \theta_j \).

Proof. In general, we can write

\[
d\theta_i = \sum_{j,k} a_{ijk} \theta_j \wedge \theta_k = 2 \sum_{j<k} a_{ijk} \theta_j \wedge \theta_k
\]

for functions \( a_{ijk} \) satisfying \( a_{ijk} = -a_{ikj} \). Being 1-forms, the \( A_{ij} \) should be of the form \( A_{ij} = \sum_k b_{ijk} \theta_k \) for some coefficients \( b_{ijk} \). Notice that (i) is equivalent to having \( b_{ijk} = -b_{ikj} \), while (ii) is equivalent to any of the following:

\[
d\theta_i = -\sum_{j,k} A_{ji} \wedge \theta_j \iff d\theta_i = -\sum_{j,k} b_{ijk} \theta_j \wedge \theta_k = \sum_{j<k} (b_{ijk} - b_{ikj}) \theta_j \wedge \theta_k
\]

\[
\iff 2a_{ijk} = b_{ijk} - b_{ikj}
\]

\[
\iff 2a_{jik} = b_{jik} - b_{jki}
\]

\[
\iff 2a_{kji} = b_{kji} - b_{kij}
\]

One can combine the latter three equations to obtain \( b_{ijk} = a_{ijk} - a_{jik} + a_{kji} \), so the \( b_{ijk} \) are all defined and \( A \) satisfies (i) and (ii). \( \square \)

Definition. The Levi-Civita connection on a Riemann manifold \( M \) is the unique connection \( \nabla \) on \( TM \) such that for any orthonormal basis \( \{e_1, \ldots, e_n\} \) with matrix of 1-forms \( A = (A_{ij}) \) satisfying the conditions of Lemma 5.3.1,

\[
\nabla e_i = \sum_{j=1}^{n} A_{ij} e_j
\]

for all \( 1 \leq i \leq n \).

Remark. By construction, the dual connection \( \nabla^* \) on \( T^*M \) satisfies \( \nabla^* \theta_i = -\sum_{j=1}^{n} A_{ji} \wedge \theta_j \) if \( \{\theta_i\} \) is the dual basis to \( \{e_i\} \).

Theorem 5.3.2. Let \( \nabla \) be the Levi-Civita connection on \( TM \). Then the composition

\[
\Gamma(M, T^*M) \xrightarrow{\nabla^*} \Gamma(M, T^*M \otimes T^*M) \to \Gamma(M, \bigwedge^2 T^*M)
\]

is equal to the exterior derivative.

Proof. Locally, if \( \alpha \in \Gamma(M, T^*M) \) can be written \( \alpha = \sum_{i=1}^{n} \alpha_i \theta_i \), then

\[
\nabla^* \alpha = \sum_{i=1}^{n} \nabla(\alpha_i \theta_i) = \sum_{i=1}^{n} (d\alpha_i \otimes \theta_i + \alpha_i \otimes \nabla \theta_i)
\]

\[
= \sum_{i=1}^{n} \left( d\alpha_i \otimes \theta_i - \alpha_i \sum_{j=1}^{n} A_{ji} \otimes \theta_j \right)
\]

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5.3 The Levi-Civita Connection

and the image of this element in $\Gamma(M, \Lambda^2 T^*M)$ is

$$
\sum_{i=1}^{n} \left( d\alpha_i \wedge \theta_i - \alpha_i \sum_{j=1}^{n} A_{ji} \wedge \theta_j \right) = \sum_{i=1}^{n} (d\alpha_i \wedge \theta_i + \alpha_i \wedge d\theta_i)
$$

$$
= \sum_{i=1}^{n} d(\alpha_i \theta_i) = d\alpha.
$$

Remark. The Levi-Civita connection is uniquely characterized by the following two properties:

1. $\nabla$ is compatible with the inner product $\langle \cdot, \cdot \rangle = g(\cdot, \cdot)$ on $M$, i.e. for all $X,Y \in \Gamma(M, TM)$, $d\langle X, Y \rangle = \langle \nabla X, Y \rangle + \langle X, \nabla Y \rangle$.

2. $\nabla$ is torsion-free, i.e. $\nabla_X Y - \nabla_Y X - [X,Y] = 0$ for any vector fields $X,Y$. 
6 Index Theory

6.1 Differential Operators

Suppose $M$ is a Riemannian manifold, $E \to M$ is a vector bundle with connection $\nabla$ and $\sigma : T^*M \otimes E \to F$ is any morphism of vector bundles over $M$. Then the composition $D = \sigma \circ \nabla : \Gamma(M, E) \xrightarrow{\nabla} \Gamma(M, T^*M \otimes E) \xrightarrow{\sigma} \Gamma(M, F)$ is an example of a first order linear differential operator $E \to F$.

**Example 6.1.1.** The Levi-Civita connection $\nabla$ on $TM$ (see Section 5.3) determines the exterior derivative

$$d : \Gamma(M, T^*M) \xrightarrow{\nabla^*} \Gamma(M, T^*M \otimes T^*M) \to \Gamma(M, \bigwedge^2 T^*M)$$

by Theorem 5.3.2.

Suppose $\nabla'$ is another connection on $E$ given by $\nabla = \nabla' + A$ for some $A \in \Gamma(M, T^*M \otimes \text{End}(E))$. Then for any bundle map $\sigma : T^*M \otimes E \to F$, the composition $\sigma \circ A$ is also a bundle map, so $D = \sigma \circ \nabla = \sigma \circ (\nabla' + A) = \sigma \circ \nabla' + \sigma \circ A$. We can view $\sigma \circ A$ as a 0th order linear differential operator $E \to F$. Thus we make the following definition.

**Definition.** Let $E, F \to M$ be vector bundles and $\nabla$ a connection on $E$. A first order linear differential operator $E \to F$ is any operator $D : \Gamma(M, E) \to \Gamma(M, F)$ of the form $D = \sigma \circ \nabla + K$ where $\sigma : T^*M \otimes E \to F$ and $K : E \to F$ are bundle maps. The corresponding map $\sigma = \sigma_D : \Gamma(M, T^*M \otimes E) \to \Gamma(M, F)$ is called the symbol of $D$.

**Example 6.1.2.** Extending the above, the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is a first order linear differential operator given by $d = \sigma \circ \nabla$ where $\nabla$ is the extension of the Levi-Civita connection to $\Gamma(M, \bigwedge^k T^*M) \to \Gamma(M, T^*M \otimes \bigwedge^k T^*M)$ and $\sigma$ is the bundle map

$$\sigma : \Gamma(M, T^*M \otimes \bigwedge^k T^*M) \to \Gamma(\bigwedge^{k+1} T^*M)$$

$$\alpha \otimes \beta \longmapsto \alpha \wedge \beta.$$

For a point $p \in M$, let $I_p \subset C^\infty(M)$ be the ideal of smooth functions vanishing at $p$.

**Definition.** A $k$th order linear differential operator $D$ from $E \to F$ is a map $D : \Gamma(M, E) \to \Gamma(M, F)$ such that $k$ is the smallest integer with the property that $D(f\sigma)(p) = 0$ for all $p \in M, \sigma \in \Gamma(E)$ and $f \in I_p^{k+1}$.

**Example 6.1.3.** A 0th order linear differential operator is just a bundle map $E \to F$. Indeed, take any $g \in C^\infty(M)$ and $p \in M$, so that $g = g(p) + f$ for some $f \in I_p$. Then a 0th order operator $D$ satisfies

$$D(g\sigma)(p) = D(g(p)\sigma + f\sigma)(p) = g(p)D(\sigma)(p) + D(f\sigma)(p) = g(p)D(\sigma)(p).$$

Thus $D$ is linear over $C^\infty(M)$, so it corresponds to a bundle map $E \to F$. 

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Definition. Let $D : \Gamma(M, E) \to \Gamma(M, F)$ be a $k$th order linear differential operator. The symbol of $D$ is the bundle map

$$\sigma_D : \text{Sym}^k(T^*M) \to \text{Hom}(E, F)$$

defined as follows. For $p \in M$ and $\omega_1, \ldots, \omega_k \in T^*_p M$, choose functions $f_1, \ldots, f_k \in I_p$ such that $d_pf_i = \omega_i$ for $1 \leq i \leq k$. For a section $\psi \in \Gamma(M, E)$, set

$$\sigma_D(\omega_1 \otimes \cdots \otimes \omega_k)(\psi)(p) = \frac{1}{k!} D(f_1 \cdots f_k \psi)(p).$$

(technically, this defines a map on sections $\Gamma(M, \text{Sym}^k(T^*M)) \to \Gamma(M, \text{Hom}(E, F))$ which is linear over $C^\infty(M)$, thus corresponding to $\sigma_D : \text{Sym}^k(T^*M) \to \text{Hom}(E, F)$.)

Lemma 6.1.4. The definition of $\sigma_D$ is independent of the choice of $f_1, \ldots, f_k$.

Proof. Clearly the order of the $f_1, \ldots, f_k$ does not matter, so suppose we have these as in the definition above and we also have $g_1 \in I_p$ with $d_p g_1 = \omega_1 = d_p f_1$. Then $f_1 - g_1 \in I^2_p$, so

$$D(f_1 \cdots f_k \psi)(p) = D(g_1 f_2 \cdots f_k \psi)(p) + D((f_1 - g_1) f_2 \cdots f_k \psi)(p)$$

$$= D(g_1 f_2 \cdots f_k \psi)(p) + 0$$

since $D$ is a $k$th order operator and $(f_1 - g_1) f_2 \cdots f_k \in I_p^{k+1}$. □

Remark. The symbol $\sigma_D$ is uniquely determined by the values of $\sigma_D(\omega \otimes \cdots \otimes \omega)$ for each $\omega \in T^*_p M$, since $\text{Sym}^d(\mathbb{R}^n)$ may be identified as the space of homogeneous polynomials of degree $d$ on $\mathbb{R}^n$.

Definition. A $k$th order linear differential operator $D : \Gamma(M, E) \to \Gamma(M, F)$ is called elliptic if its symbol is an isomorphism $\sigma_D(\omega \otimes \cdots \otimes \omega) : E_p \to F_p$ for all $p \in M, \omega \in T^*_p M$.

Example 6.1.5. Take $k = 1$ and let $\nabla : \Gamma(M, E) \to \Gamma(M, T^*M \otimes E)$ be a connection on a vector bundle $E$ (which is a first order operator). For $p \in M, f \in I_p$ and $\psi \in \Gamma(M, E)$, we have

$$\nabla(f \psi)(p) = d_pf \otimes \psi(p) + f(p) \nabla \psi(p) = d_pf \otimes \psi(p).$$

This shows that for the first order operator $D = \sigma \circ \nabla$, the symbol $\sigma_D$ is given by $\sigma_D(\omega)(\psi) = \omega \otimes \psi$.

Example 6.1.6. More generally, if $d : \Gamma(M, \wedge^k T^*M) \to \Gamma(M, \wedge^{k+1} T^*M)$ is the exterior derivative, we know $d = \wedge \circ \nabla$ where $\wedge : \alpha \otimes \beta \mapsto \alpha \wedge \beta$. For $f \in C^\infty(M)$ and $\alpha \in \Gamma(M, \wedge^k T^*M)$, $d(f \alpha) = df \wedge \alpha + f d\alpha$, so if $f \in I_p$ and $\omega = d_pf$, the symbol of $d$ is given by $\sigma_d(\omega)(\alpha) = \omega \wedge \alpha$.

Notice that if $\beta \in \wedge^{k-1} T^*M$ such that $\alpha = \omega \wedge \beta$, then $\sigma_d(\omega)(\alpha) = \omega \wedge \omega \wedge \beta = 0$, so $d$ is not elliptic (unless for example $\dim M = 1$). In general, a linear differential operator $D : \Gamma(M, E) \to \Gamma(M, F)$ can’t be elliptic unless $E$ and $F$ are vector bundles of the same dimension, so the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is certainly not elliptic in general.
Example 6.1.7. Suppose \( D \) is a \( k \)th order linear differential operator acting on functions on \( \mathbb{R}^n \) (every linear differential operator is locally so). Then

\[
D = \sum_{j \leq k, i_1 < \ldots < i_j} a_{i_1, \ldots, i_j} (p) \frac{\partial^j}{\partial x_{i_1} \ldots \partial x_{i_j}}.
\]

So if \( p \in \mathbb{R}^n \) and \( f_1, \ldots, f_k \) vanish at \( p \), then for \( \psi : \mathbb{R}^n \to \mathbb{R} \) and 1-forms \( \omega_1, \ldots, \omega_k \in T_p^* \mathbb{R}^n \), we have

\[
\sigma_D(\omega_1 \otimes \cdots \otimes \omega_k)(\psi)(p) = \frac{1}{k!} D(f_1 \cdots f_k \psi)(p)
\]

\[
= \frac{1}{k!} \sum_{j \leq k, i_1 < \ldots < i_j} a_{i_1, \ldots, i_j} (p) \left. \frac{\partial^j (f_1 \cdots f_k \psi)}{\partial x_{i_1} \cdots \partial x_{i_j}} \right|_p
\]

\[
= \frac{1}{k!} \sum_{i_1 < \ldots < i_k} a_{i_1, \ldots, i_k} (p) \frac{\partial f_{\sigma(1)}}{\partial x_{i_1}} \cdots \frac{\partial f_{\sigma(k)}}{\partial x_{i_k}} \psi(p)
\]

\[
= \frac{1}{k!} \sum_{i_1 < \ldots < i_k} a_{i_1, \ldots, i_k} (p) \omega_1 \wedge \cdots \wedge \omega_k \psi(p)
\]

\[
= \sum_{i_1, \ldots, i_k} a_{i_1, \ldots, i_k} (p) \omega_1 \wedge \cdots \wedge \omega_k \psi(p).
\]

For example, if \( D = a_{11} \frac{\partial^2}{\partial x_1^2} + a_{12} \frac{\partial^2}{\partial x_1 \partial y} + a_{21} \frac{\partial^2}{\partial y \partial x} + a_{22} \frac{\partial^2}{\partial y^2} \) is a second order operator on \( \mathbb{R}^2 \), then \( \sigma_D(\omega \otimes \eta) = a_{11} \omega^2 + (a_{12} + a_{21}) \omega \wedge \eta + a_{22} \eta^2 \). Substituting \( x = \omega \) and \( y = \eta \), we may view \( \sigma_D \) as a function on \( \mathbb{R}^2 \):

\[
\sigma_D(x, y) = a_{11} x^2 + (a_{12} + a_{21}) xy + a_{22} y^2.
\]

For \( D \) to be an elliptic operator, \( \sigma_D \) must be an isomorphism, so \( a_{11} x^2 + (a_{12} + a_{21}) xy + a_{22} y^2 \neq 0 \) for all \( (x, y) \neq (0, 0) \) in \( \mathbb{R}^2 \). In the special case that the \( a_{ij} \) are constant functions, the above holds if and only if \( a_{11} x^2 + (a_{12} + a_{21}) xy + a_{22} y^2 = c \) describes an ellipse for all \( c \neq 0 \). (Hence the name “elliptic”.)

Example 6.1.8. On \( M = \mathbb{R}^n \), let \( \Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \) be the Laplacian operator. Then \( \Delta \) is a second order linear differential operator with symbol

\[
\sigma_\Delta(\omega) = -\sum_{i=1}^n \omega_i^2 = -|\omega|^2 \cdot id
\]

if \( \omega = \omega_1 \otimes \cdots \otimes \omega_n \). Thus \( \sigma_\Delta(\omega) \neq 0 \) whenever \( \omega \neq 0 \), so \( \Delta \) is elliptic.

Definition. Let \((M, g)\) be an oriented Riemannian manifold of dimension \( n \) with induced inner product \( \langle \cdot, \cdot \rangle \) on each \( \bigwedge^k T^*M \), \( k \geq 1 \). Then the volume form of \((M, g)\) is the unique \( n \)-form \( \text{vol}_g \in \Omega^n(M) \) such that \( |\text{vol}_g| = 1 \) pointwise on \( M \) and \( \text{vol}_g \) induces the given orientation on \( M \) as a section of the orientation bundle.
Remark. Locally, \( \text{vol}_g = \varepsilon_1 \wedge \cdots \wedge \varepsilon_n \) where \( \{ \varepsilon_i \} \) is an oriented, orthonormal basis for \( T^* M \). Equivalently, if \( x_1, \ldots, x_n \) are local coordinates on \( M \), then
\[
\text{vol}_g = \sqrt{\det(g(x_i, x_j))} \, dx_1 \wedge \cdots \wedge dx_n.
\]

Definition. For a Riemannian manifold \((M, g)\), the Hodge star operator on \( M \) is the bundle map \( \star : \bigwedge^k T^* M \to \bigwedge^{n-k} T^* M \) defined uniquely by
\[
\alpha \wedge \star \beta = \langle \alpha, \beta \rangle \text{vol}_g
\]
for all \( \alpha, \beta \in \bigwedge^k T^* M \). Equivalently, for \( \{ \varepsilon_1, \ldots, \varepsilon_n \} \) an oriented normal basis for \( T^* M \), \( \star \) may be defined by
\[
\star(\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_k}) = (-1)^\sigma \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_{n-k}}
\]
where \( \{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\} \) and \( \sigma \) denotes the sign of the permutation \( (i_1, \ldots, i_k, j_1, \ldots, j_{n-k}) \) of this set.

Example 6.1.9. When \( M = \mathbb{R}^n \) with dual basis \( \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} \),
\[
\star(\varepsilon_1 \wedge \varepsilon_3) = -\varepsilon_2 \wedge \varepsilon_4, \quad \star(\varepsilon_2) = -\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_4, \quad \text{etc.}
\]

Remark. On any compact, oriented, Riemannian \( n \)-manifold \((M, g)\), the integral of an \( n \)-form can be computed locally on compatible oriented coordinate patches by ordinary iterated integration. For instance, the integral of a function \( f \) on \( M \) may be taken with respect to the volume form \( \text{vol}_g \) on \( M \) and we have
\[
\int_M f := \int_M f \, \text{vol}_g = \int_M \star f \, \text{vol}_g.
\]

Definition. Let \( E, F \to M \) be vector bundles with inner product on a closed, oriented, Riemannian manifold \((M, g)\) of dimension \( n \) and let \( D : \Gamma(M, E) \to \Gamma(M, F) \) be a linear differential operator. The formal adjoint of \( D \) is the linear differential operator \( D^* : \Gamma(M, F) \to \Gamma(M, E) \) defined uniquely by
\[
\int_M \langle D \varphi, \psi \rangle = \int_M \langle \varphi, D^* \psi \rangle
\]
for all compactly supported sections \( \varphi \in \Gamma(M, E), \psi \in \Gamma(M, F) \).

Lemma 6.1.10. The formal adjoint of a linear differential operator exists. Moreover, if \( D \) is a \( k \)-th order operator then so is \( D^* \).

Proof. Existence follows from the fact that the spaces \( \Gamma_{cpt}(M, E), \Gamma_{cpt}(M, F) \) of compactly supported sections are Hilbert spaces. The second statement is obvious. \qed
Example 6.1.11. Let \( d : \bigwedge^{k-1} T^*M \to \bigwedge^k T^*M \) be the exterior derivative on an \( n \)-manifold \( M \). Then for \( \alpha \in \Omega^{k-1}(M) \) and \( \beta \in \Omega^k(M) \), we have \( \ast \beta \in \Omega^{n-k}(M) \) so by Stokes’ theorem,
\[
0 = \int_M d(\alpha \wedge \ast \beta) = \int_M (d\alpha \wedge \ast \beta + (-1)^{k-1}\alpha \wedge d(\ast \beta))
\]
\[
= \int_M \langle d\alpha, \beta \rangle + (-1)^{k-1} \int_M (-1)^{(k-1)(n-k+1)} \alpha \wedge \ast d(\ast \beta)
\]
\[
= \int_M \left( \langle d\alpha, \beta \rangle + (-1)^{(k-1)(n-k)} \langle \alpha, \ast d(\ast \beta) \rangle \right)
\]
\[
= (-1)^{n(k-1)+1} \int_M \langle \alpha, \ast d(\ast \beta) \rangle.
\]
This shows that the formal adjoint of \( d \) is \( d^\ast = (-1)^{n(k-1)+1} \ast d \ast \).

Definition. Let \( X \in TM \) be a vector field. The \textit{interior product} (or \textit{contraction}) of \( X \) is the bundle map \( i_X : \bigwedge^k T^*M \to \bigwedge^{k-1} T^*M \) defined by
\[
(i_X \omega)(Y_1, \ldots, Y_{k-1}) = \omega(X, Y_1, \ldots, Y_{k-1}).
\]

Lemma 6.1.12. Let \( X \in TM \) be a vector field. Then
(a) \( i_X \) is formally adjoint to \( \wedge \) with respect to the identification \( TM \cong T^*M \); explicitly, if \( \alpha_X \in T^*M \) is the functional \( \alpha_X(Y) = \langle X, Y \rangle \), then
\[
\langle i_X \omega, \eta \rangle = \langle \omega, \alpha_X \wedge \eta \rangle
\]
for all \( \omega \in \Omega^k(M), \eta \in \Omega^{k-1}(M) \).
(b) \( i_X \) is a derivation:
\[
i_X(\omega \wedge \eta) = i_X \omega \wedge \eta + (-1)^k \omega \wedge i_X \eta
\]
if \( \omega \in \Omega^k(M) \).
(c) With respect to an orthonormal basis \( \{e_1, \ldots, e_n\} \) for \( TM \), the adjoint \( d^\ast \) of the exterior derivative can be written
\[
d^\ast = -\sum_{j=1}^n i_{e_j} \circ \nabla e_j
\]
where \( \nabla \) is the Levi-Civita connection on \( TM \).

Proof. Straightforward from the definitions.

Proposition 6.1.13. The operator \( D = d + d^\ast : \Omega^\ast(M) \to \Omega^\ast(M) \) is elliptic.

Proof. The symbol of \( D = d + d^\ast \) is defined for each \( \psi \in \Gamma(M, E) \) by
\[
\sigma_D(\omega)(\psi) = \omega \wedge \psi - i_\omega^\ast \psi
\]
where \( \omega^\ast \) is the tangent vector dual to \( \omega \); i.e. \( \alpha_\omega^\ast = \omega \). Now for all \( \psi \), we have
\[
[\sigma_D(\omega)]^2(\psi) = \omega \wedge (\omega \wedge \psi - i_\omega^\ast \psi) - i_\omega^\ast (\omega \wedge \psi - i_\omega^\ast \psi)
\]
\[
= 0 - \omega \wedge i_\omega^\ast \psi - i_\omega^\ast (\omega \wedge \psi) + 0 \quad \text{since } (i_\omega^\ast)^2 = 0
\]
\[
= -i_\omega^\ast \omega \wedge \psi = -|\omega|^2 \wedge \psi.
\]
So \( [\sigma_D(\omega)]^2 = -|\omega|^2 \wedge id \) which is an isomorphism unless \( \omega = 0 \). Hence \( \sigma_D^2 \) is an isomorphism, so \( \sigma_D \) is as well.
Recall from Example 6.1.8 that the Laplacian operator $\Delta$ has symbol $-|\omega|^2 \wedge id$. For this reason, a second order linear differential operator $D : \Gamma(M, E) \to \Gamma(M, F)$ with symbol $\sigma_D(\omega) = -|\omega|^2 \wedge id$ is called a generalized Laplacian operator.

**Definition.** A first order linear differential operator $D : \Gamma(M, E) \to \Gamma(M, F)$ is a Dirac-type operator if $D^2 = D \circ D$ is a generalized Laplacian operator.

**Example 6.1.14.** By Proposition 6.1.13, $D = d + d^*$ is a Dirac-type operator. The generalized Laplacian operator $\Delta = (d + d^*)^2 = dd^* + d^*d$ is called the Laplace-Beltrami operator on $M$. (Notice that it depends on the choice of metric on $M$.)

**Definition.** A $k$-form $\omega \in \Omega^k(M)$ is harmonic if $\Delta \omega = 0$. The vector space of all harmonic $k$-forms is written $\mathcal{H}^k(M)$.

**Lemma 6.1.15.** Suppose $M$ is a closed, Riemannian manifold. Then a differential form $\omega \in \Omega^k(M)$ is harmonic if and only if it is both closed ($d\omega = 0$) and co-closed ($d^*\omega = 0$).

**Proof.** ($\Longleftarrow$) is trivial.

($\Longrightarrow$) Suppose $\Delta \omega = 0$. Then

$$0 = \int_M \langle \Delta \omega, \omega \rangle = \int_M \langle dd^* \omega, \omega \rangle + \int_M \langle d^*d \omega, \omega \rangle = \int_M \langle d^* \omega, d^* \omega \rangle + \int_M \langle d \omega, d \omega \rangle.$$  

since $M$ is closed

$$= ||d^* \omega||^2 + ||d \omega||^2$$

where $|| \cdot ||$ is the operator norm. This is only possible if $||d^* \omega||^2 = ||d \omega||^2 = 0$, so both $d\omega$ and $d^* \omega$ are zero. 

### 6.2 Hodge Decomposition

The elliptic condition on a linear differential operator $D$ is really an algebraic condition: the symbol $\sigma_D$ is a linear map on vector spaces, so isomorphism has an algebraic meaning. The goal of index theory is to interpret this information topologically.

**Definition.** A linear differential operator $D : \Gamma(M, E) \to \Gamma(M, F)$ is Fredholm if $\text{im} \ D \subseteq \Gamma(M, F)$ is closed (with respect to the norm) and $\ker D$ and $\text{coker} D$ are both finite dimensional vector spaces. The index of a Fredholm operator is defined to be $\text{ind}(D) = \dim \ker D - \dim \text{coker} D$.

There is also an analytic component to index theory, as the vector spaces in question, $\ker D = \{ f \in \Gamma(M, E) \mid Df = 0 \}$ and $\text{coker} D = \ker (D^*) = \{ g \in \Gamma(M, F) \mid D^*g = 0 \}$, are spaces of homogeneous solutions to some differential equations coming from de Rham theory.
Theorem 6.2.1. Every elliptic operator is Fredholm.

Remark. A property called elliptic regularity shows that \( \ker D \) and \( \coker D \) consist of smooth functions, so we may interpret these spaces in terms of de Rham theory.

Theorem 6.2.2 (Hodge Decomposition). For a closed Riemannian manifold \( M \), there is an orthogonal decomposition

\[
\Omega^k(M) = \mathcal{H}^k(M) \oplus d\Omega^{k-1}(M) \oplus d^*\Omega^{k+1}(M)
\]

for all \( k \geq 1 \).

Proof. Observe that \( \Delta = dd^* + d^*d \) is self-adjoint, so we have a decomposition

\[
\Omega^\bullet(M) = \ker \Delta \oplus (\ker \Delta)^\perp = \ker \Delta \oplus (\ker \Delta^*)^\perp = \ker \Delta \oplus \im \Delta.
\]

Note that \( \ker \Delta = \mathcal{H}^\bullet(M) = \bigoplus_{k \geq 0} \mathcal{H}^k(M) \). On the other hand, for \( \omega \in \im \Delta \), write \( \omega = (dd^* + d^*d)\gamma \) for \( \gamma \in \Omega^\bullet(M) \). Set \( \alpha = d^*\gamma \) and \( \beta = d\gamma \). Then

\[
\int_M \langle d\gamma, d^*\eta \rangle = \int_M \langle d^2\gamma, \eta \rangle = 0
\]

for all \( \eta \) implies \( \im d \perp \im (d^*) \), so it follows that \( \alpha \) and \( \beta \) are unique. \( \square \)

Corollary 6.2.3. Every class in \( H^k_{dR}(M) \) has a unique harmonic representative.

Proof. By Theorem 6.2.2, any class \( [\omega] \in H^k_{dR}(M) \) is represented by \( \omega = h + d\alpha + d^*\beta \) for some \( h \in \mathcal{H}^k(M), \alpha \in \Omega^{k-1}(M) \) and \( \beta \in \Omega^{k+1}(M) \). Thus

\[
0 = d\omega = dh + d^2\alpha + dd^*\beta = 0 + 0 + dd^*\beta
\]

so \( d^*\beta \) is closed and co-closed, meaning \( d^*\beta \in \mathcal{H}^k(M) \cap \im (d^*) = 0 \). Hence \( \omega = h + d\alpha \), so we see that \( [\omega] = [h] \). For uniqueness, suppose \( [h] = [h'] \) for some other harmonic form \( h' \in \mathcal{H}^k(M) \). Then \( h = h' + d\alpha \) for some \( \alpha \in \Omega^{k-1}(M) \), but \( d\alpha = h - h' \) is harmonic, so \( d\alpha = 0 \). \( \square \)

Corollary 6.2.4. For any Riemannian manifold \( M \), there is an isomorphism of graded rings \( \mathcal{H}^\bullet(M) \cong H^k_{dR}(M) \).

From this, we get a primitive form of the index theorem:

Corollary 6.2.5. Let \( D = d + d^* : \Omega_{\text{even}}(M) \to \Omega_{\text{odd}}(M) \). Then \( \text{ind}(D) = \chi(M) \), the Euler characteristic of \( M \).

Proof. By definition,

\[
\text{ind}(D) = \dim \ker D - \dim \coker D \quad \text{and} \quad \chi(M) = \dim \mathcal{H}_{\text{even}}(M) - \dim \mathcal{H}_{\text{odd}}(M),
\]

so Corollary 6.2.4 gives the result. \( \square \)

Corollary 6.2.6. If \( \Delta = \Delta^* \) is a self-adjoint, Fredholm, linear differential operator, then \( \text{ind}(D) = 0 \). In particular, the Laplacian operator \( \Delta = d^*d + dd^* \) has index 0.
6.3 The Index Theorem

Suppose $D : \Gamma(X, E) \to \Gamma(X, F)$ is an elliptic linear differential operator over a Riemannian manifold $X$. If $\pi : TX \to X$ is the tangent bundle, write $\tilde{E} = \pi^*E$ and $\tilde{F} = \pi^*F$. Denote the frame bundle associated to $TX$ by $TX_0$, or equivalently, $TX_0 = TX \setminus s_0(X)$ where $s_0$ is the zero section.

**Lemma 6.3.1.** There is a bundle isomorphism $\tilde{\sigma}_D : \tilde{E}|_{TX_0} \to \tilde{F}|_{TX_0}$ such that for each $x \in X$ and $\omega \in T^*X \cong TX$, $\tilde{\sigma}|(x, \omega) = \sigma_D(\omega) : E_x \to F_x$ is the symbol of $D$.

Using the notation of Section 4.1, this defines an element of $K$-theory $\sigma_D \in L_1(TX, TX_0) \cong K(TX, TX_0)$ by Proposition 4.1.10. The pair $(TX, TX_0)$ is homotopy equivalent to the Thom space $DX/SX$, where $DX$ and $SX$ are, respectively, the unit disk and unit sphere bundle on $X$. Thus $\sigma_D \in \tilde{K}(DX/SX)$.

Using the notation in Section 4.4, let $\mathcal{C} : K(X) \to H^\bullet(X; \mathbb{Q})$ be the Chern character of $X$. Then by Theorem 4.4.6, for any complex vector bundle $E \to X$, there is an isomorphism $\Phi_K : K(X) \to K(E, E_0)$ given by $\alpha \mapsto \alpha \lambda_E$, where $\lambda_E$ is the Thom class in $K$-theory of $E \to X$. Observe that we have a diagram

$$
\begin{array}{ccc}
K(X) & \xrightarrow{\Phi_K} & K(E, E_0) \\
\mathcal{C} & \downarrow & \downarrow \mathcal{C} \\
H^\bullet(X; \mathbb{Q}) & \xrightarrow{\Phi} & H^\bullet(E, E_0; \mathbb{Q})
\end{array}
$$

where $\Phi$ is the ordinary Thom isomorphism. However, the diagram does not commute! In fact, we have $\Phi(\mathcal{C}(\alpha)) = t \mathcal{C}(\Phi_K(\alpha))$ for some $t \in H^\bullet(X; \mathbb{Q})$. Applying $i^* : H^\bullet(E, E_0; \mathbb{Q}) \to H^\bullet(X; \mathbb{Q})$ gives $i^*\Phi(\mathcal{C}(\alpha)) = t i^* \mathcal{C}(\Phi_K(\alpha))$. In particular, for $\alpha = 1$, our computations in Example 4.4.5 give us

$$
i^*u_E = t i^* \mathcal{C}(\lambda_E) = t \prod_{i=1}^n (1 - e^{x_i})
$$

where $u_E$ is the ordinary Thom class and $x_1, \ldots, x_n$ are the Chern roots of $E \to X$. On the other hand, $i^*u_E = e(E)$ is the Euler class of $E$, and by the construction in Section 3.6, $e(E) = c_n(E) = x_1 \cdots x_n$ is the top Chern class. This shows that

$$
t = \prod_{i=1}^n \frac{x_i}{1 - e^{x_i}}.
$$

This is sometimes called the Thom defect of the bundle $E$.

**Definition.** For a complex vector bundle $E \to X$, the **Todd class** of $E$ is the characteristic class $\text{Td}(E) = c_f(E) \in H^\bullet(X; \mathbb{Q})$ determined by the formal power series

$$
f(t) = \frac{t}{1 - e^{-t}} = 1 + \frac{1}{2}t + \frac{1}{12}t^2 - \frac{1}{720}t^4 + \ldots
$$
 Explicitly, the Todd class is equal to

$$\Td(E) = \prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}}$$

where $x_1, \ldots, x_n$ are the Chern roots of $E$.

**Lemma 6.3.2.** Let $t \in H^\bullet(X; \mathbb{Q})$ be the Thom defect of a vector bundle $E \to X$ of rank $n$. Then $t = (-1)^n \Td(E)$ where $E$ is the complex conjugate bundle to $E$.

**Proof.** Obvious from our computation of $t$ above. \qed

**Proposition 6.3.3.** The Todd class is a characteristic class which is uniquely characterized by the property that

$$\langle \Td(T\mathbb{C}P^n), [\mathbb{C}P^n] \rangle = \int_{\mathbb{C}P^n} \Td(T\mathbb{C}P^n) = 1$$

for all $n \geq 1$, where $[\mathbb{C}P^n]$ is the fundamental class.

We now come to the crowning achievement in index theory, the Atiyah-Singer index theorem.

**Theorem 6.3.4** (Atiyah-Singer Index Theorem). Let $X$ be a compact, closed $n$-manifold with complex vector bundles $E, F \to X$ and elliptic linear differential operator $D : \Gamma(X, E) \to \Gamma(X, F)$. Then

$$\text{ind}(D) = (-1)^n \text{ch}(\sigma_D) \Td(TX \otimes \mathbb{C})[TX]$$

where $\sigma_D \in K(TX, TX_0)$ is the $K$-theory symbol associated to $D$ and $[TX]$ is the fundamental class of the total space of the tangent bundle $TX \to X$.

**Remark.** If $\pi : H^\bullet(TX, TX_0; \mathbb{Q}) \to H^\bullet(X; \mathbb{Q})$ is the pushforward map, then the equality in the Atiyah-Singer index theorem can instead be written

$$\text{ind}(D) = (-1)^{n(n+1)/2} \pi_! \text{ch}(\sigma_D) \Td(TX \otimes X)[X].$$

We will not prove the index theorem, but simply marvel at all the information it contains: $\text{ind}(D)$ is an algebraic invariant that really comes out of the linear algebra of some differential equations describing the elliptic operator $D$; likewise, $\sigma_D$ is an algebraic expression from which we obtain a characteristic class $\text{ch}(\sigma_D)$; the Todd class is some measurement of the size of the Chern roots of the (complexified) bundle, which themselves encode the topological data of Chern classes of the bundle; and finally, $[TX]$ is a purely topological gadget. Rather than giving a proof, we will conclude this chapter with several applications of the index theorem.

First, we need the following analogue of Theorem 3.8.12 for oriented bundles:

**Proposition 6.3.5.** Let $V \to X$ be an oriented real vector bundle of dimension $2n$. Then there is a space $Y$ and a map $\pi : Y \to X$ such that

1. $\pi^* : H^\bullet(X; \mathbb{Z}) \to H^\bullet(Y; \mathbb{Z})$ is injective.
(2) \( \pi^* V \cong E_1 \oplus \cdots \oplus E_n \) for some oriented, real, dimension 2 bundles \( E_i \to Y \).

In particular,

\[
\pi^*(V \otimes \mathbb{C}) \cong \bigoplus_{i=1}^{n} (E_i \otimes \mathbb{C}) = \bigoplus_{i=1}^{n} (L_i \oplus \overline{L_i})
\]

where \( L_i \) is a complex line bundle such that \( E_i \cong (L_i)_R \).

Proof. Same as Theorem 3.8.12, replacing \( \mathbb{P}E \) with \( \widetilde{\text{Gr}}_2(E) \), the (fibrewise) Grassmannian of oriented dimension 2 subspaces of \( E \).

\( \square \)

**Definition.** Let \( D_j : \Gamma(X, E_j) \to \Gamma(X, E_{j+1}) \) be a sequence of linear differential operators on complex line bundles \( E_j \to X \) such that \( D_{j+1} \circ D_j = 0 \) for all \( j \geq 0 \). Then \( (D_j) \) is called a Fredholm sequence if \( \ker D_j/\text{im} D_{j+1} \) is finite dimensional for all \( j \geq 0 \).

**Definition.** An elliptic sequence is a sequence of linear differential operators \( (D_j) \) such that for all \( \omega \in T^*X \), the sequence

\[
E_0 \xrightarrow{\sigma_{D_0}(\omega)} E_1 \xrightarrow{\sigma_{D_1}(\omega)} E_2 \to \cdots
\]

is exact.

**Theorem 6.3.6.** Every elliptic sequence is Fredholm. Moreover, if one defines

\[
E^0 = \bigoplus_{j=0}^{\infty} E_{2j}, \quad E^1 = \bigoplus_{j=0}^{\infty} E_{2j+1}, \quad D^0 = \bigoplus_{j=0}^{\infty} D_{2j} \quad \text{and} \quad D^1 = \bigoplus_{j=0}^{\infty} D_{2j+1},
\]

then the sequence \( (D_j) \) is elliptic if and only if the linear differential operator \( D^0 + (D^1)^* : E^0 \to E^1 \) is elliptic. In this case,

\[
\text{ind}(D^0 + (D^1)^*) = \sum_{j=0}^{\infty} (-1)^j (\dim \ker D_j - \dim \text{im} D_{j+1}).
\]

**Example 6.3.7.** Consider the elliptic operator \( D = d + d^* \) on a Riemannian manifold \( X \). The symbol \( \sigma_D : \bigwedge T^*X \otimes \mathbb{C} \to \bigwedge^{i+1} T^*X \otimes \mathbb{C} \) acts as \(- \wedge \omega \) for some nonzero \( \omega \in T^*X \otimes \mathbb{C} \) and the sequence

\[
\ldots \to \bigwedge^{i-1} T^*X \otimes \mathbb{C} \xrightarrow{\sigma_D} \bigwedge^{i} T^*X \otimes \mathbb{C} \xrightarrow{\sigma_D} \bigwedge^{i+1} T^*X \otimes \mathbb{C} \to \ldots
\]

is exact. Thus on the level of \( K \)-theory, \( \sigma_D = \lambda_{T^*X \otimes \mathbb{C}} \), the Thom class. Notice that the Chern roots of \( TX \otimes \mathbb{C} \) are \( x_1, -x_1, x_2, -x_2, \ldots, x_n, -x_n \) where \( x_i = c_1(L_i) \) for the line bundles \( L_i \) in the decomposition

\[
TX \otimes \mathbb{C} \cong \bigoplus_{i=0}^{n} (L_i \oplus \overline{L_i})
\]

from Proposition 6.3.5. Hence \( i^* \text{ch}(\sigma_D) = \prod_{i=1}^{n} (1 - e^{x_i})(1 - e^{-x_i}) \). On the other hand,

\[
\text{Td}(TX \otimes \mathbb{C}) = \prod_{i=1}^{n} \frac{-x_i^2}{(1 - e^{-x_i})(1 - e^{x_i})} = (-1)^n \prod_{i=1}^{n} \frac{x_i^2}{(1 - e^{x_i})(1 - e^{-x_i})}.
\]
Also recall that $e(TX) = \prod_{i=1}^{n} x_i$. Putting this all together, we have by the index theorem:

$$\text{ind}(D) = (-1)^n \text{ch}(\sigma_D) \text{Td}(TX \otimes \mathbb{C})[TX]$$

$$= (-1)^n \left( \prod_{i=1}^{n} (1 - e^{x_i})(1 - e^{-x_i}) \right) (-1)^n \left( \prod_{i=1}^{n} \frac{x_i^2}{(1 - e^{x_i})(1 - e^{-x_i})} \right) [TX]$$

$$= \left( \prod_{i=1}^{n} x_i^2 \right) [TX] \text{ after cancelling terms}$$

$$= \left( \prod_{i=1}^{n} x_i \right) [X] \text{ by the Thom isomorphism theorem (3.4.1)}$$

$$= e(TX)[X] = \chi(X),$$

the Euler characteristic of $X$, by Theorem 3.5.6.

Going further, Theorem 3.4.1 says that every $\beta \in H^{2n}(TX, TX_0)$ is of the form $\beta = \alpha \cup u_{TX}$ where $u_{TX}$ is the Thom class, so if $s_0 : X \to TX$ is the zero section, then $s_0^* \beta = s_0^* \alpha \cup s_0^* u_{TX} = \alpha \cup e(TX)$. Thus if $e(TX) \neq 0$, we can say that $\alpha = \frac{s_0^* \beta}{e(TX)}$ and

$$\alpha \cap [TX] = (-1)^{n-1/2} \beta \cap [TX].$$

**Proposition 6.3.8.** If $D$ is an elliptic linear differential operator on a closed manifold $X$ of odd dimension, then $\text{ind}(D) = 0$.

**Proof.** The map $a : \omega \mapsto -\omega$ on $TX$ changes the sign of the fundamental class $[TX]$, but $a^* \sigma_D = \sigma_D$ in $K(TX, TX_0)$. Indeed, for $\omega \in \Omega^k(X)$ we have $\sigma_D(-\omega) = (-1)^k \sigma_D(\omega)$ but multiplication by $e^{i\pi t}$ gives a homotopy $\sigma_D(-\omega) \to \sigma_D(\omega)$. Feeding this information into the index theorem, we get $\text{ind}(D) = -\text{ind}(D)$, so the index of $D$ must be zero. \hfill \Box

### 6.4 Dolbeault Cohomology

In this section we give an application of the Atiyah-Singer index theorem to Dolbeault cohomology, which produces an analogue of the Riemann-Roch theorem in complex algebraic geometry known as the Hirzebruch-Riemann-Roch theorem.

If $V$ is a complex vector space, then the map $J : V \to V, v \mapsto iv$ is linear with $J^2 = -1$, so it has eigenvalues $i$ and $-i$. Write $V^{1,0}$ for the $i$-eigenspace of $J$ acting on $V \otimes \mathbb{C}$; likewise, write $V^{0,1}$ for the $-i$-eigenspace.

**Example 6.4.1.** If $V_\mathbb{R} = \text{Span}_\mathbb{R}\{e_1, e_2\}$ as a real vector space, with $Je_1 = e_2$, then

$$V^{1,0} = \text{Span}\{e_1 - ie_2\} \text{ and } V^{0,1} = \text{Span}\{e_1 + ie_2\}.$$  

Now suppose $X$ is a compact, complex manifold of (complex) dimension $n$. Then $TX$ is a complex vector bundle, so its complexification $TX \otimes \mathbb{C}$ can be decomposed into

$$TX \otimes \mathbb{C} = TX^{1,0} \oplus TX^{0,1}.$$
where \( TX^{1,0} \cong TX \) and \( TX^{0,1} \cong T^*X \), using the decomposition described above on each fibre. Similarly, \( T^*X \otimes \mathbb{C} = T^*X^{1,0} \oplus T^*X^{0,1} \). If \( dz_1, \ldots, dz_n \) is a local basis for \( T^*X \) with respect to \( z_j = x_j + iy_j \) for each \( 1 \leq j \leq n \), then
\[
T^*X^{1,0} = \text{Span}\{dz_1, \ldots, dz_n\} \quad \text{and} \quad T^*X^{0,1} = \text{Span}\{d\bar{z}_1, \ldots, d\bar{z}_n\}.
\]
This extends to a decomposition
\[
\bigwedge^*(T^*X \otimes \mathbb{C}) = \bigoplus_{p,q \geq 0} \bigwedge^p T^*X \otimes \bigwedge^q T^*X
\]
where \( \bigwedge^p T^*X \cong \bigwedge^p (T^*X^{1,0}) \otimes \bigwedge^q (T^*X^{0,1}) \). Set \( \Omega^{p,q}(X) = \bigwedge^p T^*X \).

**Definition. The Dolbeault operator** is the linear differential operator
\[
\partial : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)
\]
\[
\omega \mapsto \pi_{0,q+1} \circ d\omega
\]
where \( \pi_{0,r} : \Omega^r(X) \to \Omega^{0,r}(X) \) is the natural projection map.

Locally, if \( \alpha = \sum \alpha_{i_1, \ldots, i_k} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_k} \in \Omega^{0,q}(X) \), then
\[
\partial \alpha = \sum \partial \alpha_{i_1, \ldots, i_k} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_k}
\]
where \( \partial f = \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \) and \( \frac{\partial}{\partial \bar{z}_j} = -i \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \). Notice that \( \partial f = 0 \) if and only if \( f \) satisfies the Cauchy-Riemann equations, i.e. is holomorphic.

**Proposition 6.4.2.** \( \partial^2 = 0 \) on \( \Omega^{0,*}(X) \).

**Definition. The Dolbeault complex** on a complex manifold \( X \) is the complex
\[
\cdots \to \Omega^{0,q-1}(X) \xrightarrow{\partial} \Omega^{0,q}(X) \xrightarrow{\partial} \Omega^{0,q+1}(X) \to \cdots
\]

The qth Dolbeault cohomology of \( X \), written \( H^{0,q}(X) \), is the qth cohomology of this complex.

**Theorem 6.4.3 (Dolbeault).** Let \( X \) be a complex manifold with structure sheaf \( \mathcal{O}_X \). Then for all \( q \), there is an isomorphism
\[
H^{0,q}(X) \cong H^q(X, \mathcal{O}_X),
\]
where \( H^q(X, \mathcal{O}_X) \) denotes the qth sheaf cohomology.

**Remark.** One can define the Dolbeault operator \( \partial : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X) \) for any fixed \( p \), giving a similar cohomology theory \( H^{p,q}(X) \). Then the isomorphism in Theorem 6.4.3 becomes \( H^{p,q}(X) \cong H^q(X, \Omega^p(X)) \).

**Proposition 6.4.4.** The Dolbeault operator \( \partial \) is elliptic.
**Definition.** For a complex manifold \( X \), set \( h^{0,q}(X) = \dim_C H^{0,q}(X) \) and define the holomorphic Euler characteristic of \( X \) to be

\[
\chi_h(X) = \sum_{q=0}^{\infty} (-1)^q h^{0,q}(X). 
\]

**Corollary 6.4.5.** For any complex manifold \( X \), \( \text{ind}(\bar{\partial}) = \chi_h(X) \).

**Corollary 6.4.6.** \( \chi_h(X) = \text{Td}(TX)[X] \).

**Proof.** We have \( \chi_h(X) = \text{ind}(\bar{\partial}) \). The right side of the equation in the index theorem (6.3.4) is \((-1)^n \text{ch}(\sigma_\partial) \text{Td}(TX \otimes \mathbb{C})[TX] \). Note that if \( s_0 : X \to TX \) is the zero section, then we have \( s_0^* \sigma_\partial = \sum_{q=0}^{\infty} (-1)^q \wedge^q (T^* X^{0,1}) = \sum_{q=0}^{\infty} (-1)^q \wedge^q TX \)

in \( K \)-theory, so

\[
s_0^* \text{ch}(\sigma_\partial) = \prod_{i=1}^{n} (1 - e^{x_i})
\]

where \( x_i \) are the Chern roots of \( TX \). Meanwhile, the Todd class can be computed as

\[
\text{Td}(TX \otimes \mathbb{C}) = \text{Td}(TX \oplus \overline{TX}) = \text{Td}(TX) \text{Td}(\overline{TX}) \\
= \left( \prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}} \right) \left( \prod_{i=1}^{n} \frac{-x_i}{1 - e^{x_i}} \right) \\
= (-1)^n \prod_{i=1}^{n} \frac{x_i^2}{(1 - e^{x_i})(1 - e^{-x_i})}.
\]

Since \( TX \) is a real vector bundle of dimension \( 2n \), the index theorem reads (by the subsequent remark):

\[
\text{ind}(\bar{\partial}) = (-1)^{2n(2n+1)/2} s_0^* \text{ch}(\sigma_\partial) \text{Td}(TX \otimes \mathbb{C}) \frac{1}{e(X)}[X] \\
= (-1)^n \left( \prod_{i=1}^{n} (1 - e^{x_i}) \right) (-1)^n \left( \prod_{i=1}^{n} \frac{x_i^2}{(1 - e^{x_i})(1 - e^{-x_i})} \right) \left( \prod_{i=1}^{n} \frac{1}{x_i} \right) [X] \\
= \left( \prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}} \right) [X] = \text{Td}(TX)[X].
\]

**Definition.** The Todd genus of a complex manifold is \( \text{Td}(TX)[X] \).

Thus Corollary 6.4.6 says that the Todd genus equals the holomorphic Euler characteristic.
If $E \to X$ is any holomorphic vector bundle, then one can define an operator
\[ \delta_E : \Omega^{0,q}(X;E) \to \Omega^{0,q+1}(X;E) \]
in a similar fashion to the constructions in Section 5.1. Then $\delta_E^2 = 0$ and $\delta_E$ is still elliptic, but we have $\sigma_{\delta_E} = \sigma_0 [E]$ where $[E]$ is the fundamental class of the total space of $E$. This implies:

**Corollary 6.4.7.** For any holomorphic vector bundle $E \to X$, $\chi_h(X; E) = \text{ch}(E) \text{Td}(TX)[X]$.

Suppose now that $X$ is a Riemann surface, i.e. a closed, connected complex manifold of dimension 1. Then as a real manifold, $X$ is an oriented surface of genus $g = g(X)$. For any holomorphic vector bundle $E \to X$, Example 4.4.3 and the definition of the Todd class show that
\[
\text{Td}(E) = 1 + \frac{1}{2} c_1(E) + \frac{1}{12} (c_1(E)^2 + c_2(E)) + \ldots
\]
and
\[
\text{ch}(E) = \text{rank}(E) + c_1(E) + \frac{1}{2} (c_1(E)^2 - 2c_2(E)) + \ldots
\]

Also note that $\chi_h(X) = h^{0,0}(X) - h^{0,1}(X)$ and
\[ \text{Td}(X)[X] = \frac{1}{2} \langle c_1(TX), [X] \rangle = \frac{1}{2} (2 - 2g) = 1 - g. \]

This proves:

**Corollary 6.4.8.** For a Riemann surface $X$, $h^{0,0}(X) - h^{0,1}(X) = 1 - g$.

**Corollary 6.4.9.** For a Riemann surface $X$, $h^{0,1}(X) = g$.

*Proof.* By Liouville’s theorem, every global holomorphic differential 1-form on a closed Riemann surface is trivial, so $H^{0,0}(X) \cong \mathbb{C}$. \hfill \Box

If $E \to X$ is a holomorphic vector bundle on a Riemann surface, we have a similar result which is famously known as the Riemann-Roch theorem. Set $h^q(E) = \dim \mathbb{C} H^{0,q}(X,E)$.

**Corollary 6.4.10 (Riemann-Roch Theorem).** Let $X$ be a Riemann surface of genus $g$ and let $E \to X$ be a holomorphic vector bundle. Then
\[ h^0(E) - h^1(E) = \deg(D) + 1 - g \]
where $\deg(E) = \langle c_1(E), [X] \rangle$ is the degree of $E$.

*Proof.* By Corollary 6.4.7, the holomorphic Euler characteristic of $E$ can be written
\[
h^0(E) - h^1(E) = \text{ch}(E) \text{Td}(TX)[X] \\
= \left( 1 + \frac{1}{2} c_1(X) \right) (1 + c_1(E))[X] \\
= \langle c_1(E) + \frac{1}{2} c_1(X), [X] \rangle \\
= \langle c_1(E), [X] \rangle + \frac{1}{2} \langle c_1(X), [X] \rangle \\
= \deg(E) + 1 - g
\]
by Corollary 6.4.8. \hfill \Box

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Corollary 6.4.11. If \( \deg(E) + 1 - g > 0 \), then \( E \) admits a holomorphic section.

Now suppose \( X \) is a complex surface, that is, a complex manifold of dimension 2.

Corollary 6.4.12 (Noether’s Formula). If \( X \) is a complex surface, then

\[
\chi_h(X) = \frac{1}{12} (c_1(X)^2 + c_2(X))[X].
\]

In particular, \((c_1(X)^2 + c_2(X))[X]\) is an integer multiple of 12.

Proof. Apply Corollary 6.4.6.

Example 6.4.13. For \( X = \mathbb{C}P^2 \), by Example 3.6.7 we have

\[
c(\mathbb{C}P^2) = (1 + x)^3 = 1 + 3x + 3x^2
\]

for a generator \( x \in H^2(\mathbb{C}P^2) \). Thus \( c_1(\mathbb{C}P^2)^2 = 9x^2, c_2(\mathbb{C}P^2) = 3x^2 \) and we see that \((c_1(X)^2 + c_2(X))[X] = 9 + 3 = 12. In particular, \( \chi_h(\mathbb{C}P^2) = 1. \)

Example 6.4.14. For \( X = \mathbb{C}P^1 \times \mathbb{C}P^1 \), let \( c_1(\mathbb{C}P^1 \times \ast) = 2x \) and \( c_1(\ast \times \mathbb{C}P^1) = 2y \), so that \( c_1(X) = 2x + 2y \). Then \( c_1(X)^2 = (2x + 2y)^2 = 4x^2 + 8xy + 4y^2 = 8xy \), while \( c_2(X) = e(X) = 4xy \), so

\[
(c_1(X)^2 + c_2(X))[X] = (8xy + 4xy)[X] = 12
\]

and once again \( \chi_h(\mathbb{C}P^1 \times \mathbb{C}P^1) = 1. \)

Finally, when \( L \to X \) is a line bundle, we obtain the following analogue of the Riemann-Roch theorem.

Corollary 6.4.15 (Hirzebruch-Riemann-Roch Theorem). Let \( X \) be a complex surface, \( L \to X \) a complex line bundle and let \( D \) denote the Poincaré dual of \( c_1(L) \) in \( H_2(X; \mathbb{Z}) \) and \( -K \) denote the Poincaré dual of \( c_1(X) \) in \( H_2(X; \mathbb{Z}) \). Then

\[
\chi_h(X, L) = \chi_h(X) + \frac{1}{2}(D \cdot D - K \cdot D)
\]

where \( \cdot \) denotes the intersection product.

Proof. By Corollary 6.4.7, \( \chi_h(X, L) = \text{ch}(L) \text{Td}(X)[X] \), but the right side of this expression evaluates as:

\[
\text{ch}(L) \text{Td}(X)[X] = \left(1 + c_1(L) + \frac{1}{2}c_1(L)^2 \right) \left(1 + \frac{1}{2}c_1(X) + \frac{1}{12}(c_1(X)^2 + c_2(X)) \right)[X]
\]

\[
= \chi_h(X) + \frac{1}{2} \left(c_1(L)c_1(X) + c_1(L)^2 \right)[X].
\]