Analysis

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Contents

Part I

Real Analysis

Chapter 1

Introduction

The notes in Part I were compiled from a semester of lectures at Wake Forest University by Dr. Sarah Raynor. The primary text for the course is Ross's *Elementary Analysis: The Theory of Calculus*. In the introduction, we develop an axiomatic presentation for the real numbers. Subsequent chapters explore sequences, continuity, functions and finally a rigorous study of single-variable calculus.

1.1 The Natural Numbers

The following notation is standard.

$\mathbb{N}$	$= \{1, 2, 3, 4, \ldots\}$	the natural numbers
$\mathbb{Z}$	$= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$	the integers
$\mathbb{Q}$	$= \left\{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\right\}$	the rational numbers
$\mathbb{R}$	?	the real numbers

Note that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. We begin with a short axiomatic presentation of the natural numbers $\mathbb{N}$. The five axioms below are called the **Peano Axioms**.

(1) There is a special element $1 \in \mathbb{N}$, so $\mathbb{N}$ is nonempty.

(2) Every element of $\mathbb{N}$ has a **successor** which is also in $\mathbb{N}$. For $n \in \mathbb{N}$, its successor is denoted $n + 1$.

(3) 1 is not the successor of any element of $\mathbb{N}$.

(4) Suppose $S \subseteq \mathbb{N}$ is a subset. If $1 \in S$ and for all $n \in S$, $n + 1 \in S$ as well, then $S = \mathbb{N}$. This is called the Axiom of Induction.

(5) If the successors of some $m, n \in \mathbb{N}$ are the same, then m and n are the same. In other words, $m + 1 = n + 1 \implies m = n$.

Remark. Any set that satisfies all five Peano Axioms is isomorphic to the natural numbers.

1.2 The Rational Numbers

$\mathbb{Q}$ is the arithmetic completion of $\mathbb{N}$, i.e. the rationals are closed under $+, -, \times$ and $\div$. However, there are issues with $\mathbb{Q}$. For example, try finding roots of the polynomial $x^2 - 2 = 0$. The only solutions are $x = \pm\sqrt{2}$, but it turns out that these are not rational numbers:

Theorem 1.2.1. *$\sqrt{2}$ is irrational.*

Proof. Suppose $x^2 = 2$ and $x \in \mathbb{Q}$. Then $x = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$, $q \neq 0$. We may assume p and q have no common factors, i.e. $\frac{p}{q}$ is reduced. Consider

$$x^2 = \frac{p^2}{q^2} = 2 \quad \Longrightarrow \quad p^2 = 2q^2.$$

Then p^2 is even, so p must be even, i.e. there is some integer k such that $p = 2k$. This means $p^2 = 4k^2 = 2q^2 \implies q^2 = 2k^2$. Thus q^2 is even, so q is even. This shows that 2 is a common factor of p and q, which contradicts our assumption. Hence $\sqrt{2}$ is irrational. □

So it's clear that the rational numbers are missing some 'stuff' that should be there.

Definition. *A number x is* **algebaic** *if there are integers $a_0, a_1, \ldots, a_n$, with $a_n \neq 0$ and $n \geq 1$, such that $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$.*

Example 1.2.2. $\sqrt{2}$ is algebraic since it is a root of $x^2 - 2 = 0$. Is $\sqrt{2 + \sqrt{2}}$ algebraic? To see that it is, let $x = \sqrt{2 + \sqrt{2}}$ and consider

$$\begin{aligned} x^2 &= 2 + \sqrt{2} \\ (x^2 - 2)^2 &= 2 \\ x^4 - 4x^2 + 2 &= 0. \end{aligned}$$

So $a_4 = 1$, $a_2 = -4$ and $a_0 = 2$ shows that x is algebraic.

Proposition 1.2.3. *Every rational number is algebraic.*

Proof. Let $x = \frac{p}{q}$ for $p, q \in \mathbb{Z}$ and $q \neq 0$. We can rewrite this as $qx - p = 0$ so x is algebraic. □

The converse is false: note that every algebraic number is rational (see $\sqrt{2}$ above).

Definition. *The* **algebraic completion** *of $\mathbb{Q}$ is the set $\overline{\mathbb{Q}}$ of all algebraic numbers.*

Theorem 1.2.4 (Rational Roots Theorem)**.** *Let $r \in \mathbb{Q}$ with $r = \frac{p}{q}$ such that p and q have no common factors. If $a_n r^n + \ldots + a_1 r + a_0 = 0$ with $n \geq 1$, $a_n \neq 0$ and $a_0 \neq 0$, then $q \mid a_n$ and $p \mid a_0$.*

Proof. Plug in:

$$\begin{aligned} a_n\left(\frac{p}{q}\right)^n + \ldots + a_1\frac{p}{q} + a_0 = 0 \\ a_np^n + \ldots + a_1pq^{n-1} + a_0q^n = 0 \cdot q^n = 0 \\ a_np^n = -a_{n-1}p^{n-1}q - \ldots - a_1pq^{n-1} - a_0q^n. \end{aligned}$$

Since q divides everything on the right, q must be a factor of a_np^n. But q and p share no common factors, so q divides a_n. Likewise p divides a_0. □

Example 1.2.5. The rational roots theorem is particularly effective for showing that a polynomial has *no* rational roots. For example, consider $x^4 - 4x^2 + 2 = 0$. Note that $a_0 = 2$ and $a_4 = 1$, so all possible rational roots are $\pm 1, \pm 2$. But none of these are roots, so all roots of $x^4 - 4x^2 + 2$ are irrational.

Are all real numbers algebraic? In other words, does $\overline{Q} = \mathbb{R}$? It turns out that the answer is no. Examples include π, e, $\sin(1)$, etc. but these are hard to prove. In fact, it wasn't until 1844 that Liouville proved the existence of such numbers, called **transcendental numbers**. It has since been shown that $\overline{\mathbb{Q}}$ is a subset of the complex numbers $\mathbb{C}$. This means that not all real numbers are algebraic and not all algebraic numbers are real.

Next we detail an axiomatic presentation of the rationals, similar to the Peano Axioms in Section 1.1. First, there are special numbers $0, 1 \in \mathbb{Q}$ with $0 \neq 1$, so $\mathbb{Q}$ is nonempty. Define the binary operations $+, \cdot : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ that satisfy the following axioms for all $a, b, c \in \mathbb{Q}$:

(1) $(a + b) + c = a + (b + c)$ (additive associativity).

(2) $a + b = b + a$ (additive commutativity).

(3) $a + 0 = a$ (additive identity).

(4) For every a there exists an element $-a$ such that $a + (-a) = 0$ (additive inverse).

(5) $a(bc) = (ab)c$ (multiplicative associativity).

(6) $ab = ba$ (multiplicative commutativity).

(7) $a \cdot 1 = a$ (multiplicative identity).

(8) For every a there exists an element $\frac{1}{a}$ such that $a \cdot \frac{1}{a} = 1$ (multiplicative inverse).

(9) $a(b + c) = ab + ac$ (distribution).

Any set that satisfies (1) – (9) is called a **field**. $\mathbb{Q}$ is an example of a field, however this set of axioms is not a unique presentation of $\mathbb{Q}$.

Proposition 1.2.6. *Some algebraic rules regarding $a, b, c \in \mathbb{Q}$ are:*

(a) If $a + c = b + c$ then $a = b$.

(b) $a \cdot 0 = 0$.

(c) $(-a)b = -(ab)$.

(d) $(-a)(-b) = ab$.

(e) *If* $ac = bc$ *and* $c \neq 0$ *then* $a = b$.

(f) *If* $ab = 0$ *then either* $a = 0$ *or* $b = 0$.

Proof. All of these may be proven from axioms (1) – (9). We will prove (a), (b) and (e) and leave the rest for exercise.

(a) If $a + c = b + c$ for $a, b, c \in \mathbb{Q}$ then

$$\begin{aligned} (a + c) + (-c) &= (b + c) + (-c) \\ a + (c + -c) &= b + (c + -c) \qquad \text{by axiom (1)} \\ a + 0 &= b + 0 \qquad \text{by axiom (4)} \\ a &= b \qquad \text{by axiom (3).} \end{aligned}$$

(b) Take $a \in \mathbb{Q}$ and consider $a \cdot 0$. By axiom (3) we can write $0 + 0 = 0$, so $a \cdot 0 = a(0 + 0) \implies a \cdot 0 = a \cdot 0 + a \cdot 0$ by axiom (9). By axiom (4), there exists an element $-(a \cdot 0) \in \mathbb{Q}$ such that $(a \cdot 0) + -(a \cdot 0) = 0$, so

$$\begin{aligned} 0 = a \cdot 0 + -(a \cdot 0) &= (a \cdot 0 + a \cdot 0) + -(a \cdot 0) \\ &= a \cdot 0 + (a \cdot 0 + -(a \cdot 0)) \qquad \text{by association} \\ &= a \cdot 0 + 0 \qquad \text{by additive inverse} \\ &= a \cdot 0 \qquad \text{by additive identity.} \end{aligned}$$

(e) Let $a, b, c \in \mathbb{Q}$ with $c \neq 0$ and suppose $ac = bc$. Then there exists some element $\frac{1}{c} \in \mathbb{Q}$ such that $c \cdot \frac{1}{c} = 1$. Then

$$\begin{aligned} (ac) \cdot \frac{1}{c} &= (bc) \cdot \frac{1}{c} \\ a\left(c \cdot \frac{1}{c}\right) &= b\left(c \cdot \frac{1}{c}\right) \qquad \text{by association} \\ a \cdot 1 &= b \cdot 1 \qquad \text{by multiplicative inverse} \\ a &= b \qquad \text{by multiplicative identity.} \end{aligned}$$

□

To further develop the rational numbers, we state the five **Order Axioms**: for $a, b, c \in \mathbb{Q}$, we have

(10) Either $a \leq b$ or $b \leq a$ (comparability).

(11) If $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry).

(12) If $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).

(13) If $a \leq b$ then $a + c \leq b + c$.

(14) If $a \le b$ and $c \ge 0$ then $ac \le bc$.

Note that (10) – (12) endow $\mathbb{Q}$ with a **total ordering**, so that the 14 axioms together mean that $\mathbb{Q}$ is a **totally ordered field**. There are other examples of ordered fields, such as $\mathbb{R}$.

Example 1.2.7. $\mathbb{C}$ is not an ordered field: $3 + i \overset{?}{\le} 2 + 2i$ — these can't be ordered.

Proposition 1.2.8. *Some properties of an ordered field:*

(g) *If* $a \le b$ *then* $-b \le -a$.

(h) *If* $a \le b$ *and* $c \le 0$ *then* $bc \le ac$.

(i) *If* $a \ge 0$ *and* $b \ge 0$ *then* $ab \ge 0$.

(j) $a^2 \ge 0$ *for all* a.

(k) $0 < 1$.

(l) *If* $a > 0$ *then* $\frac{1}{a} > 0$.

(m) *If* $0 < a < b$ *then* $0 < \frac{1}{b} < \frac{1}{a}$.

Proof. As before, these can all be shown using the Order Axioms, as well as the first 9 axioms where necessary. We will show (i) – (k).

(i) Let $a, b \in \mathbb{Q}$ with $a, b \ge 0$. Then $a \cdot b \ge 0 \cdot b$, and Proposition 1.2.6(b) says that $b \cdot 0 = 0$, so $ab \ge 0$.

(j) If $a \ge 0$ then this is implied by (i). On the other hand, if $a < 0$ then $0 \cdot a \le a \cdot a$ by (8). Thus $0 \le a^2$.

(k) By (j) we have $0 \le 1^2$. And by multiplicative identity, $1 \cdot 1 = 1$. Thus $0 \le 1$, and since they are not equal, $0 < 1$. ☐

Definition. *For a rational number* a*, we define the* **absolute value** *of* a *to be*

$$|a| = \begin{cases} a & a \ge 0 \\ -a & a < 0. \end{cases}$$

Definition. *For any two rational numbers* a, b*, we define the* **distance** *from* a *to* b *as* $d(a, b) = |b - a|$.

Theorem 1.2.9. *Let* $a, b \in \mathbb{Q}$*. Then*

(1) $|a| \ge 0$.

(2) $|ab| = |a|\,|b|$.

(3) $|a + b| \le |a| + |b|$ *(the triangle inequality).*

Proof. (1) Suppose $a \geq 0$. Then $|a| = a \geq 0$. Conversely, if $a < 0$ we have $|a| = -a$ and by Proposition 1.2.8(g), $-a > -0 = 0$. Hence $|a| \geq 0$ in all cases.

(2) There are three cases. First suppose $a, b \geq 0$. Then $ab \geq 0 \cdot 0 = 0$, so $|ab| = ab = |a|\,|b|$ by definition. Next suppose $a \geq 0$ and $b < 0$. Then $ab \leq a \cdot 0 = 0$ by axiom (14) and property (b). So $|ab| = -ab$. Then we have

$$\begin{aligned} |a|\,|b| &= a(-b) && \text{by property (c)} \\ &= (-b)a && \text{by multiplicative commutativity} \\ &= -(ba) && \text{by (c) again} \\ &= -(ab) && \text{by commutativity again} \\ &= -ab = |ab|. \end{aligned}$$

The case when $a < 0$ and $b \geq 0$ is identical. Finally, suppose $a, b < 0$. Then $ab \geq 0 \cdot 0 = 0$ by properties (h) and (b), so $|ab| = ab$. And by (d) we have $|a|\,|b| = (-a)(-b) = ab$, so in all cases $|ab| = |a|\,|b|$.

(3) If a and b are either both positive or both negative, the proof is trivial. The interesting cases are when a and b have different signs. Suppose without loss of generality that $a \geq 0$ and $b < 0$. If $a + b \geq 0$ then

$$\begin{aligned} |a+b| = a + b \leq a + 0 &\quad \text{by axiom (13)} \\ = a &\quad \text{by additive identity} \\ = |a| &\quad \text{since } a \geq 0 \\ \leq |a| + |b| &\quad \text{since } |a|, |b| \geq 0. \end{aligned}$$

On the other hand, if $a + b \leq 0$ then

$$\begin{aligned} |a+b| = -(a+b) = (-1)(a+b) &\quad \text{by property (c)} \\ = (-1)a + (-1)b &\quad \text{by distribution} \\ = -a + -b &\quad \text{by (c) again} \\ \leq 0 + -b &\quad \text{because } -a \leq 0 \\ = -b &\quad \text{by additive identity} \\ = |b| \leq |a| + |b|. \end{aligned}$$

$\square$

Remark. As a corollary to the triangle inequality, we have $|a - c| \leq |a - b| + |b - c|$ for all $a, b, c \in \mathbb{Q}$.

1.3 The Real Numbers

In this section we give an axiomatic presentation of $\mathbb{R}$. The real numbers satisfy all 14 axioms presented in the last section, but one additional axiom is needed to single out $\mathbb{R}$ from among all ordered fields. Before stating this axiom, we need some definitions.

Definition. *If S is a nonempty subset $S \subseteq \mathbb{R}$ and $s \in S$ such that $s \geq t$ for all $t \in S$, then s is the* **maximum** *of S. The* **minimum** *of S is defined similarly.*

Examples.

1. Consider a finite set such as $S = \{1, 2, 3, 4, 5\}$. The max is 5 and the min is 1.

2. For a **closed interval** $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$, the max is b and the min is a. For a **half-open interval** $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$, the max is b but this has no minimum. Finally, an **open interval** $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ has neither a max nor a min.

3. The natural numbers $\mathbb{N}$ have a minimum value at 1, but $\mathbb{N}$ does not have a maximum.

4. The set $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}$ has a maximum value of 1, but has no minimum value because the elements get closer and closer to 0, but 0 is not in the set.

Definition. *If $S \subseteq \mathbb{R}$ is a nonempty subset, then M is an* **upper bound** *of S if for all $s \in S$, $s \leq M$. In this case we say S is* **bounded above***.*

Definition. *Similarly, m is a* **lower bound** *if for all $s \in S$, $s \geq m$, in which case S is said to be* **bounded below***.*

Remark. If S is bounded above and below, we simply say that S is bounded. For example, in 4 above, the set $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ is bounded: above by 1 and below by 0.

Definition. *If $S \subseteq \mathbb{R}$ is bounded above, a number r is the* **supremum** *of S if r is an upper bound of S, and r is greater than or equal to any other upper bound of S. In other words, r is the least upper bound.*

Definition. *Similarly, if S is bounded below, r is the* **infimum** *of S if r is the greatest lower bound of S.*

Proposition 1.3.1. *If S has a maximum value, then $\sup S = \max S$.*

Proof. If $s = \max S$ then $s \geq t$ for all $t \in S$. So s is an upper bound. If ℓ is any other upper bound, then $\ell \geq s$ because $s \in S$. Thus s is the supremum of S. □

Proposition 1.3.2. *If S has a supremum, it is unique.*

Proof omitted. □

The Completeness Axiom. *Every nonempty subset $S \subseteq \mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$.*

The real numbers are the *unique* set that satisfies all 15 axioms presented so far (axioms (1) – (9), the Order Axioms and the Completeness Axiom).

Example 1.3.3. Consider $S = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$. S is nonempty since for example $0, 1 \in S$. S is also bounded above: e.g. $4 \geq x$ for all $x \in S$. However S does not have a supremum in $\mathbb{Q}$. To prove this, take any upper bound of S that is a rational number, say $\frac{p}{q}$. Then $\frac{p^2}{q^2} \geq 2$, but since $\sqrt{2}$ is not rational, $\frac{p^2}{q^2} > 2$. Thus there's a gap between $\frac{p^2}{q^2}$ and 2, and we can always find a slightly smaller rational that's greater than 2. On the other hand, the Completeness Axiom states that S has a supremum in $\mathbb{R}$. In other words, there is some smallest real number whose square is at least 2, which implies the existence of $\sqrt{2}$.

The Completeness Axiom allows one to fill in all the gaps in $\mathbb{Q}$. Another way of saying this is that $\mathbb{R}$ has no gaps: it is **complete**. In fact, every real number r can be represented as the supremum of the set $\{x \in \mathbb{Q} \mid x \leq r\}$. For example, $\pi = \sup\{x \in \mathbb{Q} \mid x \leq \pi\} = \{3, 3.1, 3.14, 3.141, \ldots\}$.

Corollary 1.3.4. *Every nonempty subset $S \subseteq \mathbb{R}$ which is bounded below has an infimum in $\mathbb{R}$.*

Proof. Let S be such a set and define $T = \{-x \mid x \in S\}$. Then T is nonempty since S is nonempty. Say c is a lower bound of S. Then $-c$ is an upper bound of T by property (g). By the Completeness Axiom, T has a supremum in $\mathbb{R}$, say t. We claim $-t$ is the infimum of S. Since t is an upper bound of T, then as above, $-t$ is a lower bound of S. Let ℓ be any lower bound of S. Then $-\ell$ is an upper bound of T and since $t = \sup T$, $t \geq -\ell$. By (g) again, $-t \geq -(-\ell) = \ell$ so $-t$ is the greatest lower bound of S. In particular, $\inf S$ exists. □

Theorem 1.3.5 (The Archimedean Property)**.** *For every $r \in \mathbb{R}$, there is a natural number n such that $n > r$. Furthermore, for every $r > 0$ there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < r$.*

Proof. Suppose $r \in \mathbb{R}$ such that $n \leq r$ for all $n \in \mathbb{N}$. Then r is an upper bound for $\mathbb{N}$, so by Completeness, $\mathbb{N}$ has a supremum, say m. Because m is the least upper bound, $m - 1$ is not an upper bound of $\mathbb{N}$, so there is some $n \in \mathbb{N}$ such that $n > m - 1$. By the Peano Axioms, $n + 1 \in \mathbb{N}$ and $n + 1 > m - 1 + 1 = m$, but this shows that m is not an upper bound, a contradiction. Hence for every $r \in \mathbb{R}$, there exists a natural number such that $n > r$.

To obtain the second statement, use the first part with $\frac{1}{r}$ and the Order Axioms. □

Theorem 1.3.6 (Density of the Rationals in the Reals)**.** *For every $a, b \in \mathbb{R}$ such that $a < b$, there exists a rational number r such that $a < r < b$.*

Proof. Since $b - a > 0$, by the Archimedean Property there is some $n \in \mathbb{N}$ such that $b - a > \frac{1}{n}$. This can be written $(b - a)n > 1 \implies bn > 1 + an$. By the Archimedean Property again, there is some $k > \max\{|an|, |bn|\}$. Let $S = \{m \in \mathbb{Z} \mid -k \leq m \leq k \text{ and } an < m\}$. S is bounded and nonempty (e.g. $k \in S$). In particular, S is bounded below so it has an infimum, say p. Since p is the greatest lower bound of S, $p + 1$ is not a lower bound of S, so there is some $j \in S$ such that $j < p + 1$. And since $j \in S$, we have $p \leq j < p + 1$. By construction,

an is also a lower bound of S, so $an \leq p \leq j < p+1$. This implies $j-1 < p \implies j-1 \notin S$ so $j-1 \leq an \leq j$, and $j < bn-1 \implies j+1 < bn$. Thus $an < j+1 < bn$. If we let $m = j+1$, this gives us $a < \frac{m}{n} < b$. $\square$

1.4 A Note About Infinity

Positive and negative infinity are not real numbers. They cannot be added, subtracted, multiplied or divided with each other or with real numbers. However, they are useful in inequalities:

- $-\infty < x < +\infty$ for all real numbers x. In other words, $(-\infty, \infty) = \mathbb{R}$.
- $[a, \infty) = \{x \in \mathbb{R} \mid a \leq x < \infty\}$.
- $(-\infty, b) = \{x \in \mathbb{R} \mid -\infty < x < b\}$.

We may rewrite the Completeness Axiom to say that any any nonempty subset $S \subseteq \mathbb{R}$ has a supremum, where if S is not bounded above, we denote $\sup S = +\infty$. Likewise, if S is not bounded below then $\inf S = -\infty$. For example, the natural numbers are not bounded above so $\sup \mathbb{N} = +\infty$.

Chapter 2

Sequences and Series

2.1 Sequences

Definition. *A* **sequence** *of real numbers is a function* $a : \mathbb{N} \to \mathbb{R}$.

In general terms, a sequence is just a list of numbers. Normally we write the function values: $(a_1, a_2, a_3, \ldots, a_n, \ldots)$. Order matters in this notation, so a sequence is an ordered list. Repeats are allowed, so a sequence is not a set.

Examples.

1. Consider the sequence (a_n) defined by $a_n = (-1)^n$. This can be written $(-1, 1, -1, 1, \ldots)$. As a set, this would just be $\{-1, 1\}$, which is not the same as (a_n). Thus the domain of a allows us to keep track of where we are in the sequence.

2. Consider $a_n = \dfrac{1}{n^2}$. The first five values are $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$ and $\frac{1}{25}$.

3. Consider the sequence $(b_n) = \left(3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \ldots\right)$. In function notation, this can be written $b_n = \dfrac{3(-1)^{n-1}}{2^{n-1}}$.

4. The function $c_n = \cos\left(\frac{n\pi}{3}\right)$ generates $\left(\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \ldots\right)$.

5. $d_n = n^{1/n}$ generates $\left(1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \ldots\right)$.

Definition. *We say a sequence (s_n)* **converges** *to $L \in \mathbb{R}$, denoted $(s_n) \to L$ or $\lim\limits_{n\to\infty} s_n = L$, if for every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n > N$, $|s_n - L| < \varepsilon$.*

Example 2.1.1. $s_n = \dfrac{1}{n^2}$

We claim $\lim\limits_{n\to\infty} s_n = 0$. Some valid N for various choices of ε are:

$$\begin{array}{ll} \varepsilon = 0.1 & \text{N} = 4 \\ \varepsilon = 0.01 & \text{N} = 11 \\ \varepsilon = 0.0004 & \text{N} = 51. \end{array}$$

In general we need a formula for how N relates to ε. In other words, given ε, find N such that $\dfrac{1}{n^2} < \varepsilon$ whenever $n > N$. In fact, this is true if $\dfrac{1}{N^2} < \varepsilon$, and we can solve for $N = \dfrac{1}{\sqrt{\varepsilon}}$. We are now ready to prove the claim.

Proof. Let $\varepsilon > 0$ be given. Let N be any natural number greater than $\dfrac{1}{\sqrt{\varepsilon}}$, which exists since $\mathbb{N}$ is not bounded above. Also let $n > N$ be given and consider

$$|s_n - L| = \left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2}.$$

Since $n > N$, $\dfrac{1}{n^2} < \dfrac{1}{N^2}$. And by our choice of N, $\dfrac{1}{N^2} < \dfrac{1}{\left(\frac{1}{\sqrt{\varepsilon}}\right)^2} = \varepsilon \implies \dfrac{1}{n^2} < \varepsilon$. Hence $\lim\limits_{n\to\infty} s_n = 0$ as claimed. $\square$

Example 2.1.2. $s_n = \dfrac{3n+1}{7n-4}$

We claim that $\lim\limits_{n\to\infty} s_n = \dfrac{3}{7}$. To prove this, we can do some scratchwork like before:

$$\begin{aligned}
\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| = \left|\frac{7(3n+1)-3(7n-4)}{7(7n-4)}\right| &= \frac{19}{49n-28} < \varepsilon \\
\implies 19 &< \varepsilon(49n-28) \\
&= 49\varepsilon n - 28\varepsilon \\
\implies n &> \frac{19+28\varepsilon}{49\varepsilon}
\end{aligned}$$

so we have found a good choice for N. Now for the proof.

Proof. Let $\varepsilon > 0$, let N be any natural number greater than $\dfrac{19+28\varepsilon}{49\varepsilon}$, and let $n > N$ be given as well. Consider $\left|\dfrac{3n+1}{7n-4} - \dfrac{3}{7}\right| = \dfrac{19}{49n-28}$. Since $n > N > 1$, $49n - 28 > 49N - 28$, so $\dfrac{19}{49n-28} < \dfrac{19}{49N-28}$. This gives us

$$\frac{19}{49n-28} < \frac{19}{49\left(\frac{19+28\varepsilon}{49\varepsilon}\right) - 28} = \frac{19}{\frac{19}{\varepsilon} + 28 - 28} = \frac{19}{\frac{19}{\varepsilon}} = \varepsilon.$$

Thus for all $n > N$, $|s_n - L| < \varepsilon$, so $\dfrac{3n+1}{7n-4} \to \dfrac{3}{7}$ as claimed. □

Example 2.1.3. Find $\lim\limits_{n\to\infty} \dfrac{n}{n^2+1}$.

The limit should be 0 by examination. Consider

$$\left|\frac{n}{n^2+1} - 0\right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}$$

so we should choose $n > \frac{1}{\varepsilon}$ to make the limit small.

Proof. Let $\varepsilon > 0$ be given, let N be any natural number $> \frac{1}{\varepsilon}$ and let $n > N$. Consider $\left|\dfrac{n}{n^2+1} - 0\right| = \dfrac{n}{n^2+1}$. By the Archimedean Property, $\frac{1}{n} < \frac{1}{N} < \varepsilon$. Then

$$\frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

Thus $\lim\limits_{n\to\infty} \dfrac{n}{n^2+1} = 0$. □

Example 2.1.4. Compute $\lim\limits_{n\to\infty} \left(\sqrt{n^2+n} - n\right)$.

First,

$$\begin{aligned}\left(\sqrt{n^2+n}-n\right) &= \frac{\left(\sqrt{n^2+n}-n\right)\left(\sqrt{n^2+n}+n\right)}{\left(\sqrt{n^2+n}+n\right)} \\ &= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} \\ &= \frac{n}{\sqrt{n^2+n}+n} \longrightarrow \frac{1}{2}.\end{aligned}$$

So the limit should be $\frac{1}{2}$. Now consider

$$\begin{aligned}\left|\sqrt{n^2+n}-n-\frac{1}{2}\right| &= \left|\frac{n}{\sqrt{n^2+n}+n}-\frac{1}{2}\right| \\ &= \left|\frac{2n-\sqrt{n^2+n}-n}{2\sqrt{n^2+n}+2n}\right| \\ &= \left|\frac{n-\sqrt{n^2+n}}{2\left(\sqrt{n^2+n}+n\right)}\cdot\frac{n+\sqrt{n^2+n}}{n+\sqrt{n^2+n}}\right| \\ &= \left|\frac{n^2-n^2-n}{2\left(\sqrt{n^2+n}+n\right)^2}\right| \\ &= \left|\frac{-n}{2\left(\sqrt{n^2+n}+n\right)^2}\right| \\ &< \left|\frac{-n}{2n^2}\right| = \frac{n}{2n^2} = \frac{1}{2n}.\end{aligned}$$

Then letting $n > \frac{1}{2\varepsilon}$ will make this limit small.

Proof. Let $\varepsilon > 0$ be given, let $N > \frac{1}{2\varepsilon}$ and let $n > N$. Consider

$$\left|\sqrt{n^2+n}-n-\frac{1}{2}\right| = \left|\frac{-n}{2\left(\sqrt{n^2+n}+n\right)^2}\right| < \frac{n}{2n^2} = \frac{1}{2n} < \frac{1}{2N} < \frac{1}{2\left(\frac{1}{2\varepsilon}\right)} = \varepsilon.$$

Therefore $\lim\limits_{n\to\infty}\left(\sqrt{n^2+n}-n\right) = \frac{1}{2}$ as claimed. □

Example 2.1.5. Show that $\lim\limits_{n\to\infty}(-1)^n$ does not exist.

To do this, we must negate the definition of convergence, i.e. show that there is some $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there is some $n > N$ such that $|(-1)^n - L| \geq \varepsilon$ for every possible $L \in \mathbb{R}$. Given such an L, for every n, $|(-1)^n - L|$ equals either $|1-L|$ or $|-1-L|$. But we get to choose ε and n, say $\varepsilon = \frac{1}{2}$ so that $|1-L| < \varepsilon$ and $|-1-L| < \varepsilon$. By the triangle inequality,

$$2 = |1-(-1)| \leq |1-L| + |L-(-1)| = |1-L| + |-1-L| < \varepsilon + \varepsilon = \tfrac{1}{2} + \tfrac{1}{2} = 1,$$

which implies $2 < 1$, a clear contradiction. We will use this strategy in the proof below.

Proof. Suppose $(-1)^n \to L$ for some $L \in \mathbb{R}$. Then for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that $|(-1)^n - L| < \varepsilon$ for all $n > N$. In particular, if $\varepsilon = \frac{1}{2}$ we can choose N such that for all $n > N$, $|(-1)^n - L| < \varepsilon$. Let $n_1, n_2 > N$ be chosen such that $s_{n_1} = 1$ and $s_{n_2} = -1$. Then $|s_{n_1} - L| < \varepsilon \implies |1 - L| < \frac{1}{2}$ and likewise $|s_{n_2} - L| < \varepsilon \implies |-1-L| < \frac{1}{2}$. But then

$$\begin{gathered}|1-(-1)| \leq |1-L| + |L-(-1)| \\ 2 < \tfrac{1}{2} + \tfrac{1}{2} = 1,\end{gathered}$$

a contradiction. Therefore the sequence has no limit (does not converge). $\square$

Proposition 2.1.6. *If $s_n \to s$, then $\sqrt{s_n} \to \sqrt{s}$ for all $s_n \geq 0$.*

Proof sketch: Consider

$$\frac{\sqrt{s_n} - \sqrt{s}}{1} \cdot \frac{\sqrt{s_n} + \sqrt{s}}{\sqrt{s_n} + \sqrt{s}} = \frac{s_n - s}{\sqrt{s_n} + \sqrt{s}} \leq \frac{s_n - s}{\sqrt{s}}$$

since $s_n \geq 0$. Moreover, $s_n \to s$ by hypothesis so for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that $|s_n - s| < \varepsilon\sqrt{s}$ whenever $n > N$. Then we have $|\sqrt{s_n} - \sqrt{s}| \leq \dfrac{|s_n - s|}{\sqrt{s}} < \dfrac{\varepsilon\sqrt{s}}{\sqrt{s}} = \varepsilon.$ $\square$

2.2 Basic Limit Theorems

Theorem 2.2.1. *Convergent sequences are bounded.*

Proof. Suppose $(s_n) \to s$. Then given any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n > N$, $|s_n - s| < \varepsilon$. By the triangle inequality, $|s_n| - |s| < |s_n - s| < \varepsilon$. In particular, if $\varepsilon = 1$ then $|s_n| < |s| + 1$ (for $n > N$). Let $M = \max\{|s_1|, |s_2|, \ldots, |s_N|, |s| + 1\}$. Then $|s_n| < M$ for all n, so (s_n) is bounded. $\square$

Theorem 2.2.2. *If (s_n) converges to s, then for any $k \in \mathbb{R}$, (ks_n) converges to ks.*

Proof. Assume $k \in \mathbb{R}$ is nonzero; otherwise the proof is easy. Suppose $(s_n) \to s$. Then for any $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n > N$, $|s_n - s| < \varepsilon$. In particular, there is some N such that $|s_n - s| < \dfrac{\varepsilon}{|k|}$ whenever $n > N$. Then for all $n > N$, we have

$$\begin{aligned} |s_n - s| &< \frac{\varepsilon}{|k|} \\ |k|\,|s_n - s| &< \varepsilon \\ |ks_n - ks| &< \varepsilon. \end{aligned}$$

Hence (ks_n) converges to ks. $\square$

Theorem 2.2.3. *If (s_n) converges to s and (t_n) converges to t, then $(s_n + t_n)$ converges to $s + t$.*

Proof. Let $(s_n) \to s$ and $(t_n) \to t$. Then given $\varepsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that $|s_n - s| < \frac{\varepsilon}{2}$ when $n > N_1$ and $|t_n - t| < \frac{\varepsilon}{2}$ when $n > N_2$. Let $N = \max\{N_1, N_2\}$. Then for all $n > N$, the triangle inequality gives us $|(s_n + t_n) - (s + t)| = |(s_n - s) + (t_n - t)| \leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence $(s_n + t_n) \to s + t$. $\square$

Theorem 2.2.4. *If $(s_n) \to s$ and $(t_n) \to t$, then $(s_n t_n) \to st$.*

Proof. Let $(s_n) \to s$ and $(t_n) \to t$. Then given any $\varepsilon > 0$ there is some N_1 such that for all $n > N_1$, $|s_n - s| < \dfrac{\varepsilon}{2|t| + 1}$ – we choose the denominator as $2|t| + 1$ to avoid the $|t| = 0$ case. Likewise, there is some N_2 such that for all $n > N_2$, $|t_n - t| < \dfrac{\varepsilon}{2M}$, where $M > |s_n|$ for all n – this is possible since (s_n) is bounded. Note that $|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st| \leq |s_n|\,|t_n - t| + |t|\,|s_n - s|$, and for all $n > \max\{N_1, N_2\}$,

$$\begin{aligned} |s_n|\,|t_n - t| + |t|\,|s_n - s| &< |s_n|\,\frac{\varepsilon}{2M} + |t|\,\frac{\varepsilon}{2|t| + 1} \\ &< |s_n|\,\frac{\varepsilon}{2|s_n|} + \left(|t| + \tfrac{1}{2}\right)\,\frac{\varepsilon}{2|t| + 1} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore $(s_n t_n) \to st$. $\square$

Theorem 2.2.5. *If $(s_n) \to s \neq 0$ and for all n, $s_n \neq 0$, then $\left(\frac{1}{s_n}\right) \to \frac{1}{s}$.*

Proof. Let (s_n) be as stated. Then given $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n > N$, $|s_n - s| < \varepsilon|s|m$, where $m > 0$ is a value such that $|s_n| > m$ (by the Archimedean Property, since $(s_n) \not\to 0$). Thus we have

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| \leq \frac{|s - s_n|}{|s|\,|s_n|} < \frac{|s_n - s|}{|s|m} < \frac{\varepsilon|s|m}{|s|m} = \varepsilon$$

so $\left(\frac{1}{s_n}\right) \to \frac{1}{s}$. □

Proposition 2.2.6. *The following are some useful limit properties to know:*

(a) If $p > 0$ then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

(b) If $|a| < 1$ then $\lim_{n\to\infty} a^n = 0$.

(c) $\lim_{n\to\infty} n^{1/n} = 1$.

(d) If $a > 0$ then $\lim_{n\to\infty} a^{1/n} = 1$.

Proof. (a) We want to show that $\left|\frac{1}{n^p} - 0\right| < \varepsilon$, i.e. $n > \frac{1}{\sqrt[p]{\varepsilon}}$. Let $\varepsilon > 0$ be given, let $N > \frac{1}{\sqrt[p]{\varepsilon}}$ and let $n > N$. Then $\left|\frac{1}{n^p} - 0\right| = \frac{1}{n^p} < \frac{1}{N^p} < \frac{1}{\left(\frac{1}{\sqrt[p]{\varepsilon}}\right)^p} = \varepsilon$. Hence $\left(\frac{1}{n^p}\right) \to 0$ as claimed.

(b) Let $|a| < 1$. Then there is some $b > 0$ such that $|a| = \frac{1}{1+b}$, which implies

$$|a|^n = \frac{1}{(1+b)^n} < \frac{1}{1+bn}$$

by the binomial theorem. Now let $\varepsilon > 0$, $N > \frac{1}{\varepsilon b}$ and $n > N$. Then we have

$$|a^n| < \frac{1}{1+bn} < \frac{1}{bn} < \frac{1}{bN} < \frac{1}{b\left(\frac{1}{\varepsilon b}\right)} = \varepsilon.$$

So $(a^n) \to 0$ as claimed.

(c) omitted.

(d) First suppose $a \geq 1$. Then for all $n > a$, $1 \leq a^{1/n} \leq n^{1/n}$, so by (c),

$$\lim_{n\to\infty} 1 = \lim_{n\to\infty} n^{1/n} = 1.$$

Moreover, by the Squeeze Theorem, $\lim_{n\to\infty} a^{1/n} = 1$. On the other hand, suppose $a < 1$. Then $\frac{1}{a} > 1$ and by the first case, $\lim_{n\to\infty} \left(\frac{1}{a}\right)^{1/n} = 1$. By Theorem 2.2.5,

$$\lim_{n\to\infty} \frac{1}{\left(\frac{1}{a}\right)^{1/n}} = \frac{1}{\lim_{n\to\infty} \left(\frac{1}{a}\right)^{1/n}} = \frac{1}{1} = 1.$$

Thus $\lim_{n\to\infty} a^{1/n} = 1$ for all $a > 0$. □

Example 2.2.7. $\lim_{n\to\infty} \frac{n^3+6n^2+7}{4n^3+3n-4} = \frac{1}{4}$.

Proof. Using the results in this section, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^3+6n^2+7}{4n^3+3n-4} &= \lim_{n\to\infty} \frac{1+\frac{6}{n}+\frac{7}{n^3}}{4+\frac{3}{n^2}-\frac{4}{n^3}} = \frac{\lim\left(1+\frac{6}{n}+\frac{7}{n^3}\right)}{\lim\left(4+\frac{3}{n^2}-\frac{4}{n^3}\right)} \quad \text{(Theorem 2.2.5)}\\
&= \frac{\lim 1 + \lim \frac{6}{n} + \lim \frac{7}{n^3}}{\lim 4 + \lim \frac{3}{n^2} - \lim \frac{4}{n^3}} \quad \text{(Theorem 2.2.3)}\\
&= \frac{\lim 1 + 6\lim \frac{1}{n} + 7\lim \frac{1}{n^3}}{\lim 4 + 3\lim \frac{1}{n^2} - 4\lim \frac{1}{n^3}} \quad \text{(Theorem 2.2.2)}\\
&= \frac{1+6(0)+7(0)}{4+3(0)-4(0)} = \frac{1}{4} \quad \text{(Proposition 2.2.6(a))}
\end{aligned}$$

as claimed. □

Example 2.2.8. Calculate $\lim_{n\to\infty} \frac{n+5}{n^2+1}$.

By the limit theorems,

$$\lim_{n\to\infty} \frac{n+5}{n^2+1} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{5}{n^2}}{1+\frac{1}{n^2}} = \frac{\lim \frac{1}{n} + 5\lim \frac{1}{n^2}}{\lim 1 + \lim \frac{1}{n^2}} = \frac{0+5(0)}{1+0} = 0.$$

Definition. *A sequence (s_n) is said to* **diverge** *to $+\infty$ if for every $M > 0$ there is some $N \in \mathbb{N}$ such that for all $n > N$, $s_n > M$. Likewise, (s_n) diverges to $-\infty$ if for every $m < 0$ there is some $N \in \mathbb{N}$ such that for all $n > N$, $s_n < m$.*

Example 2.2.9. $\frac{n^2+1}{n+5}$ diverges to $+\infty$.

Note that $\frac{n^2+1}{n+5} > M$ if $\frac{n^2}{n+5} > M$, and for $n > 5$, $2n > n+5$. This means $\frac{n^2}{2n} = \frac{n}{2} > M$, so we should choose $n > 2M$.

Proof. Let $M > 0$ be given, choose $N \in \mathbb{N}$ such that $N > 2M$, and let $n > N$. Then $\frac{n^2+1}{n+5} > \frac{n^2}{n+5}$ and since $n > 5$, $2n > n+5$. So

$$\frac{n^2}{n+5} > \frac{n^2}{2n} = \frac{n}{2} > \frac{N}{2} > \frac{2M}{2} = M.$$

Hence $\frac{n^2+1}{n+5} \longrightarrow \infty$. □

Proposition 2.2.10. *If $(s_n) \to +\infty$ and $(t_n) \to t > 0$, then $(s_n t_n) \to +\infty$.*

Proof. Let $M > 0$ be given. Since $(t_n) \to t$, there is some $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $|t_n - t| < \frac{t}{2}$. So for all $n > N$, $-\frac{t}{2} < t_n - t < \frac{t}{2} \implies t_n > \frac{t}{2}$. Then $s_n t_n > s_n \frac{t}{2}$, and since $(s_n) \to \infty$, there is some $N_2 \in \mathbb{N}$ such that $s_n > \frac{2M}{t}$ for all $n > N_2$. Let $N = \max\{N_1, N_2\}$. Then $s_n t_n > \frac{2M}{t} \cdot \frac{t}{2} = M$. Therefore $(s_n t_n)$ diverges. $\square$

Proposition 2.2.11. *Suppose $s_n > 0$ for all $n \in \mathbb{N}$. Then $(s_n) \to \infty \iff \left(\frac{1}{s_n}\right) \to 0$.*

Proof. First suppose $(s_n) \to \infty$. Then given $\varepsilon > 0$, $\dfrac{1}{\varepsilon} > 0$ as well so there is some $N \in \mathbb{N}$ such that for all $n > N$, $s_n > \dfrac{1}{\varepsilon}$. This gives us $\dfrac{1}{s_n} < \dfrac{1}{\frac{1}{\varepsilon}} = \varepsilon$ so $\left(\frac{1}{s_n}\right) \to 0$. Conversely, suppose $\left(\frac{1}{s_n}\right) \to 0$. Then given $M > 0$, there is some $N \in \mathbb{N}$ such that $\left|\dfrac{1}{s_n}\right| < \dfrac{1}{M}$ for all $n > N$. This means $\dfrac{1}{s_n} < \dfrac{1}{M} \implies s_n > \dfrac{1}{\frac{1}{M}} = M$. Hence (s_n) diverges. $\square$

2.3 Monotone Sequences

Definition. *A sequence (s_n) is* **non-decreasing** *if for all $n \in \mathbb{N}$, $s_{n+1} \geq s_n$. Likewise, (s_n) is* **non-increasing** *if $s_{n+1} \leq s_n$ for all n. A sequence is called* **monotone** *if it is either non-decreasing or non-increasing.*

Examples.

1. $(s_n) = \left(\dfrac{1}{n}\right)_{n=1}^{\infty}$ is monotone non-increasing.

2. For $(s_n) = \left(1 + \dfrac{1}{n}\right)^n$, consider

$$\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} = \frac{\left(1+\frac{1}{n+1}\right)^n\left(1+\frac{1}{n+1}\right)}{\left(1+\frac{1}{n}\right)^n} = \frac{\left(\frac{n+2}{n+1}\right)^n\left(1+\frac{1}{n+1}\right)}{\left(\frac{n+1}{n}\right)^n} = \frac{(n+2)^{n+1}n^n}{(n+1)^{2n+1}} > 1.$$

   So (s_n) is monotone non-decreasing.

3. Notice that in $n^{1/n} = \left(1, \sqrt{2}, \sqrt[3]{3}, \sqrt{2}, \ldots\right)$ the second and fourth terms of the sequence are equal, so the terms either increase then decrease, or the opposite. Either way, $n^{1/n}$ is not monotone.

Theorem 2.3.1 (Monotone Convergence Theorem)**.** *Every bounded, monotone sequence converges.*

Proof. Without loss of generality, assume (s_n) is non-decreasing. Since (s_n) is bounded, the set $S = \{s_n \mid n \in \mathbb{N}\}$ is bounded above. By the Completeness Axiom, S has a supremum s. In particular, s is an upper bound, so $s_n \leq s$ for all $n \in \mathbb{N}$. And s is the least upper bound, so for any $\varepsilon > 0$, $s - \varepsilon$ is not an upper bound. This means that there is some $N \in \mathbb{N}$ such that $s_N > s - \varepsilon$. By monotonicity, for all $n > N$, $s_n \geq s_N > s - \varepsilon$. Then we have

$$s + \varepsilon > s \geq s_n > s - \varepsilon$$

i.e. $|s_n - s| < \varepsilon$ for all $n > N$. Therefore (s_n) converges to s. □

Corollary 2.3.2. *Every bounded, monotone non-decreasing sequence converges to* $\sup S$*, where S is as above. Likewise, every bounded, monotone non-increasing sequence converges to* $\inf S$*.*

Theorem 2.3.3. *Every unbounded, non-decreasing sequence diverges to $+\infty$, and similarly every unbounded, non-increasing sequence diverges to $-\infty$.*

Proof. First suppose (s_n) is non-decreasing. Let $M > 0$. Since (s_n) is unbounded above, there is some $N \in \mathbb{N}$ such that $s_N > M$. By monotonicity, $s_n \geq s_N > M$ for all $n > N$. Hence $(s_n) \to +\infty$. The proof for non-increasing sequences is the same. □

Example 2.3.4. A common type of sequence is a *series*, which is a sequence of sums (see Section 7.2). For example, $0.\bar{9} = 0.999\ldots$ can be expressed as the limit of a sequence of partial sums: $\left(\frac{9}{10}, \frac{99}{100}, \frac{999}{1000}, \frac{9999}{10000}, \ldots\right)$ which is usually written $\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \ldots + \frac{9}{10^n}$. Formally, a geometric series is of the form

$$s_N = \sum_{n=0}^{N-1} ar^n = a\left(\frac{1-r^N}{1-r}\right).$$

It turns out that (s_N) is bounded by $\dfrac{a}{1-r}$ and this is the supremum. Clearly (s_N) is non-decreasing, since each term is the sum of nonnegative values. Thus by the monotone convergence theorem, $(s_N) \to \dfrac{a}{1-r}$. In the decimal example, $(s_N) \to \dfrac{9/10}{1 - 9/10} = \dfrac{9/10}{9/10} = 1$. Thus $.\bar{9}$ is a monotone series that converges to 1.

A nonrepeating decimal $0.r_1r_2r_3r_4r_5\ldots r_n\ldots = \frac{r_1}{10} + \frac{r_2}{100} + \frac{r_3}{1000} + \ldots$ can be represented by a bounded, non-decreasing sequence of partial sums. For each $n \in \mathbb{N}$, $r_n \le 9$ so we see that $0.r_1r_2r_3\ldots \le 0.999\ldots = 1$. Therefore by MCT, every decimal represents a real number (its limit) between 0 and 1.

Not all sequences are monotone, as we've seen. If (s_n) is bounded but not necessarily monotone, we can still get information from the MCT: suppose (s_n) is bounded. For each $N \in \mathbb{N}$, define $U_N = \inf\{s_n \mid n \ge N\}$ and $V_N = \sup\{s_n \mid n \ge N\}$. Then U_N is a non-decreasing sequence and V_N is a non-increasing sequence. By MCT, $\lim\limits_{N\to\infty} U_N$ and $\lim\limits_{N\to\infty} V_N$ exist. Moreover, for any $n \ge N$, $U_N \le s_n \le V_N$ so if $\lim\limits_{n\to\infty} s_n$ exists, then for all N, $U_N \le \lim\limits_{n\to\infty} s_n \le V_N$. In fact, the limits of U_N and V_N bound the sequential limit as well: $\lim\limits_{N\to\infty} U_N \le \lim\limits_{n\to\infty} s_n \le \lim\limits_{N\to\infty} V_N$.

Definition. *For a sequence* (s_n), *the* **limit inferior** *is* $\liminf\limits_{n\to\infty} s_n = \lim\limits_{N\to\infty} \inf\{s_n \mid n \ge N\}$. *Similarly, the* **limit superior** *of* (s_n) *is* $\limsup\limits_{n\to\infty} s_n = \lim\limits_{N\to\infty} \sup\{s_n \mid n \ge N\}$.

By the above remarks, every bounded sequence has a lim inf and a lim sup.

Example 2.3.5. $s_n = (-1)^n = (-1, 1, -1, 1, -1, \ldots)$

Fix $N \in \mathbb{N}$. Then

$$\begin{aligned} U_N &= \inf\{s_n \mid n \ge N\} = \inf\{-1, 1\} = -1 \\ V_N &= \sup\{s_n \mid n \ge N\} = \sup\{-1, 1\} = 1 \end{aligned}$$

and these are not equal.

Example 2.3.6. $t_n = \frac{1}{n} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$

Fix $N \in \mathbb{N}$. Then

$$\begin{aligned} U_N &= \inf\{t_n \mid n \ge N\} = \inf\left\{\frac{1}{N}, \frac{1}{N+1}, \ldots\right\} = 0 \\ V_N &= \sup\{t_n \mid n \ge N\} = \sup\left\{\frac{1}{N}, \frac{1}{N+1}, \ldots\right\} = \frac{1}{N}. \end{aligned}$$

We see that $\liminf t_n = 0$ and $\limsup t_n = \lim_{N\to\infty} V_N = 0$. This makes sense since the sequential limit is also 0.

Theorem 2.3.7. *Let (s_n) be a bounded sequence. Then $\lim_{n\to\infty} s_n = L$ for some $L \in \mathbb{R}$ if and only if $\liminf s_n = \limsup s_n = L$.*

Proof. ($\implies$) Suppose $(s_n) \to L \in \mathbb{R}$. Let $\varepsilon > 0$ be given. Then there is some $N \in \mathbb{N}$ such that for all $n > N$,

$$\begin{aligned} |s_n - L| &< \frac{\varepsilon}{2} \\ -\frac{\varepsilon}{2} < s_n - L &< \frac{\varepsilon}{2} \\ L - \frac{\varepsilon}{2} < s_n &< L + \frac{\varepsilon}{2}. \end{aligned}$$

So $\sup\{s_n \mid n > N\} \leq L + \frac{\varepsilon}{2} \implies V_{N+1} \leq L + \frac{\varepsilon}{2}$. Thus (V_k) is monotone non-increasing. So for all $k > N$, $V_k \leq V_{N+1} \leq L + \frac{\varepsilon}{2} \implies \limsup s_n = \lim V_k \leq L + \frac{\varepsilon}{2}$. Similarly, $\liminf s_n = \lim U_k \geq L - \frac{\varepsilon}{2}$. So $\liminf s_n \geq \limsup s_n - \varepsilon$. But $\liminf s_n \leq \limsup s_n$ for all n, so as $\varepsilon \to 0$ it follows that $\liminf s_n = \limsup s_n = L$.

($\impliedby$) Now suppose for some $L \in \mathbb{R}$, $\liminf s_n = \limsup s_n = L$. Let $\varepsilon > 0$ be given. Then there is some $N_1 \in \mathbb{N}$ such that for all $N > N_1$, $|U_N - L| < \varepsilon$. U_N is non-decreasing so for all N, $U_N < L$. In particular, $U_N > L - \varepsilon$ for all $N > N_1$, so $L - \varepsilon < U_N < \varepsilon$. Thus $s_n > L - \varepsilon$ for all $n > N_1$. Likewise, there is some $N_2 \in \mathbb{N}$ such that for all $n > N_2$, $s_n < L + \varepsilon$. Now let $N_3 = \max\{N_1, N_2\}$ and let $n > N_3$. Then $s_n > L - \varepsilon$ and $s_n < L + \varepsilon \implies |s_n - L| < \varepsilon$. Hence $(s_n) \to L$. $\square$

Definition. *A sequence (s_n) is called* **Cauchy** *if there is some $N \in \mathbb{N}$ such that for all $n, m > N$, $|s_n - s_m| < \varepsilon$.*

Theorem 2.3.8 (Cauchy Convergence Theorem)**.** *A sequence of real numbers is Cauchy if and only if it converges.*

Proof. ($\impliedby$) Suppose (s_n) converges to some $L \in \mathbb{R}$. Then given $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n > N$, $|s_n - L| < \frac{\varepsilon}{2}$. Let $n, m > N$. Then $|s_n - L| < \frac{\varepsilon}{2}$ and $|s_m - L| < \frac{\varepsilon}{2}$. By the triangle inequality,

$$|s_n - s_m| \leq |s_n - L| + |L - s_m| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Hence (s_n) is Cauchy.

($\implies$) This is the real content of the Cauchy convergence theorem. We first prove that Cauchy sequences are bounded. Let $\varepsilon = 1$. Then there is some $N \in \mathbb{N}$ such that $|s_n - s_m| < 1$ for all $n, m > N$. So if $m \geq N + 1$ then $|s_m - s_{N+1}| < 1$, which implies

$$\begin{aligned} -1 < s_m - s_{N+1} &< 1 \\ s_m &< 1 + s_{N+1} \\ \text{and} \qquad -1 < s_{N+1} - s_m &< 1 \\ -1 - s_{N+1} &< s_m. \end{aligned}$$

Thus $|s_m| < |s_{N+1} + 1| \leq |s_{N+1}| + 1$ for all $m > N$. Now set

$$M = \max\{|s_1|, |s_2|, \ldots, |s_N|, |s_{N+1}| + 1\}.$$

If $n \leq N$ then $|s_n| \leq \max\{|s_1|, |s_2|, \ldots, |s_N|\} \leq M$, or if $n > N$ then $|s_n| < |s_{N+1}| + 1 \leq M$. Therefore all Cauchy sequences are bounded.

Now recall that (s_n) converges if $\liminf s_n = \limsup s_n$, which exist since (s_n) is bounded. Then given $\varepsilon > 0$, choose N such that for all $n, m > N$, $|s_n - s_m| < \varepsilon$. Then for all $n, m > N$ we have

$$\begin{aligned} s_n &< s_m + \varepsilon \\ \sup\{s_n \mid n > N\} &\leq s_m + \varepsilon \\ \lim_{N\to\infty} \sup\{s_n \mid n > N\} &\leq s_m + \varepsilon \\ \lim_{N\to\infty} \sup\{s_n \mid n > N\} &\leq \inf\{s_m \mid m > N\}. \end{aligned}$$

So $\limsup s_n \leq \liminf s_n + \varepsilon$, and as $\varepsilon \to 0$, $\limsup s_n \leq \liminf s_n$. But we know that for any bounded sequence, $\liminf s_n \leq \limsup s_n$, so we conclude that $\liminf s_n = \limsup s_n$ and (s_n) converges. $\square$

2.4 Subsequences

Definition. *Let (s_n) be a sequence. A* **subsequence** *of (s_n) is a sequence (t_k) such that for all $k \in \mathbb{N}$, there is an $n_k \in \mathbb{N}$ such that $t_k = s_{n_k}$ and $n_k < n_{k+1}$.*

Example 2.4.1. Let $s_n = \frac{1}{n}$ and consider the subsequence $t_k = s_{2k-1} = \frac{1}{2k-1}$

$$s_n = \left(1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6}, \tfrac{1}{7}, \ldots\right)$$

$$t_n = \left(1, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7}, \ldots\right)$$

Subsequences cannot:

- jump out of order, i.e. $(s_1, s_5, s_4, s_{10}, s_2, \ldots)$
- repeat values, i.e. $(s_1, s_2, s_2, s_4, s_4, s_4, \ldots)$
- add new terms, i.e. $\left(1, \frac{1}{3}, \frac{1}{5}, \frac{1}{\pi}, \frac{1}{7}, \ldots\right)$

The Completeness Axiom can be stated in terms of subsequences:

Proposition 2.4.2. *It's possible to write all rational numbers in a list (i.e. the rationals are countable). Then if $(r_n)_{n=1}^{\infty}$ is the sequence of rational numbers and $x \in \mathbb{R}$, there is a subsequence of (r_n) converging to x.*

Proposition 2.4.3. *Suppose (s_n) is a sequence of positive numbers with $\inf s_n = 0$. Then there is a subsequence (s_{n_k}) of (s_n) that converges to 0.*

Proof. Since $\inf s_n = 0$, 1 is not a lower bound for s_n. This means there is some n_1 such that $0 < s_{n_1} < 1$. And since $s_j > 0$ for all $j \in \mathbb{N}$, $0 < \min\left\{s_1, s_2, \ldots, s_n, \frac{1}{2}\right\}$, so $\frac{1}{2}$ is not a lower bound for (s_n). Then there is some n_2 such that $s_{n_2} < s_{n_1}$, $s_{n_2} < \frac{1}{2}$ and $n_2 > n_1$. Continue to select $s_{n_1}, s_{n_2}, \ldots, s_{n_k}, \ldots$ so that $0 < \min\left\{s_1, \ldots, s_{n_k}, \frac{1}{k}\right\}$. In this way, we construct a subsequence (s_{n_k}) such that for all k, $0 < s_{n_k} < \frac{1}{k}$. By the Squeeze Theorem, $(s_{n_k}) \to 0$. □

Theorem 2.4.4. *Suppose (s_n) is a convergent sequence. Then every subsequence of (s_n) also converges to $\lim\limits_{n\to\infty} s_n$.*

Proof. Let $\varepsilon > 0$ be given. Since $(s_n) \to L \in \mathbb{R}$, there is some $N \in \mathbb{N}$ such that for all $n > N$, $|s_n - L| < \varepsilon$. Let (s_{n_k}) be any subsequence of (s_n). Then there is some K such that $n_K > N$ and for all $k > K$, $n_k > n_K > N$. Hence for all $k > K$, $|s_{n_k} - L| < \varepsilon$ so the subsequence converges to L as well. □

Theorem 2.4.5 (Bolzano-Weierstrass)**.** *Every bounded sequence has a convergent subsequence.*

Proof. We will prove that every sequence has a monotone subsequence, and then apply MCT to any bounded sequence to obtain the result. First, we say that an entry s_n in a sequence (s_n) is **dominant** if for all $m > n$, $s_n > s_m$. Then there are three cases.

Case 1: There are infinitely many dominant terms. Then if $n_1 < n_2$ and s_{n_1} and s_{n_2} are both dominant, $s_{n_2} < s_{n_1}$. Thus $\{s_{n_k} \mid s_{n_k} \text{ is dominant}, n_i < n_{i+1}\}$ is a non-increasing subsequence of (s_n).

Case 2: There are finitely many dominant terms. Then there is some M such that s_M is the last dominant term, and for all $N > M$, s_N is not dominant. This means that for each $N > M$, there is some $n > N$ such that $s_n \geq s_N$. To choose the subsequence, let $n_1 > M$. If $s_{n_1}, \ldots, s_{n_k}$ have been chosen, choose $s_{n_{k+1}}$ such that $s_{n_{k+1}} \geq s_{n_k}$ and $n_{k+1} > n_k$. Then (s_{n_k}) is non-decreasing.

Case 3: There are no dominant terms. Then let $M = 1$ and construct the same sequence as in Case 2. Hence every sequence has a monotone subsequence. $\square$

Corollary 2.4.6. *If (s_n) is any sequence, then there is a monotone subsequence that converges to $\limsup s_n$ and a monotone subsequence that converges to $\liminf s_n$.*

Proof. We will prove the existence of a subsequence that converges to $\limsup s_n$ and note that the proof for $\liminf s_n$ is symmetrical. Recall the sequence $V_N = \sup\{s_n \mid n \geq N\}$, where $(V_N) \to \limsup s_n$. Note that $V_1 \geq V_2 \geq \cdots \geq V_N \geq \cdots$. If $V_N \to -\infty$ then $s_n \to -\infty$ so every subsequence converges to $-\infty$ by Theorem 2.4.4. Now assume $\limsup s_n \neq -\infty$. As in the proof of the Bolzano-Weierstrass theorem, we can break the proof up by how many dominant terms are in the sequence.

Case 1: (s_n) has infinitely many dominant terms. As before, choose (s_{n_k}) to be the subsequence of dominant terms of (s_n). Then for all $j > k$, $s_{n_j} \geq s_{n_k}$. Thus $s_{n_k} = V_N$, and $(s_{n_k}) \to \lim V_N = \limsup s_n$.

Case 2: (s_n) has finitely many dominant terms. For each $n \in \mathbb{N}$, choose n_k such that $s_{n_k} > \max\left\{V - \frac{1}{n}, s_{n_1}, \ldots, s_{n_{k-1}}\right\}$, where $V = \lim V_N = \limsup s_n$. Then (s_{n_k}) is a subsequence of (s_n) such that $V - \frac{1}{k} < s_{n_k} < V$ for each $k \in \mathbb{N}$. Thus $(s_{n_k}) \to V$ and the subsequence is monotone by construction. $\square$

Definition. *Let (s_n) be any sequence. A* **subsequential limit** *of (s_n) is a number s, or $\pm\infty$, such that there is some subsequence (s_{n_k}) of (s_n) that converges to s.*

Definition. *For a set A, a is a* **limit point** *of A if there is some sequence $(s_n) \subset A$ converging to a.*

Definition. *A is a* **closed set** *if it contains all of its limit points.*

Example 2.4.7. $\mathbb{Q}$ is not closed because any $r \in \mathbb{R}$ is a limit point, and an infinite number of those are not rational.

Let S be the set of subsequential limits of (s_n). Some examples are shown below.

(s_n)	S	$\limsup s_n$	$\liminf s_n$
$(-1)^n$	$\{-1, 1\}$	1	-1
$(-1)^n n^2$	$\{-\infty, \infty\}$	$+\infty$	$-\infty$
$\frac{1}{n}$	$\{0\}$	0	0
$\cos\left(\frac{n\pi}{3}\right)$	$\left\{\frac{1}{2}, -\frac{1}{2}, -1, 1\right\}$	1	-1
$\mathbb{Q} \cap [0,1]$	$[0,1]$	1	0

(The last one is shown by the density of rationals in the reals.) This table suggests a few properties of subsequential limits:

Proposition 2.4.8. *Let (s_n) be a sequence of real numbers and let S be the set of subsequential limits of (s_n). Then*

(1) $\limsup s_n, \liminf s_n \in S$.

(2) $S = \{s\}$ *if and only if* $\lim_{n\to\infty} s_n = s$.

(3) S *is nonempty.*

(4) S *always has a maximum and minimum element. In fact,* $\limsup s_n = \max S$ *and* $\liminf s_n = \min S$.

(5) S *is a closed set.*

Proof. (1) Recall that every sequence (s_n) contains both a monotone subsequence converging to $\limsup s_n$ and a monotone subsequence converging to $\liminf s_n$. Thus $\limsup s_n$ and $\liminf s_n \in S$.

(2) Recall that by Theorem 2.3.7, $\limsup s_n = \liminf s_n$ if and only if $(s_n) \to L \in \mathbb{R}$. Then the statement follows from (4).

(3) By the proof of the Bolzano-Weierstrass theorem (2.4.5), every sequence (s_n) has a monotone subsequence (s_{n_k}), which has a limit by the monotone convergence theorem (18.1.5). Therefore S is nonempty.

(4) Take any sequence (s_n). Let $t \in S$ and suppose $(s_{n_k}) \to t$. Consider $\liminf s_{n_k} = \lim_{K\to\infty} \inf\{s_{n_k} \mid k \geq K\}$. There is some n such that $s_{n_k} = s_n$ and as $n \to \infty$, $k \to \infty$ as well. Then $\liminf s_{n_k} \geq \lim_{N\to\infty} \inf\{s_n \mid n \geq N\} = \liminf s_n$. A similar argument shows that $\limsup s_{n_k} \leq \limsup s_n$. But (s_{n_k}) so $\liminf s_{n_k} = \limsup s_{n_k} = t$. Hence $\liminf s_n \leq t \leq \limsup s_n$ for all $t \in S$.

(5) Let $(t_n) \subset S$ be a sequence converging to t. We will show that $t \in S$. Let $m \in \mathbb{N}$. Then since $t_m \in S$, there is some subsequence $(s_{n_k})_m \to t_m$. Choose s_{n_1} to be any element of $(s_{n_k})_1$ such that $|s_{n_1} - t_1| < 1$; choose s_{n_2} to be any element of $(s_{n_k})_2$ such that $|s_{n_2} - t_2| < \frac{1}{2}$ and $n_2 > n_1$; in general, choose s_{n_i} to be any element of $(s_{n_k})_i$ such that $|s_{n_i} - t_i| < \frac{1}{i}$. Let $\varepsilon > 0$ be given and let $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $|t_n - t| < \frac{\varepsilon}{2}$ by convergence of (t_n).

Also let $N_2 \in \mathbb{N}$ such that for all $n > N_2$, $\frac{1}{n} < \frac{\varepsilon}{2}$ by the Archimedean Property (1.3.5). Let $k > \max\{N_1, N_2\}$. Then

$$|s_{n_k} - t| \leq |s_{n_k} - t_k| + |t_k - t| < \frac{1}{k} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $(s_{n_k})_t \to t$, so $t \in S$. $\square$

2.5 lim inf and lim sup

Theorem 2.5.1. *Suppose (s_n) and (t_n) are sequences such that $(s_n) \to s$ and $s_n > 0$ for all n. Then* $\limsup s_n t_n = s \limsup t_n$.

NOTE: This is *not* the same as $\limsup s_n t_n = (\limsup s_n)(\limsup t_n)$.

Proof. Let $\beta = \limsup t_n$; we start by proving $\limsup s_n t_n \geq s\beta$. First consider $\beta = +\infty$. Then there is some subsequence (t_{n_k}) diverging to $+\infty$. Let (s_{n_k}) be the subsequence of (s_n) with the same indices as (t_{n_k}). Let $M > 0$. Then there is some $K_1 \in \mathbb{N}$ such that for all $k > K_1$, $t_{n_k} > \frac{2M}{s}$. Since $(s_n) \to s$, $(s_{n_k}) \to s$ as well. So for $\varepsilon = \frac{s}{2}$, there is some $N \in \mathbb{N}$ such that $|s_n - s| < \frac{s}{2}$ for all $n > N$. In particular, $-\frac{s}{2} < s_n - s < \frac{s}{2} \implies s_n > \frac{s}{2}$. Then there exists a K_2 such that $n_k > n$ for all $k > K_2$. Now let $K_3 = \max\{K_1, K_2\}$ and let $k > K_3$. Then $t_{n_k} > \frac{2M}{s}$ and $s_{n_k} > \frac{s}{2}$, so $s_{n_k} t_{n_k} > \frac{s}{2} \cdot \frac{2M}{s} = M$. Hence $(s_{n_k} t_{n_k}) \to +\infty$, and since lim sup is the largest possible subsequential limit, $\limsup s_n t_n = +\infty$. The case when $\beta = -\infty$ is easy: everything is $\geq -\infty$, so $\limsup s_n t_n \geq -\infty$. Finally if $\beta \in \mathbb{R}$, there is a subsequence (t_{n_k}) of (t_n) converging to β. Let (s_{n_k}) be as above. Then $(s_{n_k} t_{n_k}) \to s\beta$, so by lim sup properties, $\limsup s_n t_n \geq s\beta$.

On the other hand, consider $t_n = \dfrac{1}{s_n}(s_n t_n)$. Since s_n is nonzero for all but possibly the first few terms, $\dfrac{1}{s_n}$ is defined for large enough n. Then by limit properties, $\lim \dfrac{1}{s_n} = \dfrac{1}{\lim s_n} = \dfrac{1}{s}$. By the preceding paragraph, $\limsup t_n \geq \dfrac{1}{s} \limsup s_n t_n \implies \limsup s_n t_n \leq s \limsup t_n = s\beta$. Therefore $\limsup s_n t_n = s \limsup t_n$. □

Theorem 2.5.2. *Suppose (s_n) is a sequence such that $s_n \neq 0$ for all n. Then*

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|.$$

Proof. The middle inequality is obvious. We will prove the lim sup part of the statement and note that the lim inf case is proven similarly. Let $\alpha = \limsup |s_n|^{1/n}$ and $L = \limsup \left|\dfrac{s_{n+1}}{s_n}\right|$. Explicitly, $L = \lim\limits_{N\to\infty} \sup \left\{\left|\dfrac{s_{n+1}}{s_n}\right| : n > N\right\}$. Let $L_1 > L$. Then there is some $N \in \mathbb{N}$ such that $\sup \left\{\left|\dfrac{s_{n+1}}{s_n}\right| : n > N\right\} < L_1$. So for all $n > N$, $\left|\dfrac{s_{n+1}}{s_n}\right| < L_1 \implies |s_{n+1}| < L_1 |s_n|$. Then by induction, for all $n > N$, $|s_n| < L_1^n \dfrac{|s_n|}{L_1^N}$. Let $a = \dfrac{|s_N|}{L_1^N}$, so that $|s_n| \leq a L_1^n$. This implies $|s_n|^{1/n} \leq a^{1/n} L_1$. And since $\lim\limits_{n\to\infty} a^{1/n} = 1$ by limit properties, $|s_n|^{1/n} \leq L_1$. Thus $\alpha \leq L_1$, and since $L_1 > L$ was arbitrary, this proves that $\alpha \leq L$. □

This last theorem will be useful in the next section when we prove the relation between the Ratio and Root Tests.

2.6 Series

Definition. *A* **series** *is a sequence of partial sums of some list of numbers (another sequence). Explicitly, if (a_n) is a sequence,*

$$s_N = \sum_{n=1}^{N} a_n$$

is a series. An **infinite series** $\sum_{n=1}^{\infty} a_n$ *is just a formal way of representing an infinite sequence of partial sums, and if* $\lim_{N\to\infty} s_N$ *exists, this is how we write it.*

Definition. *We say* $\sum_{n=1}^{\infty} a_n$ **converges** *if* $\lim_{N\to\infty} \sum_{n=1}^{N} a_n$ *exists. If* $\lim_{N\to\infty} \sum_{n=1}^{N} a_n = \pm\infty$*, the series is said to* **diverge to** $\pm\infty$*. And if the limit does not exist, we say the series* **diverges***.*

Recall that a **geometric series** is an infinite series of the form

$$\begin{aligned}\sum_{n=0}^{\infty} ar^n &= \lim_{N\to\infty} \sum_{n=0}^{N} ar^n \\ &= \lim_{N\to\infty} \frac{a(1-r^{N+1})}{1-r} = \frac{a}{1-r}.\end{aligned}$$

Then a geometric series converges if $|r| < 1$ and diverges if $|r| \geq 1$.

Example 2.6.1. A geometric series with $a = 1, r = \frac{1}{2}$

This is the series $s_N = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{n=0}^{\infty} \frac{1}{2^n}$. Then we have

$$\lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{2^n} = \lim_{N\to\infty} \left(1 - \frac{1}{2^N}\right) = 1$$

so the geometric series converges to 1. This is confirmed by the formula.

Theorem 2.6.2 (Convergence Tests for Series)**.** *The following are some important criteria for series convergence.*

(1) A geometric series $\sum_{n=0}^{\infty} ar^n$ *converges if and only if* $|r| < 1$*. In this case, the series converges to* $\frac{a}{1-r}$*.*

(2) A p-series is a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$*. A p-series converges if and only if* $p > 1$*.*

Remark. If p is a positive, even integer, there is a formula for the sum:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k+1} b_{2k} (2\pi)^{2k}}{2(2k)!}$$

where b_{2k} is the $2k$th Bernoulli number (see any text on analytic number theory, specifically Riemann's Zeta Function). Examples include:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

However, there is no known formula if p is positive and odd.

Definition. *A series* $\sum_{n=0}^{\infty} a_n$ *satisfies the* **Cauchy criterion** *if the sequence of partial sums is Cauchy, i.e. if for every* $\varepsilon > 0$ *there is some* $N \in \mathbb{N}$ *such that for all* $n, m > N$,

$$\left| \sum_{k=0}^{n} a_k - \sum_{k=0}^{m} a_k \right| < \varepsilon.$$

Without loss of generality, we may assume $n > m$ *and write this inequality as*

$$\left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon.$$

The following is a corollary to the Cauchy convergence theorem (2.3.8):

Corollary 2.6.3. *A series converges if and only if it satisfies the Cauchy criterion.*

For the rest of the section we focus on developing a toolbox of convergence/divergence tests for series.

Theorem 2.6.4 (Divergence Test). *If* $\sum_{n=0}^{\infty} a_n$ *converges then* $|a_n| \to 0$. *The contrapositive is a test for divergence.*

Proof. Suppose $\sum_{n=0}^{\infty} a_n$ converges. Then it satisfies the Cauchy criterion, so given $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n, m > N$, $\left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon$. Let $m > N$ and $n = m + 1$. Then $|a_{m+1}| < \varepsilon$ so $|a_k| \to 0$. □

Example 2.6.5. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called a **harmonic series**. This series diverges (e.g. by a comparison test with a p-series; see below). But $\frac{1}{n} \to 0$. This shows that the converse to the Divergence Test fails in general.

Theorem 2.6.6 (Comparison Test)**.** *Suppose that for all $n \in \mathbb{N}$, $a_n \geq 0$.*

(1) If $\sum_{n=0}^{\infty} a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum_{n=0}^{\infty} b_n$ converges.

(2) If $\sum_{n=0}^{\infty} a_n$ diverges and $|b_n| \geq a_n$ for all n, then $\sum_{n=0}^{\infty} b_n$ diverges.

Proof. (1) Suppose $\sum_{n=0}^{\infty} a_n$ converges. By the Cauchy criterion, for any $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that for all $n > m > N$, $\left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon$. Then by hypothesis we have

$$\left|\sum_{k=m+1}^{n} b_k\right| \leq \sum_{k=m+1}^{n} |b_k| \leq \sum_{k=m+1}^{n} a_k < \varepsilon.$$

So $\sum_{n=0}^{\infty} b_n$ satisfies the Cauchy criterion, hence it converges.

(2) Let $s_n = \sum_{k=0}^{n} a_k$ and $t_n = \sum_{k=0}^{n} b_k$. Then by hypothesis, $t_n \geq s_n$ for all n, and $(s_n) \to +\infty$. By an earlier result, $(t_n) \to +\infty$ as well, so $\sum_{k=0}^{\infty} b_k$ diverges. □

Corollary 2.6.7 (Absolute Convergence)**.** *If $\sum_{n=0}^{\infty} |a_n|$ converges then $\sum_{n=0}^{\infty} a_n$ converges.*

Proof. Apply (1) of the Comparison Test with $a_n = |a_n|$ and $b_n = a_n$. □

Definition. *A series $\sum_{n=0}^{\infty} a_n$ is said to* **converge absolutely** *if $\sum_{n=0}^{\infty} |a_n|$ converges. If the series converges but the series of absolute values does not converge, the series is said to* **converge conditionally**.

Theorem 2.6.8 (Root Test)**.** *For a series $\sum_{n=0}^{\infty} a_n$, let $\alpha = \limsup |a_n|^{1/n}$. Then*

(1) If $\alpha < 1$, the series converges absolutely.

(2) If $\alpha > 1$, the series diverges.

Otherwise the Root Test is inconclusive.

Proof. (1) Suppose $\alpha < 1$. Choose $\varepsilon > 0$ such that $\alpha + \varepsilon < 1$. Then there is some $N \in \mathbb{N}$ such that for all $n > N$, $|a_n|^{1/n} - \alpha < \varepsilon$ since $\alpha = \limsup |a_n|^{1/n}$. So for all $n > N$, $|a_n|^{1/n} < \alpha + \varepsilon \implies |a_n| < (\alpha+\varepsilon)^n$. Now $(\alpha + \varepsilon) < 1$ so $\sum_{n=0}^{\infty} (\alpha+\varepsilon)^n$ converges, and by the Comparison Test, $\sum_{n=0}^{\infty} |a_n|$ converges as well.

(2) Now suppose $\alpha > 1$. Then there is some subsequence $(a_{n_k})^{1/nk}$ converging to α. So for sufficiently large k, $a_{n_k}^{1/nk} > 1$. Then $(a_{n_k}) \not\to 0$, so $(a_n) \not\to 0$. Thus by the Divergence Test, $\sum_{n=0}^{\infty} a_n$ diverges. □

Theorem 2.6.9 (Ratio Test)**.** *If (a_n) is a sequence of nonzero values, then*

(1) $\sum_{n=0}^{\infty} a_n$ *converges absolutely if* $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$.

(2) $\sum_{n=0}^{\infty} a_n$ *diverges if* $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$.

Otherwise the Ratio Test is inconclusive.

Proof. Recall from Theorem 2.5.2 that

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \le \liminf |a_n|^{1/n} \le \limsup |a_n|^{1/n} \le \limsup \left|\frac{a_{n+1}}{a_n}\right|.$$

Then the Ratio Test follows directly from the Root Test. □

Examples.

1. $\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n$ is a geometric series with $r = -\frac{1}{3}$. By the geometric series test, this series will converge if $|r| < 1$. But $\left|-\frac{1}{3}\right| = \frac{1}{3} < 1$, so the series converges to $\dfrac{1}{1+\frac{1}{3}} = \dfrac{3}{4}$.

2. Consider $\sum_{n=1}^{\infty} \dfrac{1}{n^2+1}$. We will do a comparison test with $a_n = \dfrac{1}{n^2}$ and $b_n = \dfrac{1}{n^2+1}$. Then $|a_n| \ge |b_n|$ for all n, and $\sum |a_n|$ converges by p-series test. Hence by a comparison test, $\sum |b_n| = \sum \dfrac{1}{n^2+1}$ converges.

3. Consider $\sum_{n=1}^{\infty} \dfrac{n}{n^2+3}$. We will do another comparison test, this time letting $a_n = \dfrac{1}{n+3}$ and $b_n = \dfrac{n}{n^2+3}$. First, $\sum_{n=1}^{\infty} \dfrac{1}{n+3} = \sum_{k=4}^{\infty} \dfrac{1}{k}$ which diverges. Now observe that

$$\frac{1}{n+3} \le \frac{n}{n^2+3} \quad\iff\quad n^2+3 \le n^2+3n \quad\iff\quad 1 \le n$$

which holds for all $n \in \mathbb{N}$. Hence by a comparison test, $\sum b_n = \sum_{n=1}^{\infty} \frac{n}{n^2+3}$ diverges.

4. Consider $\sum_{n=1}^{\infty} \left(\frac{2}{(-1)^n - 3} \right)^n$. Let's try the root test:

$$\limsup_{n\to\infty} \left| \left(\frac{2}{(-1)^n - 3} \right)^n \right|^{1/n} = \limsup_{n\to\infty} \left| \frac{2}{(-1)^n - 3} \right| = \sup \left\{1, \tfrac{1}{2}\right\} = 1.$$

The lim sup is 1, so the root test is indeterminate. However, notice that

$$\sum_{n=1}^{\infty} \left(\frac{2}{(-1)^n - 3} \right)^n = -\tfrac{1}{2} + 1 - \tfrac{1}{8} + 1 - \tfrac{1}{32} + 1 - \dots$$

so $a_n \not\to 0$. By the divergence test, the series diverges.

5. Consider $\sum_{n=1}^{\infty} 2^{(-1)^n - n} = \frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \frac{1}{8} + \frac{1}{64} + \frac{1}{32} + \dots$ Notice that this is just a rearranged geometric series. If we rearrange terms, we can write it as $\frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \dots$ so

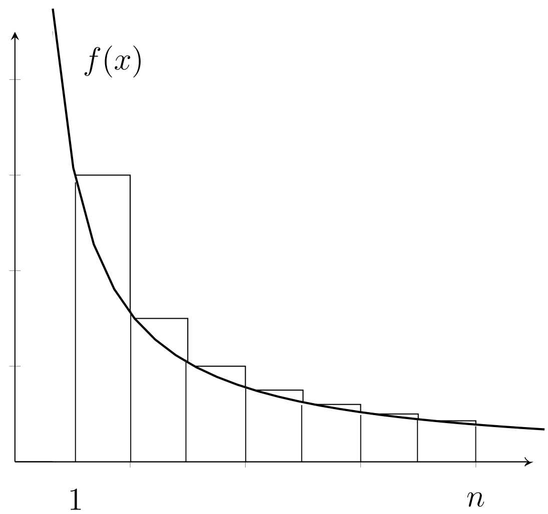
$$\sum_{n=1}^{\infty} 2^{(-1)^n - n} = \sum_{n=1}^{\infty} \frac{3}{4^n}$$

which is geometric with $a = 3$ and $r = \frac{1}{4} < 1$. Hence this series converges to 4.

6. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is an example of an alternating series. We need another test for convergence because we think this converges, albeit too slowly for the ratio or root test to detect.

2.7 The Integral Test

Consider the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. Below we plot the values of $f(n) = \frac{1}{n}$ as rectangles, along with the function $f(x) = \frac{1}{x}$ for comparison.

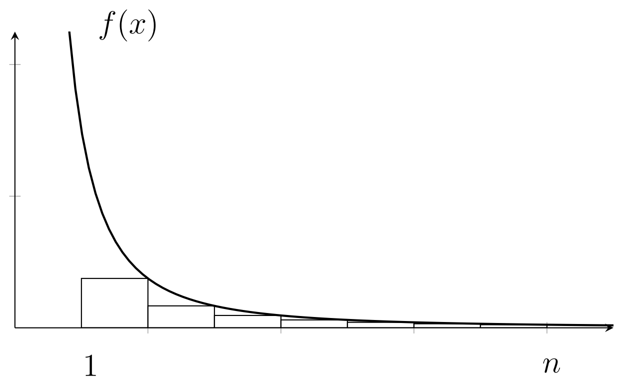


Then $\sum_{n=1}^{\infty} \frac{1}{n}$ is equal to the area of all the rectangles. But this is larger than the area under the curve, which is $\int_1^{\infty} \frac{1}{x}\,dx = \infty$, so the series appears to diverge. In fact, if we fix $N \in \mathbb{N}$ then

$$\sum_{n=1}^{N} \frac{1}{n} \geq \int_1^{N+1} \frac{1}{x}\,dx$$

(this can be proven using Riemann sums, but the picture is enough to indicate it in this case). Then $\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} \geq \lim_{N\to\infty} \int_1^{N+1} \frac{1}{x}\,dx = \infty$ so the series diverges by something akin to a comparison test. This is the foundation for the integral test, which we will state shortly.

Let us also examine a case when the series converges. Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$.



Notice here that, with the right-hand rule, we can write

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \leq \int_1^{\infty} \frac{1}{x^2}\, dx = 1$$

so it appears that the series converges to something less than 1. Fixing $N \in \mathbb{N}$, we have

$$\lim_{N\to\infty} \sum_{n=2}^{N} \frac{1}{n^2} \leq \lim_{N\to\infty} \int_1^{N} \frac{1}{x^2}\, dx = 1$$

so $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges to something less than 1, which further implies that $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}$ converges to something less than 2. Indeed this p-series converges to $\frac{\pi^2}{6} \approx 1.645$.

Consider a p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ in general. If $p > 1$ then

$$\sum_{n=2}^{\infty} \frac{1}{n^p} < \int_1^{\infty} \frac{1}{x^p}\, dx$$

which converges, so p-series converge when $p > 1$ (as we have already seen). And if $p \leq 1$, $\frac{1}{n^p} \geq \frac{1}{n} > 0$ for all n, so by a comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$, p-series diverge whenever $p \leq 1$.

Theorem 2.7.1 (Integral Test)**.** *Suppose (a_n) is a sequence such that $a_n = f(n)$ for some continuous function $f : \mathbb{R} \to \mathbb{R}$ and $f(x)$ is a nonnegative, decreasing function on all of $[1, \infty)$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\, dx$ converges.*

2.8 Alternating Series

An alternating series is one that can be written in the form $\sum_{n=1}^{\infty}(-1)^n a_n$.

Theorem 2.8.1 (Alternating Series Test)**.** *Suppose (a_n) is a sequence of nonnegative terms that is monotone non-increasing with $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n=0}^{\infty}(-1)^n a_n$ converges.*

Proof. We will show convergence via the Cauchy criterion, and in fact it suffices to show that if $n > m \geq N$ then $\left|\sum_{k=m+1}^{n}(-1)^k a_k\right| \leq a_N$. This will imply the theorem because if $\varepsilon > 0$ we can choose N such that for all $n > N-1$, $|a_n -)| < \varepsilon$ (since $(a_n) \to 0$), and then for all $n > m \geq N$, $\left|\sum_{k=m+1}^{n}(-1)^k a_k\right| \leq a_N < \varepsilon$. Now to prove the claim: let $n > m \geq N$ and set $A = a_{m+1} - a_{m+2} + a_{m+3} - \ldots \pm a_n$. Then $\left|\sum_{k=m+1}^{n}(-1)^k a_k\right| = \pm A$. On one hand, if $n-m$ is even, the last term is negative. Then

$$\begin{aligned} A &= (a_{m+1} - a_{m+2}) + (a_{m+3} - a_{m+4}) + \ldots + (a_{n-1} - a_n) \\ &\geq 0 + 0 + \ldots + 0 \end{aligned}$$

by monotonicity. Also note that

$$\begin{aligned} A &= a_{m+1} - (a_{m+2} - a_{m+3}) - \ldots - (a_{n-2} - a_{n-1}) - a_n \\ &\leq a_{m+1} - 0 - \ldots - 0 - 0 = a_{m+1}. \end{aligned}$$

So $0 \leq A \leq a_{m+1} \leq a_N$ by monotonicity again, which implies $\left|\sum_{k=m+1}^{n}(-1)^k a_k\right| = A \leq a_N$.

On the other hand, if $n-m$ is odd, the last term is positive and we have

$$\begin{aligned} A &= (a_{m+1} - a_{m+2}) + \ldots + (a_{n-2} - a_{n-1}) + a_n \\ &\geq 0 + \ldots + 0 + 0 = 0 \\ \text{and} \qquad A &= a_{m+1} - (a_{m+2} - a_{m+3}) - \ldots - (a_{n-1} - a_n) \\ &\leq a_{m+1} - 0 - \ldots - 0 = a_{m+1}. \end{aligned}$$

Again this shows that $0 \leq A \leq a_{m+1} \leq a_N$, and as above this proves the claim. □

Examples.

1. Consider the **alternating harmonic series** $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$. Let $a_k = \frac{1}{k}$. Then for all k, $a_k > a_{k+1}$ and $\lim_{k\to\infty}\frac{1}{k} = 0$. By the Alternating Series Test, the series converges.

However, $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so the alternating harmonic series converges conditionally.

2. Let $\displaystyle\sum_{k=1}^{\infty}(-1)^k a_k$ be the alternating series where

$$a_k = \begin{cases} \frac{1}{k} & k \text{ is even} \\ -\frac{1}{k^2} & k \text{ is odd.} \end{cases}$$

The positive terms are $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2k}$ which is a subsequence of the harmonic series, so they diverge. On the other hand, the negative terms are $-1-\frac{1}{9}-\frac{1}{25}-\ldots-\frac{1}{(2k+1)^2}$ which is a subsequence of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ and thus converges by Theorem 2.4.4. So what's the behavior of the whole series? The negative terms converge, but the positive terms continue to diverge so the whole series must diverge. This shows that the monotonicity requirement in the Alternating Series Test is necessary.

3. $\displaystyle\sum_{n=1}^{\infty}\frac{\sin(n)}{n}$ oscillates as $n\to\infty$, but it's not properly alternating. So far we have no test to determine this series' convergence.

Chapter 3

Functions

3.1 Continuous Functions

Definition. *Let f be a real-valued function $f : A \to \mathbb{R}$, where A is any subset of $\mathbb{R}$. The* **natural domain** *of f is the largest subset of $\mathbb{R}$ on which $f(x)$ can have values.*

Definition. *A function $f(x)$ is* **continuous** *at a point x_0 in the domain of f if for all sequences $(x_n) \to x_0$, $f(x_n) \to f(x_0)$. (This is known as the sequential definition of continuity.) For any subset S of the domain of f, we say f is continuous on S if f is continuous at every point $x_0 \in S$. If f is continuous on its entire domain, we simply say f is a continuous function.*

Example 3.1.1. Show that $f(x) = x^2$ is continuous.

Proof. The domain of f is $\mathbb{R}$ so let $x_0 \in \mathbb{R}$. Suppose (x_n) is a sequence converging to x_0. Then by limit properties,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n^2 = \lim_{n\to\infty} x_n \cdot x_n = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} x_n = x_0 \cdot x_0 = x_0^2 = f(x_0).$$

Hence f is continuous. □

Example 3.1.2. Show $f(x) = \sqrt{x}$ is continuous.

Proof. Here the domain is $\mathbb{R}^+ = [0, \infty)$, so let $x_0 \geq 0$. Suppose (x_n) is a sequence such that $x_n \geq 0$ for all n, and $(x_n) \to x_0$. Then by Proposition 2.1.6,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sqrt{x_n} = \sqrt{x_0}.$$

So $f(x_n) \to f(x_0)$ and the function is continuous. □

The following proposition introduces the more common "epsilon - delta definition of continuity".

Proposition 3.1.3. *A function f is continuous at x_0 if and only if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - x_0| < \delta$ and $x \in dom(f)$ then $|f(x) - f(x_0)| < \varepsilon$.*

Proof. ($\Longleftarrow$) Suppose the epsilon - delta definition holds for a function f at a point $x_0 \in \mathrm{dom}(f)$. Let (x_n) be any sequence in $\mathrm{dom}(f)$ that converges to x_0. Then for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x_n - x_0| < \delta$ then $|f(x_n) - f(x_0)| < \varepsilon$. Since $(x_n) \to x_0$, there is some N such that for all $n > N$, $|x_n - x_0| < \delta$. It follows that for all $n > N$, $|f(x_n) - f(x_0)| < \varepsilon$. So $f(x_n) \to f(x_0)$ and f is sequentially continuous at x_0.

($\Longrightarrow$) We will prove the forward direction by contrapositive. Suppose there is some $\varepsilon > 0$ such that for all $\delta > 0$, there is some x with $|x - x_0| < \delta$ but $|f(x) - f(x_0)| \geq \varepsilon$. We construct a sequence as follows: first let $\delta_1 = 1$ and pick x_1 such that $|x_1 - x_0| < 1$ but $|f(x_1) - f(x_0)| \geq \varepsilon$. Next let $\delta_2 = \frac{1}{2}$ and pick x_2 such that $|x_2 - x_0| < \frac{1}{2}$ but $|f(x_2) - f(x_0)| \geq \varepsilon$. Continue in this way, letting $\delta_n = \frac{1}{n}$ and picking x_n so that $|x_n - x_0| < \frac{1}{n}$ but $|f(x_n) - f(x_0)| \geq \varepsilon$. Then $(x_n) \to x_0$, but $f(x_n) \not\to f(x_0)$ because $|f(x_n) - f(x_0)| \geq \varepsilon$ for all n. Hence f is not sequentially continuous at x_0. □

Example 3.1.4. Show the continuity of $f(x) = x^2$ using the $\varepsilon - \delta$ definition of continuity.

First we do some scratchwork. Let $|x - x_0| < \delta$. Then

$$|x^2 - x_0^2| = |(x - x_0)(x + x_0)| < \delta|x + x_0|.$$

Note that $|x - x_0| < \delta$, so $x_0 - \delta < x < x_0 + \delta$. If δ is small enough, we will have $x < 2|x_0| < 2|x_0| + 1$ – we use the latter inequality to avoid the possibility of $x_0 = 0$. Then for $\delta|x + x_0|$ to be less than ε, we want $\delta < \frac{\varepsilon}{2|x_0|+1}$. Now for the proof:

Proof. Let $\varepsilon > 0$ and let $\delta < \min\left\{\max\{|x_0|, 1\}, \frac{\varepsilon}{2|x_0|+1}\right\}$. Let x be chosen so that $|x - x_0| < \delta$. Then

$$\begin{aligned}
|f(x) - f(x_0)| &= |x^2 - x_0^2| \\
&= |(x - x_0)(x + x_0)| \\
&< \delta|x + x_0| \\
&< \frac{\varepsilon}{2|x_0| + 1}|x + x_0| \\
&< \frac{\varepsilon}{2|x_0| + 1}(2|x_0| + 1) \qquad \text{by scratchwork} \\
&= \varepsilon.
\end{aligned}$$

Thus f is continuous. □

Example 3.1.5. Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$ Is f continuous at $x_0 = 0$?

If $|x| < \delta$ then $\left|x^2 \sin\left(\frac{1}{x}\right) - 0\right| = |x|^2 \left|\sin\left(\frac{1}{x}\right)\right|$. Since $\sin(x) \leq 1$ for all x, we have $|x|^2 \left|\sin\left(\frac{1}{x}\right)\right| \leq |x|^2 < \delta^2$ so if f is to be continuous, we should let $\delta = \sqrt{\varepsilon}$.

Proof. Let $\varepsilon > 0$ and let $\delta = \sqrt{\varepsilon}$. Let x be chosen such that $|x| < \delta$. Then

$$\begin{aligned}
\left|x^2 \sin\left(\frac{1}{x}\right) - 0\right| &= |x|^2 \left|\sin\left(\frac{1}{x}\right)\right| \\
&\leq |x|^2 \\
&< \delta^2 \\
&= (\sqrt{\varepsilon})^2 = \varepsilon.
\end{aligned}$$

Hence f is continuous at 0. □

Example 3.1.6. Is $f(x) = \begin{cases} \frac{1}{x} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ continuous?

We think it's not. To show this, we need a sequence $(x_n) \to 0$ such that $\sin\left(\frac{1}{x_n}\right) = 1$ for example. This occurs precisely when $\frac{1}{x_n} = \frac{\pi}{2}(4n + 1)$, so let $x_n = \frac{2}{\pi(4n+1)}$. By limit laws, $(x_n) \to 0$, but $f(x_n) = \frac{\pi}{2}(4n + 1) \sin\left(\frac{\pi}{2}(4n + 1)\right) = \frac{\pi}{2}(4n + 1)$ which tends to ∞ as $n \to \infty$. Hence $f(x_n)$ diverges, so f is not continuous at $x_0 = 0$.

Theorem 3.1.7. *Suppose f is continuous at x_0. Then*

(1) $|f|$ is continuous at x_0.

(2) If $k \in \mathbb{R}$, then kf is continuous at x_0.

Proof. (1) Given $\varepsilon > 0$, there is some $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$. By the triangle inequality, $||f(x)| - |f(x_0)|| \leq |f(x) - f(x_0)|$, so when $|x - x_0| < \delta$, $||f(x)| - |f(x_0)|| \leq |f(x) - f(x_0)| < \varepsilon$. Thus $|f|$ is continuous.

(2) Let $k \in \mathbb{R}$. First note that if $k = 0$, $kf = 0$ which is continuous, so we may assume $k \neq 0$. Then given $\varepsilon > 0$, there is some $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \frac{\varepsilon}{|k|}$. This means $|kf(x) - kf(x_0)| = |k|\,|f(x) - f(x_0)| < |k| \cdot \frac{\varepsilon}{|k|} = \varepsilon$, so kf is continuous. □

Theorem 3.1.8. *If f and g are continuous at x_0, then the following are also continuous at x_0:*

(1) $f + g$

(2) $f - g$

(3) fg

(4) $\dfrac{f}{g}$ if $g(x_0) \neq 0$.

Proof. Let $(x_n) \to x_0$. Since f and g are continuous, $f(x_n) \to f(x_0)$ and $g(x_n) \to g(x_0)$. Then by limit laws,

$$(f(x_n) + g(x_n)) \longrightarrow f(x_0) + g(x_0) \tag{3.1}$$

$$(f(x_n) - g(x_n)) \longrightarrow f(x_0) - g(x_0) \tag{3.2}$$

$$(f(x_n)g(x_n)) \longrightarrow f(x_0)g(x_0) \tag{3.3}$$

$$\text{and if } g(x_0) \neq 0, \left(\frac{f(x_n)}{g(x_n)}\right) \longrightarrow \frac{f(x_0)}{g(x_0)}. \tag{3.4}$$

Thus the continuity of (1) – (4) is proven by the sequential definition. □

Proposition 3.1.9. *The following are some useful examples of continuous functions.*

(1) $f(x) = x$ is continuous on $\mathbb{R}$.

(2) All polynomials are continuous on $\mathbb{R}$.

(3) All rational functions are continuous on their domains.

(4) If f and g are continuous, then $\max(f, g)$ is continuous.

Proof. (1) is simple. To show (2), consider $f(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + a_n x^n$. This is a sum of products of constants and powers of x, so by Theorem 3.1.8, $f(x)$ is continuous on $\mathbb{R}$.

(3) Let $f(x) = \dfrac{a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + a_n x^n}{b_0 + b_1 x + \ldots + b_{m-1}x^{m-1} + b_m x^m}$. This is a quotient of polynomials, so the numerator and denominator are both continuous on $\mathbb{R}$ by (2). The domain of f is thus all points where the denominator is nonzero. Hence $f(x)$ is continuous on its domain.

(4) If f and g are continuous, then

$$\begin{aligned}\max(f, g) &= \begin{cases} f(x) & f(x) \geq g(x) \\ g(x) & g(x) \geq f(x) \end{cases} \\ &= \tfrac{1}{2}(f(x) + g(x)) + \tfrac{1}{2}|f(x) - g(x)|.\end{aligned}$$

By Theorem 3.1.8, this formula is continuous, so $\max(f, g)$ is continuous. $\square$

3.2 Properties of Continuous Functions

Theorem 3.2.1 (Extreme Value Theorem)**.** *If f is continuous on a closed interval $[a, b]$, then f is bounded on this interval and there exist $x_0, y_0 \in [a, b]$ such that $f(x_0) = \min\{f(x) \mid x \in [a, b]\}$ and $f(y_0) = \max\{f(x) \mid x \in [a, b]\}$.*

Proof. Suppose f is not bounded above (the proof is the same for bounded below). Then for all n there exists some x_n such that $f(x_n) > n$. Consider the sequence of these x_n. By construction (x_n) is a bounded sequence. Then by the Bolzano-Weierstrass theorem (2.4.5), (x_n) has a subsequence (x_{n_k}) which converges to x_0. The interval $[a, b]$ is closed, so $x_0 \in [a, b]$. Since $(x_{n_k}) \to x_0$ and f is continuous, $f(x_{n_k}) \to f(x_0)$. But $f(x_{n_k}) > n_k$ for all k. This implies $f(x_{n_k}) \to +\infty$, a contradiction. Hence f is bounded.

Now let $M = \sup f(x)$, which exists since f is bounded. For all $n \in \mathbb{N}$, $M - \frac{1}{n}$ is not an upper bound of $\{f(x) \mid x \in [a, b]\}$, so there exists an x_n such that $f(x_n) > M - \frac{1}{n}$. Again consider the sequence of these x_n, which is bounded and has a convergent subsequence (x_{n_k}). Let $y_0 = \lim\limits_{k\to\infty} x_{n_k}$. Then $f(y_0) = \lim\limits_{k\to\infty} f(x_{n_k}) \le M$ by boundedness, and $f(y_0) \ge M - \frac{1}{n}$ for all n. Thus $f(y_0) = M = \max\{f(x) \mid x \in [a, b]\}$. The proof is the same for the minimum. $\square$

Note that the extreme value theorem is highly dependent on the interval $[a, b]$ being closed.

The second important theorem for continuous functions is the Intermediate Value Theorem. In plain English, it says that when you draw a function on a continuous interval, you don't pick up your pencil.

Theorem 3.2.2 (Intermediate Value Theorem)**.** *Suppose f is continuous on some interval I. Then for all $a, b \in I$ with $a < b$, if either $f(a) < y < f(b)$ or $f(a) > y > f(b)$, then there is some x between a and b such that $f(x) = y$.*

Proof. Without loss of generality, assume $f(a) < y < f(b)$. Let $S = \{x \in [a, b] \mid f(x) < y\}$. Note that $a \in S$ so S is nonempty, and S is bounded above by b. Then the Completeness Axiom says S has a supremum, call it x_0. Then $x_0 - \frac{1}{n}$ is not an upper bound for S for any n, so there exists an $x_n \in S$ such that $x_0 - \frac{1}{n} < x_n \le x_0$. By the Squeeze Theorem, $(x_n) \to x_0$, and since f is continuous, $f(x_n) \to f(x_0)$. Now since $x_n \in S$ for all n, $f(x_n) < y$, so $f(x_0) = \lim\limits_{n\to\infty} f(x_n) \le y$. On the other hand, x_0 is an upper bound for S, so $x_0 + \frac{1}{n} \notin S$ for any n. Let $z_n = \min\left\{x_0 + \frac{1}{n}, b\right\}$. Then $f(z_n) \ge y$ and $(z_n) \to x_0 \implies f(z_n) \to f(x_0)$ by continuity. Therefore $f(x_0) \ge y$ by the above reasoning. Hence $f(x_0) = y$. $\square$

The intermediate value theorem says that if f is continuous on an interval, then the image of the interval is also an interval. One use of the intermediate value theorem is in fixed point problems:

Theorem 3.2.3. *Suppose $f : [0, 1] \to [0, 1]$ is continuous. Then f has a fixed point x such that $f(x) = x$.*

Proof. Consider $g(x) = f(x) - x$. By Theorem 3.1.8, g is continuous on $[0, 1]$. Suppose $f(0) \neq 0$ and $f(1) \neq 1$. Then $f(0) > 0$ so $g(0) > 0$, and $f(1) < 1$ so $g(1) = f(1) - 1 < 0$. This gives us $g(1) < 0 < g(0)$, and the intermediate value theorem says there exists an $x \in (0, 1)$ such that $g(x) = 0$. $\square$

Proposition 3.2.4. *Suppose $y > 0$, $m \in \mathbb{N}$ and $m \geq 2$. Then there is some x such that $x^m = y$, that is $\sqrt[m]{y}$ exists.*

Proof. Consider the continuous function $f(x) = x^m$. Note that $f(0) = 0 < y$. If $y > 1$ then let $b = y$ so that $f(b) = y^m > y$. Or if $y < 1$, let $b = 1$ so that $f(b) = 1 > y$, i.e. $f(b) > y$ in both cases. We have $0 < y < f(b)$, so by the intermediate value theorem there exists $x \in (0, b)$ such that $f(x) = x^m = y$. □

The third important theorem of this section states the existence of an inverse function f^{-1} under certain conditions.

Theorem 3.2.5 (Inverse Function Theorem)**.** *If f is continuous and strictly increasing on an interval I, then $J = f(I)$ is an interval and $f^{-1} : J \to I$ is continuous and strictly increasing.*

Proof. First, if f is strictly increasing it is clearly one-to-one, so the inverse $f^{-1} : J \to I$ exists. The fact that $J = f(I)$ is an interval is provided by the intermediate value theorem. Since $x < y \implies f(x) < f(y)$, we have $x < y \implies f^{-1}(x) < f^{-1}(y)$, so f^{-1} is strictly increasing.

We now prove that $g = f^{-1}$ is continuous; this is a partial converse to the intermediate value theorem. Let $x_0 \in J$ and $\varepsilon > 0$. Assume x_0 is not an endpoint of J; then $f^{-1}(x_0)$ is not an endpoint of I by monotonicity. Then there is some ε_0 such that $(g(x_0)-\varepsilon_0, g(x_0)+\varepsilon_0) \subset J$. We may make ε small enough so that $\varepsilon \leq \varepsilon_0$. Then there exists an x_1 such that $g(x_1) = g(x_0) - \varepsilon$ because $g(x_0) - \varepsilon \in (g(x_0) - \varepsilon_0, g(x_0) + \varepsilon_0)$. Similarly, there exists an x_2 such that $g(x_2) = g(x_0) + \varepsilon$. Let $\delta = \min\{x_2 - x_0, x_0 - x_1\}$ and let $x \in J$ such that $|x - x_0| < \delta$. Then $x_0 - \delta < x < x_0 + \delta$ which implies $x_0 + \delta \leq x_2$ and $x_0 - \delta \geq x_1$. This gives us

$$\begin{aligned} x_1 < x < x_2 &\implies g(x_1) < g(x) < g(x_2) \qquad \text{by monotonicity} \\ &\implies g(x_0 - \varepsilon < g(x) < g(x_0) + \varepsilon \\ &\implies |g(x) - g(x_0)| < \varepsilon. \end{aligned}$$

Hence $g = f^{-1}$ is continuous on J. □

Example 3.2.6. $f(x) = \sin\left(\frac{1}{x}\right)$ is not continuous on $\mathbb{R}$ (for example, f is not continuous at $x = 0$), but it does satisfy the intermediate value property for its entire domain: the image of the interval between *any* two nonzero points, even including $a < 0 < b$, is an interval. In fact, if $a < 0 < b$ then $f((a, b)) = [-1, 1]$ because there are an infinite number of whole oscillations of the sine curve on the interval (a, b). This is a counterexample to the converse of the inverse function theorem which shows why the partial converse requires the condition that g is strictly increasing.

Theorem 3.2.7. *If f is one-to-one and continuous on an interval I, then f is strictly monotone.*

Proof. Without loss of generality, assume there are $a_0, b_0 \in I$ with $a_0 < b_0$ such that $f(a_0) < f(b_0)$. To contradict, suppose $f(b) \geq \max\{f(a), f(c)\}$. Since f is one-to-one, $f(b) > \max\{f(a), f(c)\}$. Without loss of generality, assume $f(a) < f(c)$. Then by the intermediate value theorem, there exists an $x_0 \in (a,b)$ such that $f(x_0) = f(c)$. But $a < x_0 < b < c$ so $x_0 \neq c$ which contradicts injectivity. By the same argument, we can't have $f(c) < f(a) < f(b)$. Thus $f(b) < \max\{f(a), f(c)\}$.

Now let $x_1, x_2 \in I$ such that $x_1 < x_2$. If $a_0 < x_2 < b_0$ then $f(a_0) < f(b_0)$ implies $f(a_0) < f(x_2) < f(b_0)$. If $a_0 < x_1 < x_2$ then $f(a_0) < f(x_1) < f(x_2)$, so $f(x_1) < f(x_2) \ldots$ (lots of cases later ...) x_1 and x_2 are two reference points that indicate whether f is increasing or decreasing on I. Since x_1 and x_2 were chosen arbitrarily, f must be strictly monotone on the entire interval. $\square$

3.3 Uniform Continuity

Recall that f is continuous on a domain S if for all $x_0 \in S$ and for all $\varepsilon > 0$, there is a $\delta > 0$ such that if $x \in S$ and $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$. In this case the δ depends on x_0 and ε. There is a stronger notion of continuity that only depends on the ε chosen:

Definition. *A function f is* **uniformly continuous** *on S if for all $\varepsilon > 0$ there is some $\delta > 0$ such that for all $x, y \in S$ with $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$.*

Examples.

1. $f(x) = \frac{1}{x}$ on $S = (0, \infty)$

   Claim. *f is continuous on $(0, \infty)$.*

   Proof. Let $x \in S$, $\varepsilon > 0$ and $\delta = \min\left\{\frac{x}{2}, \frac{\varepsilon x^2}{2}\right\}$. If $y > 0$ and $|y - x| < \delta$ then

$$\left|\frac{1}{y} - \frac{1}{x}\right| = \left|\frac{x - y}{xy}\right| = \frac{|x - y|}{|x|\,|y|}.$$

   Since $|y - x| < \delta \le \frac{x}{2}$, then $y > \frac{x}{2}$, so

$$\frac{|y - x|}{|x|\,|y|} < \frac{2|y - x|}{|x|^2} < \frac{2\delta}{|x^2|} \le \frac{2}{x^2}\left(\frac{\varepsilon x^2}{2}\right) = \varepsilon.$$

   Hence f is continuous on S. □

   But is f uniformly continuous? No, because as $x \to 0$ we get larger and larger values for ε:

   Proof. Let $\varepsilon = 1$ and let $\delta > 0$ be given. Choose $x = \delta$ and $y = \frac{\delta}{2}$. Then $|x - y| = \frac{\delta}{2} < \delta$, but $|f(x) - f(y)| = \left|\frac{1}{\delta} - \frac{2}{\delta}\right| = \frac{1}{\delta}$. If $\delta < 1$ this is a contradiction. Hence f is not uniformly continuous. □

2. $f(x) = \frac{1}{x}$ *is* uniformly continuous on $[a, \infty)$ for all $a > 0$. Look at

$$\begin{aligned} \frac{|y - x|}{|y|\ |x|} &\le \frac{|y - x|}{a^2} && \text{since } x, y \in [a, \infty) \\ &\le \frac{\delta}{a^2} \end{aligned}$$

   so we should let $\delta = \varepsilon a^2$.

   Proof. Let $\varepsilon > 0$, $\delta = \varepsilon a^2$ and $x, y \in [a, \infty)$ such that $|x - y| < \delta$. Then

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|y - x|}{|x|\,|y|} \le \frac{\delta}{a^2} = \frac{\varepsilon a^2}{a^2} = \varepsilon.$$

   Hence f is uniformly continuous on $[a, \infty)$. □

The idea here is as long as you avoid the asymptote at $x = 0$, f remains uniformly continuous.

3. Is $f(x) = x^2$ uniformly continuous on $\mathbb{R}$? In the proof of regular continuity, we see that

$$|x^2 - y^2| = |x - y|\,|x + y| < \varepsilon \implies |x - y| < \frac{\varepsilon}{|x + y|}.$$

If $\delta < 1$, $y < x + 1$ so we choose $\delta = \frac{\varepsilon}{2|x|+1}$. The problem is that this δ depends on x, so as $x \to \infty$, the graph gets steeper and steeper. Thus f is not uniformly continuous on $\mathbb{R}$:

Proof. Let $\varepsilon = 1$ and let $\delta > 0$ be given. Let $x = \frac{1}{\delta}$ and $y = \frac{1}{\delta} + \frac{\delta}{2}$. Then

$$|f(x) - f(y)| = |x^2 - y^2| = \left|\frac{1}{\delta^2} - \frac{1}{\delta^2} - 1 - \frac{\delta^2}{4}\right| = \left|1 + \frac{\delta^2}{4}\right| \geq 1.$$

Hence f is not uniformly continuous. □

Claim. *$f(x) = x^2$ is uniformly continuous on $[-a, a]$ for all $a > 0$.*

Proof. Let $a > 0$, $\varepsilon > 0$ and $x, y \in [-a, a]$ such that $|x - y| < \frac{\varepsilon}{2a}$. Then $|x|, |y| \leq a$ so we have

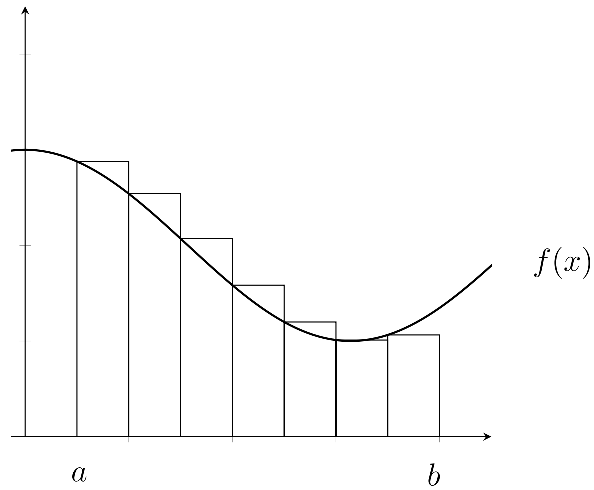
$$|f(x) - f(y)| = |x^2 - y^2| = |x - y|\,|x + y| \leq |x - y|\,2a < \delta(2a) = \left(\tfrac{\varepsilon}{2a}\right)(2a) = \varepsilon.$$

Hence f is uniformly continuous on $[-a, a]$. □

Theorem 3.3.1. *If a function f is continuous on a closed interval $[a, b]$ then f is uniformly continuous on $[a, b]$.*

Proof. Suppose f is continuous on $[a, b]$ but not uniformly continuous on $[a, b]$. Then there is some $\varepsilon > 0$ such that for all $\delta > 0$, there are $x, y \in [a, b]$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$. Let $\delta = \frac{1}{n}$ and define sequences (x_n) and (y_n) where for each n, $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon$. Consider (x_n) which is bounded. By the Bolzano-Weierstrass theorem (2.4.5), there is a convergent subsequence (x_{n_k}). Then $|y_{n_k} - x_{n_k}| \leq \frac{1}{n_k}$ for all k, so $x_{n_k} - \frac{1}{n_k} < y_{n_k} < x_{n_k} + \frac{1}{n_k}$. Since $\frac{1}{n_k} \to 0$, the Squeeze Theorem gives us $\lim y_{n_k} = \lim x_{n_k} := x$. Then $(x_{n_k}) \to x$, and $x \in [a, b]$ because it's a closed interval. Moreover, since f is continuous on $[a, b]$, $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(x)$. Thus $|f(x_{n_k}) - f(y_{n_k})| \to |f(x) - f(x)| = 0 < \varepsilon$, contradicting the assumption that f is not uniformly continuous. □

Example 3.3.2. Riemann sums



Suppose $f(x)$ is continuous on $[a,b]$. A problem is to calculate $\int_a^b f(x)\,dx$ by taking a limit:

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Let R_U be the upper Riemann sum (i.e. the biggest the sum can be, which depends on the choices of x_i^*) and let R_L be the lower Riemann sum. How bad is $R = R_U - R_L$? Consider

$$R_U - R_L = \sum \left(\max_{\substack{x \text{ in} \\ \text{subinterval}}} f(x) - \min_{\substack{x \text{ in} \\ \text{subinterval}}} f(x) \right) \Delta x.$$

The limit above, which equals the integral, exists if $R_U - R_L \to 0$ as $\Delta x \to 0$. It turns out that if f is uniformly continuous on $[a,b]$ then $R_U - R_L \to 0$ as $\Delta x \to 0$. Hence by Theorem 3.3.1, it is sufficient to show that f is continuous on $[a,b]$. See Section 4.5 for more on integration.

Theorem 3.3.3. *Suppose $f : S \to \mathbb{R}$ is uniformly continuous. If $(x_n) \subset S$ is a Cauchy sequence then $f(x_n)$ is a Cauchy sequence in $\mathbb{R}$.*

Proof. Let $\varepsilon > 0$ and choose $\delta > 0$ such that if $|x-y| < \delta$, $|f(x) - f(y)| < \varepsilon$ using uniform continuity. Choose $N \in \mathbb{N}$ such that for all $n, m > N$, $|x_n - x_m| < \delta$ by (x_n) Cauchy. Then for all $n, m > N$, $|x_n - x_m| < \delta \implies |f(x_n) - f(x_m)| < \varepsilon$ so $f(x_n)$ is Cauchy. $\square$

Example 3.3.4. Recall $f(x) = \frac{1}{x}$ on $(0,1)$. The sequence $x_n = \frac{1}{n}$ is Cauchy, but $f(x_n) = \frac{1}{\frac{1}{n}} = n$ is not Cauchy. Thus f is not uniformly continuous on $(0,1)$.

Definition. *A function $\widetilde{f}$ is an* **extension** *of f if $dom(f) \subset dom(\widetilde{f})$ and for all $x \in dom(f)$, $f(x) = \widetilde{f}(x)$.*

Theorem 3.3.5. *A function $f : (a,b) \to \mathbb{R}$ is uniformly continuous if and only if there exists an extension $\widetilde{f}$ of f such that $\widetilde{f} : [a,b] \to \mathbb{R}$ and $\widetilde{f}$ is continuous on $[a,b]$.*

Proof. ($\Longleftarrow$) Suppose $\widetilde{f}$ exists. Then by Theorem 3.3.1, $\widetilde{f}$ is uniformly continuous on $[a, b]$, and since $\operatorname{dom}(f) \subset \operatorname{dom}(\widetilde{f})$, f is uniformly continuous on (a, b).

($\Longrightarrow$) Suppose f is uniformly continuous on $[a, b]$. Let (x_n) be any sequence on the interval (a, b) that converges to a. Then (x_n) is Cauchy, so $f(x_n)$ is Cauchy by Theorem 3.3.3. By the Cauchy convergence theorem (2.3.8), $f(x_n)$ converges. Let $f(a) = \lim\limits_{n\to\infty} f\left(a + \frac{1}{n}\right)$. Let $(x_n) \to a$ and consider the sequence

$$y_n = \left(x_1, a+1, x_2, a + \frac{1}{2}, x_3, a + \frac{1}{3}, \dots\right).$$

Then $(y_n) \to a$. By the above, $f(y_n)$ converges too, so it must converge to $f(a)$ since half of its terms converge to a – one can make this rigorous using subsequences. Finally, we have $f(x_n) \to f(a)$ since $f(x_n)$ is a subsequence of $f(y_n)$. □

Examples.

4. Consider $f(x) = x \sin\left(\frac{1}{x}\right)$ on the interval $(0, 1]$. Is it possible to extend f to $[0, 1]$? Define

$$\widetilde{f}(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & 0 < x \leq 1 \\ 0 & x = 0. \end{cases}$$

Claim. *$\widetilde{f}$ is continuous on* $[0, 1]$.

Proof. Let $x_0 \in [0, 1]$. If $x_0 > 0$ then $\widetilde{f}(x_0) = f(x_0)$ and there there is some $\varepsilon_0 > 0$ such that $\widetilde{f}(x) = f(x)$ for all $x \in (x_0 - \varepsilon_0, x_0 + \varepsilon_0)$. Since f is continuous at x_0, for all $\varepsilon > 0$ if $|x - x_0| < \varepsilon_0$ then $|\widetilde{f}(x) - \widetilde{f}(x_0)| = |f(x) - f(x_0)|$. So if we let $\delta < \varepsilon_0$, then $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon \implies |\widetilde{f}(x) - \widetilde{f}(x_0)| < \varepsilon$.

Now suppose $x_0 = 0$. Let $\delta = \varepsilon$ and suppose $|x - 0| = |x| < \delta$. Then $|\widetilde{f}(x) - \widetilde{f}(0)| = \left|x \sin\left(\frac{1}{x}\right)\right| < |x|$ since $0 < \sin\theta < 1$. Then $|x| < \delta = \varepsilon$, so $\widetilde{f}$ is continuous on $[0, 1]$ in all cases. □

Now note that because $\widetilde{f}$ is continuous on $[0, 1]$, f is uniformly continuous on $(0, 1]$ by Theorem 3.3.5.

5. Consider $g(x) = \sin\left(\frac{1}{x}\right)$ on $(0, 1]$. Note that it's always possible to extend g onto a closed interval by picking any value you want for the endpoint. We could let

$$\widetilde{g}(x) = \begin{cases} g(x) & 0 < x \leq 1 \\ 0 & x = 0. \end{cases}$$

Is $\widetilde{g}(x)$ continuous on $[0, 1]$? Let $x_n = \frac{2}{\pi(4n+1)}$. Then for all n,

$$\sin(x_n) = \sin\left(\frac{1}{\frac{2}{\pi(4n+1)}}\right) = \sin\left(\frac{4\pi n + \pi}{2}\right) = \sin\left(2\pi n + \tfrac{\pi}{2}\right) = 1.$$

So $\widetilde{g}(x_n) \to 1$ but $(x_n) \to 0$. Therefore $\widetilde{g}$ is not continuous on $[0, 1]$.

6 Is $h(x) = \dfrac{\sin(x)}{x}$ uniformly continuous on $\mathbb{R} \setminus \{0\}$? We first consider h on $[-1, 1] \setminus \{0\}$. Since $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x} = 1$, define

$$\widetilde{h}(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0. \end{cases}$$

(This function is sometimes denoted $\operatorname{sinc}(x)$.) Note that $\widetilde{h}(x)$ is continuous at 0 because if $(x_n) \to 0$, $\dfrac{\sin(x_n)}{x_n} \to 1$. Thus $h(x)$ is uniformly continuous on $[-1, 1] \setminus \{0\}$.

To show $h(x)$ is uniformly continuous on $\mathbb{R} \setminus \{0\}$, we must show that $\operatorname{sinc}(x)$ is uniformly continuous on all of $\mathbb{R}$. In Section 4.3, we will prove the Mean Value Theorem (4.3.3), which states that for all $a < b$ there is some c such that $a < c < b$ and $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. If $|f'(c)| \leq M$ then $|f(b) - f(a)| \leq M|b - a|$. So letting $\delta = \frac{\varepsilon}{M}$ finishes the proof.

3.4 Limits of Functions

Definition. *Let $S \subseteq \mathbb{R}$ and let $a \in \mathbb{R}$ or $a = \pm\infty$. We say the* **limit** *of a function f at a is L, written $\lim\limits_{x \to a^S} f(x) = L$, if $dom(f) \supset S$ and for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in S$ with $|x - a| < \delta$, $|f(x) - L| < \varepsilon$.*

The above is known as the epsilon – delta definition of a limit. Equivalently, the sequential definition of a limit says that $\lim\limits_{x \to a^S} f(x) = L$ if for all sequences $(x_n) \subset S$, $(x_n) \to a$ implies $f(x_n) \to L$.

Lemma 3.4.1. *Sequential limits are unique.*

Proof. Suppose $(s_n) \to s$ and $(s_n) \to t$, with $s \neq t$. Let $\varepsilon = \frac{|s-t|}{2}$. Choose N_1 such that for all $n > N_1$, $|s_n - s| < \varepsilon$. Likewise choose N_2 such that for all $n > N_2$, $|s_n - t| < \varepsilon$. Then for all $n > N = \max\{N_1, N_2\}$ we have

$$\begin{aligned} |s-t| &= |s - s_n + s_n - t| \\ &\leq |s_n - s| + |s_n - t| \qquad \text{by triangle inequality} \\ &< \varepsilon + \varepsilon \\ &= \frac{|s-t|}{2} + \frac{|s-t|}{2} = |s-t|, \end{aligned}$$

a clear contradiction. Hence $s = t$. □

Remarks.

(i) f is continuous at $a \in \mathbb{R} \iff \lim\limits_{x \to a^S} f(x) = L$, where $S = (a - \varepsilon_0, a + \varepsilon_0)$ and $L = f(a)$.

(ii) If $\lim\limits_{x \to a^S} f(x)$ exists, it is unique.

Proof. Suppose $\lim\limits_{x \to a^S} f(x) = L_1 = L_2$. Take a sequence $(x_n) \subset S$ with $(x_n) \to a$. Then $f(x_n) \to L_1$ and $f(x_n) \to L_2$. By Lemma 3.4.1, sequential limits are unique, which implies $L_1 = L_2$. □

(iii) Suppose $\lim\limits_{x \to a^S} f(x) = L$. If $S = (a - \varepsilon_0, a + \varepsilon_0) \setminus \{a\}$ then we still have that for all $\varepsilon > 0$ there is a $\delta > 0$ such that $) < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$. In other words, the normal definition of a limit holds even when a is not included in the domain.

(iv) If $S = (a - \varepsilon_0, a)$ then we write $\lim\limits_{x \to a^S} f(x) = \lim\limits_{x \to a^-} f(x)$.

(v) Likewise, if $S = (a, a + \varepsilon_0)$ then we write $\lim\limits_{x \to a^S} f(x) = \lim\limits_{x \to a^+} f(x)$.

Examples.

1. Calculate $\lim\limits_{x\to 4} x^3$. Note that $f(x) = x^3$ is continuous on $\mathbb{R}$. So we may let $S = (3,5) \smallsetminus \{4\}$, and remark (i) tells us that $L = f(4) = 64$.

2. Calculate $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$. The domain of $f(x) = \dfrac{x^2-4}{x-2}$ is $\mathbb{R} \smallsetminus \{2\}$ and f is continuous on its entire domain. Let
$$g(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 6 & x = 2. \end{cases}$$
Then g is a continuous extension of f onto $\mathbb{R}$. Thus $\lim\limits_{x\to 2} f(x)$ must be 6.

3. Consider $f(x) = \frac{1}{x}$. Is $\lim\limits_{x\to 0} f(x) = \infty$? No: $\lim\limits_{x\to 0^+} f(x) = +\infty$ but $\lim\limits_{x\to 0^-} f(x) = -\infty$, i.e. the right- and left-sided limits are not equal, so the function does not have a limit at $x = 0$.

Like diverging sequences in the last chapter, functions may have one of the following types of **asymptotic behavior**:

- f has a **vertical asymptote** at a, written $\lim\limits_{x\to a} f(x) = +\infty$, if $(x_n) \to a$ implies $f(x_n) \to +\infty$. An equivalent definition is if for all $M > 0$, there is some $\delta > 0$ such that $|x - a| < \delta \implies f(x) > M$. (This definition can be adapted for $-\infty$.)
- f has a **horizontal asymptote**, written $\lim\limits_{x\to \infty} f(x) = L$ if $(x_n) \to \infty$ implies that $f(x_n) \to L$. An equivalent definition is if for all $\varepsilon > 0$, there is some $M > 0$ such that for all $x > M$, $|f(x) - L| < \varepsilon$. (This can also occur as $x \to -\infty$.)
- $\lim\limits_{x\to \infty} f(x) = \infty$ if $(x_n) \to \infty$ implies $f(x_n) \to \infty$, or if for all $M > 0$, there is some $N > 0$ such that for all $x > N$, $f(x) > M$.

Theorem 3.4.2. *Suppose f_1 and f_2 are continuous on S with $\lim\limits_{x\to a^S} f_1(x) = L_1$ and $\lim\limits_{x\to a^S} f_2(x) = L_2$ for some $L_1, L_2 \in \mathbb{R}$. Then*

(1) $\lim\limits_{x\to a^S} (f_1(x) + f_2(x)) = L_1 + L_2$.

(2) $\lim\limits_{x\to a^S} (f_1(x) - f_2(x)) = L_1 - L_2$.

(3) $\lim\limits_{x\to a^S} f_1(x) f_2(x) = L_1 L_2$.

(4) If $L_2 \neq 0$, $\lim\limits_{x\to a^S} \dfrac{f_1(x)}{f_2(x)} = \dfrac{L_1}{L_2}$.

Proof. Use the sequence laws. ☐

Theorem 3.4.3. *Suppose $\lim\limits_{x\to a^S} f(x) = L$ exists, g is defined on $\{L\} \cup \{y \mid y = f(x), x \in S\}$ and g is continuous at $y_0 = L$. Then $\lim\limits_{x\to a^S} g \circ f(x) = g(L)$.*

Proof omitted. $\square$

Example 3.4.4. Let $f(x) = 1 + x\sin\left(\frac{\pi}{x}\right)$ on $\mathbb{R} \setminus \{0\}$ and

$$g(x) = \begin{cases} 4 & x \neq 1 \\ -4 & x = 1. \end{cases}$$

Then $\lim_{x\to 0} f(x) = 1$ and $\lim_{y\to 1} g(y) = 4$, BUT g is not continuous at $y_0 = 1$. Thus $\lim_{x\to 0} g \circ f(x)$ does not exist.

3.5 Power Series

This section introduces the notion of power series as a way to represent functions. This is an important connection between Chapter 2 and the study of calculus.

Definition. *A series of the form* $\sum_{n=0}^{\infty} a_n x^n$ *is called* **power series** *centered at* 0. *In general, a power series centered at* x_0 *has the form* $\sum_{n=0}^{\infty} a_n (x - x_0)^n$.

Theorem 3.5.1. *Let* $\beta = \limsup |a_n|^{\frac{1}{n}}$ *and let* $R = \frac{1}{\beta}$, *called the* **radius of convergence** *of the power series. Then the power series converges absolutely if* $|x| < R$ *and diverges if* $|x| > R$.

Proof. Let $x \in \mathbb{R}$. By the Root Test, the series converges if $\limsup |a_n x^n|^{\frac{1}{n}} < 1$, diverges if $\limsup |a_n x^n|^{\frac{1}{n}} > 1$ and is inconclusive otherwise. We compute

$$\begin{aligned} \limsup |a_n x^n|^{\frac{1}{n}} &= \limsup |a_n|^{\frac{1}{n}} |x^n|^{\frac{1}{n}} \\ &= \limsup |a_n|^{\frac{1}{n}} |x| \\ &= |x| \limsup |a_n|^{\frac{1}{n}} \qquad \text{since } |x| \text{ does not depend on } n \\ &= |x| \cdot \beta. \end{aligned}$$

The the above characterization by the Root Test says the series converges if $|x| \cdot \beta < 1 \iff |x| < \frac{1}{\beta}$ and diverges if $|x| \cdot \beta > 1 \iff |x| > \frac{1}{\beta}$. □

There are three cases for the radius of convergence of a power series:

- If $\beta = 0$ and $R = \infty$, then for all $x \in \mathbb{R}$, $|x| < R$ so the power series converges for all x.
- If β is a positive number, then the power series converges for all $|x| < R$ and diverges for all $|x| > R$. In this case we must check the endpoints $x = x_0 \pm R$ separately.
- If $\beta = \infty$ and $R = 0$, the power series only converges at x_0. Note that *every* power series at minimum converges at the point about which it is centered.

Examples.

1. For $\sum_{n=0}^{\infty} n^n x^n$, we compute $\limsup |n^n|^{\frac{1}{n}} = \limsup |n| = \infty$. So $\beta = \infty$ and $R = 0$, and thus the series only converges at $x_0 = 0$.

(2) Consider the series $\displaystyle\sum_{x=0}^{\infty} \frac{x^n}{n!}$; this is an example of a Taylor series for e^x (see Section 4.4). Here we have

$$\begin{aligned}
\beta &= \limsup |a_n|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \qquad \text{if this limit exists} \\
&= \lim_{n\to\infty} \left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \lim_{n\to\infty} \left|\frac{n!}{(n+1!)}\right| = \lim_{n\to\infty} \left|\frac{1}{n+1}\right| = 0.
\end{aligned}$$

Then $R = 0$ so the series converges for all x.

(3) A geometric series $\displaystyle\sum_{n=0}^{\infty} x^n$ converges for $|x| < 1$ and diverges otherwise. There's no need to check the endpoints because the geometric series test (Theorem 2.6.2) tells us it diverges at $|x| = 1$.

(4) Consider $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$. As above, we compute

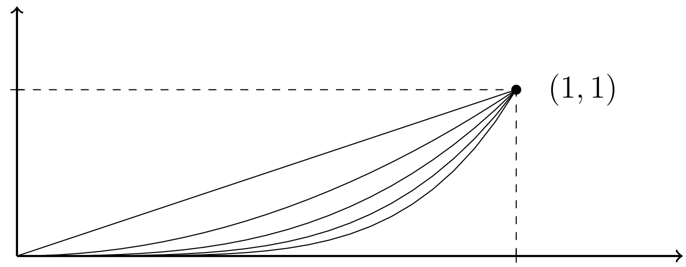
$$\begin{aligned}
\beta &= \lim \left|\frac{a_{n+1}}{a_n}\right| \qquad \text{if it exists} \\
&= \lim_{n\to\infty} \left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = 1.
\end{aligned}$$

Thus $R = 1$. Theorem 3.5.1 says that this series converges if $|x| < 1$ and diverges if $|x| > 1$, but what happens at $|x| = 1$? If $x = -1$, the series becomes the alternating harmonic series, which converges by the alternating series test (Theorem 2.8.1). On the other hand, if $x = 1$ then the series is the regular harmonic series, which we know diverges. Thus the *interval of convergence* of this series is $[-1, 1)$.

(5) Now consider $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$. By a similar argument as above, $R = 1$ so the series converges for $|x| < 1$ and diverges for $|x| > 1$. Here however, the series converges at $x = 1$ by p-series, and also still converges at $x = -1$ by the alternating series test (Theorem 2.8.1). Hence the interval of convergence is closed this time: $[-1, 1]$.

(6) The geometric series $\displaystyle\sum_{n=0}^{\infty} 2^{-n} x^{3n}$ converges for $|x|^3 < 2$, i.e. $|x| < \sqrt[3]{2}$ and diverges at the endpoints. Thus the interval of convergence is $\left(-\sqrt[3]{2}, \sqrt[3]{2}\right)$.

Why should a power series $\displaystyle\sum_{n=0}^{\infty} a_n (x - x_0)^n$ be continuous? The partial sums are polynomials, so they are all continuous. However, not all limits of continuous functions are continuous. Consider for example $f_n(x) = x^n$.



On $0 \leq x < 1$, $\lim x^n = 0$, but for $x \geq 1$, $\lim x^n > 0$. For example, at $x = 1$ the limit is 1 as seen in the picture above. It turns out that we need uniform convergence to make the limit of a sequence of continuous funtions continuous.

3.6 Uniform Convergence

Definition. *Let $S \subseteq \mathbb{R}$ and let f_n be a sequence of functions $S \to \mathbb{R}$. We say f_n* **converges pointwise** *to $f : S \to \mathbb{R}$ if for all $x \in S$, $\lim\limits_{n\to\infty} f_n(x) = f(x)$; that is, for all $x \in S$ and for all $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n > N$, $|f_n(x) - f(x)| < \varepsilon$.*

In the previous example with $f_n(x) = x^n$ on $S = [0, 1]$, we do have pointwise convergence. For $x < 1$, $f_n(x)$ is a geometric sequence with ratio < 1, so $f_n(x) \to 0$. If $x = 1$, then $f_n(x) = 1^n = 1$ so $f_n(1) \to 1$. Thus $f_n \to f$ pointwise, where

$$f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1. \end{cases}$$

Definition. *Let $f_n : S \to \mathbb{R}$ be a sequence of functions. Then f_n* **converges uniformly** *to f on S if for all $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n > N$ and for all $x \in S$, $|f_n(x) - f(x)| < \varepsilon$.*

The word *uniform* has the same meaning as in Section 3.3. It means that the 'rate' of convergence cannot depend on x.

Examples.

1. Does $f_n(x) = x^n$ converge uniformly on $[0, 1]$? First, if $x = 1$ then $|1-1| < \varepsilon \implies 0 < \varepsilon$ so any N works. Now if $x < 1$ then we have

$$\begin{aligned} & |x^n - 0| < \varepsilon \\ \implies & \quad |x|^n < \varepsilon \\ \implies & \log |x|^n < \log \varepsilon \\ \implies & \quad n > \frac{\log \varepsilon}{\log |x|}. \end{aligned}$$

But $\log |x| \to 0$ as $x \to 1$, so $N \to \infty$. So there's a bad spot at $x = 1$. Say we cut out $x = 1$. We claim that f_n is uniformly convergent to $f(x) = 0$ on $[0, a]$ for any $a < 1$.

Proof. Let $0 < a < 1$, $\varepsilon > 0$, $N > \frac{\log \varepsilon}{\log a}$, $n > N$ and $x \in [0, a]$. Then

$$|f_n(x) - f(x)| = |x^n - 0| = x^n \leq a^n < a^N < a^{\frac{\log \varepsilon}{\log a}} = a^{\log_a \varepsilon} = \varepsilon.$$

Hence $f_n \to 0$ on $[0, a]$. □

2. Does $g_n(x) = \frac{\cos(nx)}{n}$ converge uniformly on $\mathbb{R}$? As $n \to \infty$, $g_n(x) \to 0$ so let $g(x) = 0$. Since $cos(nx)$ is bounded, we have

$$\left| \frac{\cos(nx)}{n} - 0 \right| = \left| \frac{\cos(nx)}{n} \right| \leq \frac{1}{n}.$$

Then our choice of N should be bigger than $\frac{1}{\varepsilon}$.

Proof. Let $\varepsilon > 0$, $N > \frac{1}{\varepsilon}$, $n > N$ and $x \in \mathbb{R}$. Then

$$|g_n(x) - g(x)| = \left|\frac{\cos(nx)}{n} - 0\right| = \frac{|\cos(nx)|}{n} \leq \frac{1}{n} < \frac{1}{N} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon.$$

Hence g_n converges uniformly to 0 on $\mathbb{R}$. □

Note that our choice of N doesn't depend on x, and only on ε. This is how uniform convergence works in general.

The next theorem establishes that if a sequence of continuous functions converges uniformly, it must converge to a continuous function. The proof is a classic in analysis known as the "three epsilon proof".

Theorem 3.6.1. *The uniform limit of a sequence of continuous functions is continuous; that is, if f_n converges uniformly to some f on S and f_n is continuous at $x_0 \in S$ for all n, then f is also continuous at x_0.*

Proof. Let S, f_n, f and x_0 be as the theorem states. Let $\varepsilon > 0$. Since $f_n \to f$ uniformly, there exists an N such that for all $n > N$ and for all $x \in S$, $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$. Let $n > N$. Since f_n is continuous at x_0, there is some $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$. Let $|x - x_0| < \delta$. Then

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Therefore f is continuous at x_0. □

Remark. An equivalent way to express uniform convergence is if

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in S\} = 0.$$

Proof. First suppose $f_n \to f$ uniformly. Then given $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > N, x \in S$. We want to show that $|\sup\{|f_n(x) - f(x)|\} - 0| < \varepsilon$. By uniform continuity, there exists an N' such that for all $n > N$, $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$. Then for all $n > \max\{N, N'\}$ and $x \in S$, we have $\sup\{|f_n(x) - f(x)|\} \leq \frac{\varepsilon}{2} < \varepsilon$. Thus $\lim\limits_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in S\} = 0$.

Conversely, suppose $\lim\limits_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in S\} = 0$. Then given $\varepsilon > 0$, there is some N such that for all $n > N$, $|\sup\{|f_n(x) - f(x)| : x \in S\} - 0| < \varepsilon$. Moreover, for all $x \in S$ we have $|f_n(x) - f(x) \leq \sup\{|f_n(x) - f(x)|\} < \varepsilon$. Thus f_n converges to f uniformly. □

Examples.

3. Consider $h_n(x) = \frac{x}{n}$ on $[0, \infty)$. h_n converges pointwise to 0. However, observe that $\sup\{|h_n(x) - 0| : x \geq 0\} = +\infty$, so $\lim \sup\{|h_n(x)| : x \geq 0\} = +\infty$. Therefore $\frac{x}{n}$ does not converge to 0 uniformly.

4 Let $f_n(x) = \frac{x}{1+nx^2}$, which is continuous on $\mathbb{R}$. First, $f_n(x) \to 0$ as $n \to \infty$ for $x \neq 0$, and $f_n(0) = 0$ for all n, so f_n converges pointwise to $f(x) = 0$. Does it converge uniformly? First let's find the minimum and maximum values of $f_n(x)$:

$$f_n'(x) = \frac{1 + nx^2 - 2nx^2}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}.$$

Setting this equal to 0, we have $1 - nx^2 = 0 \implies x = \pm\frac{1}{\sqrt{n}}$. So $\sup\{|f_n(x) - f(x)|\} = f_n\left(\frac{1}{\sqrt{n}}\right)$ or $-f_n\left(-\frac{1}{\sqrt{n}}\right)$. Observe that

$$f_n\left(\frac{1}{\sqrt{n}}\right) = \frac{\frac{1}{\sqrt{n}}}{1 + n\left(\frac{1}{\sqrt{n}}\right)^2} = \frac{\frac{1}{\sqrt{n}}}{1+1} = \frac{1}{2\sqrt{n}} \longrightarrow 0$$

as $n \to \infty$. Then by the remark, f_n converges to 0 uniformly.

5 Consider $f_n(x) = n^2x^n(1-x)$ on $S = [0,1]$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2x^{n+1}(1-x)}{n^2x^n(1-x)}\right| = |x|\left|\frac{n+1}{n}\right|^2 \longrightarrow |x|$$

as $n \to \infty$. Thus f_n converges pointwise for $|x| < 1$. To check the endpoint $x = 1$, we have $f_n(1) = n^2 1^n(1-1) = 0$ so f_n converges pointwise to 0 on the interval $[0,1]$. Does $f_n \to 0$ uniformly? Again we calculate the maximum and minimum function values:

$$\begin{aligned} f_n'(x) &= n^3x^{n-1}(1-x) - n^2x^n = n^2x^{n-1}[n(1-x) - x] \\ &= n^2x^{n-1}(n - (n+1)x). \end{aligned}$$

Setting this equal to 0, we have $0 = n - (n+1)x$ and $0 = x^{n-1}$, so the critical points are $x = \frac{n}{n+1}$ and $x = 0$. Thus $\sup\{|f_n(x) - f(x)| : 0 \le x \le 1\} = f(0)$ or $f\left(\frac{n}{n+1}\right)$. But $f(0) = 0$ and

$$\begin{aligned} f\left(\frac{n}{n+1}\right) &= n^2\left(\frac{n}{n+1}\right)^n\left(1 - \frac{n}{n+1}\right) \\ &= n^2\left(\frac{n}{n+1}\right)^n\left(\frac{n+1-n}{n+1}\right) \\ &= \frac{n^{n+2}(1)}{(n+1)^{n+1}} = n\left(\frac{n}{n+1}\right)^{n+1} \end{aligned}$$

which tends to ∞ as $n \to \infty$. Thus f_n does not converge to 0 uniformly.

3.7 Applications of Uniform Convergence

In this section we develop some important results in analysis that deal with uniform convergence.

Theorem 3.7.1. *Suppose f_n is a sequence of continuous functions converging uniformly to f on a domain $S = [a, b]$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx \;=\; \int_a^b \left(\lim_{n\to\infty} f_n(x)\right)\, dx \;=\; \int_a^b f(x)\, dx.$$

To prove the theorem, we need the following facts about Riemann integrals:

(a) If $f(x) \le g(x)$ for all $x \in [a, b]$ then

$$\int_a^b f(x)\, dx \;\le\; \int_a^b g(x)\, dx.$$

(b) $\displaystyle \left| \int_a^b g(x)\, dx \right| \;\le\; \int_a^b |g(x)|\, dx.$

(c) All continuous functions are Riemann integrable.

Proof. First note that since $f_n \to f$ uniformly and f_n is a sequence of continuous of functions, Theorem 3.6.1 states that f is also continuous. In particular, (c) implies that f is Riemann integrable. Let $\varepsilon > 0$ be given. Then there is some $N \in \mathbb{N}$ such that for all $n > N$ and for all $x \in [a, b]$, $|f_n(x) - f(x)| < \frac{\varepsilon}{b-a}$. Using the facts above, we compute

$$\begin{aligned}
\left| \int_a^b f_n(x)\, dx - \int_a^b f(x)\, dx \right| &= \left| \int_a^b [f_n(x) - f(x)]\, dx \right| \\
&\le \int_a^b |f_n(x) - f(x)|\, dx \\
&\le \int_a^b \frac{\varepsilon}{b-a}\, dx = \frac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon.
\end{aligned}$$

Therefore $\displaystyle \lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$ □

Definition. *A sequence of functions f_n is* **uniformly Cauchy** *if for all $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n, m > N$ and for all $x \in S$, $|f_n(x) - f_m(x)| < \varepsilon$.*

The above definition comes from the Cauchy criterion for series: a series is uniformly Cauchy if for all $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that for all $n \ge m > N$, $\displaystyle \left| \sum_{k=m}^{n} a_k \right| < \varepsilon.$

Theorem 3.7.2. *Every uniformly Cauchy sequence is uniformly convergent.*

Proof. Let f_n be a sequence of continuous functions that is uniformly Cauchy on S. Fix $x \in S$. Then $(f_n(x))$ is a Cauchy sequence of real numbers, so by the Cauchy convergence theorem (2.3.8), $(f_n(x))$ converges. For each $x \in S$, define $f(x) = \lim_{n\to\infty} f_n(x)$. Clearly $f_n \to f$ pointwise. To show uniform convergence, let $\varepsilon > 0$. By uniformly Cauchy, there is some $N \in \mathbb{N}$ such that for all $n, m > N, x \in S$, $|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$. Fix $x \in S$ again and also fix $m > N$. Then for any $n > N$, $|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$, so $f_m(x) - \frac{\varepsilon}{2} < f_n(x) < f_m(x) + \frac{\varepsilon}{2}$. As $n \to \infty$, we have

$$\begin{aligned} f_m(x) - \tfrac{\varepsilon}{2} < f_n(x) < f_m(x) + \tfrac{\varepsilon}{2} &\implies f_m(x) + \tfrac{\varepsilon}{2} \leq f(x) \leq f_m(x) + \tfrac{\varepsilon}{2} \\ &\implies |f(x) - f_m(x)| \leq \tfrac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Since $x \in S$ and $m > N$ were arbitrary, this shows that $f_m \to f$ uniformly. □

Example 3.7.3. Power series are uniformly convergent on their interval of convergence. For instance, consider $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{1+x^n}$ on the interval $(-1, 1)$. First suppose $0 < a < 1$; we will show the series is uniformly convergent on $[0, a]$. By the triangle inequality, we have

$$\left|\sum_{k=m}^{n} \frac{x^k}{1+x^k}\right| \leq \sum_{k=m}^{n} \frac{|x|^k}{|1+x^k|} \leq \sum_{k=m}^{n} \frac{a^k}{1} = \frac{a^m - a^n}{1-a} < \frac{a^m}{1-a} \longrightarrow 0 < \varepsilon.$$

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\dfrac{a^N}{1-a} < \varepsilon$ (which is possible since this tends to 0). Let $n \geq m > N$. By the work above,

$$\left|\sum_{k=m}^{n} \frac{x^k}{1+x^k}\right| < \frac{a^m}{1-a} < \frac{a^N}{1-a} < \varepsilon.$$

Therefore the series is uniformly Cauchy on $[0, a]$. By Theorem 3.7.2, $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{1+x^n}$ represents a uniformly convergent sequence for all $x \in [0, a]$. Moreover, a was chosen arbitrarily between 0 and 1, so it follows that $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{1+x^n}$ is a continuous function on $[0, 1)$. The argument can be repeated for $-1 < b < 0$ to conclude that it's continuous on all of $(-1, 1)$.

Theorem 3.7.4 (Weierstrass M-Test)**.** *Suppose (M_k) is a nonnegative sequence of numbers such that $\displaystyle\sum_{k=0}^{\infty} M_k$ converges. Then if g_k is a sequence of continuous functions on some domain S such that $|g_k(x)| \leq M_k$ for all $x \in S$, then $\displaystyle\sum_{k=0}^{\infty} g_k(x)$ converges uniformly on S, and the limit function represented by this series is continuous on S.*

The Weierstrass M-test is sort of like a "uniform comparison test".

Proof. Let $\varepsilon > 0$. By the Cauchy criterion, there is an $N \in \mathbb{N}$ such that for all $n \geq m > N$, $\left|\sum_{k=m}^{n} M_k\right| < \varepsilon$. This implies that for all $n \geq m > N$,

$$\left|\sum_{k=m}^{n} g_k(x)\right| \;\leq\; \sum_{k=m}^{n} |g_k(x)| \;\leq\; \sum_{k=m}^{n} M_k \;<\; \varepsilon.$$

Thus $\sum_{k=0}^{\infty} g_k(x)$ satisfies the uniform Cauchy criterion, so it is uniformly convergent. $\square$

Example 3.7.5. Consider the geometric series $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$. The radius of convergence is 2 and the interval of convergence is $(-2, 2)$. Pick some $a < 2$ and look at $S = [-a, a]$. For the sequence $M_k = \left(\frac{a}{2}\right)^k$, we have

$$\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} \left(\frac{a}{2}\right)^k = \frac{1}{1 - \frac{a}{2}} < +\infty.$$

Also, for the sequence of functions $g_k(x) = \left(\frac{x}{2}\right)^k$,

$$|g_k(x)| \leq \frac{|x|^k}{2^k} \leq \frac{a^k}{2^k} = \left(\frac{a}{2}\right)^k = M_k$$

so $g_k(x) \leq M_k$ for all $x \in [-a, a]$. By the Weierstrass M-test, $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$ converges uniformly on $[-a, a]$, and furthermore the series converges to a continuous function on $(-2, 2)$.

Proposition 3.7.6. *If* $\sum_{k=0}^{\infty} g_k(x)$ *converges uniformly on* S *then* $\lim_{k\to\infty} \sup\{|g_k(x)| : x \in S\} = 0$.

Proof. Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ such that for all $n \geq m > N$, $\left|\sum_{k=m}^{n} g_k(x)\right| < \varepsilon$. In particular, for $n = m$, $\left|\sum_{k=m}^{m} g_k(x)\right| < \varepsilon$. Then for all $m > N$, $|g_m(x)| < \varepsilon$ for all $x \in S$. Hence $\limsup g_m(x) = 0$. $\square$

Chapter 4

Calculus

4.1 Differentiation and Integration of Power Series

Lemma 4.1.1. *Suppose* $\sum_{n=0}^{\infty} a_n x^n$ *has a positive radius of convergence* R *and suppose* $0 < R_1 < R$. *Then* $\sum_{n=0}^{\infty} a_n x^n$ *converges uniformly on* $[-R_1, R_1]$.

Proof. Since R_1 is inside the radius of convergence, $\sum_{n=0}^{\infty} a_n R_1^n < +\infty$. So for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq m > N$, $\left|\sum_{k=m}^{n} a_k R_1^k\right| < \varepsilon$. In fact, $\sum_{n=0}^{\infty} a_n R_1^n$ converges absolutely so $\sum_{k=m}^{n} |a_k| R_1^k < \varepsilon$. Now for all $x \in [-R_1, R_1]$,

$$\begin{aligned}
\left|\sum_{k=m}^{n} a_k x^k\right| &\leq \sum_{k=m}^{n} |a_k|\,|x|^k && \text{by triangle inequality} \\
&\leq \sum_{k=m}^{n} |a_k| R_1^k && \text{since } |x| \leq R_1 \\
&< \varepsilon && \text{by the above.}
\end{aligned}$$

Hence $\sum_{n=0}^{\infty} a_n x^n$ satisfies the Cauchy criterion, so it converges uniformly on $[-R_1, R_1]$. □

Corollary 4.1.2. $\sum_{n=0}^{\infty} a_n x^n$ *converges to a continuous function on* $(-R, R)$.

Lemma 4.1.3. *The* **derivative** $\sum_{n=1}^{\infty} n a_n x^{n-1}$ *and the* **integral** $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ *have the same radius of convergence as the power series* $\sum_{n=0}^{\infty} a_n x^n$.

Proof. Define the following values:

$$\begin{aligned}
\alpha &= \limsup |a_n|^{1/n} \\
\beta &= \limsup |n a_n|^{1/n} \\
\gamma &= \limsup \left|\frac{a_n}{n+1}\right|^{1/n}.
\end{aligned}$$

By limit properties (Section 2.2),

$$\beta = \limsup |n a_n|^{1/n} = \limsup |n|^{1/n} \cdot \limsup |a_n|^{1/n} = 1 \cdot \limsup |a_n|^{1/n} = \alpha.$$

On the other hand, we have

$$\gamma = \limsup \left|\frac{a_n}{n+1}\right|^{1/n} = \limsup |a_n|^{1/n} \cdot \limsup \left|\frac{1}{n+1}\right|^{1/n} = \limsup |a_n|^{1/n} \cdot 1 = \alpha.$$

Thus $\alpha = \beta = \gamma$. Notice that the radii of convergence for the series, the derivative and the integral are $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$, respectively, so these are all equal as claimed. □

Theorem 4.1.4. *Suppose* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *on some radius of convergence* $R > 0$*. Then*

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

for all $|x| < R$*.*

Proof. Without loss of generality assume $-R < x < 0$ (the case when $x = 0$ is trivial). By Lemma 4.1.1, $\sum_{n=0}^{\infty} a_n t^n$ converges uniformly to $f(t)$ on $[-x, 0]$, so we may switch the order of the integral and summation in the following steps:

$$\begin{aligned}
\int_x^0 \lim_{N\to\infty} \sum_{n=0}^{N} a_n t^n\,dt &= \lim_{N\to\infty} \int_x^0 \sum_{n=0}^{N} a_n t^n\,dt \\
\implies \int_x^0 f(t)\,dt &= \lim_{N\to\infty} \sum_{n=0}^{N} \int_x^0 a_n t^n\,dt \\
&= \lim_{N\to\infty} \sum_{n=0}^{N} \left[a_n \frac{x^{n+1}}{n+1}\right]_x^0 \\
&= \lim_{N\to\infty} \sum_{n=0}^{N} -a_n \frac{x^{n+1}}{n+1}.
\end{aligned}$$

Then we have $\int_0^x f(t)\,dt = -\lim_{N\to\infty} -a_n \frac{x^{n+1}}{n+1} = \lim_{N\to\infty} a_n \frac{x^{n+1}}{n+1}$ as claimed. □

Theorem 4.1.5. *Suppose* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *on some radius of convergence* $R > 0$*. Then* f *is* **differentiable** *(see Section 4.2) on* $(-R, R)$ *with derivative*

$$f'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

for all $|x| < R$*.*

Proof. Let $g(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$; we will show that g is the derivative of f on $(-R, R)$. By Lemma 4.1.3, g has the same radius of convergence R as the series for f, and by Theorem 4.1.4,

$$\int_0^x g(t)\,dt = \sum_{n=1}^{\infty} a_n n \frac{x^n}{n} = \sum_{n=1}^{\infty} a_n x^n.$$

For $0 < R_1 < R$, there is some k such that

$$\int_{-R_1}^{x} g(t)\,dt + k = f(x).$$

Then by the Fundamental Theorem of Calculus, $f'(x)$ exists and $f'(x) = g(x)$ on $[-R_1, R_1]$ for all $R_1 < R$. Therefore f' exists and equals g on $(-R, R)$ and the theorem is proved. $\square$

Example 4.1.6. Consider $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ on $(-1, 1)$. By Theorem 4.1.4,

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \int \frac{1}{1-x}\,dx = -\log(1-x)$$

on $(-1, 1)$. Note that if $x = -1$, $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} = -\log(2)$ even though -1 is an endpoint of the series' interval of convergence.

Example 4.1.6 suggests the following theorem, which is a famous result due to Abel.

Theorem 4.1.7 (Abel's Theorem)**.** *If* $\sum_{n=0}^{\infty} a_n x^n$ *has radius of convergence* R *and converges at the endpoint* $x = R$ *(resp.* $x = -R$*), then the series is continuous at* $x = R$ *(resp.* $x = -R$*).*

Proof omitted. $\square$

4.2 The Derivative

Definition. *Suppose $f : S \to \mathbb{R}$ where S is an open interval containing a. Then f is said to be* **differentiable** *at a if $\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists (is finite). If so, we define the* **derivative** *of f at a in two ways:*

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Examples.

1. For $f(x) = x^2$, fix $a \in \mathbb{R}$. Then we compute the derivative of f at a to be

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^2-a^2}{x-a} = \lim_{x\to a}(x+a) = 2a.$$

2. Likewise, let $g(x) = \sqrt{x}$ and for a fixed $a > 0$, compute

$$\begin{aligned}
\lim_{h\to 0} \frac{g(a+h)-g(a)}{h} &= \lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h} \cdot \frac{\sqrt{a+h}+\sqrt{a}}{\sqrt{a+h}+\sqrt{a}} \\
&= \lim_{h\to 0} \frac{a+h-a}{h\left(\sqrt{a+h}+\sqrt{a}\right)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{a+h}+\sqrt{a}} = \frac{1}{2\sqrt{a}}.
\end{aligned}$$

This also shows that the domain of g' is $\{a > 0\}$.

3. Let $h(x) = x^n$ for some $n \in \mathbb{N}$ and fix $a \in \mathbb{R}$. Then we compute

$$\begin{aligned}
\lim_{k\to 0} \frac{h(a+k)-h(a)}{k} &= \lim_{k\to 0} \frac{(a+k)^n - a^n}{k} \\
&= \lim_{k\to 0} \frac{\left(\sum_{i=0}^{n} \binom{n}{i} a^i k^{n-i}\right) - a^n}{k} \\
&= \lim_{k\to 0} \frac{1}{k} \sum_{i=0}^{n-1} \binom{n}{i} a^i k^{n-i} = \lim_{k\to 0} \sum_{i=0}^{n-1} \binom{n}{i} a^i k^{n-i-1} \\
&= \binom{n}{n-1} a^{n-1} + 0 + 0 + 0 + \ldots = na^{n-1}.
\end{aligned}$$

Theorem 4.2.1. *Suppose $f : S \to \mathbb{R}$ is differentiable at some $a \in S$. Then f is continuous at a.*

Proof. Suppose f is differentiable at a. Then $\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists and is finite. Note that $\lim\limits_{x\to a}(x-a) = 0$. Then we have

$$\lim_{x\to a} \left(\frac{f(x)-f(a)}{x-a} \cdot (x-a) \right) = \lim_{x\to a} \left(\frac{f(x)-f(a)}{x-a} \right) \lim_{x\to a}(x-a) = f'(a) \cdot 0 = 0.$$

This implies $\lim_{x\to a}(f(x)-f(a)) = 0 \implies \lim_{x\to a} f(x) = f(a)$, so f is continuous at a. $\square$

Theorem 4.2.2 (Properties of Derivatives)**.** *Suppose $c \in \mathbb{R}$, and $f, g : S \to \mathbb{R}$ are functions that are differentiable at $a \in S$. Then*

(1) cf is differentiable at a with $(cf)'(a) = cf'(a)$.

(2) $f+g$ is differentiable at a with $(f+g)'(a) = f'(a) + g'(a)$.

(3) fg is differentiable at a with $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$.

(4) If $g(a) \neq 0$ then $\dfrac{f}{g}$ is differentiable at a with $\left(\dfrac{f}{g}\right)'(a) = \dfrac{g(a)f'(a) - f(a)g'(a)}{g(a)^2}$.

Proof. (1) Consider

$$\lim_{x\to a} \frac{cf(x) - cf(a)}{x-a} = \lim_{x\to a} c\frac{f(x)-f(a)}{x-a} = c\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = cf'(a).$$

Thus $(cf)'(a) = cf'(a)$.

(2) By limit properties (Section 2.2),

$$\begin{aligned}
\lim_{x\to a}\frac{(f+g)(x)-(f+g)(a)}{x-a} &= \lim_{x\to a}\frac{f(x)+g(x)-f(a)-g(a)}{x-a} \\
&= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} + \lim_{x\to a}\frac{g(x)-g(a)}{x-a} \\
&= f'(a) + g'(a).
\end{aligned}$$

(3) Consider

$$\begin{aligned}
(fg)'(a) &= \lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a} \\
&= \lim_{x\to a}\frac{f(x)g(x)-f(x)g(a)+f(x)g(a)-f(a)g(a)}{x-a} \\
&= \lim_{x\to a}\left[\frac{f(x)(g(x)-g(a))+g(a)(f(x)-f(a))}{x-a}\right] \\
&= \lim_{x\to a} f(x)\,\frac{g(x)-g(a)}{x-a} + \lim_{x\to a} g(a)\,\frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} f(x)\,\lim_{x\to a}\frac{g(x)-g(a)}{x-a} + \lim_{x\to a} g(a)\,\lim_{x\to a}\frac{f(x)-f(a)}{x-a}.
\end{aligned}$$

Since f is differentiable at a, Theorem 4.2.1 says that f is continuous at a, so $\lim_{x\to a} f(x) = f(a)$. Thus $(fg)'(a) = f(a)g'(a) + g(a)f'(a)$.

(4) Finally, consider

$$\begin{aligned}
\left(\frac{f}{g}\right)'(a) &= \lim_{x\to a}\frac{\frac{f(x)}{g(x)}-\frac{f(a)}{g(a)}}{x-a} = \lim_{x\to a}\frac{f(x)g(a)-f(a)g(x)}{(x-a)g(x)g(a)} \\
&= \lim_{x\to a}\frac{f(x)g(a)-f(a)g(a)+f(a)g(a)-f(a)g(x)}{(x-a)g(x)g(a)} \\
&= \lim_{x\to a}\frac{1}{g(x)}\cdot\lim_{x\to a}\frac{1}{g(a)}\cdot\left[\lim_{x\to a} g(a)\,\frac{f(x)-f(a)}{x-a}-\lim_{x\to a} f(a)\,\frac{g(x)-g(a)}{x-a}\right] \\
&= \frac{1}{[g(a)]^2}\left(g(a)f'(a)-f(a)g'(a)\right) \\
&= \frac{g(a)f'(a)-f(a)g'(a)}{[g(a)]^2}.
\end{aligned}$$

□

Theorem 4.2.3 (Chain Rule)**.** *If $f : S \to \mathbb{R}$ is differentiable at $x_0 \in S$ and $g : T \to \mathbb{R}$ is differentiable at $f(x_0) \in T$, then $g \circ f : S \to \mathbb{R}$ is differentiable at x_0 with derivative*

$$(g\circ f)'(x_0) = g'(f(x_0))\cdot f'(x_0).$$

Proof. The trick with this proof is that we can expand the limit:

$$\lim_{x\to x_0}\frac{g(f(x))-g(f(x_0))}{x-x_0} = \left(\lim_{x\to x_0}\frac{g(f(x))-g(f(x_0))}{f(x)-f(x_0)}\right)\cdot\left(\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\right)$$

but we are not guaranteed to have $f(x) \neq f(x_0)$. To remedy this, define

$$h(y) = \begin{cases} \dfrac{g(y)-g(f(x_0))}{y-f(x_0)} & y \neq f(x_0) \\ g'(y) & y = f(x_0). \end{cases}$$

Since g is differentiable at $f(x_0)$, h is well-defined. In order for h to be continuous at $f(x_0)$, we must have

$$\lim_{y\to f(x_0)} h(y) = \lim_{y\to f(x_0)}\frac{g(y)-g(f(x_0))}{y-f(x_0)} = h(f(x_0))$$

but this is true by continuity of g at $f(x_0)$. This further implies that the above limit equals $g'(f(x_0))$ by definition of h. Thus for each $y \neq f(x_0)$, $h(y)(y-f(x_0)) = g(y)-g(f(x_0))$, and for $y = f(x_0)$ the equation becomes $0 = 0$. So the equation holds for all y, and in particular $h(f(x))(f(x)-f(x_0)) = g(f(x))-g(f(x_0))$. Now we compute

$$\begin{aligned}
\lim_{x\to x_0}\frac{h(f(x))(f(x)-f(x_0))}{x-x_0} &= \lim_{x\to x_0}\frac{g(f(x))-g(f(x_0))}{x-x_0} \\
\implies \lim_{x\to x_0} h(f(x))\,\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} &= \lim_{x\to x_0}\frac{g(f(x))-g(f(x_0))}{x-x_0} \\
\implies h(f(x_0))f'(x_0) &= g'(f(x_0)).
\end{aligned}$$

Since h is continuous at x_0, $g(f(x_0)) \cdot f'(x_0) = (g \circ f)'(x_0)$, proving the Chain Rule. □

Examples.

4. Consider $f(x) = \sin(x^2)$. By the Chain Rule, $f'(x) = 2x\cos(x^2)$.

5. For $g(x) = \sin^2(x)$, the Chain Rule tells us that the derivative is $g'(x) = 2\sin(x)\cos(x)$.

4.3 The Mean Value Theorem

In this section we prove one of the "Main Theorems" in calculus, the Mean Value Theorem. The route we take is standard; we first prove the special case called Rolle's Theorem and then use it to prove the general result. Later in the section we prove several corollaries, including two other "Main Theorems".

Theorem 4.3.1. *Suppose that for some $x_0 \in (a, b)$, $f(x_0)$ is either a maximum or minimum value of f on (a, b). If f is differentiable at x_0 then $f'(x_0) = 0$.*

Proof. We will prove the maximum case and remark that the minimum case is identical up to symmetry. First suppose $f'(x_0) > 0$. Then

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} > 0$$

so there is some $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies $\dfrac{f(x) - f(x_0)}{x - x_0} > 0$. So if $x - x_0 > 0$ we have $f(x) - f(x_0) > 0$, i.e. $f(x) > f(x_0)$, contradiction the maximality of $f(x_0)$. Similarly, if $f'(x_0) < 0$ then

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} < 0.$$

In this case, there is some $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies $\dfrac{f(x) - f(x_0)}{x - x_0} < 0$. So if $x - x_0 < 0$ then $f(x) - f(x_0) > 0 \implies f(x) > f(x_0)$. Both cases produce contradictions, so $f'(x_0)$ must be 0. $\square$

Theorem 4.3.2 (Rolle's Theorem)**.** *Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then if $f(a) = f(b)$, there is some $x_0 \in (a, b)$ such that $f'(x_0) = 0$.*

Proof. By the extreme value theorem (3.2.1), there exist $x_0, y_0 \in [a, b]$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a, b]$. Case 1 is if $f(a) = f(b) = f(x_0) = f(y_0)$. In this case, f is constant, so $f'(x) = 0$ for the whole interval and any point between a and b will suffice.

Case 2 is when one of $f(x_0), f(y_0)$ is not equal to $f(a) = f(b)$. Then either $x_0 \in (a, b)$ or $y_0 \in (a, b)$. Without loss of generality, suppose it's x_0. By Theorem 4.3.1, $f'(x_0) = 0$. This proves Rolle's Theorem in all cases. $\square$

Theorem 4.3.3 (Mean Value Theorem)**.** *Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists some $x_0 \in (a, b)$ such that $f'(x_0) = \dfrac{f(b) - f(a)}{b - a}$.*

Proof. Define

$$g(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a} (x - a) + f(a) \right).$$

(The part in parentheses is the secant line from a to b.) Then $g(a) = g(b) = 0$, so by Rolle's Theorem, there is some $x_0 \in (a, b)$ such that $g'(x_0) = 0$. But we calculate that

$$g'(x_0) = f'(x_0) - \left(\frac{f(b) - f(a)}{b - a} \cdot 1 + 0 \right)$$

which shows that $f'(x_0) = \dfrac{f(b) - f(a)}{b - a}$ as claimed. $\square$

Corollary 4.3.4. *Suppose f is differentiable on (a, b) and $f'(x) = 0$ for all $x \in (a, b)$. Then there is some $c \in \mathbb{R}$ such that $f(x) = c$ for all $x \in (a, b)$, i.e. f is a constant function on the interval (a, b).*

Proof. Suppose there exist $y, z \in (a, b)$, $y < z$, such that $f(y) \neq f(z)$. Without loss of generality, say $f(y) < f(z)$. By the mean value theorem, there exists an $x \in (a, b)$ such that $y < x < z$ and $f'(x) = \dfrac{f(z) - f(y)}{z - y} > 0$, contradicting our assumption on f'. Therefore $f(x)$ is constant across (a, b). $\square$

Corollary 4.3.5. *Suppose f and g are differentiable on (a, b) and $f'(x) = g'(x)$ for all $x \in (a, b)$. Then there is some $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for all $x \in (a, b)$, i.e. f and g differ by a constant on the whole interval.*

Proof. Let $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) = 0$ by hypothesis, so by Corollary 4.3.4, $h(x) = c$ for all $x \in (a, b)$ and for some constant c. Therefore $f(x) = g(x) + c$. $\square$

Corollary 4.3.6. *Let f be a continuous function on some interval (a, b).*

(1) f' strictly positive $\implies$ f strictly increasing.

(2) f' nonnegative $\implies$ f nondecreasing.

(3) f' strictly negative $\implies$ f strictly decreasing.

(4) f' nonpositive $\implies$ f nonincreasing.

Proof. We will prove (1) and leave the similar proofs for (2) – (4) for exercise. Suppose f is not strictly increasing on (a, b). Then there exist $x, y \in (a, b)$ such that $x < y$ but $f(x) \geq f(y)$. By the mean value theorem, there is some $z \in (a, b)$ such that $x < z < y$ and $f'(z) = \dfrac{f(y) - f(x)}{y - x} \leq 0$. By contrapositive, f' strictly positive implies that f is strictly increasing on (a, b). $\square$

Theorem 4.3.7 (Intermediate Value Theorem for Derivatives)**.** *Suppose f is differentiable on (a, b) and $a < x_1 < x_2 < b$ with $f'(x_1) < f'(x_2)$. Then for all c between $f'(x_1)$ and $f'(x_2)$ there is some $x_0 \in (x_1, x_2)$ such that $f'(x_0) = c$.*

Note that the intermediate value property holds for derivatives *even if* they are not continuous on the interval. This is sometimes known as Darboux's Theorem.

Proof. Let $g(x) = f(x) - cx$ for any c between $f'(x_1)$ and $f'(x_2)$. Then $g'(x_1) = f'(x_1) - c < 0$ and $g'(x_2) = f'(x_2) - c > 0$. This means that on the interval $[x_1, x_2]$, g must have an absolute minimum, but by Theorem 4.3.1, x_1 and x_2 cannot be minima. Thus for some $x_0 \in (x_1, x_2)$, $f(x_0)$ is a minimum. Then we have $g'(x_0) = f'(x_0) - c = 0$ and hence $f'(x_0) = c$. $\square$

Theorem 4.3.8 (Inverse Function Theorem)**.** *Suppose f is one-to-one and continuous on an interval I. Let $J = f(I)$. If f is differentiable at some $x_0 \in I$ and $f'(x_0) \neq 0$, then the inverse function f^{-1} is differentiable at $y_0 = f(x_0)$ and $(f^{-1})'(y_0) = \dfrac{1}{f'(x_0)}$.*

Proof. First note that if I is an open interval, so is J by the intermediate value theorem (3.2.2). Then $y_0 = f(x_0) \in J$ so $f^{-1}(y_0)$ is well-defined. Moreover,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \implies \frac{1}{f'(x_0)} = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)}.$$

Let $y = f(x)$. Then we can write the above as

$$\frac{1}{f'(x_0)} = \lim_{x \to x_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0}.$$

This is almost the derivative of f^{-1}. By our previous work proving the intermediate value theorem (3.2.2), $f^{-1}(y)$ is continuous at y_0, so for all $\delta > 0$ there exists an $\eta > 0$ such that $0 < |y - y_0| < \eta$ implies $0 < |x - x_0| < \delta$. Suppose $|y - y_0| < \eta$. Then $|x - x_0| < \delta$ implies by the above that

$$\left| \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} - \frac{1}{f'(x_0)} \right| < \varepsilon$$

for any $\varepsilon > 0$ we choose. Therefore

$$\frac{1}{f'(x_0)} = \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = (f^{-1})'(y_0)$$

and the theorem is proved. □

Example 4.3.9. Find $\frac{d}{dx}\left(x^{1/n}\right)$.

To compute this derivative (without using the power rule on this function), let $f^{-1}(x) = x^{1/n}$, so that $f(x) = x^n$. Now the power rule becomes useful: $f'(x) = nx^{n-1}$ as we know. Then the inverse function theorem says that

$$\frac{d}{dx}\left(x^{1/n}\right) = (f^{-1})'(x) = \frac{1}{n\left(x^{1/n}\right)^{n-1}} = \frac{1}{nx^{1/n-1}}.$$

Example 4.3.10. Calculate $\frac{d}{dx}(\arcsin(x))$.

The inverse function theorem is useful for calculating derivatives of inverse trig functions. Let $f^{-1}(x) = \arcsin(x)$, so that $f(x) = \sin(x)$ and $f'(x) = \cos(x)$. Then

$$\begin{aligned}
\frac{d}{dx}(\arcsin(x)) = (f^{-1})'(x) &= \frac{1}{\cos(\sin^{-1}(x))} \\
&= \frac{1}{\sqrt{\cos 2(\sin^{-1}(x))}} \\
&= \frac{1}{\sqrt{1 - [\sin(\sin^{-1}(x))]^2}} \\
&= \frac{1}{\sqrt{1 - x^2}}.
\end{aligned}$$

4.4 Taylor's Theorem

The notes in this section are taken from a presentation I gave for Dr. Sarah Raynor's analysis course in Fall 2012, using Ross as a primary reference. The section covers Taylor series, Taylor's Theorem and a corollary to the theorem.

Recall the derivative theorem for power series (Theorem 4.1.5): if $f(x) = \sum_{k=0}^{\infty} a_k x^k$ with radius of convergence $R > 0$, the derivative of f is given by

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

and has the same radius of convergence as $f(x)$. By the same theorem, we can calculate the second derivative:

$$f''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

We obtain the nth derivative of $f(x)$ with the following formula:

$$f^{(n)}(x) = \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} a_k x^{k-n}.$$

Notice that if we plug in $x = 0$, we have $f^{(n)}(0) = n!\, a_n$ — this holds even for $n = 0$, since $f^{(0)} = f$ and $0! = 1$. This allows us to write the **Taylor series** expansion of f about 0:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \quad \text{for all } x \text{ such that } |x| < R.$$

Notice that for this definition to make sense, f must be defined on some open interval containing 0 and have derivatives of all orders defined at 0.

Next, define the nth remainder with respect to f by:

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k.$$

Then $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ if and only if $\lim_{n\to\infty} R_n(x) = 0$. We are now prepared to prove the main theorem on Taylor series.

Theorem 4.4.1 (Taylor's Theorem)**.** *Let f be a function on (a, b) where $a < 0 < b$ and suppose $f^{(n)}$ exists on (a, b). Then for all $x \neq 0$ between a and b, there is some y between 0 and x such that*

$$R_n(x) = \frac{f^{(n)}(y)}{n!} x^n.$$

Proof. Fix $x \neq 0$ in (a, b) and $n \in \mathbb{N}$. Then

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + R_n(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{Mx^n}{n!}$$

for a unique M. We will show that $f^{(n)}(y) = M$ for some y between 0 and x. Consider the function

$$g(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k + \frac{Mt^n}{n!} - f(t).$$

Then $g(0) = 0$ and for $k < n$, $g^{(k)}(0) = 0$. Since M is unique, this means that $g(x) = 0$ as well. Now we can apply Rolle's Theorem (4.3.2) to find some x_1 between 0 and x such that $g'(x_1) = 0$. And since $g'(0) = 0$, we can choose some x_2 between 0 and x_1 such that $g''(x_2) = 0$. Continuing in this way, we find x_n between 0 and x_{n-1} such that $g^{(n)}(x_n) = 0$. Note that for all $t \in (a, b)$, $g^{(n)}(t) = M - f^n(t)$. Then $M = f^{(n)}(x_n)$. Finally by letting $y = x_n$, we have

$$R_n(x) = \frac{Mx^n}{n!} = \frac{f^{(n)}(y)}{n!} x^n$$

which proves Taylor's Theorem. □

Corollary 4.4.2. *Let f be defined on (a, b) where $a < 0 < b$. If $f^{(n)}$ is defined on (a, b) for all n and $f^{(n)}$ is bounded by a single constant C, then $R_n(x) \to 0$ for all $x \in (a, b)$. In other words, if all derivatives are bounded on (a, b) then $f(x)$ equals its Taylor series on the interval.*

Proof. Let $x \in (a, b)$. By Taylor's Theorem,

$$|R_n(x)| = \frac{|f^{(n)}(y)|}{n!} |x|^n.$$

Since $f^{(n)}$ is bounded, we have

$$|R_n(x)| \leq \frac{C}{n!} |x|^n.$$

Therefore $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$ implies $\lim_{n\to\infty} R_n(x) = 0$. □

Example 4.4.3. The Taylor series for $f(x) = e^x$ about 0 is $\sum_{k=0}^{\infty} \frac{x^k}{k!}$. Recall from calculus that the nth derivative of e^x is e^x for all n. Then on the interval $(-M, M) \subset \mathbb{R}$ for some fixed, positive M, f is bounded by e^M. Then Corollary 4.4.2 implies that the radius of convergence for the Taylor series of e^x is ∞, i.e. e^x is represented by its Taylor series expansion on all of $\mathbb{R}$.

Example 4.4.4. Let $g(x) = \sin(x)$. We calculate

$$g^{(n)}(x) = \begin{cases} \sin(x) & n \equiv 0 \pmod 4 \\ \cos(x) & n \equiv 1 \pmod 4 \\ -\sin(x) & n \equiv 2 \pmod 4 \\ -\cos(x) & n \equiv 3 \pmod 4. \end{cases}$$

So for all n, $|g^{(n)}(x)| \leq 1$. Thus Corollary 4.4.2 says that $\sin(x)$ is represented by its Taylor series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1}$ on all of $\mathbb{R}$.

An alternate version of Taylor's Theorem is given here.

Theorem 4.4.5. *Given f defined on (a,b) with $a < 0 < b$ and continuous derivatives $f^{(n)}$ on the whole interval, there is some $x \in (a,b)$ such that*

$$R_n(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\, f^{(n)}(t)\, dt.$$

Proof. Shown using induction and integration by parts. □

4.5 The Integral

In this section we forego a routine discussion of the standard Riemann integral, which can be found in any undergraduate calculus text (cf. Stewart). Instead, we describe the Darboux integral, which turns out to be equivalent to the Riemann integral.

Definition. *Suppose f is bounded on $[a, b]$. Then for a subset $S \subseteq [a, b]$, we define*

$$M(f, S) = \sup\{f(x) \mid x \in S\}$$
$$m(f, S) = \inf\{f(x) \mid x \in S\}.$$

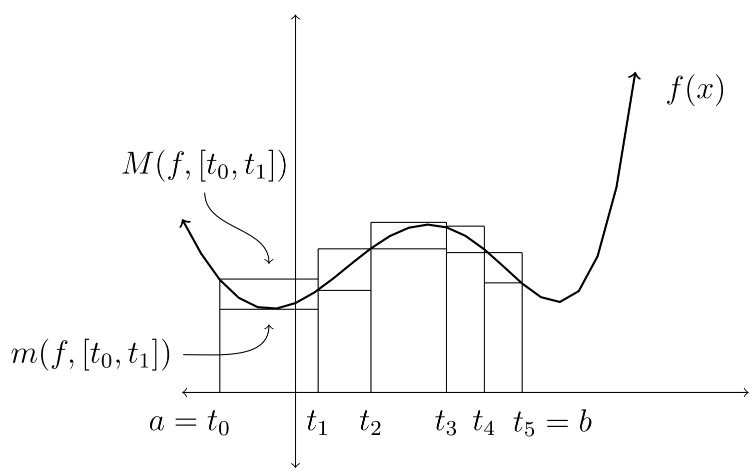
Definition. *A* **partition** *of $[a, b]$ is a finite sequence of numbers $\{t_0, t_1, \ldots, t_n\}$ such that $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$. Partition may also refer to the collection of intervals $[t_i, t_{i+1}]$ that compose the full interval $[a, b]$.*

Definition. *For any partition P of $[a, b]$, the* **upper Darboux sum** *of f with respect to P is given by*

$$U(f, P) = \sum_{i=1}^{n} M(f, [t_{i-1}, t_i]) \cdot (t_i - t_{i-1})$$

and the **lower Darboux sum** *is*

$$L(f, P) = \sum_{i=1}^{n} m(f, [t_{i-1}, t_i]) \cdot (t_i - t_{i-1}).$$



Notice that for any partition P,

$$m(f, [a, b]) \cdot (b - a) \leq L(f, P) \leq U(f, P) \leq M(f, [a, b]) \cdot (b - a)$$

so the set $\{L(f, P) \mid P \text{ is a partition of } [a, b]\}$ is bounded, and it has a sup which is finite. Likewise, the set of $U(f, P)$ has an inf.

Definition. *The* **lower Darboux integral** *and* **upper Darboux integral** *are defined as*

$$L(f) = \sup\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$$
$$U(f) = \inf\{U(f,P) \mid P \text{ is a partition of } [a,b]\}.$$

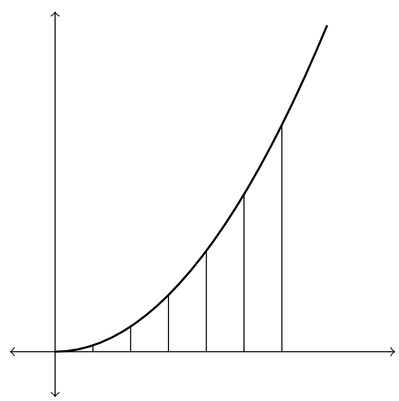
If $L(f) = U(f)$ then we say f is **Darboux integrable**. *In this case, the* **Darboux integral** *of f on $[a,b]$ is given by*

$$\int_a^b f(x)\,dx = L(f) = U(f).$$

Proposition 4.5.1. *The Darboux and Riemann integrals are equivalent.*

Proof omitted. □

Example 4.5.2. Consider $f(x) = x^2$ on $[0,1]$.



Let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition. Then

$$U(f,P) = \sum_{i=1}^{n} M(f,[t_{i-1},t_i]) \cdot (t_i - t_{i-1}) = \sum_{i=1}^{n} (t_i)^2 (t_i - t_{i-1})$$

since f is monotone increasing. If we take $P_1 = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$, then for all i, we have $t_i - t_{i-1} = \frac{1}{n}$ and

$$U(f,P_1) = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \left(\frac{1}{n}\right) = \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + 3n^2 + n}{6n^3}.$$

Taking the limit as $n \to \infty$, we see that $U(f,P_1) \to \frac{1}{3}$. This means

$$U(f) = \inf U(f,P) \leq \inf U(f,P_1) = \tfrac{1}{3}.$$

On the other hand,

$$\begin{aligned} L(f,P_1) &= \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \left(\frac{1}{n}\right) = \frac{1}{n^3} \sum_{i=1}^{n} (i-1)^2 \\ &= \frac{1}{n^3} \cdot \frac{(n-1)\,n\,(2n-1)}{6} \end{aligned}$$

which also converges to $\frac{1}{3}$. So $L(f) = \sup L(f,P) \geq \sup L(f,P_1) = \frac{1}{3}$. We have shown that $U(f) \leq \frac{1}{3} \leq L(f)$, and at the end of the section we will prove that $L(f) \leq U(f)$ holds in general, so $U(f) = L(f)$. Hence $f(x) = x^2$ is Darboux integrable.

Example 4.5.3. Define the following function on $[0,1]$:

$$f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ 1 & x \in \mathbb{Q}. \end{cases}$$

Let P be a partition of $[0,1]$. By definition,

$$U(f,P) = \sum_{i=1}^{n} M(f,[t_{i-1},t_i]) \cdot (t_i - t_{i-1}).$$

By the density of $\mathbb{Q}$ in the reals (Theorem 1.3.6), there is some $r_i \in \mathbb{Q}$ such that $t_{i-1} < r_i < t_i$ for each i. Thus for all i, $M(f,[t_{i-1},t_i]) = 1$. Then $U(f,P) = \displaystyle\sum_{i=1}^{n} 1 \cdot (t_i - t_{i-1}) = 1$. Likewise, $L(f,P) = \displaystyle\sum_{i=1}^{n} 0 \cdot (t_i - t_{i-1}) = 0$. This shows that f is not Darboux integrable.

Lemma 4.5.4. *If $P \subset Q$ are partitions then $L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$.*

Proof. This can be proven by induction on the number of points in Q. We will prove the base case when Q has one more point than P. Suppose

$$\begin{aligned} &P = \{t_0, t_1, \ldots, t_k, t_{k+1}, \ldots, t_n\} \\ \text{and}\quad &Q = \{t_0, t_1, \ldots, t_k, s, t_{k+1}, \ldots, t_n\}. \end{aligned}$$

Then $U(f,P) = \displaystyle\sum_{i=1}^{n} M(f,[t_{i-1},t_i]) \cdot (t_i - t_{i-1})$ and

$$\begin{aligned} U(f,Q) &= \sum_{j=1}^{n+1} M(f,[s_{j-1},s_j]) \cdot (s_j - s_{j-1}) \\ &= \sum_{i=1}^{n} M(f,[t_{i-1},t_i]) \cdot (t_i - t_{i-1}) - M(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1}) \\ &\quad + M(f,[t_{k-1},s]) \cdot (s - t_{k-1}) + M(f,[s,t_k]) \cdot (t_k - s). \end{aligned}$$

But $M(f,[t_{k-1},s]) \leq M(f,[t_{k-1},t_k])$ and $M(f,[s,t_k]) \leq M(f,[t_{k-1},t_k])$, so

$$\begin{aligned} U(f,Q) \leq U(f,P) &- M(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1}) \\ &+ M(f,[t_{k-1},s]) \cdot (s - t_{k-1}) + M(f,[s,t_k]) \cdot (t_k - s) \end{aligned}$$

which implies $U(f,Q) \leq U(f,P)$. By similar calculations using $m(f,[t_{k-1},t_k])$, we can show that $L(f,Q) \geq L(f,P)$. The inductive step is nearly identical, and the middle inequality comes from the definition, so we have proved the lemma. $\square$

Lemma 4.5.5. *If P and Q are any partitions of $[a, b]$, then $L(f, P) \leq U(f, Q)$.*

Proof. Let $R = P \cup Q$. Then $P \subset R$ and $Q \subset R$, so by Lemma 4.5.4,

$$\begin{aligned} & L(f, P) \leq L(f, R) \leq U(f, R) \\ \text{and} \quad & L(f, R) \leq U(f, R) \leq U(f, Q). \end{aligned}$$

Stringing these all together gives us $L(f, P) \leq U(f, Q)$ as claimed. □

Theorem 4.5.6. *For all f, $U(f) \geq L(f)$.*

Proof. We may treat $U(f)$ as the infimum of $\{U(f, P) \mid \text{partitions } P\}$, and $L(f)$ as the supremum of $\{L(f, Q) \mid \text{partitions } Q\}$. By Lemma 4.5.5, $L(f, Q) \leq U(f, P)$ for any partitions P, Q, so $L(f, Q)$ is a lower bound for $\{U(f, P) \mid \text{partitions } P\}$. This means that

$$L(f, Q) \leq \inf\{U(f, P) \mid \text{partitions } P\} = U(f)$$

so we see that $U(f)$ is an upper bound of $\{L(f, Q) \mid \text{partitions } Q\}$. This in turn implies that $U(f) \geq \sup\{L(f, Q) \mid \text{partitions } Q\} = L(f)$. Hence for all f, $U(f) \geq L(f)$. □

Theorem 4.5.7. *f is Darboux integrable on $[a, b]$ if and only if for all $\varepsilon > 0$, there is a partition P of $[a, b]$ such that $U(f, P) < L(f, P) + \varepsilon$.*

Proof. ($\Longleftarrow$) Let $\varepsilon > 0$ and choose P so that $U(f, P) < L(f, P) + \varepsilon$. For all partitions P, $L(f, P) \leq L(f)$, so with this particular partition, $U(f, P) < L(f) + \varepsilon$. Then as $\varepsilon \to 0$, this gives us $U(f) = L(f)$. Hence f is integrable.

($\Longrightarrow$) Conversely, suppose f is integrable. Then $U(f) = L(f)$. Since $U(f, P) \leq U(f)$, for any $\varepsilon > 0$ we may choose a partition P so that $U(f, P) < U(f) + \frac{\varepsilon}{2}$. Thus $U(f) + \frac{\varepsilon}{2}$ is not a lower bound of the set $\{U(f, P) \mid \text{partitions } P\}$. Similarly, we can choose a partition Q so that $L(f, Q) > L(f) - \frac{\varepsilon}{2}$ since $L(f, Q) \geq L(f)$. Let $R = P \cup Q$; we claim that R is the partition we are looking for. Using Lemma 4.5.4 and our work so far, we can write

$$\begin{aligned} L(f) - \frac{\varepsilon}{2} < L(f, Q) \leq L(f, R) \leq U(f, R) \leq U(f, P) < U(f) + \frac{\varepsilon}{2} = L(f) + \frac{\varepsilon}{2} \\ \Longrightarrow L(f) - \frac{\varepsilon}{2} < L(f, R) \leq U(f, R) < L(f) + \frac{\varepsilon}{2} \\ \Longrightarrow U(f, R) - L(f, R) < L(f) - L(f, R) + \frac{\varepsilon}{2} \\ < L(f) - \left(L(f) - \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence $U(f, R) < L(f, R) + \varepsilon$. □

4.6 Properties of Integrals

Theorem 4.6.1. *Every monotone function on $[a,b]$ is integrable.*

Proof. Without loss of generality suppose f is nondecreasing. Then for all $x \in [a,b]$, $f(a) \leq f(x) \leq f(b)$ so f is bounded. Let $\varepsilon > 0$. By Theorem 4.5.7, it suffices to find a partition P such that $U(f,P) - L(f,P) < \varepsilon$. Choose P such that

$$\max\{t_k - t_{k-1}\}_{k=1,\dots,n} < \frac{\varepsilon}{f(b)-f(a)}.$$

This maximum is called the **mesh** of f on $[a,b]$, i.e. the widest base of a partition on the interval. Consider

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} M(f,[t_{i-1},t_i]) \cdot (t_i - t_{i-1}) - \sum_{i=1}^{n} m(f,[t_{i-1},t_i]) \cdot (t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} (M-m) \cdot (t_i - t_{i-1}). \end{aligned}$$

Since $f(x) \leq f(b)$ for all x, $M \leq f(t_i)$ for all i. Similarly, $m \geq f(t_{i-1})$. So $M - m \leq f(t_i) - f(t_{i-1})$. This implies

$$\begin{aligned} U(f,P) - L(f,P) &\leq \sum_{i=1}^{n} (f(t_i) - f(t_{i-1})) \cdot (t_i - t_{i-1}) \\ &< \sum_{i=1}^{n} (f(t_i) - f(t_{i-1})) \cdot \frac{\varepsilon}{f(b)-f(a)} \\ &= \frac{\varepsilon}{f(b)-f(a)} (f(b) - f(a)) \qquad \text{by telescoping sum} \\ &= \varepsilon. \end{aligned}$$

Hence f is integrable on $[a,b]$. □

Note that in Theorem 4.6.1, f need not even be continuous.

Theorem 4.6.2. *Every continuous function on $[a,b]$ is integrable.*

Proof. Let f be continuous on $[a,b]$. By Theorem 3.3.1, f is uniformly continuous on $[a,b]$, so given $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in [a,b]$ with $|x-y| < \delta$, $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$. Let P be any partition with mesh less than δ. Then

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} \left(M(f,[t_{i-1},t_i]) - m(f,[t_{i-1},t_i]) \right) \cdot (t_i - t_{i-1}).$$

Since f is continuous on $[t_{i-1},t_i]$, the extreme value theorem (3.2.1) says that there exist x_0, y_0 such that for all $x \in [t_{i-1},t_i]$, $f(x_0) \leq f(x) \leq f(y_0)$. So $m = f(x_0)$ and $M = f(y_0)$.

By our choice of P, $|x_0 - y_0| \leq |t_i - t_{i-1}| < \delta$, so $f(y_0) - f(x_0) < \frac{\varepsilon}{b-a}$. This gives us

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} (M - m) \cdot (t_i - t_{i-1}) \\ &< \sum_{i=1}^{n} \frac{\varepsilon}{b-a} (t_i - t_{i-1}) \\ &= \frac{\varepsilon}{b-a} (b-a) \qquad \text{by telescoping sum} \\ &= \varepsilon. \end{aligned}$$

Hence f is integrable on $[a,b]$. □

Remark. It's simple to tweak the last two theorems to show that piecewise monotone functions and piecewise continuous functions are all integrable.

Proposition 4.6.3. *Let f be a bounded function on $[a,b]$ and suppose there exist sequences (U_n) and (L_n) of upper and lower Darboux sums for f such that $\lim_{n\to\infty} (U_n - L_n) = 0$. Then f is integrable and $\int_a^b f = \lim U_n = \lim L_n$.*

Proof. Note that U_n is bounded. Then by the Bolzano-Weierstrass theorem (2.4.5) U_n has a monotone subsequence U_{n_k}. Similarly, L_{n_k} has a monotone subsequence $L_{n_{k_j}}$. By the monotone convergence theorem (18.1.5), $U_{n_{k_j}}$ and $L_{n_{k_j}}$ converge, so

$$\lim_{j\to\infty} (U_{n_{k_j}} - L_{n_{k_j}}) = \lim_{j\to\infty} U_{n_{k_j}} - \lim_{j\to\infty} L_{n_{k_j}}$$

and this must be 0. Let $I = \lim U_{n_{k_j}} = \lim L_{n_{k_j}}$. We claim that $I = \int_a^b f$. By definition,

$$\begin{aligned} U(f) &= \inf\{U(f,P) \mid \text{partitions } P\} \\ &\leq \inf\{U_n \mid n \in \mathbb{N}\} \\ &\leq \lim U_{n_{k_j}} = I. \end{aligned}$$

On the other hand, for all j and for all partitions P, $U(f,P) \geq L_{n_{k_j}}$ so $\inf U(f,P) \geq L_{n_{k_j}}$. This shows that for all j, $U(f) \geq L_{n_{k_j}} \geq \lim L_{n_{k_j}} = I$. Hence $U(f) = I$. A similar proof shows that $L(f) = I$. Hence f is Darboux integrable with $U(f) = L(f) = I$ as claimed. □

Proposition 4.6.4. *Let f and g be integrable functions on $[a,b]$. Then*

(1) $f + g$ is integrable on $[a,b]$.

(2) $f - g$ is integrable on $[a,b]$.

(3) fg is integrable on $[a,b]$.

(4) $\max(f,g)$ and $\min(f,g)$ are integrable on $[a,b]$.

Part II

Complex Analysis

Chapter 5

Introduction

The contents of Part II come from a semester course on complex analysis taught by Dr. Richard Carmichael at Wake Forest University during the fall of 2010. The main topics covered include

- Complex numbers and their properties
- Complex-valued functions
- Line integrals
- Derivatives and power series
- Cauchy's Integral Formula
- Singularities and the Residue Theorem

The primary reference for the course and throughout these notes is Fisher's *Complex Variables*, 2^{nd} edition.

Chapter 6

The Complex Plane

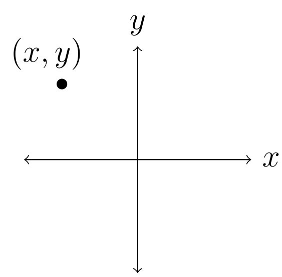
6.1 A Formal View of Complex Numbers

We begin with a description of the complex number system. In the 16th century, mathematicians sought solutions to polynomial equations such as $x^3 + x + 1$, but struggled to find a 'complete' way of describing the solutions. Recall for instance that the roots of a quadratic polynomial $ax^2 + bx + c$ is given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Of course if $b^2 - 4ac < 0$ this has no real solutions. This led Gerolamo Cardano to create the imaginary value $i = \sqrt{-1}$ to compensate for a perceived lack of completeness of solutions.

Formally, complex numbers are numbers of the form $z = x + iy$ where x and y are real numbers. These numbers lie on what is known as the *complex plane*, denoted $\mathbb{C}$.



In this way we can view the **real part** x and the **imaginary part** y of $x + iy$ separately. The set of all complex numbers is denoted $\mathbb{C}$, and they form an algebraic field under the operations

- Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$.
- Scaling: $k(x, y) = (kx, ky)$ where k is a real scalar.
- Multiplication: $(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$. Note that this multiplication differs from the usual multiplication on $\mathbb{R}$, as in Euclidean geometry.

In this class we will freely use both notations for a complex number, that is $x + iy = (x, y)$. For example,

$$\begin{aligned} x &= (x, 0) \\ i &= (0, 1) \\ i^2 &= (0, 1)(0, 1) = (-1, 0). \end{aligned}$$

For $z = x + iy$ we will also denote the real and imaginary parts by $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. As a vector space, $\mathbb{C}$ has the following special attributes for each vector (complex number).

Definition. *For a complex number $z = x + iy$, the* **modulus** *or absolute value of z is $|z| = \sqrt{x^2 + y^2}$ and the* **complex conjugate** *of z is $\bar{z} = x - iy$.*

Note that $|z|$ and $|\bar{z}|$ are always equal. Geometrically, the modulus represents the distance in the complex plane from the origin $(0,0)$ to (x,y).

Proposition 6.1.1. *For $z, w \in \mathbb{C}$,*

(i) $|zw| = |z|\,|w|$.

(ii) $\overline{zw} = \bar{z}\bar{w}$.

Since $\mathbb{C}$ is a field, there is also a notion of divisibility for complex numbers. In particular if $x+iy, u+iv \in \mathbb{C}$ and $u+iv \neq 0$, we define

$$\frac{x+iy}{u+iv} = \frac{xu+yv+i(yu-xv)}{u^2+v^2}.$$

One can check that this is the appropriate formula by multiplying and dividing $\frac{x+iy}{u+iv}$ by the conjugate $u-iv$.

As in the xy-plane, there is a polar coordinate system for complex numbers: if $z = x+iy$ then we set $r = |z|$, $x = r\cos\theta$ and $y = r\sin\theta$ where $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. This gives us

$$z = |z|(\cos\theta + i\sin\theta).$$

Multiplication is compatible with polar representations, for if $z = |z|(\cos\theta + i\sin\theta)$ and $w = |w|(\cos\psi + i\sin\psi)$ we have

$$\begin{aligned} zw &= |z|\,|w|(\cos\theta + i\sin\theta)(\cos\psi + i\sin\psi) \\ &= |z|\,|w|(\cos\theta\cos\psi - \sin\theta\sin\psi) + i(\cos\theta\sin\psi + \sin\theta\cos\psi) \\ &= |z|\,|w|(\cos(\theta+\psi) + i\sin(\theta+\psi)). \end{aligned}$$

Likewise, $\frac{z}{w} = \frac{|z|}{|w|}(\cos(\theta-\psi) + i\sin(\theta-\psi))$.

Taking powers of complex numbers, e.g. z^n, is sometimes difficult to compute, since multiplication isn't quite as straightforward in the complex plane. However, there is a result which utilizes the polar representation of a complex number to simplify the expression.

Theorem 6.1.2 (De Moivre's Theorem)**.** *For all integers n, $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.*

Proof. We prove this using induction on n. For the base case $n = 1$, we simply have

$$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta.$$

Now assume De Moivre's Theorem holds for n. Then we have

$$\begin{aligned} (\cos\theta + i\sin\theta)^{n+1} &= (\cos\theta + i\sin\theta)^n(\cos\theta + i\sin\theta) \\ &= (\cos(n\theta) + i\sin(n\theta))(\cos\theta + i\sin\theta) \\ &= (\cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta) + i(\sin\theta\cos(n\theta) + \cos\theta\sin(n\theta)) \\ &= \cos((n+1)\theta) + i\sin((n+1)\theta). \end{aligned}$$

□

Definition. *When we write $z = |z|(\cos\theta + i\sin\theta)$, the angle θ is called the* **argument** *of z, denoted* $\arg z$.

We often want to restrict our attention to a single, canonical value of θ for any z. Thus we define the **principal argument** $\theta = \operatorname{Arg} z$, where $-\pi \leq \theta \leq \pi$.

Proposition 6.1.3. $\operatorname{Arg}(zw) = \operatorname{Arg} z + \operatorname{Arg} w$, *where these may differ by a multiple of 2π.*

Example 6.1.4. Let $z = -1 + i$ and $w = i$. Then $zw = -1 - i$, $\operatorname{Arg}(zw) = -\frac{3\pi}{4}$ and

$$\operatorname{Arg} z + \operatorname{Arg} w = \frac{3\pi}{4} + \frac{\pi}{2} = \frac{5\pi}{4} \equiv -\frac{3\pi}{4} \mod 2\pi.$$

6.2 Properties of Complex Numbers

Continuing with the geometric parallels between Euclidean space and the complex plane, we have the important **triangle inequality** for complex numbers:

$$|z + w| \leq |z| + |w|.$$

There is also a related inequality, sometimes called the reverse triangle inequality:

$$||z| - |w|| \leq |z - w|.$$

The original purpose of complex numbers was to compute roots of all polynomials, so it will be desirable to be able to compute roots of complex numbers. In other words, if $w = |w|(\cos\psi + i\sin\psi)$, what is $w^{1/n}$? Let $z = w^{1/n}$, so that $z^n = w$. Then using De Moivre's Theorem (6.1.2) we have

$$|w|(\cos\psi + i\sin\psi) = (|z|(\cos\theta + i\sin\theta))^n = |z|^n(\cos(n\theta) + i\sin(n\theta)).$$

Solving for θ, we see that

$$\cos\psi = \cos(n\theta) \implies n\theta = \psi + 2\pi k \implies \theta = \frac{\psi + 2\pi k}{n}$$

for some integer k. Hence our expression for $w^{1/n}$ is

$$z = w^{1/n} = |w|^{1/n}\left(\cos\left(\frac{\psi + 2\pi k}{n}\right) + i\sin\left(\frac{\psi + 2\pi k}{n}\right)\right).$$

For the nth root of w, that is $w^{1/n}$, this formula gives all possible roots. In fact there are n distinct roots; all others are repeated values.

Recall that the equation of a circle in $\mathbb{R}^2$ is $\sqrt{(x - x_0)^2 + (y - y_0)^2} = r$ for $r > 0$. In the complex plane, this is expressed by $|z - z_0| = r$.

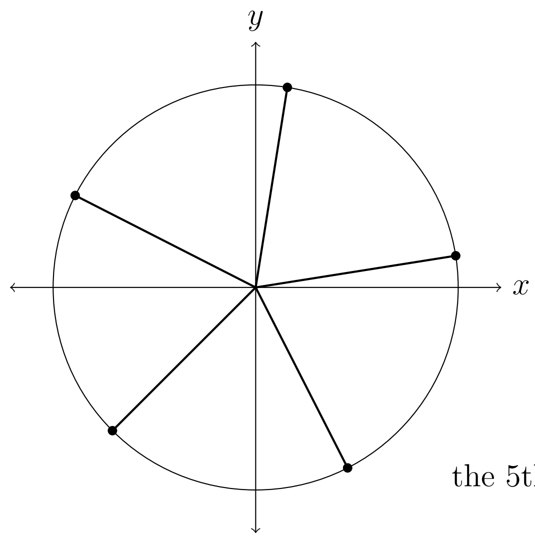
Example 6.2.1. Let's find the 5th roots of $z = 1 + i$. The polar representation of $1 + i$ is

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

The modulus of all the 5th roots of unity is $2^{1/10} \approx 1.07171$. Our work above gives all of these roots as

$$(1 + i)^{1/5} = 2^{1/10}\left(\cos\left(\frac{\pi}{20} + \frac{2\pi k}{5}\right) + i\sin\left(\frac{\pi}{20} + \frac{2\pi k}{5}\right)\right).$$

These are shown on the circle of radius $2^{1/10}$ below.



the 5th roots of $1+i$

Example 6.2.2. Consider the equation $z^4 - 4z^2 + 4 - 2i = 0$. This may be rewritten as $(z^2-2)^2 = 2i = (1+i)^2$ which has solutions

$$z^2 - 2 = \pm(1+i) \implies z^2 = \begin{cases} 3+i \\ 1-i. \end{cases}$$

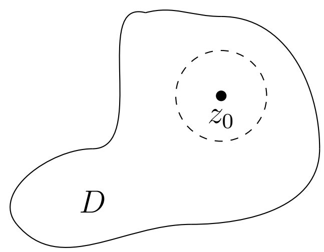
Using the expression for roots above, this yields the following solutions to the original equation:

$$\begin{aligned} z &= \pm\sqrt[4]{10}\left(\cos\left(\frac{1}{2}\arctan\frac{1}{3}\right) + i\sin\left(\frac{1}{2}\arctan\frac{1}{3}\right)\right) \\ \text{and } \ z &= \pm\sqrt[4]{2}\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right). \end{aligned}$$

6.3 Subsets of the Complex Plane

In Chapter 7 we will define functions on the complex plane, i.e. functions whose domain and range are subsets of the complex plane. The following topological terms will be useful.

Definition. *A subset $D \subseteq \mathbb{C}$ is* **open** *if all its points are interior points, that is, any circle drawn around a point (called a neighborhood of the point) lies entirely within D.*



Circles are actually a specific case of a more general notion of 'neighborhood' or open set in topology. Since the open disks (sometimes called open balls) $B(z_0, \varepsilon) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ form a basis for $\mathbb{C}$ (see any introductory topology text, e.g. Adams and Franzosa or Munkres) it suffices to consider open sets as those 'composed' of smaller open balls.

Example 6.3.1. The half plane $H = \{z \in \mathbb{C} \mid \text{Re}(z) > 0\}$ is an open set. Likewise, for any $a \in \mathbb{R}$, $\{z \in \mathbb{C} \mid \text{Re}(z) > a\}$ and $\{z \in \mathbb{C} \mid \text{Re}(z) < a\}$ are open sets, and the same is true for $\text{Im}(z)$.

Definition. *A point z_0 in a set D is called a* **boundary point** *if every neighborhood of z_0 contains both interior and exterior points. D is said to be* **closed** *if it contains its boundary, or the set of all boundary points of D.*

Definition. *An open set D is* **connected** *if all points in D may be joined by a series of contiguous, direct line segments, each of which is completely contained within D. Furthermore, D is* **convex** *if it is connected and* any *single line segment joining two points in D also lies in D.*

Chapter 7

Complex-Valued Functions

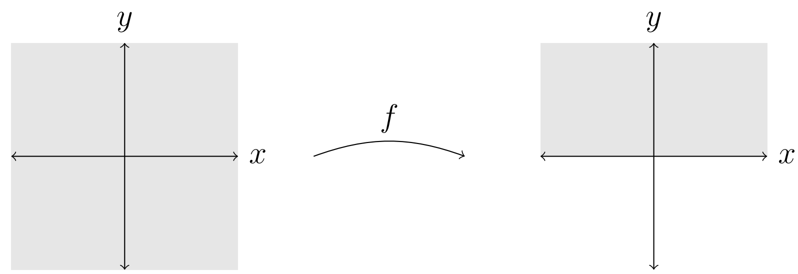
7.1 Functions and Limits

In this section we introduce functions that have values in the complex plane.

Definition. *A* **function** *of a complex variable z is a map $f : D \to \mathbb{C}$ for some subset $D \subseteq \mathbb{C}$, i.e. f assigns a complex number to each $z \in D$.*

Definition. *The* **domain** *of a complex-valued function f is the set of all values z for which the function operates; this is usually denoted D. The* **range** *is all possible values of the function, denoted $\operatorname{Im} f$ or $f(D)$.*

Example 7.1.1. Let $f(z) = z^2$. The domain of f is all of $\mathbb{C}$, while the range of f is the closed upper half plane $\{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq 0\}$.



Example 7.1.2. $f(z) = \frac{1}{z-1}$ has domain $D = \{z \in \mathbb{C} \mid z \neq 1\}$ and range $f(D) = \{z \in \mathbb{C} \mid z \neq 0\}$.

Definition. *A* **sequence** *is a complex-valued function whose domain is the set of positive integers, written $(z_n) = (z_1, z_2, z_3, \ldots)$ where each z_i is a complex number.*

Definition. *A sequence (z_n) is said to have a* **limit** *L if, given any $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that $|z_n - L| < \varepsilon$ for all $n \geq N$. In this case we write $\lim_{n\to\infty} z_n = L$ and say that (z_n)* **converges** *to L. If no such L exists, then (z_n) is said to* **diverge**.

The definitions of sequence and limit are nearly identical to their counterparts in real analysis. However, in the complex plane every number has a real and an imaginary part. The following proposition helps us relate the definition of a complex limit to its real and imaginary parts.

Proposition 7.1.3. *Let $z_n = x_n + iy_n$ and $z = x + iy$. Then $\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.*

Proof. ($\Longrightarrow$) If $\lim_{n\to\infty} z_n = z$ then the inequalities $|x_n - x| \leq |z_n - z|$ and $|y_n - y| \leq |z_n - z|$ directly imply that (x_n) and (y_n) converge to x and y, respectively.

($\Longleftarrow$) On the other hand, suppose $(x_n) \to x$ and $(y_n) \to y$. If $\varepsilon > 0$ is given, we may choose N_1 and N_2 such that $|x_n - x| < \frac{\varepsilon}{2}$ for all $n \geq N_1$ and $|y_n - y| < \frac{\varepsilon}{2}$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$ the triangle inequality gives us

$$|z_n - z| \leq |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence (z_n) converges to $z = x + iy$. $\square$

As a result, we have

Corollary 7.1.4. *If $z_n \to z$ then $|z_n| \to |z|$.*

The converse to this is generally false. For example, the sequence $|i^n|$ converges to 1 since $|i^n| = |i|^n = 1^n = 1$ for all n; however, $i^n = (i, -1, -i, 1, i, -1, \ldots)$ and this fluctuates infinitely often between these four values, so the sequence diverges.

Proposition 7.1.5. *Suppose $\lim_{n\to\infty} z_n = z$. Then*

(i) For any complex scalar $k \neq 0$, $\lim_{n\to\infty} kz_n = kz$.

(ii) If $z_n \neq 0$ for any n and $z \neq 0$, then $\lim_{n\to\infty} \frac{1}{z_n} = \frac{1}{z}$.

Proof. (i) Let $\varepsilon > 0$ be given. By convergence of (z_n) there exists a positive integer N such that $|z_n - z| < \frac{\varepsilon}{|k|}$. Then for all $n \geq N$,

$$|kz_n - kz| = |k|\,|z_n - z| < |k|\,\frac{\varepsilon}{|k|} = \varepsilon.$$

Hence $(kz_n) \to kz$.

(ii) First we can choose an N_1 such that $|z_n - z| < \frac{|z|}{2}$ for all $n \geq N_1$. Note that by the reverse triangle inequality,

$$|z_n| \geq |z| - |z_n - z| > |z| - \frac{|z|}{2} = \frac{|z|}{2}.$$

We use this to control the $|z_n|$ term in the calculations below. Next for any $\varepsilon > 0$ there is an N_2 such that for all $n \geq N_2$, $|z_n - z| < \frac{|z|^2\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. Then for any $n \geq N$,

$$\left|\frac{1}{z_n} - \frac{1}{z}\right| = \left|\frac{z - z_n}{z_n z}\right| = \frac{|z_n - z|}{|z_n|\,|z|} \leq \frac{2}{|z|}\,\frac{1}{|z|}\,|z_n - z| < \frac{2}{|z|^2}\,\frac{|z|^2\varepsilon}{2} = \varepsilon.$$

Hence $\left(\frac{1}{z_n}\right) \to \frac{1}{z}$. □

This shows that limits of complex sequences behave as expected (by which we mean they behave as their counterparts do in the real case). We also have

Theorem 7.1.6. *If (z_n) converges to z and (w_n) converges to w, then the sequence $(z_n w_n)$ converges to zw.*

Proof omitted. □

Definition. *Given a function $f(z)$ with domain D and a point z_0 either in D or in the boundary ∂D of D, we say f has a* **limit** *at z_0 if*

$$\lim_{z\to z_0} f(z) = L$$

for some $L \in \mathbb{C}$. Explicitly, $f(z)$ has limit L at z_0 if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies $|f(z) - L| < \varepsilon$.

Definition. *$f(z)$ is* **continuous** *at a point z_0 in its domain if $\lim_{z\to z_0} f(z)$ exists and it equals $f(z_0)$. In particular, $f(z)$ is continuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \varepsilon$.*

Example 7.1.7. The function $f(z) = |z|^2$ is continuous on its domain $\mathbb{C}$. For example, $f(z)$ has limit 4 at $z_0 = 2i$. To see this, let $\varepsilon > 0$ and define $\delta_1 = 1$, $\delta_2 = \frac{\varepsilon}{5}$ and $\delta = \min\{\delta_1, \delta_2\}$. Note that by the reverse triangle inequality, $|z| \leq |z - 2i| + |2i| < 1 + 2 = 3$; we will use this below. Then if $0 < |z - 2i| < \delta$ we have

$$\begin{aligned} |f(z) - f(2i)| &= ||z|^2 - 4| \\ &= ||z| + 2| \cdot ||z| - 2| \\ &= (|z| + 2)|z - 2i| \\ &< (3 + 2)\frac{\varepsilon}{5} = \varepsilon. \end{aligned}$$

Hence $\lim_{z\to 2i} f(z) = 4$ as claimed.

Example 7.1.8. Consider the function $f(z) = \dfrac{z}{\bar{z}}$ where $z = x + iy \neq 0$ and $\bar{z} = x - iy$, its complex conjugate. Does $\lim_{z\to 0} f(z)$ exist? Well consider this limit along two different paths in the complex plane:

$$\begin{aligned} \lim_{(x,y)\to(0,y)} f(z) &= \frac{0 + iy}{0 - iy} = -1 \\ \lim_{(x,y)\to(x,0)} f(z) &= \frac{x + i0}{x - i0} = 1. \end{aligned}$$

Since these limits are different, the limit of the function must not exist. Hence $\dfrac{z}{\bar{z}}$ is not continuous at $z_0 = 0$.

Definition. *A function $f(z)$ has a* **limit at infinity***, denoted $\lim_{z\to\infty} f(z) = L$, if for any $\varepsilon > 0$ there is a (large) number M such that $|f(z) - L| < \varepsilon$ whenever $|z| \geq M$. Note that there is no restriction on $\arg z$; only $|z|$ is required to be large.*

Example 7.1.9. The family of functions $f(z) = \frac{1}{z^m}$ has a limit $L = 0$ as $z \to \infty$ for all $m = 1, 2, 3, \ldots$. To see this, let $\varepsilon > 0$ and choose $M = \frac{1}{\varepsilon^{1/m}}$. Then if $|z| \geq M$,

$$\left|\frac{1}{z^m}\right| = \left(\frac{1}{|z|}\right)^m \geq \left(\frac{1}{M}\right)^m = (\varepsilon^{1/m})^m = \varepsilon.$$

By properties of limits, we have

Proposition 7.1.10.

1) Every polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ is continuous on the complex plane.

2) If $p(z)$ and $q(z)$ are polynomials, then their quotient $\frac{p(z)}{q(z)}$ is continuous at all points such that $q(z) \neq 0$.

Every complex-valued function $f(z)$ can be written as $f(z) = u(z) + iv(z)$, where u and v are each real-valued functions. This allows us to view every complex function by its real and imaginary parts. It is easy to see that all of the results on continuity for functions of the real numbers now apply for complex-valued functions. In particular,

Proposition 7.1.11. *Let $f = u + iv$ be a complex-valued function. Then f is continuous at z_0 if and only if u and v are both continuous at z_0.*

7.2 Infinite Series

In this section we briefly review infinite series, since they carry over to the complex case nearly identically.

Definition. *For complex numbers $z_1, z_2, \ldots$ their nth* **partial sum** *is* $\sum_{j=1}^{n} z_j = z_1 + \ldots + z_n$.

Definition. *An* **infinite series** *of complex numbers is a limit of partial sums*

$$\sum_{j=1}^{\infty} z_j = \lim_{n\to\infty} \sum_{j=1}^{n} z_j.$$

Definition. *We say an infinite series of partial sums* $s_n = \sum_{j=1}^{n} z_j$ **converges** *if* $s = \lim_{n\to\infty} s_n$ *exists. Otherwise, the series* **diverges**.

In the complex case, we can write each $z_j = x_j + iy_j$ so every infinite series may be written as the sum of a real and imaginary series:

$$\sum_{j=1}^{\infty} z_j = \sum_{j=1}^{\infty} x_j + i \sum_{j=1}^{\infty} y_j.$$

As with functions, the series $\sum z_j$ converges if and only if $\sum x_j$ and $\sum y_j$ converge. In other words, $\lim_{n\to\infty} s_n$ only converges when $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ both exist.

Definition. *A series* $\sum_{j=1}^{\infty} z_j$ *has* **absolute convergence** *if* $\sum_{j=1}^{\infty} |z_j|$ *converges. If* $\sum_{j=1}^{\infty} z_j$ *converges but the absolute series does not converge, we say the series* **converges conditionally**.

Notice that if $\sum_{j=1}^{\infty} z_j$ converges (absolutely) then both $\sum_{j=1}^{\infty} x_j$ and $\sum_{j=1}^{\infty} y_j$ converge (absolutely) as well. The triangle inequality for series looks like

$$\left| \sum_{j=1}^{\infty} z_j \right| \leq \sum_{j=1}^{\infty} |z_j|.$$

Example 7.2.1. As in the real case, a geometric series $\sum_{j=1}^{\infty} \alpha^j$ converges to $\dfrac{1}{1-\alpha}$ if $|\alpha| < 1$ and diverges otherwise. The value α is sometimes called the ratio of the series.

Example 7.2.2. Consider the series $\sum_{j=1}^{\infty} j\left(\frac{1+2i}{3}\right)^j$. Absolute convergence is useful in complex analysis since we can reduce complex numbers to purely real-valued expressions. In this case, we see that

$$\sum_{j=1}^{\infty}\left|j\left(\frac{1+2i}{3}\right)^j\right| = \sum_{j=1}^{\infty} j\left|\frac{1+2i}{3}\right|^j = \sum_{j=1}^{\infty} j\left(\frac{\sqrt{5}}{3}\right)^j$$

which converges by the ratio test, for example. Hence the original series converges absolutely.

Example 7.2.3. The series $\sum_{n=1}^{\infty}\frac{i^n}{n}$ converges even though the similar-looking harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. To see this, notice that we can write

$$\sum_{n=1}^{\infty}\frac{i^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{2n} + i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$$

and both parts converge by the alternating series test.

7.3 Exponential and Logarithmic Functions

Recall from single-variable calculus the exponential function e^x. This function has many definitions, with the two most important being

$$e^x = \lim_{t\to\infty}\left(1+\frac{x}{t}\right)^t$$
$$\text{and}\quad e^x = \sum_{n=1}^{\infty}\frac{x^n}{n!}.$$

In complex analysis, we define

Definition. *For $z = x + iy$, the* **complex exponential function** e^z *is defined by*

$$e^z = e^x(\cos y + i\sin y).$$

The special case $e^{it} = \cos t + i\sin t$ is called **Euler's formula**. Euler was the first to realize the connection between the exponential function and sine and cosine. This amazing identity, called "the most remarkable formula in mathematics" by Feynman, has been around since 1748 and has far-reaching implications in many branches of mathematics and physics.

The following proposition shows that this definition captures all of the nice properties of e^x from the real case. We will see in a moment that in the complex plane, the exponential function has even deeper properties and an essential connection to the geometry of $\mathbb{C}$.

Proposition 7.3.1. *For complex numbers z and w,*

(a) $e^{z+w} = e^z e^w$.

(b) $\frac{1}{e^z} = e^{-z}$.

(c) $e^{z+2\pi i} = e^z$, *that is, the complex exponential function is periodic with period* $2\pi i$.

(d) If $z = x + iy$, $|e^z| = e^x$ *and therefore* $|e^{iy}| = 1$.

(e) $e^z \neq 0$ *for any* $z \in \mathbb{C}$.

Proof. (a) Let $z = x + iy$ and $w = x' + iy'$. Then

$$\begin{aligned} e^{z+w} &= e^{(x+x')+i(y+y')} = e^{x+x'}(\cos(y+y') + i\sin(y+y')) \\ &= e^x e^{x'}(\cos y + i\sin y)(\cos y' + i\sin y') = e^z e^w \end{aligned}$$

(the last part uses a trick similar to the one used in the proof of De Moivre's Theorem (6.1.2)).

(b) follows from (a) and trig properties.

(c) follows directly from the definition of e^z.

(d) follows from the fact that for any θ, $|\cos\theta + i\sin\theta| = 1$.

(e) By part (d), $|e^{x+iy}| = e^x$, and x is real so e^x is always nonzero. Therefore $|e^z| \neq 0$ which implies $e^z \neq 0$. $\square$

We will see in Chapter 15 that e^z also satisfies one of the nicest properties of the exponential function in the real case: $\frac{d}{dz}e^z = e^z$

Note that part (c) of Proposition 7.3.1 implies that $f(z) = e^z$ is *not* a one-to-one function on the complex plane. This is unfortunate, since that was one of the nice attributes of e^x in the real case, as it allowed us to define an inverse, the logarithm $\log x$. We next show how to construct a partial solution to this problem.

Let $w = e^{x+iy}$. We seek a function F such that $F(w) = x + iy$ and $e^{F(x+iy)} = x + iy$. Note that since $|w| = e^x$ and these are real numbers, we have $x = \ln |w|$. This allows us to define

Definition. *The* **formal logarithm** *is written* $\log z = \ln |z| + i \arg z$.

This is *not* a function (meaning it is not well-defined), since $\arg z$ represents a set of values which differ by $2k\pi$ for integers k.

We remedy this by making *branch cuts* of the complex plane. This is done by taking a ray from the origin, say with angle θ and defining the branch $(\theta, \theta + 2\pi]$ so that $\log z$ is well-defined on this domain. The most important branch is

Definition. *Let* $\operatorname{Arg} z$ *denote the argument of* z *in the branch* $(-\pi, \pi]$*; this is called the* **principal branch**. *Then we define the* **principal logarithm** *by*

$$\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z.$$

Proposition 7.3.2. *On the principal branch,* $\operatorname{Log} e^z = e^{\operatorname{Log} z} = z$.

Proof. Let $z = x + iy$ with $\operatorname{Arg} z = \theta \in (-\pi, \pi]$. Then on one hand,

$$\operatorname{Log} e^z = \ln |e^z| + i \operatorname{Arg} e^z = \ln e^x + iy = x + iy = z$$

and on the other hand,

$$e^{\operatorname{Log} z} = e^{\ln |z| + i \operatorname{Arg} z} = e^{\ln |z|}(\cos\theta + i \sin\theta) = |z|(\cos\theta + i\sin\theta) = z.$$

Note that these require that we restrict our attention to a single branch (it may not even be the principal branch) for the expressions to be well-defined. □

Recall that $f(z) = u(z) + iv(z)$ is continuous if and only if u and v are continuous. Well $\operatorname{Arg} z$ has no limit at values along the negative real axis. Therefore $\operatorname{Log} z$ is not continuous at any point $\operatorname{Re}(z) \leq 0$. However, making a different branch cut allows us to define a function with different continuity.

As in the real case, exponentials for bases other than e are permitted. They relate to the logarithm by

$$a^z = e^{z \log a}$$

where $\log a$ is defined on a fixed branch of the logarithm.

Example 7.3.3. Let's use the complex logarithm to evaluate $(-1)^i$. Note that $(-1)^i = e^{i\log(-1)}$ where log is defined appropriately. We also have

$$\log(-1) = \ln|-1| + i(\arg(-1) + 2k\pi) = 0 + i(-\pi + 2k\pi).$$

Then $e^{i\log(-1)} = e^{-(-\pi+2k\pi)} = e^{\pi-2k\pi}$ for any integer k. The principal value of $(-1)^i$ is e^{π}, which is found by

$$(-1)^i = e^{i\,\mathrm{Log}(-1)} = e^{i(-\pi i)} = e^{\pi}.$$

Example 7.3.4. We can use logarithms to solve an equation such as $z^{1+i} = 4$. First consider $(1+i)\log z = \log 4 = \ln|4| + 2k\pi i$. This gives us

$$\begin{aligned}\log z &= \frac{\ln|4| + 2k\pi i}{1+i}\left(\frac{1-i}{1-i}\right)\\ &= \frac{(\ln|4| + 2k\pi) - i\ln|4| + 2k\pi i}{2}\\ &= (\ln|2| + k\pi) + i(-\ln|2| + k\pi).\end{aligned}$$

Taking the exponential of both sides yields

$$\begin{aligned}z = e^{\log z} &= e^{(\ln 2+k\pi)+i(-\ln 2+k\pi)}\\ &= 2e^{k\pi}((-1)^k\cos(\ln 2) + i(-1)^{k+1}\sin(\ln 2))\\ &= (-1)^k 2e^{k\pi}(\cos(\ln 2) - i\sin(\ln 2)).\end{aligned}$$

Example 7.3.5. To simplify an expression such as $(1+i)^i$, use the logarithm to write $(1+i)^i = e^{i\log(1+i)}$. Then

$$\begin{aligned}\log(1+i) = \ln|1+i| + i(\arg(1+i) + 2\pi k) &= \frac{\ln 2}{2} + i\left(\frac{\pi}{4} + 2\pi k\right)\\ \implies e^{i\log(1+i)} = e^{-\left(\frac{\pi}{4}+2\pi k\right)+i\frac{\ln 2}{2}} &= e^{-\frac{\pi}{4}}\left(\cos\left(\frac{\ln 2}{2}\right) + i\sin\left(\frac{\ln 2}{2}\right)\right).\end{aligned}$$

7.4 Trigonometric Functions

The complex trigonometric functions are defined in terms of e^z. This should come as no surprise, given the relation we have seen between exponential and trig functions. By the end of the section we will see that this connection runs even deeper.

Definition. *The* **complex cosine** *and* **complex sine** *functions are defined by*

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}) \qquad \textit{and} \qquad \sin z = \tfrac{1}{2i}(e^{iz} - e^{-iz}).$$

Note that the complex trig functions coincide with their real counterparts, for if $x \in \mathbb{R}$ we have

$$\begin{aligned} \tfrac{1}{2}(e^{ix} + e^{-ix}) &= \tfrac{1}{2}(\cos x + i\sin x + \cos(-x) + i\sin(-x)) \\ &= \tfrac{1}{2}(\cos x + i\sin x + \cos x - i\sin x) = \cos x \\ \text{and} \quad \tfrac{1}{2i}(e^{ix} - e^{-ix}) &= \tfrac{1}{2i}(\cos x + i\sin x - (\cos(-x) + i\sin(-x))) \\ &= \tfrac{1}{2i}(\cos x + i\sin x - \cos x + i\sin x) = \sin x. \end{aligned}$$

The complex cosine and sine functions are also periodic, with period 2π like the real-valued cosine and sine. Using the fact that e^z is periodic, we can write

$$\begin{aligned} \cos(z + 2\pi) &= \tfrac{1}{2}(e^{i(z+2\pi)} + e^{-i(z+2\pi)}) \\ &= \tfrac{1}{2}(e^{iz}e^{2\pi i} + e^{-iz}e^{-2\pi i}) \\ &= \tfrac{1}{2}(e^{iz} + e^{-iz}) = \cos z \\ \text{and} \quad \sin(z + 2\pi) &= \tfrac{1}{2i}(e^{i(z+2\pi)} - e^{-i(z+2\pi)}) \\ &= \tfrac{1}{2i})(e^{iz}e^{2\pi i} - e^{-iz}e^{-2\pi i}) \\ &= \tfrac{1}{2i}(e^{iz} - e^{-iz}) = \sin z. \end{aligned}$$

Many other properties of the real trig functions carry over the complex case. Just to name a few,

(a) $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$

(b) $\sin\left(z + \frac{\pi}{2}\right) = \cos z$ and $\cos\left(z + \frac{\pi}{2}\right) = -\sin z$

(c) $\sin(z + w) = \sin z \cos w + \cos z \sin w$

(d) $\cos(z + w) = \cos z \cos w - \sin z \sin w$

(e) $\cos^2 z + \sin^2 z = 1$

(f) $\cos^2 z - \sin^2 z = \cos(2z)$

(g) When we define the derivative of a complex-valued function in Section 15.1, we will see that the derivatives of $\cos z$ and $\sin z$ are similar to the real case.

Example 7.4.1. It is easy to see from the definition of cosine that $\cos z = 0$ if and only if $z = \frac{\pi}{2} + \pi k$ for any integer k.

Example 7.4.2. Complex conjugation commutes with trig and exponential functions:

$$e^{\overline{z}} = \overline{e^z} \qquad \cos \overline{z} = \overline{\cos z} \qquad \sin \overline{z} = \overline{\sin z}.$$

Using the definitions of $\cos z$ and $\sin z$, we can define the other four main trig functions.

$$\begin{aligned}
\tan z &= \frac{\sin z}{\cos z} = -i\frac{e^{2iz}-1}{e^{2iz}+1} \\
\sec z &= \frac{1}{\cos z} \\
\csc z &= \frac{1}{\sin z} \\
\cot z &= \frac{\cos z}{\sin z} = i\frac{e^{2iz}+1}{e^{2iz}-1}.
\end{aligned}$$

Chapter 8

Calculus in the Complex Plane

8.1 Line Integrals

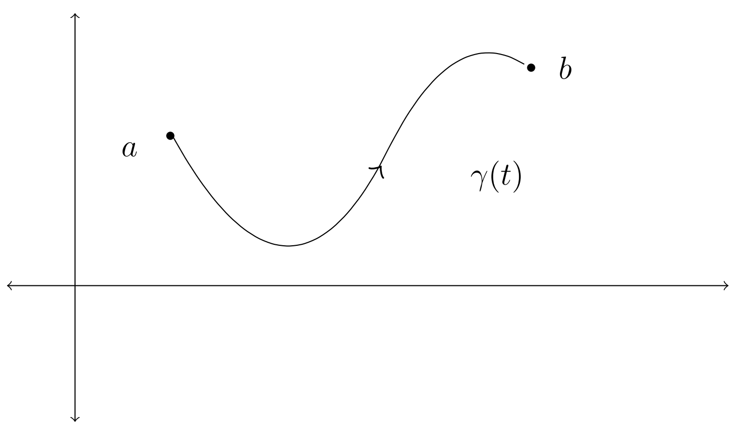
If $f : [a,b] \to \mathbb{C}$ is a complex-valued function which is continuous on some interval $[a,b]$ where $a, b \in \mathbb{R}$, then the integral of f over $[a,b]$ is simply

$$\int_a^b f(t)\,dt = \int_a^b \mathrm{Re}(f(t))\,dt + i\int_a^b \mathrm{Im}(f(t))\,dt.$$

For functions that take on values over some region in the complex plane, we integrate over curves.

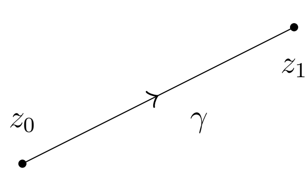
Definition. *Let $f(z)$ be a complex-valued function which is continuous on some region $D \subseteq \mathbb{C}$ and let γ be a smooth curve contained in D that is parametrized by $\gamma(t), a \le t \le b$. Then the* **line integral** *of f over γ is*

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$



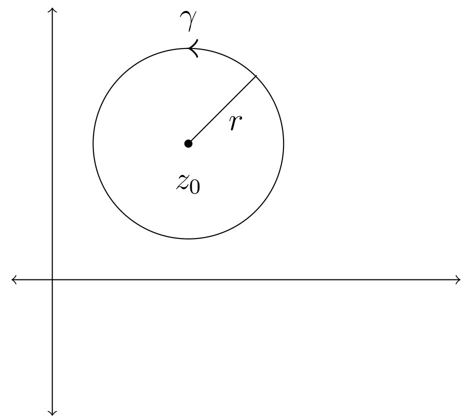
Remember that a curve is **smooth** if its first derivative $\gamma'(t)$ exists and is continuous on $[a,b]$. Since the curves are all functions on a real interval $[a,b]$, we need not worry about complex derivatives yet; $\gamma'(t)$ is just the first derivative in the normal sense. Some important examples of parametrizations in the complex plane are

Example 8.1.1. A curve γ is **simple** if $\gamma(t_1) \ne \gamma(t_2)$ whenever $a < t_1 < t_2 < b$. In plain language, a simple curve does not intersect itself; it is an embedding of the interval $[a,b]$ into $\mathbb{C}$. The easiest simple curve to parametrize is a line:



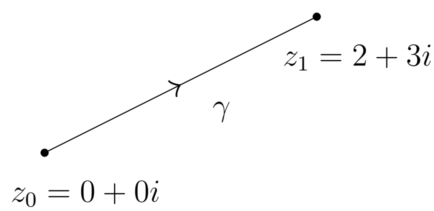
If γ is the line between z_0 and z_1, then we parametrize it by $\gamma(t) = z_0 + t(z_1 - z_0)$ for $0 \le t \le 1$.

Example 8.1.2. A curve γ is **closed** if $\gamma(a) = \gamma(b)$, i.e. it starts and ends in the same location. The canonical example of a simple closed curve is a circle:



This is parametrized by $\gamma(t) = z_0 + re^{it}$ for $0 \le t \le 2\pi$.

Example 8.1.3. Let's compute the line integral $\displaystyle\int_\gamma z^2\,dz$ over the line from $(0,0)$ to $(2,3)$ in the complex plane.



We parametrize the curve by $\gamma(t) = 2t + 3it, 0 \le t \le 1$. Then using the formula above, we compute

$$\begin{aligned}
\int_\gamma z^2\,dz &= \int_0^1 \gamma(t)^2\gamma'(t)\,dt = \int_0^1 (2t+3it)^2(2+3i)\,dt \\
&= \int_0^1 (4t^2 - 9t^2 + 12it^2)(2+3i)\,dt = \int_0^1 (-5t^2 + 12it^2)(2+3i)\,dt \\
&= \int_0^1 (-46t^2 + 9it^2)\,dt = -\frac{46}{3}t^3\bigg|_0^1 + 3it^3\big|_0^1 = -\frac{46}{3} + 3i.
\end{aligned}$$

Example 8.1.4. Just as reversing the order of a and b in a real integral changes the integral by -1, one can reverse the orientation of a smooth curve γ to switch the sign of the line integral along γ. Let $-\gamma$ denote the curve γ with orientation reversed. Then

$$\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz.$$

Example 8.1.5. Next let's change the path of integration to be the semicircle $\gamma(t) = e^{it}, 0 \le t \le \pi$. We will write $\gamma(t) = \cos t + i\sin t$ so that the derivative may be written

$\gamma'(t) = -\sin t + i\cos t$. Then we compute

$$\begin{aligned}
\int_\gamma z^2\,dz &= \int_0^\pi (\cos t + i\sin t)^2(-\sin t + i\cos t)\,dt \\
&= \int_0^\pi (\cos^2 t - \sin^2 t + 2i\cos t\sin t)(-\sin t + i\cos t)\,dt \\
&= \int_0^\pi (\sin^3 t - \cos^2 t\sin t - 2\cos^2 t\sin t)\,dt + i\int_0^\pi (\cos^3 t - \sin^2 t\cos t - 2\sin^2 t\cos t)\,dt \\
&= \int_0^\pi (\sin t - \cos^2 t\sin t - 3\cos^2 t\sin t)\,dt + i\int_0^\pi (\cos t - \sin^2 t\cos t - 3\sin^2 t\cos t)\,dt \\
&= \int_0^\pi (\sin t - 4\cos^2 t\sin t)\,dt + i\int_0^\pi ((\cos t - 4\sin^2 t\cos t)\,dt \\
&= \left[-\cos t + \frac{4}{3}\cos^3 t\right]_0^\pi + i\left[\sin t - \frac{4}{3}\sin^3 t\right]_0^\pi = -\frac{2}{3}.
\end{aligned}$$

Example 8.1.6. Compute the line integral $\int_\gamma (z^2 - 3|z| + \operatorname{Im} z)\,dz$ where γ is parametrized by $\gamma(t) = 2e^{it}, 0 \le t \le \frac{\pi}{2}$. First note that $\gamma'(t) = 2ie^{it}$. Then

$$\begin{aligned}
\int_\gamma (z^2 - 3|z| + \operatorname{Im} z)\,dz &= \int_0^{\frac{\pi}{2}} (4e^{2it} - 3|2e^{it}| + \operatorname{Im}(2e^{it}))\cdot 2ie^{it}\,dt \\
&= \int_0^{\frac{\pi}{2}} (8ie^{3it} - 12ie^{it} + 4ie^{it}\sin t)\,dt \\
&= \int_0^{\frac{\pi}{2}} \left(8ie^{3it} - 12ie^{it} + 4ie^{it}\left(\frac{1}{2i}(e^{it} - e^{-it})\right)\right)\,dt \\
&= \left[\frac{8}{3}e^{3it} - 12ie^{it} + \frac{1}{2}\sin(2t) - \frac{i}{2}\cos(2t) - 2t\right]_0^{\frac{\pi}{2}} \\
&= \frac{28}{3} - \frac{\pi}{2} - \frac{44}{3}i.
\end{aligned}$$

The definition of line integrals can be extended to *piecewise* smooth curves by

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \ldots + \int_{\gamma_k} f(z)\,dz$$

where each γ_i is a smooth curve on an interval $[a_i, b_i] \subset [a, b]$, $\gamma_1(a) = \gamma(a)$, $\gamma_k(b) = \gamma(b)$ and $\gamma_i(b_i) = \gamma_{i+1}(b_i)$ for all i.

Definition. *The* **length of a curve** γ *is given by the integral*

$$\int_a^b |\gamma'(t)|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

where $\gamma(t) = x(t) + iy(t), a \le t \le b$ is a parametrization of γ.

Example 8.1.7. Let γ be the unit circle, which has the parametrization $\gamma(t) = e^{it}, 0 \leq t \leq 2\pi$. Let's verify the circumference of the circle with the formula for the length of γ:

$$\int_0^{2\pi} |\gamma'(t)|\, dt = \int_0^{2\pi} |ie^{it}|\, dt = \int_0^{2\pi} dt = 2\pi.$$

The next proposition contains some useful properties of the line integral.

Proposition 8.1.8. *Suppose γ is a smooth curve and f and g are continuous, complex-valued functions on a domain containing γ.*

(a) $\displaystyle\int_\gamma (f(z) + g(z))\, dz = \int_\gamma f(z)\, dz + \int_\gamma g(z)\, dz.$

(b) For any $c \in \mathbb{C}$, $\displaystyle\int_\gamma cf(z)\, dz = c\int_\gamma f(z)\, dz.$

(c) If τ is a curve whose initial point is the terminal point of γ, then $\gamma\tau$ is defined to be the curve obtained by following γ and then τ. The integral over $\gamma\tau$ is given by

$$\int_{\gamma\tau} f(z)\, dz = \int_\gamma f(z)\, dz + \int_\tau f(z)\, dz.$$

(d) $\displaystyle\left|\int_\gamma f(z)\, dz\right| \leq \max_{z\in\gamma} |f(z)| \cdot \text{length}(\gamma).$

8.2 Differentiability

Recall that the function $f(z) = \dfrac{z}{\bar{z}}$ is not continuous at $z_0 = 0$. This points to the fact that complex functions are somehow different than their real brethren, and in particular the convergence of a function in $\mathbb{C}$ is much stronger than convergence in $\mathbb{R}$.

Definition. *The* **derivative** *of a complex function $f(z)$ at a point $z_0 \in \mathbb{C}$ is defined by*

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

If these limits exist, we say $f(z)$ is **differentiable** *at z_0.*

This definition is the same as in the real case, although as discussed above the notion of a limit is much stronger in $\mathbb{C}$. In the complex world, we have a further notion of differentiability:

Definition. *A complex function $f(z)$ is* **holomorphic** *at $z_0 \in \mathbb{C}$ if $f(z)$ is differentiable on some open disk centered at z_0. Functions which are holomorphic on the whole complex plane $\mathbb{C}$ are called* **entire**.

Example 8.2.1. Many familiar functions from real analysis have the same derivative in the complex plane. For example, $f(z) = z^2$ has derivative $2z$ which may be confirmed by computing either of the above limits. In fact this holds for all $z \in \mathbb{C}$ so z^2 is an entire function.

Example 8.2.2. $f(z) = \bar{z}^2$ is differentiable at 0 and nowhere else, which means $f(z)$ is *not* holomorphic at 0. To see this, write $z = z_0 + re^{i\theta}$. Then the difference quotient can be written

$$\begin{aligned}
\frac{\bar{z}^2 - \bar{z}_0^2}{z - z_0} &= \frac{(\bar{z}_0 + re^{-i\theta})^2 - \bar{z}_0^2}{re^{i\theta}} \\
&= \frac{\bar{z}_0^2 + 2\bar{z}_0 re^{-i\theta} + r^2e^{-2i\theta} - \bar{z}_0^2}{re^{i\theta}} \\
&= \frac{2\bar{z}_0 re^{-i\theta} + r^2e^{-2i\theta}}{re^{i\theta}} = 2\bar{z}_0 e^{-2i\theta} + re^{-3i\theta}.
\end{aligned}$$

If $r \neq 0$ then we get different answers for the limit $z \to z_0$ (e.g. take $\theta = 0$ and $\theta = \frac{\pi}{2}$) which shows that $f(z)$ is not differentiable at any point other than the origin. At $z_0 = 0$, we see that

$$\lim_{z\to z_0} \frac{\bar{z}^2 - \bar{z}_0^2}{z - z_0} = \lim_{z\to z_0} \frac{\bar{z}^2}{z} = 0.$$

Example 8.2.3. Complex conjugation is not differentiable at any $z_0 \in \mathbb{C}$ since

$$\lim_{z\to z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0} = \lim_{z\to z_0} \frac{\overline{z - z_0}}{z - z_0} = \lim_{z\to 0} \frac{\bar{z}}{z}$$

does not exist as we have seen.

Most of the nice properties of real derivatives carry over to the complex place.

Proposition 8.2.4. *Let f and g be differentiable at $z \in \mathbb{C}$.*

(a) $(f(z) + g(z))' = f'(z) + g'(z)$.

(b) For any $c \in \mathbb{C}$, $(cf)'(z) = cf'(z)$.

(c) $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.

(d) If $g(z) \neq 0$ *then* $\left(\dfrac{f(z)}{g(z)}\right)' = \dfrac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$.

(e) $(z^n)' = nz^{n-1}$. *In particular this means that polynomials are entire.*

(f) If g is differentiable at $f(z)$ then $(g(f(z)))' = g'(f(z))f'(z)$.

The fundamental property in this section is a pair of equations called the Cauchy-Riemann Equations, which relate the derivative $f'(z)$ to the partial derivatives with respect to the real and imaginary parts of z.

Theorem 8.2.5 (Cauchy-Riemann Equations)**.** *Let $f(z) = u(x, y) + iv(x, y)$ be a complex function which is continuous at $z_0 = x_0 + iy_0$. Then $f(z)$ is differentiable at z_0 if and only if the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exist, are continuous and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

on some neighborhood of z_0.

Proof. ($\Longrightarrow$) If $f(z)$ is differentiable at $z_0 = x_0 + iy_0$ then

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

First consider approaching z along the line $(x_0 + h) + iy_0$:

$$\begin{aligned}\lim_{h\to 0} \frac{f((x_0+h) + iy_0) - f(x_0 + iy_0)}{h} &= \lim_{h\to 0} \frac{u(x_0 + h, y_0) + iv(x_0 + h, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h} \\ &= \lim_{h\to 0} \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i\frac{v(x_0 + h, y_0) - v(x_0, y_0)}{h} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = f'(z_0).\end{aligned}$$

Next, approach along $x_0 + i(y_0 + h)$:

$$\begin{aligned}\lim_{ih\to 0} \frac{f(x_0 + i(y_0 + h)) - f(x_0 + iy_0)}{ih} &= \lim_{ih\to 0} \frac{u(x_0, y_0 + h) + iv(x_0, y_0 + h) - u(x_0, y_0) - iv(x_0, y_0)}{ih} \\ &= \lim_{h\to 0} \frac{u(x_0, y_0 + h) - u(x_0, y_0)}{ih} + i\frac{v(x_0, y_0 + h) - v(x_0, y_0)}{ih} \\ &= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = f'(z_0).\end{aligned}$$

Setting these two expressions for $f'(z_0)$ equal gives the result, since the real and imaginary parts of the resulting expression must be equal.

($\Longleftarrow$) The converse requires a little more care. We will show that $f(z)$ is differentiable at z_0 with derivative $f'(z_0) = \frac{\partial f}{\partial x}(z_0) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0)$. We first break up the difference quotient, using $h = h_x + ih_y$:

$$\begin{aligned}\frac{f(z_0+h)-f(z_0)}{h} &= \frac{f(z_0+h)-f(z_0+h_x)+f(z_0+h_x)-f(z_0)}{h}\\ &= \frac{f(z_0+h_x+ih_y)-f(z_0+h_x)}{h} + \frac{f(z_0+h_x)-f(z_0)}{h}\\ &= \frac{h_y}{h}\cdot\frac{f(z_0+h_x+ih_y)-f(z_0+h_x)}{h_y} + \frac{h_x}{h}\cdot\frac{f(z_0+h_x)-f(z_0)}{h_x}.\end{aligned}$$

Elsewhere, we have

$$\frac{\partial f}{\partial x}(z_0) = \frac{h_y}{h}\cdot\frac{\partial f}{\partial y}(z_0) + \frac{h_x}{h}\cdot\frac{\partial f}{\partial x}(z_0).$$

Now we subtract these two expressions and take a limit, which gives

$$\begin{aligned}\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} - \frac{\partial f}{\partial x}(z_0) &= \lim_{h\to 0}\left[\frac{h_y}{h}\left(\frac{f(z_0+h_x+ih_y)-f(z_0+h_x)}{h_y} - \frac{\partial f}{\partial y}(z_0)\right)\right]\\ &\quad + \lim_{h\to 0}\left[\frac{h_x}{h}\left(\frac{f(z_0+h_x)-f(z_0)}{h_x} - \frac{\partial f}{\partial x}(z_0)\right)\right].\end{aligned}$$

If we can show that the limits on the right are both 0, then we're done. The ratios $\frac{h_x}{h}$ and $\frac{h_y}{h}$ are both bounded by the triangle inequality, so it suffices to prove the the expressions in parentheses tend to 0. The second term goes to 0 since by definition,

$$\frac{\partial f}{\partial x}(z_0) = \lim_{h_x\to 0}\frac{f(z_0+h_x)-f(z_0)}{h_x}.$$

The other expression is more problematic, since it involves both h_x and h_y. However, the Mean Value Theorem from real analysis gives us real numbers $0 < a, b < 1$ such that

$$\frac{u(x_0+h_x, y_0+h_y) - u(x_0+h_x, y_0)}{h_y} = u_y(x_0+h_x, y_0+ah_y)$$

and

$$\frac{v(x_0+h_x, y_0+h_y) - v(x_0+h_x, y_0)}{h_y} = v_y(x_0+h_x, y_0+bh_y).$$

Substituting these expressions into the first term above gives us

$$\begin{aligned}\frac{f(z_0+h_x+ih_y)-f(z_0+h_x)}{h_y} - \frac{\partial f}{\partial y}(z_0) &= u_y(x_0+h_x, y_0+ah_y) + iv_y(x_0+h_x, y_0+bh_y)\\ &\quad - u_y(x_0,y_0) - iv_y(x_0,y_0)\\ &= (u_y(x_0+h_x, y_0+ah_y) - u_y(x_0,y_0))\\ &\quad + i(v_y(x_0+h_x, y_0+bh_y) - v_y(x_0,y_0)).\end{aligned}$$

Finally, these two pieces each tend to 0 since u_y and v_y are assumed to be continuous at $z_0 = x_0 + iy_0$. This finishes the proof. $\square$

Example 8.2.6. Consider the function

$$f(z) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} + i\dfrac{x^3 + y^3}{x^2 + y^2} & z \neq 0 \\ 0 & z = 0. \end{cases}$$

It is easy to see that the Cauchy-Riemann equations hold for $f(z)$ at $z_0 = 0$, but the complex derivative $f'(0)$ does not exist. This is not a failure of the theorem, however, since the partial derivatives u_x, u_y, v_x and v_y are not continuous at any point but 0.

Example 8.2.7. Consider $f(z) = \operatorname{Log} z$ using the principal branch D as its domain. We may write this as

$$f(z) = \ln|z| + i \operatorname{Arg} z = \tfrac{1}{2}\ln(x^2 + y^2) + i \arctan\left(\tfrac{y}{x}\right).$$

So one sees that $u(x, y) = \frac{1}{2}\ln(x^2 + y^2)$ and $v(x, y) = \arctan\left(\frac{y}{x}\right)$. We calculate the partials:

$$\begin{aligned} u_x &= \frac{x}{x^2 + y^2} & v_x &= -\frac{y}{x^2}\frac{1}{1 + \left(\frac{y}{x}\right)^2} = \frac{-y}{x^2 + y^2} \\ u_y &= \frac{y}{x^2 + y^2} & v_y &= \frac{1}{x}\frac{1}{1 + \left(\frac{y}{x}\right)^2} = \frac{x}{x^2 + y^2}. \end{aligned}$$

Hence $u_x = v_y$ and $u_y = -v_x$ so $f(z)$ satisfies the Cauchy-Riemann equations on D, meaning it is differentiable. Moreover, we can write its derivative as

$$f'(z) = u_x + iv_x = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} = \frac{x - iy}{x^2 + y^2} = \frac{\bar{z}}{|z|^2} = \frac{1}{|z|}.$$

This is a striking, yet perhaps predictable result that reassures us that our definition of the complex logarithm captures the real case.

8.3 Power Series

Definition. *A* **power series** *is an infinite series of the form*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Such a series is said to be centered about z_0.

Example 8.3.1. Power series are really a generalization of a geometric series

$$\sum_{n=0}^{\infty} z^n$$

centered about $z_0 = 0$, where all the coefficients are 1. We know from Section 7.2 that this series converges to $\dfrac{1}{1-r}$ exactly when $|z| < 1$. We will see that power series behave in similar ways, and when they converge, they converge to complex functions that we may be interested in.

For a power series $\displaystyle\sum_{n=0}^{\infty} a_n (z - z_0)^n$ we have three cases for convergence:

(1) The series only converges at $z = z_0$. In this case, the radius of convergence of the series is 0.

(2) The series converges for all z in a disc of finite radius R centered at z_0.

(3) The series converges for all $z \in \mathbb{C}$, in which case we say the series has an infinite radius of convergence.

Examples.

1. Consider the series $\displaystyle\sum_{n=0}^{\infty} n!\, z^n$. By the ratio test,

$$\lim_{n\to\infty} \left| \frac{(n+1)!\, z^{n+1}}{n!\, z^n} \right| = \lim_{n\to\infty} |z|(n+1) = \infty$$

so the series diverges for all positive radii. This is an example of case 1, i.e. the series has no radius of convergence.

2. For $\displaystyle\sum_{n=0}^{\infty} 5^n (z-i)^n$, the ratios test gives us

$$\lim_{n\to\infty} \left| \frac{5^{n+1}(z-i)^{n+1}}{5^n (z-i)^n} \right| = \lim_{n\to\infty} 5|z-i|.$$

So the series converges (absolutely) whenever $5|z-i| < 1 \implies |z-i| < \frac{1}{5}$. This is an example of case 2, where the series has positive radius of convergence $R = \frac{1}{5}$.

(3) The power series $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is an example of case 3, since it converges (absolutely) for all z as shown again by the ratio test:

$$\lim_{n\to\infty} \left| \frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{z}{n+1} \right| = 0 < 1.$$

A power series with positive or infinite radius of convergence represents a function that is holomorphic within the disc of convergence of the series. This is one of the most important facts in complex analysis, so we take a moment to formalize it here.

Theorem 8.3.2. *Suppose* $\displaystyle\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *has a positive or infinite radius of convergence* R. *Then it represents a function* $f(z)$ *which is holomorphic on* $D = \{z \in \mathbb{C} : |z - z_0| < R\}$.

Proof. This will be proven in Section 8.6. □

Now that we know that power series are holomorphic (differentiable) on their discs of convergence, we can take derivatives.

Theorem 8.3.3. *Suppose* $\displaystyle\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *has a positive or infinite radius of convergence* R. *Then its derivative is also a power series:*

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

which has radius of convergence R.

This can be applied repeatedly to obtain the **Taylor series expansion** of $f(z)$ about z_0:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Example 8.3.4. The Taylor series for the exponential function is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Using the formulas for $\cos z$ and $\sin z$ from Section 7.4, we can derive their Taylor series as well:

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(z - z_0)^{2n}$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(z - z_0)^{2n+1}.$$

8.4 Cauchy's Theorem

We now arrive at a theorem of central importance in complex analysis. The statement of the theorem is simple, but as we will see, this result has far-reaching implications in the complex world.

Theorem 8.4.1 (Cauchy's Theorem)**.** *Let $f(z)$ be a complex function that is holomorphic on domain D, and suppose γ is any piecewise smooth, simple, closed curve in D. Then*

$$\int_\gamma f(z)\,dz = 0.$$

Proof. By assumption $f'(z)$ is continuous on D and γ has interior Ω within D. We compute

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_\gamma (u+iv)(dx+i\,dy) = \int_\gamma (u\,dx - v\,dy + i(v\,dx + u\,dy)) \\
&= \int_\gamma (u\,dx - v\,dx) + i\int_\gamma (v\,dx + u\,dy) \\
&= \iint_\Omega (-v_x - u_y)\,dxdy + i\iint_\Omega (u_x - v_y)\,dxdy \quad \text{by Green's Theorem} \\
&= \iint_\Omega (-v_x + v_x)\,dxdy + i\iint_\Omega (u_x - u_x)\,dxdy \quad \text{by Cauchy-Riemann equations} \\
&= 0 + i0 = 0.
\end{aligned}$$

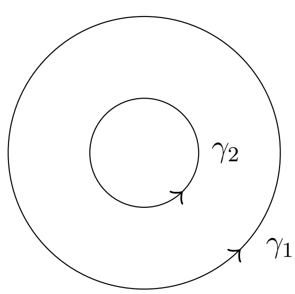
□

Some immediate consequences of Cauchy's Theorem are

Corollary 8.4.2 (Independence of Path)**.** *If γ_1 and γ_2 are curves with the same initial and terminal points lying in a domain on which $f(z)$ is holomorphic, then*

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

Corollary 8.4.3 (Deformation of Path)**.** *Suppose γ_1 and γ_2 are two simple, closed curves with the same orientation, with γ_2 lying on the interior of γ_1.*



If $f(z)$ is holomorphic on the region between γ_1 and γ_2 then

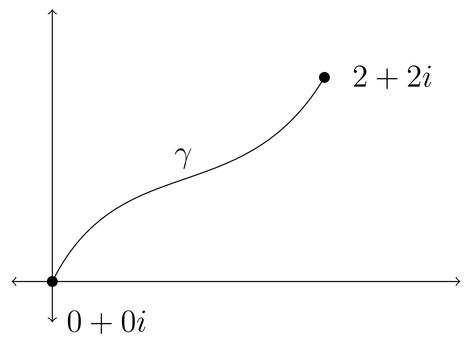
$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

Corollary 8.4.4 (Fundamental Theorem of Calculus)**.** *If $f(z)$ is holomorphic on a simply-connected domain D, then there is a holomorphic function F satisfying*

$$F(z) = \int_\gamma f(z)\,dz$$

for any γ lying in D. Equivalently, F satisfies $F'(z) = f(z)$ on all of D.

Example 8.4.5. Now it's easy to solve an integral such as $\int_\gamma e^z\,dz$ where γ is some path from 0 to $2+2i$:



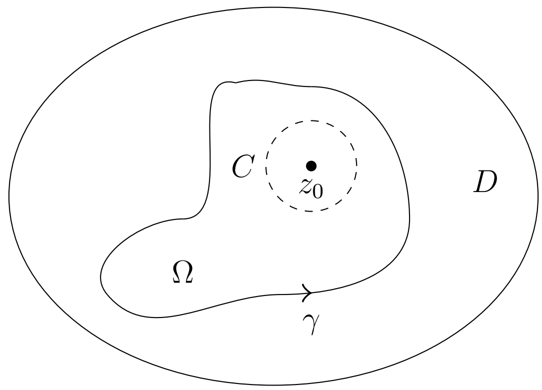
$$\int_\gamma e^z\,dz = e^z|_{2+2i} - e^z|_{0+0i} = e^2(\cos 2 + i\sin 2) - 1.$$

The most important application of Cauchy's Theorem is Cauchy's Integral Formula, which is described in the next section.

8.5 Cauchy's Integral Formula

Theorem 8.5.1 (Cauchy's Integral Formula)**.** *Suppose f is holomorphic on a domain D and γ is a simple closed curve on D, with positive orientation and interior Ω. Then for all $z \in \Omega$,*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$



Proof. Fix $z \in \Omega$ and let C be a circle with center z contained in Ω. Note that for any $z \in D$, $\dfrac{f(\zeta)}{\zeta - z}$ is holomorphic on $D \smallsetminus \{z\}$. By deformation of path,

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

We parametrize C by $z + re^{it}$ for $0 \le t \le 2\pi$ and write

$$\begin{aligned} \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + re^{it})}{re^{it}}\, ire^{it}\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it})\, dt. \end{aligned}$$

Now take the limit as $r \to 0$. Since $f(z)$ is continuous, we can bring the limit inside the integral:

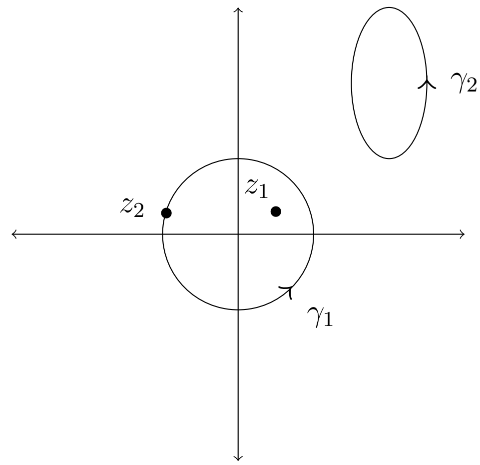
$$\lim_{r \to 0} \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it})\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(z)\, dt.$$

Notice that $f(z)$ doesn't depend on t, so we can integrate this easily and see that it equals $f(z)$. This proves the theorem. $\square$

Example 8.5.2. Cauchy's integral formula allows us to solve path integrals that were previously inaccessible. For example, if γ is a circle about the origin of radius 1, then $z = \frac{1}{2}$ is on its interior and $\dfrac{e^z}{z - \frac{1}{2}}$ is not holomorphic on the interior of γ. However, Cauchy's integral formula lets us compute

$$\int_\gamma \frac{e^z}{z - \frac{1}{2}}\, dz = 2\pi i e^{1/2}.$$

Example 8.5.3. Consider the following contours



First, Cauchy's Theorem (8.4.1) makes it easy to evaluate integrals around γ_2, since z_1 and z_2 are not on the interior of this curve. For example,

$$\int_{\gamma_2} \frac{e^z}{z - z_1}\, dz = 0 \qquad \text{and} \qquad \int_{\gamma_2} \frac{e^z}{z - z_2}\, dz = 0.$$

When a point is on the interior of a curve, we use Cauchy's integral formula (8.5.1):

$$\int_{\gamma_1} \frac{e^z}{z - z_1}\, dz = 2\pi i e^{z_1}.$$

Unfortunately, since z_2 lies directly on γ_1, the integral

$$\int_{\gamma_1} \frac{e^z}{z - z_2}\, dz$$

must be evaluated by hand, e.g. by parametrization.

Example 8.5.4. Using our integration formulas so far, we can break complicated contours down into simple pieces. For example, consider

$$\int_{|z+1|=2} \frac{-z^2}{(z-2)(z+2)}\, dz.$$

The contour of integration is the circle of radius 2 centered at $z_0 = -1$, which contains $z_1 = -2$ on its interior but not $z_2 = 2$. By partial fraction decomposition, we can write

$$\begin{aligned}\int_{|z+1|=2} \frac{-z^2}{(z-2)(z+2)}\, dz &= \int_{|z+1|=2} \left(\frac{-1}{z-2} + \frac{1}{z+2} \right) dz \\ &= \int_{|z+1|=2} \frac{1}{z+2}\, dz - \int_{|z+1|=2} \frac{1}{z-2}\, dz.\end{aligned}$$

The second of these integrals is 0 by Cauchy's Theorem (8.4.1). The first evaluates to $2\pi i$ by Cauchy's integral formula (8.5.1), so we see that the original integral is equal to $2\pi i$.

We can see this another way, by setting $f(z) = \dfrac{-z^2}{z-2}$ and noticing that f is holomorphic on $|z+1| = 2$. Then Cauchy's integral formula (8.5.1) tells us that

$$\int_{|z+1|=2} \frac{-z^2}{(z-2)(z+2)}\, dz = 2\pi i f(-2) = 2\pi i \frac{-4}{-4} = 2\pi i.$$

The next theorem shows that Cauchy's Integral Formula is intimately related to complex power series.

Theorem 8.5.5. *Let f be holomorphic on a domain D and suppose z_0 is a point in D such that the circle $|z - z_0| < R$ for some real R lies in D. Let γ be a simple closed curve lying within this circle and containing z_0 on its interior. Then*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad \textit{where} \qquad a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta$$

Proof. Let $\Delta = \{z : |z - z_0| < R\}$. By deformation of path, it suffices to consider when γ is a circle. For a fixed $r < R$, we take γ to be the positively-oriented circle $\gamma : |z - z_0| = r$. By Cauchy's Integral Formula (8.5.1),

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for any z on the interior of γ. For any one of these z's, let $s = |z - z_0|$ so that $s < r$. Consider

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}.$$

Note that $\dfrac{|z - z_0|}{|\zeta - z_0|} = \dfrac{s}{r} < 1$. This allows us to introduce the series as a convergent geometric series:

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^k.$$

Using this and the expression given by Cauchy's integral formula above, we are able to write

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^k d\zeta \\ &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} (z - z_0)^k \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta. \end{aligned}$$

□

Corollary 8.5.6. *If $f(z)$ is holomorphic on D, f has derivatives of all orders on D and each derivative is holomorphic on D.*

Proof. By Theorem 8.5.5, $f(z)$ can be written as a power series with positive radius of convergence,

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad \text{with} \quad a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta,$$

for some γ about z_0. We will see in Section 8.6 that we can differentiate (and antidifferentiate) power series, so $f(z)$ is infinitely differentiable on the region of convergence of the power series. $\square$

8.6 Analytic Functions

Theorem 8.5.5 suggests a powerful connection between power series and holomorphic functions in the complex plane. In this section we prove that every power series represents a holomorphic function on its region of convergence and every holomorphic function has a power series representation on its domain. First, we need a converse to Cauchy's Theorem (8.4.1).

Theorem 8.6.1 (Morera's Theorem)**.** *Suppose $f(z)$ is continuous on a domain D and*

$$\int_\gamma f(z)\,dz = 0$$

for all smooth, closed curves γ in D. Then f is holomorphic on D.

Proof. We may assume D is connected; otherwise the proof can be repeated on each connected component of D. Fix $z_0 \in D$ and define $F(z) = \displaystyle\int_\gamma f(\zeta)\,d\zeta$ where γ is any smooth curve connecting z_0 and z. By independence of path, $F(z)$ is well-defined for all $z \in D$. Since all *closed* curves γ give $F = 0$ and $f(z)$ is continuous, it follows that $F'(z) = f(z)$, that is, F is an antiderivative of f. Then $F(z)$ is holomorphic on D, which by Corollary 8.5.6 implies that $f(z)$ is also holomorphic on D. □

We prove the first direction of the power series-holomorphic function connection below.

Theorem 8.6.2. *Suppose $f(z) = \displaystyle\sum_{k=0}^\infty a_k(z - z_0)^k$ has a positive radius of convergence R. Then f is a holomorphic function on the domain $D = \{z \in \mathbb{C} : |z - z_0| < R\}$.*

Proof. Given any closed curve γ in D,

$$\int_\gamma \sum_{k=0}^\infty a_k(z - z_0)^k\,dz = 0$$

by continuity of the power series on its region of convergence. Then Morera's Theorem says that $f(z)$ is holomorphic on D. □

Now we know that power series are differentiable on their region of convergence. The next result says that we can differentiate power series term-by-term, just as in the real case.

Theorem 8.6.3. *Suppose $f(z) = \displaystyle\sum_{k=0}^\infty a_k(z - z_0)^k$ has positive radius of convergence R. Then $f(z)$ is differentiable with*

$$f'(z) = \sum_{k=1}^\infty k a_k(z - z_0)^{k-1}$$

which also has radius of convergence R.

Example 8.6.4. In this example we verify the derivatives for $e^z, \cos z$ and $\sin z$. In Example 8.3.4 we saw that the Taylor series expansions for these functions are

$$\begin{aligned} e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ \cos z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(z-z_0)^{2n} \\ \text{and} \quad \sin z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(z-z_0)^{2n+1}. \end{aligned}$$

Differentiating the power series for e^z term-by-term shows that

$$\frac{d}{dz}e^z = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z.$$

We can use the definitions of $\cos z$ and $\sin z$ in terms of the complex exponential function (Section 7.4) to prove that their derivatives are

$$\frac{d}{dz}\cos z = -\sin z \qquad \text{and} \qquad \frac{d}{dz}\sin z = \cos z.$$

We can repeatedly apply Theorem 8.6.3 to subsequent derivatives of f to obtain a statement of Taylor's Theorem for complex functions:

Theorem 8.6.5. *Suppose* $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ *has a positive radius of convergence. Then*

$$a_k = \frac{f^{(k)}(z_0)}{k!}.$$

We now turn to the other connection between holomorphic functions and power series. Well actually, we have already proven (Corollary 8.5.6) that holomorphic functions have power series representations, which we recall here.

Theorem 8.6.6. *Let f be holomorphic on a domain D. Then*

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k \qquad \text{for} \qquad a_k = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z_0)^{k+1}}\,d\zeta$$

where $z_0 \in D$ and γ is a simple closed curve lying in D and containing z_0 on its interior.

We immediately obtain the following generalization of Cauchy's integral formula (8.5.1).

Corollary 8.6.7. *Suppose f is holomorphic on a domain D and γ is a simple closed curve in D, positively oriented and with interior Ω. Then for all $z \in \Omega$ and $n \in \mathbb{N}$,*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta.$$

We now define what it means for a function to be analytic on a certain region in the complex plane.

Definition. *A function $f(z)$ that is continuous on a region $D \subseteq \mathbb{C}$ is* **analytic** *at $z_0 \in D$ if f equals its Taylor series expansion about z_0 and f is analytic on D if it is analytic at every point in D.*

The following theorem summarizes everything we have learned so far about holomorphic functions in the complex plane.

Theorem 8.6.8. *For a complex function $f(z)$ which is continuous on a domain D, the following are equivalent:*

(1) $f(z)$ is differentiable on some open disk centered at $z_0 \in D$, that is, f is holomorphic at z_0.

(2) The Taylor series expansion of $f(z)$ about z_0 converges to $f(z)$ with positive radius of convergence, i.e. f is analytic.

(3) $f(z)$ satisfies the Cauchy-Riemann equations on some neighborhood of z_0.

(4) $\displaystyle\int_\gamma f(z)\,dz = 0$ for every simple closed curve γ inside D with z_0 on its interior (Cauchy's Theorem and Morera's Theorem).

We conclude with a consequence of the generalized Cauchy's integral formula to entire functions that are bounded.

Theorem 8.6.9 (Liouville's Theorem)**.** *If $f(z)$ is entire and there exists a constant M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$, then f is a constant function.*

Proof. Let $z_0 \in \mathbb{C}$ and take C_r to be the circle centered at z_0 with radius $r > 0$. By Corollary 8.6.7,

$$f'(z_0) = \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^2}\,d\zeta.$$

Parametrize the circle by $C_r : z_0 + re^{it}, 0 \leq t \leq 2\pi$. Then

$$\begin{aligned} f'(z_0) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{it})}{r^2 e^{2it}} ire^{it}\,dt \\ &= \frac{1}{2\pi r}\int_0^{2\pi} \frac{f(z_0 + re^{it})}{e^{it}}\,dt. \end{aligned}$$

Taking the modulus of both sides and applying the triangle inequality for integrals, we have

$$\begin{aligned} |f'(z_0)| &\leq \frac{1}{2\pi r}\int_0^{2\pi} \left|\frac{f(z_0 + re^{it})}{e^{it}}\right| dt \\ &= \frac{1}{2\pi r}\int_0^{2\pi} \frac{|f(z_0 + re^{it})|}{|e^{it}|}\,dt \\ &\leq \frac{1}{2\pi r}\int_0^{2\pi} M\,dt. \end{aligned}$$

As we take $r \to 0$, this expression tends to 0 as well, showing $|f'(z_0)| = 0$. Since z_0 was arbitrary, we have shown that $f(z)$ is constant. $\square$

8.7 Harmonic Functions

There is a certain class of holomorphic functions which are important in physics. We study them here.

Definition. *A complex function $f = u+iv$ is* **harmonic** *on a domain D if it has continuous second partial derivatives on D that satisfy the* **Laplace equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The next result says that the real and imaginary parts of a holomorphic function are harmonic.

Proposition 8.7.1. *Suppose $f = u+iv$ is a holomorphic function on a domain D. Then u and v are harmonic on D.*

Proof. Since f is holomorphic, it is infinitely differentiable and so are u and v. In particular, u and v have continuous second partial derivatives. Moreover, f satisfies the Cauchy-Riemann equations:

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x$$

which imply $u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$ since these are continuous. Hence u is harmonic. The proof is the same for v. ☐

Given a harmonic function u, one may be interested in finding a **harmonic conjugate** of u, i.e. another harmonic function v such that $f = u + iv$ is holomorphic in some region of the complex plane.

Example 8.7.2. Consider the function $u(x,y) = \dfrac{x}{x^2+y^2}$. We first show that u is harmonic by computing second partials.

$$\begin{aligned}
u_x &= \frac{-x^2+y^2}{(x^2+y^2)^2} \\
u_{xx} &= \frac{-2x(x^2+y^2)^2 - 4x(x^2+y^2)(-x^2+y^2)}{(x^2+y^2)^4} \\
&= \frac{(x^2+y^2)\cdot 2x(-x^2-y^2+x^2-y^2)}{(x^2+y^2)^4} \\
&= \frac{-4xy^2}{(x^2+y^2)^3} \\
\text{and} \qquad u_y &= \frac{-2xy}{(x^2+y^2)^2} \\
u_{yy} &= \frac{-2x(x^2+y^2)^2 - 4y(x^2+y^2)(-2xy)}{(x^2+y^2)^4} \\
&= \frac{(x^2+y^2)\cdot 2x(4y^2-x^2-y^2)}{(x^2+y^2)^4} \\
&= \frac{4xy^2}{(x^2+y^2)^3}.
\end{aligned}$$

Thus $u_{xx} + u_{yy} = 0$ so $u(x, y)$ is harmonic. Now for $f = u + iv$ to be a holomorphic function, it will need to satisfy the Cauchy-Riemann equations, so $u_x = v_y$ and $u_y = -v_x$. The above shows that we must have $v_x = \dfrac{2xy}{(x^2+y^2)^2}$. Integrating with respect to x,

$$v = \int 2xy(x^2+y^2)^{-2}\,dx = \frac{-y}{x^2+y^2} + y\Psi(y)$$

for some function $\Psi(y)$. Now if we differentiate this with respect to y, we have

$$v_y = \frac{-(x^2+y^2) - 2y(-y)}{(x^2+y^2)^2} + y\Psi'(y) + \Psi(y) = \frac{-x^2+y^2}{(x^2+y^2)^2} + y\Psi'(y) + \Psi(y).$$

By the expression for u_x determined above, we must have $y\Psi'(y) + \Psi(y) = 0$. A general solution to this differential equation is $\Psi(y) = \frac{c}{|y|}$, which gives us

$$v(x, y) = \frac{-y}{x^2+y^2} + y\frac{c}{|y|} = \frac{-y}{x^2+y^2} \pm c$$

and this is holomorphic for all $(x, y) \in \mathbb{C}$ such that $y \neq 0$.

Proposition 8.7.3. *If $u(x, y) = k$ is a constant function, then it has a harmonic conjugate $v(x, y)$ which is also constant.*

Proof. To begin with, u clearly satisfies the Laplace equation so it is harmonic. A harmonic conjugate v must satisfy $v_x = v_y = 0$ by the Cauchy-Riemann equations, so

$$\begin{aligned} v &= \int v_y\,dy = k_1 + \chi(x) \\ \text{and} \qquad v &= \int v_x\,dx = k_2 + \psi(y). \end{aligned}$$

Setting these equal, we have $k_1 + \chi(x) = k_2 + \psi(y)$, so $\chi_x(x) = \psi_y(y) = 0$, showing that each of the functions must be constant, say $\chi(x) = c_1$ and $\psi(y) = c_2$. Therefore $v(x, y) = k_1 + c_1 = k_2 + c_2$, showing v is a constant function. □

In general, the existence of harmonic conjugates is characterized by

Theorem 8.7.4. *Suppose $u(x, y)$ is a harmonic function on the simply connected region $D \subseteq \mathbb{C}$. Then there exists a harmonic conjugate $v(x, y)$ such that $f = u + iv$ is holomorphic on D.*

Proof. Fix $(x_0, y_0) \in D$ and define $v(x, y)$ by

$$v(x, y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x, t)\,dt - \int_{x_0}^{x} \frac{\partial u}{\partial y}(t, y_0)\,dt.$$

Then $f = u + iv$ is holomorphic in D since it satisfies the Cauchy-Riemann equations:

$$\begin{aligned}
&\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)\,dt = \frac{\partial u}{\partial x} \\
\text{and}\quad &\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)\,dt - \frac{\partial^2 u}{\partial x \partial y}(x,y_0) = \int_{y_0}^{y} \frac{\partial^2 u}{\partial x^2}(x,t)\,dt - \frac{\partial^2 u}{\partial x \partial y}(x,y_0) \\
&\quad = -\int_{y_0}^{y} \frac{\partial^2 u}{\partial y^2}(x,t)\,dt - \frac{\partial^2 u}{\partial x \partial y}(x,y_0) \quad \text{using the Laplace equation} \\
&\quad = -\frac{\partial u}{\partial y}(x,y) + \frac{\partial u}{\partial y}(x,y_0) - \frac{\partial u}{\partial y}(x,y_0) = -\frac{\partial u}{\partial y}(x,y).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= \frac{\partial}{\partial x}\int_{y_0}^{y} \frac{\partial^2 u}{\partial x^2}(x,t)\,dt - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}(x,y_0)\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}(x,y)\right) - \frac{\partial}{\partial y}\int_{x_0}^{x} \frac{\partial^2 u}{\partial y^2}(t,y_0)\,dt \\
&= -\frac{\partial}{\partial x}\int_{y_0}^{y} \frac{\partial^2 u}{\partial y^2}(x,t)\,dt - \frac{\partial^2 u}{\partial x \partial y}(x,y_0) + \frac{\partial^2 u}{\partial y \partial x}(x,y) + \frac{\partial}{\partial y}\int_{x_0}^{x} \frac{\partial^2 u}{\partial x^2}(t,y_0)\,dt \\
&= -\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}(x,y) - \frac{\partial u}{\partial y}(x,y_0)\right) - \frac{\partial^2 u}{\partial x \partial y}(x,y_0) + \frac{\partial^2 u}{\partial y \partial x}(x,y) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}(x,y_0) - \frac{\partial u}{\partial x}(x_0,y_0)\right) \\
&= 0.
\end{aligned}$$

So $v(x,y)$ is indeed a harmonic conjugate of $u(x,y)$. □

Corollary 8.7.5. *Every harmonic function is infinitely differentiable on its domain.*

Chapter 9

Meromorphic Functions and Singularities

9.1 Laurent Series

With Theorem 8.6.6, we saw that an analytic function can be written

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad \text{where} \qquad a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta$$

for all z in its domain D. This is highly useful, but when $f(z)$ is *not* analytic on a domain D we still want a way of representing f as a series. This motivates the introduction and application of Laurent series:

Definition. *A* **Laurent series** *is a series expansion of a function $f(z)$ about a point z_0 not in the domain of f in terms of two infinite power series, a positive and negative one:*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k + \sum_{k=1}^{\infty} b_k (z - z_0)^{-k} = \sum_{k \in \mathbb{Z}} c_k (z - z_0)^k.$$

Remark. A Laurent series converges if and only if both the positive and negative series converge. Absolute and uniform convergence are defined analagously. Notice that any Taylor series is a Laurent series whose negative part vanishes.

Example 9.1.1. $f(z) = e^{1/z}$ is not analytic at $z_0 = 0$, but we can write its Laurent series expansion

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-k}.$$

In this case only the $k = 0$ term of the positive series is nonzero.

Example 9.1.2. Consider the function $f(z) = \frac{z^3+z^2}{(z-1)^2}$ about $z_0 = 1$. First we write the regular Taylor series expansion of the numerator about z_0:

$$z^3 + z^2 = \sum_{k=0}^{\infty} a_n (z-1)^k = 2 + 5(z-1) + \frac{8}{2!}(z-1)^2 + \frac{6}{3!}(z-1)^3.$$

Dividing by $(z-1)^2$ yields

$$\frac{z^3 + z^2}{(z-1)^2} = \frac{2}{(z-1)^2} + \frac{5}{z-1} + 4 + (z-1)$$

which is a Laurent series for $f(z)$ about $z_0 = 1$. The coefficients are $b_2 = 2, b_1 = 5, a_0 = 4, a_1 = 1$ and the rest are zero.

Example 9.1.3. Similarly, we use the Taylor series for $\sin z$ to write the Laurent series for $f(z) = \frac{\sin z}{z^3}$ about $z_0 = 0$ as

$$\frac{\sin z}{z^3} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \dots$$

We should take a moment to explicitly describe the region of convergence of a Laurent series. Suppose

$$\sum_{k\in\mathbb{Z}} c_k(z-z_0)^k = \sum_{k=0}^{\infty} a_k(z-z_0)^k + \sum_{k=1}^{\infty} b_k(z-z_0)^{-k}.$$

The positive series has some radius convergence R_1, that is, the series converges on the region $\{z \in \mathbb{C} : |z-z_0| < R_1\}$. Similarly, the negative series is just a power series in $\frac{1}{z-z_0}$ so it has radius of convergence $\frac{1}{R_2}$, i.e. it converges when $\frac{1}{|z-z_0|} < \frac{1}{R_2}$. This can be written as the complement of a closed disk, $\{z \in \mathbb{C} : |z-z_0| > R_2\}$. Thus we see that the Laurent series is convergent on an annular region $\{z \in \mathbb{C} : R_2 < |z-z_0| < R_1\}$ (as long as $R_2 < R_1$). By Theorem 8.6.2, the Laurent series represents an analytic function $f(z)$ on the region $D = \{z \in \mathbb{C} : R_2 < |z-z_0| < R_1\}$. This is made explicit in the next theorem.

Theorem 9.1.4. *Suppose f is a holomorphic function on $D = \{z \in \mathbb{C} : R_1 < |z-z_0| < R_2\}$. Then f is equal to its Laurent series expansion about z_0 which can be written*

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k + \sum_{k=1}^{\infty} b_k(z-z_0)^{-k}$$

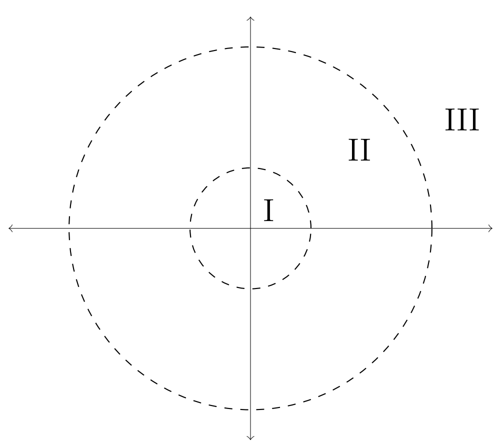
$$\text{where} \qquad a_k = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{(\zeta-z_0)^{k+1}}\,d\zeta \quad \text{and} \quad b_k = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{(\zeta-z_0)^{-k+1}}\,d\zeta$$

for circles C_1 and C_2 centered at z_0 with radii R_1 and R_2, respectively.

Proof. Apply Cauchy's Theorem (8.4.1) and related results to both series. □

Remark. By the definition of their coefficients in terms of the integrals above, Laurent series expansions are unique.

Example 9.1.5. Consider $f(z) = \frac{1}{(z-1)(z-3)}$ on three different regions centered about the origin:



The three regions are given by I : $\{z \in \mathbb{C} : |z| < 1\}$, II : $\{z \in \mathbb{C} : 1 < |z| < 3\}$ and III : $\{z \in \mathbb{C} : |z| > 3\}$. We want to compute Laurent series for $f(z)$ in each of the regions. First we use partial fraction decomposition to write

$$\frac{1}{(z-1)(z-3)} = \frac{-1/2}{z-1} + \frac{1/2}{z-3}.$$

On various regions, we compute the following using geometric series:

$$-\frac{1}{2}\cdot\frac{1}{z-1}=\frac{1}{2}\cdot\frac{1}{1-z}=\frac{1}{2}\sum_{n=0}^{\infty}z^n,\quad |z|<1$$
$$\frac{1}{2}\cdot\frac{1}{z-3}=-\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{1-{}^2\!/_3}=-\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n,\quad |z|<3$$
$$-\frac{1}{2}\cdot\frac{1}{z-1}=-\frac{1}{2}\cdot\frac{1}{z}\cdot\frac{1}{1-{}^1\!/_z}=-\frac{1}{2}\cdot\frac{1}{z}\sum_{n=0}^{\infty}z^{-n},\quad |z|>1$$
$$\frac{1}{2}\cdot\frac{1}{z-3}=\frac{1}{2}\cdot\frac{1}{z}\frac{1}{1-{}^3\!/_z}=\frac{1}{2}\cdot\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n,\quad |z|>3.$$

Putting these together into Laurent series on each region, we have

$$\begin{aligned}
\mathrm{I}: f(z)&=\frac{1}{2}\sum_{n=0}^{\infty}z^n-\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\\
&=\sum_{n=0}^{\infty}\left(\frac{1}{2}-\frac{1}{6}\cdot\frac{1}{3^n}\right)z^n\\
&=\sum_{n=0}^{\infty}\frac{1}{2}(1-3^{-n-1})z^n\\
\mathrm{II}: f(z)&=-\frac{1}{2}\cdot\frac{1}{z}\sum_{n=0}^{\infty}z^{-n}-\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\\
&=\sum_{n=0}^{\infty}-\frac{1}{2}3^{-n-1}z^n+\sum_{n=0}^{\infty}-\frac{1}{2}z^{-n-1}\\
\mathrm{III}: f(z)&=-\frac{1}{2}\cdot\frac{1}{z}\sum_{n=0}^{\infty}z^{-n}+\frac{1}{2}\cdot\frac{1}{z}\sum_{n=0}^{\infty}3^n z^{-n}\\
&=\sum_{n=0}^{\infty}\frac{1}{2}(3^n-1)z^{-n-1}.
\end{aligned}$$

In these we see examples of a Laurent series that is a Taylor series (I), corresponding to a disk on which $f(z)$ is holomorphic; a Laurent series with both positive and negative parts (II), which is holomorphic on an annulus; and a Laurent series with only negative part (III), holomorphic on the complement of a disk.

Laurent series give us a way to deal with 'holes' in the domain of a function which is otherwise holomorphic on the region. Such functions have a special name:

Definition. *A complex function $f(z)$ is* **meromorphic** *on a domain D if it is holomorphic on $D \setminus \{z_1, z_2, \dots, z_r\}$ where r is finite.*

9.2 Isolated Singularities

A singularity is the name we give to a 'hole' in the domain of a complex function. Below we describe the three different types of singularities a function may have.

Definition. *If $f(z)$ is holomorphic on the punctured disk $D = \{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ for some $R > 0$ (R may be infinite) but not at z_0 then z_0 is called an* **isolated singularity** *of f. The three types of isolated singularities are*

(a) z_0 *is a* **removable singularity** *if there is a function g which is holomorphic on the disk $D \cup \{z_0\} = \{z \in \mathbb{C} : |z - z_0| < R\}$ such that $f(z) = g(z)$ for all $z \in D$.*

(b) z_0 *is a* **pole** *if $\lim_{z \to z_0} |f(z)| = \infty$. In particular, z_0 is a* **pole of order m** *if z_0 is a root of $\frac{1}{f(z)}$ with multiplicity m. Equivalently, m is the smallest integer such that $\lim_{z \to z_0} (z - z_0)^{m+1} f(z) = 0$.*

(c) z_0 *is an* **essential singularity** *if it is neither removable nor a pole.*

The isolated singularities of a function may be characterized in terms of Laurent series expansions of the function.

Proposition 9.2.1. *Let z_0 be an isolated singularity of $f(z)$ and suppose $f(z)$ has a Laurent series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

in the region $0 < |z - z_0| < R$.

(a) z_0 is a removable singularity if and only if $b_n = 0$ for all n and there is a function g,

$$g(z) = \begin{cases} f(z) & z \neq z_0 \\ a_0 & z = z_0, \end{cases}$$

which is analytic in $|z - z_0| < R$.

(b) z_0 is a pole of $f(z)$ if and only if all but a finite number of the b_n vanish. Specifically, if $b_n = 0$ for all $n > m$ then z_0 is a pole of order m and f can be written

$$f(z) = \frac{b_m}{(z - z_0)^m} + \frac{b_{m-1}}{(z - z_0)^{m-1}} + \ldots + \frac{b_1}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

(c) z_0 is an essential singularity if and only if infinitely many of the b_n are nonzero.

Examples. We examine functions with each type of isolated singularity.

1. The function $f(z) = \frac{\sin z}{z}$ has a Laurent series which is a Taylor series:

$$\frac{\sin z}{z} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n}$$

on the region $\{z \in \mathbb{C} : |z| > 0\}$. Therefore $z_0 = 0$ is removable and the function g that removes the singularity is

$$g(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0. \end{cases}$$

Note that $g(z)$ is analytic everywhere; it is an entire function. This shows that $f(z)$ is meromorphic on $\mathbb{C} \setminus \{0\}$.

2. Consider $f(z) = \frac{\sin z}{z^4}$ whose Laurent series is given by

$$\frac{\sin z}{z^4} = \frac{1}{z^4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n-3} = \frac{1}{z^3} - \frac{1/6}{z} + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+5)!}z^{2n+1}.$$

This shows that $z_0 = 0$ is a pole of order 3. Moreover, by Theorem 9.1.4 we can use the coefficients of the Laurent series to integrate $f(z)$ around some contour C containing $z_0 = 0$ on its interior:

$$\int_C \frac{\sin z}{z^4}\,dz = b_1 \cdot 2\pi i = \left(-\frac{1}{6}\right)2\pi i = -\frac{\pi i}{3}.$$

3. $\sin\left(\frac{1}{z}\right), \cos\left(\frac{1}{z}\right)$ and $e^{1/z}$ are all functions with essential singularities at $z_0 = 0$. For example, consider the Laurent series expansion of $f(z) = e^{1/z}$:

$$e^{1/z} = \sum_{n=0}^{\infty}\frac{1}{n!}z^{-n} = 1 + z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{6}z^{-3} + \ldots.$$

Although there is not a nice extension of $e^{1/z}$ to an analytic function about $z_0 = 0$, we can still use the b_1 coefficient of its Laurent series to compute contour integrals:

$$\int_C e^{1/z}\,dz = b_1 \cdot 2\pi i = (1)2\pi i = 2\pi i.$$

The next result is rather neat. It says that if $f(z)$ has an essential singularity at z_0 then the image $f(D)$ of any disk D centered at z_0 is dense in $\mathbb{C}$ (in the topological sense).

Theorem 9.2.2 (Casorati-Weierstrass)**.** *If z_0 is an essential singularity of $f(z)$ and $D = \{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ for some positive R, then for any $z \in \mathbb{C}$ and $\varepsilon > 0$, there is some $z' \in D$ such that $|z - f(z')| < \varepsilon$.*

Proof. To contradict, suppose there is some $z \in \mathbb{C}$ and an $\varepsilon > 0$ such that for all $z' \in D$, $|z - f(z')| \geq \varepsilon$. Then $g(z) = \frac{1}{f(z')-z}$ is bounded as $z \to z_0$, so

$$\lim_{z \to z_0} (z - z_0) g(z) = \lim_{z \to z_0} \frac{z - z_0}{f(z') - z} = 0.$$

By Proposition 9.2.1, g has a removable singularity at z_0, and therefore

$$\lim_{z \to z_0} \left| \frac{f(z') - z}{z - z_0} \right| = \infty.$$

This implies that $\frac{f(z')-z}{z-z_0}$ has a pole at z_0, say of order m. By definition,

$$\lim_{z \to z_0} (z - z_0)^{m+1} \frac{f(z') - z}{z - z_0} = \lim_{z \to z_0} (z - z_0)^n (f(z') - z) = 0.$$

Finally, this shows that $f(z') - z$ has a pole or removable singularity at z_0 which implies the same of $f(z)$, but this cannot be the case since z_0 was essential. Hence $f(D)$ must be dense. $\square$

9.3 The Residue Theorem

The examples in Section 9.2 illustrate the connection between the coefficients of the negative part of the Laurent series of a function and contour integrals of the function about its singularities. The coefficient b_1 is of particular importance, so much so that it has a special name.

Definition. *Let z_0 be an isolated singularity of $f(z)$. The* **residue** *of f at z_0 is*

$$Res(f; z_0) := \frac{1}{2\pi i}\int_C f(z)\,dz$$

where $C : |z - z_0| = r$ for some $0 < r < R$, the radius of convergence of the Laurent series for f. This is in turn equal to the b_1 coefficient of the Laurent series.

There is a nice formula for the residues of removable singularities and poles.

Proposition 9.3.1. *Suppose z_0 is a nonessential singularity of $f(z)$.*

(a) If z_0 is a removable singularity, $Res(f; z_0) = 0$.

(b) If z_0 is a pole of order m, then

$$Res(f; z_0) = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}(z - z_0)^m f(z).$$

Proof. (a) follows from Cauchy's Theorem (8.4.1), and (b) is a simple application of Taylor's Theorem to the series

$$(z - z_0)^m f(z) = \sum_{n=-m}^{\infty} c_n (z - z_0)^{n+m}.$$

The formula for $\mathrm{Res}(f; z_0)$ follows from the identification of the residue and b_1. □

Example 9.3.2. Let $f(z) = \dfrac{e^z}{z^2(z - i\pi)^4}$. Then f has a pole of order 2 at $z_0 = 0$, so we define $g(z) = z^2 f(z)$ which is analytic on a small enough neighborhood of 0 (so that it avoids $i\pi$). By Proposition 9.3.1,

$$\mathrm{Res}(f; 0) = \frac{1}{(2-1)!}\lim_{z\to 0}\frac{d^{2-1}}{dz^{2-1}}g(z) = \lim_{z\to 0} g'(z).$$

The first derivative of g is

$$\begin{aligned} g'(z) &= \frac{e^z(z - i\pi)^4 - e^z\cdot 4(z - i\pi)^3}{(z - i\pi)^8} \\ &= \frac{e^z(z - i\pi)^3(z - i\pi - 4)}{(z - i\pi)^8} \\ &= \frac{e^z(z - i\pi - 4)}{(z - i\pi)^5}. \end{aligned}$$

Then the formula for the residue above allows us to compute

$$\operatorname{Res}(f;0) = \lim_{z\to 0} \frac{e^z(z-i\pi-4)}{(z-i\pi)^5} = \frac{-i\pi-4}{(-i\pi)^5} = \frac{1}{\pi^4} + \frac{4}{i\pi^5}.$$

Proposition 9.3.3. *Suppose f and g are analytic on $|z-z_0|<r$ for some $z_0 \in \mathbb{C}$ and $r>0$, and suppose $g(z_0)=0$ but $g'(z_0)\neq 0$. Then*

$$Res\left(\frac{f}{g}; z_0\right) = \frac{f(z_0)}{g'(z_0)}.$$

Proof. Let $g(z)$ have the following power series centered at z_0 (by assumption the series has no c_0 coefficient):

$$g(z) = \sum_{k=1}^{\infty} c_k(z-z_0)^k = (z-z_0)\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

where $a_k = c_{k-1}$; call the analytic function represented by this new series $h(z)$. Note that $h(z_0) = c_1 \neq 0$, so

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z-z_0)h(z)}$$

and $\frac{f}{h}$ is analytic at z_0. Using the definition of residue in terms of the Laurent series coefficients, the residue of $\frac{f}{g}$ is equal to the constant term of the series for $\frac{f}{h}$ (the $n=-1$ term of the series for $\frac{f}{g}$). This is computed to be $\frac{f(z_0)}{h(z_0)}$, but by the way we defined h, $h(z_0) = g'(z_0)$. Hence

$$\operatorname{Res}\left(\frac{f}{g}; z_0\right) = \frac{f(z_0)}{g'(z_0)}.$$

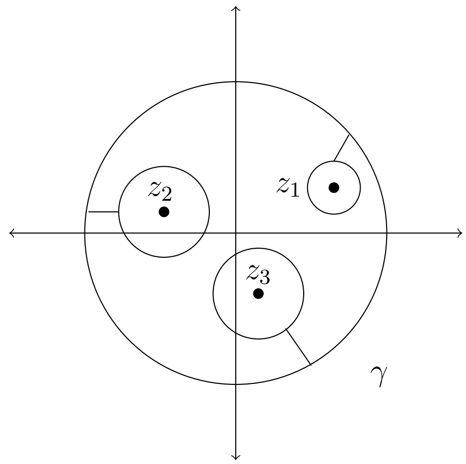
□

We finally arrive at the central theorem in basic complex analysis: the Residue Theorem.

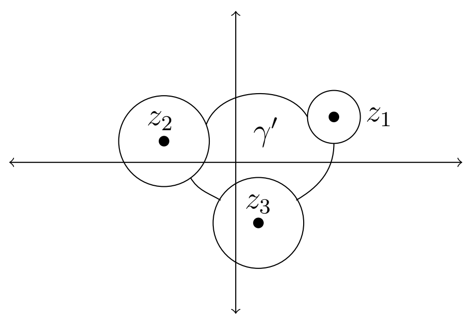
Theorem 9.3.4 (The Residue Theorem)**.** *Suppose $f(z)$ is meromorphic on a region D; let $z_1,\dots,z_n$ be the isolated singularties of f inside D. If γ is a piecewise smooth, positively oriented, simple closed curve lying in D that does not pass through any of the z_i then*

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{i=1}^{n} Res(f;z_i).$$

Proof. Draw a positively-oriented circle C_i around each singularity z_i such that z_i is the only singularity of f on its interior. The case where $n=3$ is illustrated below.



Then γ is contractible to a curve γ' which connects the C_i together and otherwise contains no singularities on its interior. Such a contraction is shown in the next figure.

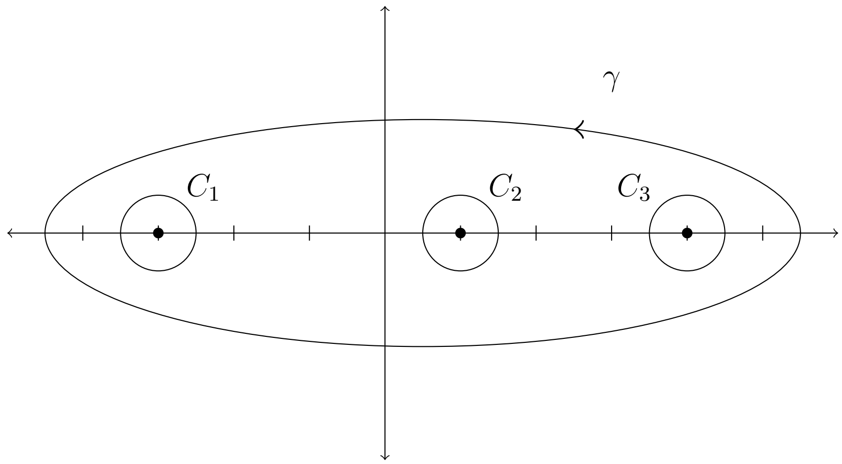


Then $\displaystyle\int_\gamma f(z)\,dz = \int_{\gamma'} f(z)\,dz + \sum_{i=1}^{n} \int_{C_i} f(z)\,dz$ but by construction, $f(z)$ is holomorphic on the interior of γ', so by Cauchy's Theorem (8.4.1) this part equals 0. Evaluate the remaining terms using the definition of residue to produce the main summation formula:

$$\int_\gamma f(z)\,dz = \sum_{i=1}^{n} \int_{C_i} f(z)\,dz = \sum_{i=1}^{n} 2\pi i \operatorname{Res}(f; z_i).$$

□

Example 9.3.5. Evaluate $\displaystyle\int_\gamma \frac{z^2-2z+1}{(z-1)(z-4)(z+3)}\,dz$ about the given contour.



Set $f(z) = \dfrac{z^2 - 2z + 1}{(z-1)(z-4)(z+3)}$. By the Residue Theorem we may evalute the integral of f over γ as

$$\int_\gamma \frac{z^2 - 2z + 1}{(z-1)(z-4)(z+3)}\, dz = 2\pi i(\mathrm{Res}(f; -3) + \mathrm{Res}(f; 1) + \mathrm{Res}(f; 4)).$$

First, note that the function $g(z) = \frac{z-1}{(z-4)(z+3)}$ is holomorphic on C_2 and $f(z) = g(z)$ on the interior of C_1 minus 1. Thus $z_2 = 1$ is a removable singularity, so by Proposition 9.3.1, $\mathrm{Res}(f; 1) = 0$. Next, it is easy to see that $z_1 = -3$ and $z_3 = 4$ are both simple poles, so we compute their residues using the pole formula (Proposition 9.3.1):

$$\mathrm{Res}(f; -3) = \lim_{z \to -3} (z+3) f(z) = \lim_{z \to -3} \frac{z^2 - 2z + 1}{(z-1)(z-4)} = \frac{4}{7}$$
$$\mathrm{Res}(f; 4) = \lim_{z \to 4} (z-4) f(z) = \lim_{z \to 4} \frac{z^2 - 2z + 1}{(z-1)(z+3)} = \frac{3}{7}.$$

Putting this together, we have

$$\int_\gamma \frac{z^2 - 2z + 1}{(z-1)(z-4)(z+3)}\, dz = 2\pi i \left(\frac{4}{7} + 0 + \frac{3}{7}\right) = 2\pi i.$$

9.4 Some Fourier Analysis

Techniques in Fourier analysis are vital in many areas of mathematics and the physical sciences, especially when signal or wave data needs to be broken down into simple components. By studying heat diffusion and wave equations, Joseph Fourier discovered that every continuous function can be approximated with arbitrarily small error by a series of the form

$$\sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

Extending these functions to the complex plane, we can take advantage of Euler's formula $e^{it} = \cos t + i \sin t$.

Definition. *Let f be an integrable, complex-valued function defined on $\mathbb{R}$ (or defined on $\mathbb{C}$ but restricted to $\mathbb{R}$ for this section). The* **Fourier transform** *of F is*

$$\hat{f}(w) = \int_{-\infty}^{\infty} f(x)e^{2\pi iwx}\, dx.$$

The reason the Residue Theorem (9.3.4) is important to the study of Fourier series becomes evident in the next theorem.

Theorem 9.4.1. *Let $f(z)$ be analytic on the half-plane $\mathfrak{H} : \mathrm{Im}(z) \geq 0$ except possibly at a finite number of singularities $\{z_1, \ldots, z_n\}$, all of which have positive imaginary part. Suppose $|f(z)|$ gets arbitrarily small for all $z \in \mathfrak{H}$ with sufficiently large modulus, i.e.*

$$\lim_{R\to\infty} \max_{\substack{|z|=R \\ \mathrm{Im}(z)\geq 0}} |f(z)| = 0.$$

Then for all real numbers $w > 0$,

$$\hat{f}(w) = 2\pi i \sum_{j=0}^{n} Res(f(z)e^{2\pi iwz}; z_j).$$

Similarly, if all of the above conditions hold for the negative half-plane $\mathfrak{H}' : \mathrm{Im}(z) \leq 0$, then

$$\hat{f}(w) = -2\pi i \sum_{j=1}^{n} Res(f(z)e^{2\pi iwz}; z_j).$$

Example 9.4.2. Consider the real-valued function $f(x) = \dfrac{1}{1+x^2}$. We can extend this to a complex function $f(z) = \dfrac{1}{1+z^2}$ which clearly satisfies

$$\lim_{R\to\infty} \max_{\substack{|z|=R \\ \mathrm{Im}(z)\geq 0}} |f(z)| = 0.$$

Then Theorem 9.4.1 above tells us we can compute the Fourier transform of $f(x)$ in terms of residues of $f(z)$: $\hat{f}(w) = 2\pi i \operatorname{Res}(f(z)e^{2\pi i w z}; i)$. Since $f(z)$ only has a simple pole in the upper half-plane at $z_0 = i$, we use Proposition 9.3.1 to compute this residue:

$$\operatorname{Res}(f(z)e^{2\pi i w z}; i) = \lim_{z \to i}(z-i)\frac{e^{2\pi i w z}}{(z-i)(z+i)} = \lim_{z \to i}\frac{e^{2\pi i w z}}{z+i} = \frac{e^{-2\pi w}}{2i}.$$

Hence the Fourier transform of $f(z)$ is $\hat{f}(w) = \pi e^{-2\pi w}$.

Chapter 10

Riemann Surfaces

Riemann surfaces are a mix of the topology of covering spaces and the complex analysis of analytic continuation. The main problem one encounters in the latter setting is that a holomorphic function does not always admit a uniquely defined analytic continatuion. The normal strategy then is to employ ‘branch cuts’, but this tactic seems ad hoc and not suited to generalization. Riemann’s idea was to replace the branches of a function with a covering space on which the analytic continuation is an actual function.

10.1 Holomorphic and Meromorphic Maps

Definition. *Let X be a surface, i.e. a two-dimensional manifold. A* **complex atlas** *on X is a choice of open covering $\{U_i\}$ of X together with homeomorphisms $\varphi_i : U_i \to \varphi_i(U_i) \subseteq \mathbb{C}$ such that for each pair of overlapping charts U_i, U_j, the transition map*

$$\varphi_{ij} := \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$

and its inverse are holomorphic. A **complex structure** *on X is the choice of a complex atlas, up to holomorphic equivalence of charts, defined by a similar condition to the above. A connected surface which admits a complex structure is called a* **Riemann surface***.*

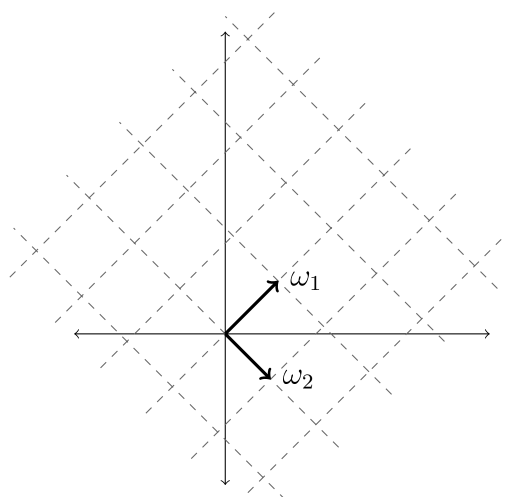
Example 10.1.1. The complex plane $\mathbb{C}$ is a trivial Riemann surface. Any connected open subset U in $\mathbb{C}$ is also a Riemann surface via the given embedding $U \hookrightarrow \mathbb{C}$.

Example 10.1.2. The complex projective line $\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ admits a complex structure defined by the open sets $U_0 = \mathbb{P}^1 \smallsetminus \{\infty\} = \mathbb{C}$ and $U_1 = \mathbb{P}^1 \smallsetminus \{0\} = \mathbb{C}^\times \cup \{\infty\}$, together with charts

$$\varphi_0 : U_0 \to \mathbb{C}, z \mapsto z \quad \text{and} \quad \varphi_1 : U_1 \to \mathbb{C}, z \mapsto \frac{1}{z},$$

where $\frac{1}{\infty} = 0$ by convention. Note that $\varphi_1 \circ \varphi_0^{-1}$ is the function $z \mapsto \frac{1}{z}$ on $\mathbb{C}^\times$ which is holomorphic.

Example 10.1.3. Let $\Lambda \subseteq \mathbb{C}$ be a lattice with basis $[\omega_1, \omega_2]$.



Then the quotient $\mathbb{C}/\Lambda$ admits a complex structure as follows. Let $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ be the quotient map and suppose $\Pi \subseteq \mathbb{C}$ is a *fundamental domain* for Λ, meaning no two points in Π are equivalent mod Λ. Set $U = \pi(\Pi) \subseteq \mathbb{C}/\Lambda$. Then $\pi|_\Pi : \Pi \to U$ is a homeomorphism, so let $\varphi : U \to \Pi$ be its inverse. Letting $\{U_i\}$ be the collection of all images under π of fundamental domains for Λ, we get a complex atlas on $\mathbb{C}/\Lambda$ (one can easily check that the transition functions between the U_i are locally constant, hence holomorphic). Topologically, $\mathbb{C}/\Lambda$ is homeomorphic to a torus.

Definition. *A function $f : U \to \mathbb{C}$ on an open subset U of a Riemann surface X is* **holomorphic** *if for every complex chart $\varphi : V \to \varphi(V) \subseteq \mathbb{C}$, the function $f \circ \varphi^{-1} : \varphi(U \cap V) \to U \cap V \to \mathbb{C}$ is holomorphic.*

Let $\mathcal{O}(U)$ denote the set of all holomorphic functions $U \to \mathbb{C}$.

Lemma 10.1.4. *For any open set U of a Riemann surface X, $\mathcal{O}(U)$ is a commutative $\mathbb{C}$-algebra.*

Proposition 10.1.5 (Holomorphic Continuation)**.** *For any open set $U \subseteq X$ of a Riemann surface and any $x \in U$, if $f \in \mathcal{O}(U \smallsetminus \{x\})$ is bounded in a neighborhood of x, then f extends uniquely to some $\tilde{f} \in \mathcal{O}(U)$.*

More generally, we can define holomorphic maps between two Riemann surfaces.

Definition. *A continuous map $f : X \to Y$ between Riemann surfaces is called* **holomorphic** *if for every pair of charts $\varphi : U \to \varphi(U) \subseteq \mathbb{C}$ on X and $\psi : V \to \psi(V) \subseteq \mathbb{C}$ on Y such that $f(U) \subseteq V$, the map*

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to U \to V \to \psi(V)$$

is holomorphic. We say f is **biholomorphic** *if it is a bijection and its inverse f^{-1} is also holomorphic. In this case X and Y are said to be* **isomorphic as Riemann surfaces***.*

Lemma 10.1.6. *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are holomorphic maps between Riemann surfaces, then $g \circ f : X \to Z$ is also holomorphic.*

Proposition 10.1.7. *Let $f : X \to Y$ be a holomorphic map. Then for all open $U \subseteq X$, there is an induced $\mathbb{C}$-algebra homomorphism*

$$\begin{aligned} f^* : \mathcal{O}(U) &\longrightarrow \mathcal{O}(f^{-1}(U)) \\ \psi &\longmapsto f^*\psi := \psi \circ f. \end{aligned}$$

Proof. The fact that $f^*\psi$ is an element of $\mathcal{O}(f^{-1}(U))$ follows from the above definitions of $\mathcal{O}$ and a holomorphic map between Riemann surfaces. The ring axioms are also easy to verify. □

Theorem 10.1.8. *Suppose $f, g : X \to Y$ are holomorphic maps between Riemann surfaces such that there exist a set $A \subseteq X$ containing a limit point $a \in A$ and $f|_A = g|_A$. Then $f = g$.*

Proof. Let $U \subseteq X$ be the set of all $x \in X$ with an open neighborhood W on which $f|_W = g|_W$. Then U is open and $a \in U$; we will show it is also closed. If $x \in \partial U$, we have $f(x) = g(x)$ since f and g are continuous. Choose a neighborhood $W \subseteq X$ of x and charts $\varphi : W \to \varphi(W) \subseteq \mathbb{C}$ and $\psi : W' \to \psi(W') \subseteq \mathbb{C}$ in Y with $f(W) \subseteq W'$ and $g(W) \subseteq W'$. Consider

$$F = \psi \circ f \circ \varphi^{-1} : \varphi(W) \to \psi(W') \quad \text{and} \quad G = \psi \circ g \circ \varphi^{-1} : \varphi(W) \to \psi(W').$$

Then F and G are holomorphic and $W \cap U \neq \varnothing$, so we must have $F = G$. Therefore $f|_W = g|_W$, so $x \in U$ after all. This implies $U = X$. □

Definition. *A* **meromorphic function** *on an open set $U \subseteq X$ consists of an open subset $V \subseteq U$ and a holomorphic function $f : V \to \mathbb{C}$ such that $U \smallsetminus V$ contains only isolated points, called the* **poles** *of f, and $\lim_{x\to p} |f(x)| = \infty$ for every pole $p \in U \smallsetminus V$.*

Denote the set of meromorphic functions on U by $\mathcal{M}(U)$. Then $\mathcal{M}(U)$ is a $\mathbb{C}$-algebra, where $f + g$ and fg are defined by meromorphic continuation.

Example 10.1.9. Any polynomial $f(z) = c_0 + c_1 z + \ldots + c_n z^n$ is a holomorphic function $\mathbb{C} \to \mathbb{C}$. Viewing $\mathbb{C} \subseteq \mathbb{P}^1$, f is a meromorphic function on $\mathbb{P}^1$ with only a pole at ∞ of order n (assuming $c_n \neq 0$).

Example 10.1.10. Any meromorphic function $f \in \mathcal{M}(X)$ may be represented by a Laurent series expansion about any of its poles p by choosing a complex chart $U \to \mathbb{C}$ containing p, lifting z to a parameter t on U and writing

$$f(t) = \sum_{n=-N}^{\infty} c_n t^n \text{for some } c_n \in \mathbb{C}.$$

Theorem 10.1.11. *Suppose X is a Riemann surface. Then the set of meromorphic functions $\mathcal{M}(X)$ is in bijection with the set of holomorphic maps $X \to \mathbb{P}^1$.*

Proof. If $f \in \mathcal{M}(X)$ is a meromorphic function, then setting $f(p) = \infty$ for every pole p of f defines a holomorphic map $f : X \to \mathbb{P}^1$. Indeed, it is clear that f is continuous. Let P be the set of its poles. If $\varphi : U \to \varphi(U) \subseteq \mathbb{C}$ is a chart on X and $\psi : V \to \psi(V) \subseteq \mathbb{C}$ is a chart on $\mathbb{P}^1$ with $f(U) \subseteq V$, then since f is holomorphic on $X \smallsetminus P$, $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is holomorphic on $\varphi(U) \smallsetminus \varphi(P)$. By Proposition 10.1.5, $\psi \circ f \circ \varphi^{-1}$ is actually holomorphic on $\varphi(U)$, so f is a holomorphic map of Riemann surfaces.

Conversely, if $g : X \to \mathbb{P}^1$ is holomorphic, then by Theorem 10.1.8, either $g(X) = \{\infty\}$ or $g^{-1}(\infty)$ is a set of isolated points in X. It is then easy to see that $g : X \smallsetminus g^{-1}(\infty) \to \mathbb{C}$ is meromorphic. □

Corollary 10.1.12 (Meromorphic Continuation)**.** *For any open set $U \subseteq X$ and any $x \in U$, if $f \in \mathcal{M}(U \smallsetminus \{x\})$ is bounded in a neighborhood of x, then f extends uniquely to some $\tilde{f} \in \mathcal{M}(U)$.*

Proof. Apply Proposition 10.1.5 and Theorem 10.1.11. □

Corollary 10.1.13. *Any nonzero function in $\mathcal{M}(X)$ has only isolated zeroes. In particular, $\mathcal{M}(X)$ is a field.*

Theorem 10.1.14. *Let $f : X \to Y$ be a nonconstant holomorphic map between Riemann surfaces. Then for every $x \in X$ with $y = f(x) \in Y$, there exists $k \in \mathbb{N}$ and complex charts $\varphi : U \to \varphi(U) \subseteq \mathbb{C}$ of X and $\psi : V \to \psi(V) \subseteq \mathbb{C}$ of Y with $f(U) \subseteq V$ such that*

(1) $x \in U$ with $\varphi(x) = 0$ and $y \in V$ with $\psi(y) = 0$.

(2) $F = \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is given by $F(z) = z^k$ for all $z \in \varphi(U)$.

Proof. It is easy to arrange (1) by replacing (U, φ) with another chart obtained by composing φ with an automorphism of $\mathbb{C}$ taking $\varphi(x) \mapsto 0$. So without loss of generality assume (1) is satisfied. By Theorem 10.1.8, $F = \psi \circ f \circ \varphi^{-1}$ is nonconstant. Thus since $f(0) = 0$, we may write $F(z) = z^k g(z)$ for some $k \geq 1$ and some $g \in \mathcal{O}(\varphi(U))$ with $g(0) \neq 0$. Then $g(z) = h(z)^k$ for some holomorphic function h on $\varphi(U)$, and $H(z) = zh(z)$ defines a biholomorphic map α of some open neighborhood $W \subseteq \varphi(U)$ of 0 onto another open neighborhood of 0. Finally, replace (U, φ) by $(\varphi^{-1}(W), \alpha \circ \varphi)$. By construction, $F = \psi \circ f\varphi^{-1}$ is now of the form $F(z) = z^k$. □

Definition. *The integer k for which F can be written $F(z) = z^k$ about $x \in X$ is called the* **multiplicity** *of f at x.*

Corollary 10.1.15. *If $f : X \to Y$ is a nonconstant holomorphic map between Riemann surfaces, then f takes open sets to open sets.*

Corollary 10.1.16. *If $f : X \to Y$ is an injective holomorphic map, then f is biholomorphic $X \to f(X)$.*

Proof. If f is injective, then locally $F(z) = z^k$ with $k = 1$. Hence f^{-1} is holomorphic. □

Corollary 10.1.17 (Maximum Principle)**.** *Suppose X is a Riemann surface and $f : X \to \mathbb{C}$ is a nonconstant holomorphic function. Then $|f|$ does not attain its maximum.*

Proof. Suppose $x_0 \in X$ exists such that $|f(x_0)| = \sup\{|f(x)| : x \in X\}$. Set

$$D = \{z \in \mathbb{C} : |z| \leq |f(x_0)|\}$$

so that $f(x_0)$ lies in the boundary of D. Then $f(X) \subseteq D$, but by Corollary 10.1.15, $f(X)$ is open in D, contradicting $f(x_0) \in \partial D$. □

Theorem 10.1.18. *If $f : X \to Y$ is a nonconstant holomorphic map and X is compact, then Y is also compact and f is surjective.*

Proof. By Corollary 10.1.15, $f(X)$ is open but since X is compact, $f(X)$ is also compact and in particular closed. Therefore $f(X) = Y$. □

Corollary 10.1.19 (Fundamental Theorem of Algebra)**.** *Every nonconstant polynomial $f(z) = c_0 + c_1 z + \ldots + c_n z^n$ with $c_i \in \mathbb{C}$ has a root.*

Proof. Such an f extends to a holomorphic map $f : \mathbb{P}^1 \to \mathbb{P}^1$ by setting $f(\infty) = \infty$. Since $\mathbb{P}^1$ is compact, Theorem 10.1.18 says f is surjective, so $f(z) = 0$ for some $z \in \mathbb{C}$. □

Corollary 10.1.20. *Every holomorphic function on a compact Riemann surface is constant.*

Proof. $\mathbb{C}$ is not compact, so Theorem 10.1.18 implies that every holomorphic function from a compact space into $\mathbb{C}$ must be constant. □

Corollary 10.1.21. *Every meromorphic function on $\mathbb{P}^1$ is rational.*

Proof. First, note that the only way for such an $f \in \mathcal{M}(\mathbb{P}^1)$ to have infinitely many poles is if it had a limit point, but then Theorem 10.1.8 would imply $f \equiv \infty$. Thus f has finitely many poles, say $a_1, \ldots, a_n \in \mathbb{P}^1$; we may assume ∞ is not one of the poles, or else consider the function $\frac{1}{f}$ instead. For $1 \le i \le n$, expand f as a Laurent series about a_i:

$$f_i(z) = \sum_{j=1}^{m_i} c_{ij}(z - a_i)^{-j} \quad \text{for } c_{ij} \in \mathbb{C}.$$

Then $g = f - (f_1 + \ldots + f_n)$ is holomorphic on $\mathbb{P}^1$ and thus constant by Corollary 10.1.20 since $\mathbb{P}^1$ is compact. This shows f is rational. □

Corollary 10.1.20 gives another proof of Liouville's Theorem (8.6.9):

Corollary 10.1.22 (Liouville's Theorem)**.** *Every bounded holomorphic function on $\mathbb{C}$ is constant.*

Proof. By Proposition 10.1.5, f has a holomorphic continuation to $\tilde{f} : \mathbb{P}^1 \to \mathbb{C}$, but by Corollary 10.1.20, $\tilde{f}$ must be constant. □

10.2 Covering Spaces

The idea in this section is to relate holomorphic maps between Riemann surfaces to covering space theory. Recall the following definition from topology.

Definition. *A map $p : Y \to X$ between connected, Hausdorff spaces is a* **covering map** *if each point $x \in X$ has a neighborhood U such that $p^{-1}(U) \subseteq Y$ is a nonempty disjoint union $p^{-1}(U) = \coprod U_\alpha$ such that the restriction $p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism for each U_α. Such a neighborhood U is called an* **evenly covered** *neighborhood of x, and the U_α are called the* **sheets** *of the cover over x. The domain space Y is called a* **covering space** *of X.*

Theorem 10.2.1. *If $p : Y \to X$ is a nonconstant holomorphic map between Riemann surfaces then p is open and has discrete fibres.*

Proof. By Corollary 10.1.15, p is open and Theorem 10.1.8 implies each fibre is discrete. □

Let $p : Y \to X$ be a cover of Riemann surfaces. Traditionally, holomorphic functions $f : Y \to \mathbb{C}$ are treated as multi-valued functions on X by setting $f(x) = \{f(y_1), \ldots, f(y_n)\}$ where $p^{-1}(x) = \{y_1, \ldots, y_n\}$.

Example 10.2.2. Let $\exp : \mathbb{C} \to \mathbb{C}^\times$ be the exponential map $z \mapsto e^z$ and $f = id : \mathbb{C} \to \mathbb{C}$ the identity map. Then the resulting multi-valued function $\mathbb{C}^\times \to \mathbb{C}$ is the complex logarithm, which is only defined as a function after making a particular choice of *branch* of the function. We can describe this idea more cleanly with Riemann surfaces and branched covers.

Definition. *Suppose $p : Y \to X$ is a nonconstant holomorphic map. A* **ramification point** *of p is a point $y \in Y$ such that for every neighborhood $V \subseteq Y$ of y, $p|_V : V \to p(V)$ is not injective. The image $x = p(y)$ is called a* **branch point** *of p. If p has no ramification points (and hence no branch points), then we call p an* **unramified map**.

Theorem 10.2.3. *A nonconstant holomorphic map $p : Y \to X$ is unramified if and only if it is a local homeomorphism.*

Proof. Suppose p is unramified. Then for any $y \in Y$, there exists a neighborhood $V \subseteq Y$ of y such that $p|_V : V \to p(V)$ is injective and open. Therefore $p|_V$ is a homeomorphism onto $p(V)$. The converse follows from basically the same argument. □

Example 10.2.4. For each $n \geq 2$, the map $p_n : \mathbb{C} \to \mathbb{C}$ defined by $p_n(z) = z^n$ is ramified at $0 \in \mathbb{C}$ and unramified everywhere else. Therefore $p_n : \mathbb{C}^\times \to \mathbb{C}$ is an unramified cover. Moreover, Theorem 10.1.14 says that every ramified cover of Riemann surfaces $Y \to X$ is locally of the form $\mathbb{C} \to \mathbb{C}, z \mapsto z^n$.

Example 10.2.5. The exponential map $\exp : \mathbb{C} \to \mathbb{C}^\times, z \mapsto e^z$ is an unramified cover. In fact, as in the topological case, exp gives a universal cover of $\mathbb{C}$ via the inverse system of the covers p_n.

Example 10.2.6. The quotient map $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ from Example 10.1.3 is an unramified cover of Riemann surfaces.

Theorem 10.2.7. *Suppose $p : Y \to X$ is a local homeomorphism of Hausdorff topological spaces and X is a Riemann surface. Then Y admits a unique complex structure making p a holomorphic map.*

Proof. Let $\varphi : V \to \mathbb{C}$ be a chart of X. Then there exists an open subset $U \subseteq V$ over which $p|_U : p^{-1}(U) \to U$ is a homeomorphism. Set $\widetilde{U} = p^{-1}(U)$ and $\widetilde{\varphi} = \varphi \circ p|_U : \widetilde{U} \to \mathbb{C}$. Then $\widetilde{\varphi}$ is a complex chart on Y and the collection $\{\widetilde{U}, \widetilde{\varphi}\}$ obtained in this way forms a complex atlas on Y. Since $p : Y \to X$ is locally biholomorphic by construction, it is a holomorphic map between Riemann surfaces. Uniqueness is easy to check. □

Example 10.2.8. Now that we can view nonconstant holomorphic maps as local homeomorphisms, and in most cases covering spaces, we can rephrase the language of branch cuts as a lifting problem. For example, let $\exp : \mathbb{C} \to \mathbb{C}^\times$ be the exponential map and suppose $f : X \to \mathbb{C}^\times$ is a holomorphic map of Riemann surfaces, with X simply connected. Then by covering space theory, for each fixed $x_0 \in X$ and $z_0 \in \mathbb{C}$ such that $f(x_0) = e^{z_0}$, there exists a unique lift $F : X \to \mathbb{C}$ making the diagram

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ X \arrow[ur, dashed, "F"] \arrow[r, "f"'] & \mathbb{C}^\times \end{tikzcd}$$

commute. Theorem 10.2.7 can be used to show that any such F is holomorphic. Moreover, any other lift G of f differs from F by $2\pi i n$ for some $n \in \mathbb{Z}$. For the special case of a simply connected open set $X \subseteq \mathbb{C}^\times$, any lift F is a branch of the complex logarithm on X.

Example 10.2.9. Similarly, one can construct the complex root functions $z \mapsto z^{1/n}, n \geq 2$, as lifts along the cover $p_n : \mathbb{C}^\times \to \mathbb{C}$.

Let $f : Y \to X$ be a nonconstant holomorphic map that is *proper*, i.e. the preimage of any compact set in X is compact in Y. For each $x \in X$, define the *multiplicity* of f at x to be

$$\mathrm{ord}_x(f) = \sum_{y \in f^{-1}(x)} v_y(f)$$

where $v_y(f)$ is the multiplicity of f at y.

Example 10.2.10. If $f : Y \to X$ is unbranched at $x \in X$, then $p^{-1}(x) = \{y_1, \ldots, y_n\}$ for some n and $v_{y_i}(f) = 1$ for each $1 \leq i \leq n$. Thus $\mathrm{ord}_x(f) = n$.

Theorem 10.2.11. *If $f : Y \to X$ is a proper, nonconstant holomorphic map between Riemann surfaces, then there exists a number $n \in \mathbb{N}$ such that for every $x \in X$, $\mathrm{ord}_x(f) = n$.*

Proof. By Theorem 10.2.1, the set B of ramification points of f is a closed, discrete subset of Y. Let $A = f(B) \subseteq X$. Then since f is proper, A is also closed and discrete. The restriction $f|_{Y \smallsetminus B} : Y \smallsetminus B \to X \smallsetminus A$ is unramified, so it is a finite-sheeted covering space; say n is the number of sheets of $f|_{Y \smallsetminus B}$, i.e. the size of any fibre $f^{-1}(x)$ for an unbranched point

$x \in X$. By the above example, f has multiplicity n at every $y \in Y \setminus B$. Suppose $a \in A$ and write $f^{-1}(a) = \{b_1, \ldots, b_k\} \subseteq B$ and $m_i = v_{b_i}(f)$. For each $1 \leq i \leq k$, we may choose neighborhoods $V_i \subset Y$ of b_i and $U_i \subset X$ of a such that for all $x \in U_i \setminus \{a\}$, $f^{-1}(x) \cap V_i$ consists of exactly m_i points. Then there is a neighborhood $U \subseteq U_1 \cap \cdots \cap U_k$ of a such that $f^{-1}(U) \subseteq V_1 \cup \cdots \cup U_k$ and for every $x \in U \cap (X \setminus A)$, $f^{-1}(x)$ consists of exactly $m_1 + \ldots + m_k$ points. However we showed that $|f^{-1}(x)| = n$, so $n = m_1 + \ldots + m_k$ as required. $\square$

Corollary 10.2.12. *Let X be a compact Riemann surface and $f : X \to \mathbb{C}$ a nonconstant meromorphic function. Then the number of zeroes of f equals the number of poles of f, counted with multiplicity.*

Proof. View f as a holomorphic function $X \to \mathbb{P}^1$. Since X and $\mathbb{P}^1$ are compact, f is a proper map so $\mathrm{ord}_0(f) = \mathrm{ord}_\infty(f)$. But $\mathrm{ord}_0(f)$ is precisely the number of zeroes of f, while $\mathrm{ord}_\infty(f)$ is the number of poles. $\square$

Corollary 10.2.13. *Any complex polynomial $f(z) \in \mathbb{C}[z]$ of degree n has exactly n zeroes, counted with multiplicity.*

Proof. We may view f as a holomorphic map $\mathbb{P}^1 \to \mathbb{P}^1$. Then it is easy to see $\mathrm{ord}_\infty(f) = n$, so once again $\mathrm{ord}_0(f) = n$. $\square$

Chapter 11

Elliptic Functions

In this chapter we review the classical theory of Jacobians for complex curves, starting with the construction and basic properties of elliptic functions, their connection to elliptic curves and their Jacobians, and then describing the construction in arbitrary dimension.

11.1 Elliptic Functions

Let $\Lambda \subseteq \mathbb{C}$ be a lattice, i.e. a free abelian subgroup of rank 2. Then Λ can be written

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \quad \text{for some } \omega_1, \omega_2 \in \mathbb{C} \text{ such that } \frac{\omega_1}{\omega_2} \notin \mathbb{R}.$$

Definition. *A function* $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ *is* **doubly periodic** *with* **lattice of periods** Λ *if* $f(z + \ell) = f(z)$ *for all* $\ell \in \Lambda$ *and* $z \in \mathbb{C}$.

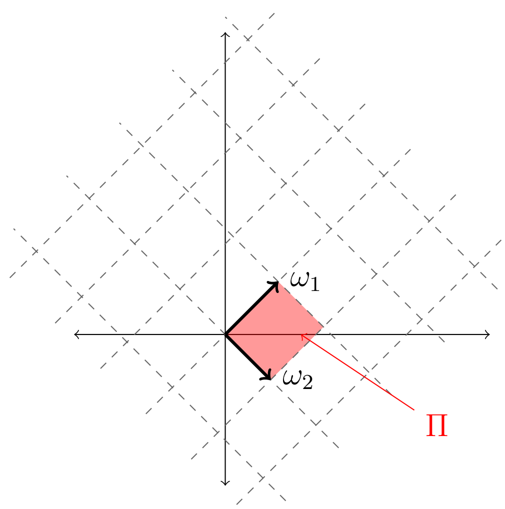
Definition. *An* **elliptic function** *is a function* $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ *that is meromorphic and doubly periodic.*

It is not obvious that doubly periodic functions even exist! We will prove this shortly.

Definition. *Let* $\Lambda \subseteq \mathbb{C}$ *be a lattice. The set*

$$\Pi = \Pi(\omega_1, \omega_2) = \{t_1\omega_1 + t_2\omega_2 \mid 0 \le t_i < 1\}$$

is called the **fundamental parallelogram***, or* **fundamental domain***, of* Λ*. We say a subset* $\Phi \subseteq \mathbb{C}$ *is* **fundamental** *for* Λ *if the quotient map* $\mathbb{C} \to \mathbb{C}/\Lambda$ *restricts to a bijection on* Φ*.*



Lemma 11.1.1. *For any choice of basis* $[\omega_1, \omega_2]$ *of* Λ*,* $\Pi(\omega_1, \omega_2)$ *is fundamental for* Λ*.*

Lemma 11.1.2. *Let* Λ *be a lattice. Then*

(a) If Π *is the fundamental domain of* Λ*, then for any* $\alpha \in \mathbb{C}$*,* $\Pi_\alpha := \Pi + \alpha$ *is fundamental for* Λ*.*

(b) If Φ *is fundamental for* Λ*, then* $\mathbb{C} = \bigcup_{\ell \in \Lambda} \Phi + \ell$*.*

Corollary 11.1.3. *Suppose* f *is an elliptic function with lattice of periods* Λ *and* Φ *fundamental for* Λ*. Then* $f(\mathbb{C}) = f(\Phi)$*.*

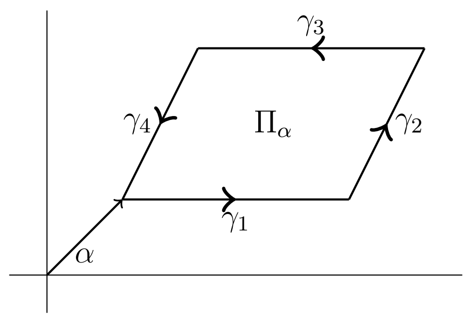
Proposition 11.1.4. *A holomorphic elliptic function is constant.*

Proof. Let f be such an elliptic function and let Φ be the fundamental domain for its lattice of periods. Then $\overline{\Pi}$ is compact and hence $f(\overline{\Pi})$ is as well. In particular, $f(\mathbb{C}) = f(\Pi) \subseteq f(\overline{\Pi})$ is bounded, so by Liouville's theorem, f is constant. □

Proposition 11.1.5. *Let f be an elliptic function. If $\alpha \in \mathbb{C}$ is a complex number such that $\partial\Pi_\alpha$ does not contain any of the poles of f, then the sum of the residues of f inside $\partial\Pi_\alpha$ equals 0.*

Proof. Fix a basis $[\omega_1, \omega_2]$ of Λ and set $\Delta = \partial\Pi_\alpha$. By the residue theorem, it's enough to show $\int_\Delta f(z)\,dz = 0$. We parametrize the boundary of Π as follows:

$$\begin{aligned}
\gamma_1 &= \alpha + t\omega_1\\
\gamma_2 &= \alpha + \omega_1 + t\omega_2\\
\gamma_3 &= \alpha + (1-t)\omega_1 + \omega_2\\
\gamma_4 &= \alpha + (1-t)\omega_2.
\end{aligned}$$



We show that $\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz = 0$ and leave the proof that $\int_{\gamma_2} f(z)\,dz + \int_{\gamma_2} f(z)\,dz = 0$ for exercise. Consider

$$\begin{aligned}
\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz &= \int_0^1 f(\alpha + t\omega_1)(\omega_1\,dt) + \int_0^1 f(\alpha + (1-t)\omega_1 + \omega_2)(-\omega_1\,dt)\\
&= \omega_1 \int_0^1 f(\alpha + t\omega_1)\,dt + \omega_1 \int_1^0 f(\alpha + s\omega_1)\,ds \quad \text{since } f \text{ is elliptic}\\
&= \omega_1 \left(\int_0^1 f(\alpha + t\omega_1)\,dt - \int_0^1 f(\alpha + s\omega_1)\,ds \right) = 0.
\end{aligned}$$

Hence the sum of the residues equals 0. □

Corollary 11.1.6. *Any elliptic function has either a pole of order at least 2 or two poles on the fundamental domain of its lattice of periods.*

Proposition 11.1.7. *Suppose f is an elliptic function with fundamental domain Π and $\alpha \in \mathbb{C}$ such that $\Delta = \partial\Pi_\alpha$ does not contain any zeroes or poles of f. Let $\{a_j\}_{j=1}^n$ be a finite set of zeroes and poles in Π_α, with m_j the order of the pole a_j. Then $\sum_{j=1}^n m_j = 0$.*

Proof. For a pole z_0, we can write $f(z) = (z - z_0)^m g(z)$ for some holomorphic function $g(z)$, with $g(z_0) \neq 0$. Then

$$\frac{f'(z)}{f(z)} = (z - z_0)^{-1} \left(m + (z - z_0) \frac{g'(z)}{g(z)} \right).$$

Hence $\operatorname{Res}\left(\frac{f'}{f}; z_0\right) = m$. Then the statement follows from Proposition 11.1.5. □

Proposition 11.1.7 has an analogue in algebraic geometry: if f is a rational function on an algebraic curve C, the formal sum $(f) = \sum m_j a_j$, where the a_j and m_j are defined as above, is called the *principal divisor* associated to f and its *degree* is $\deg(f) = \sum m_j$. Then one can prove that $\deg(f) = 0$.

Continuing in the complex setting, let f be an elliptic function and let $a_1, \ldots, a_r$ be the poles and zeroes of f in the fundamental domain of Λ. Write $\operatorname{ord}_{a_i} f = m_i$ if a_i is a pole of order $-m_i$ or if a_i is a zero of multiplicity m_i. The sum $\operatorname{ord}(f) = \sum_{i=1}^r m_i$ is called the *order* of f. Then Corollary 11.1.6 says that there are no elliptic functions of order 1. We will show that the field of elliptic functions with period lattice Λ is generated by an order 2 and an order 3 function.

Let f be elliptic and $z_0 \in \mathbb{C}$ with $\operatorname{ord}_{z_0} f = m$. Then for any $\ell \in \Lambda$, $\operatorname{ord}_{z_0+\ell} f = m$ as well. Indeed, if z_0 is a zero then

$$0 = f(z_0) = f(z_0) = \ldots = f^{(m-1)}(z_0)$$

but $f^{(k)}(z)$ is also elliptic for all $k \geq 1$. If z_0 is a pole of f, the same result can be obtained using $\frac{1}{f}$ instead of f.

If Φ_1 and Φ_2 are any two fundamental domains for Λ, then for all $a_1 \in \Phi_1$, there is a unique $a_2 \in \Phi_2$ such that $a_2 = a_1 + \ell$ for some $\ell \in \Lambda$. Thus Propositions 11.1.5 and 11.1.7 hold for any fundamental domain of Λ, so it follows that $\operatorname{ord}(f)$ is well-defined on the quotient $\mathbb{C}/\Lambda$.

Now given any meromorphic function $f(z)$ on $\mathbb{C}$, we would like to construct an elliptic function $F(z)$ with lattice Λ. Put

$$F(z) = \sum_{\ell \in \Lambda} f(z + \ell).$$

There are obvious problems of convergence and (in a related sense) the order of summation. It turns out we can do this construction with $f(z) = \frac{1}{z^m}, m \geq 3$ though. First, we need the following result, which can be proven using Cauchy's integral formula (8.5.1) and Morera's theorem (8.6.1).

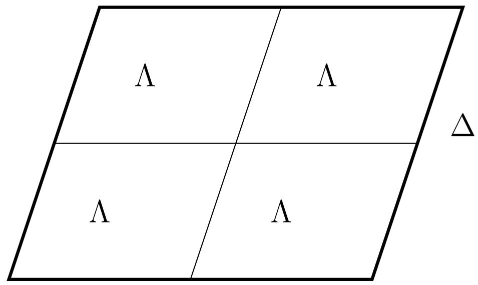
Lemma 11.1.8. *Let $U \subseteq \mathbb{C}$ be an open set and suppose (f_n) is a sequence of holomorphic functions on U such that $f_n \to f$ uniformly on every compact subset of U. Then f is holomorphic on U and $f_n' \to f'$ uniformly on every compact subset of U.*

Proposition 11.1.9. *Let Λ be a lattice with basis $[\omega_1, \omega_2]$. Then the sum*

$$\sum_{\omega \in \Lambda \smallsetminus \{0\}} \frac{1}{|\omega|^s}$$

converges for all $s > 2$.

Proof. Extend the fundamental domain by translation by the vectors ω_1, ω_2 and $\omega_1 + \omega_2$, and call the boundary of the resulting region Δ:



Then Δ is compact, so there exists $c > 0$ such that $|z| \geq c$ for all $z \in \Delta$. We claim that for all $m, n \in \mathbb{Z}$,

$$|m\omega_1 + n\omega_2| \geq c \cdot \max\{|m|, |n|\}.$$

The cases when $m = 0$ or $n = 0$ are trivial, so without loss of generality assume $m \geq n > 0$. Then

$$|m\omega_1 + n\omega_2| = |m| \left|\omega_1 + \frac{n}{m}\omega_2\right| \geq |m|c.$$

Hence the claim holds. Set $M = \max\{|m|, |n|\}$ and arrange the sum in question so that the $\frac{1}{|\omega|^s}$ are added in order of increasing M values. Then the sum can be estimated by

$$\sum_{\omega \in \Lambda \smallsetminus \{0\}} \frac{1}{|\omega|^s} \leq \sum_{M=1}^{\infty} \frac{8M}{c^s M^s} \sim \sum_{M=1}^{\infty} \frac{1}{M^{s-1}}.$$

This converges for $s > 2$ by p-series. □

Proposition 11.1.10. *Let $n \geq 3$ and define*

$$F_n(z) = \sum_{\omega \in \Lambda} \frac{1}{(z - \omega)^n}.$$

Then $F_n(z)$ is holomorphic on $\mathbb{C} \smallsetminus \Lambda$ and has poles of order n at the points of Λ. Moreover, F_n is doubly periodic and hence elliptic.

Proof. Fix $r > 0$ and let $B_r = B_r(0)$ be the open complex r-ball centered at the origin in $\mathbb{C}$. Let $\Lambda_r = \Lambda \cap \overline{B}_r$ be the lattice points contained in the closed r-ball. Then the function

$$F_{n,r}(z) = \sum_{\omega \in \Lambda \smallsetminus \Lambda_r} \frac{1}{(z - \omega)^n}$$

is holomorphic on B_r. To see this, one has $\frac{1}{|z-\omega|^n} \leq \frac{C}{|\omega|^n}$ for some constant C and for all $z \in B_r, \omega \in \Lambda \smallsetminus \Lambda_r$. Then $\frac{C}{|\omega|^n}$ converges by Proposition 11.1.9, so by the Weierstrass M-test, $\frac{1}{|z-\omega|^n}$ converges uniformly and hence $F_{n,r}(z)$ is holomorphic. It follows from the definition that F_n has a pole of order n at each $\omega \in \Lambda$. Finally, for $\ell \in \Lambda$, we have

$$F_n(z + \ell) = \sum_{\omega \in \Lambda} \frac{1}{(z + \ell - \omega)^n} = \sum_{\eta \in \Lambda} \frac{1}{(z - \eta)^n} = F_n(z)$$

since the series is absolutely convergent and we can rearrange the terms. □

This shows that elliptic functions exist and more specifically that for each $n \geq 3$, there is at least one elliptic function of order n. Unfortunately the previous proof won't work to construct an elliptic function of order 3. However, Weierstrass discovered the following elliptic function.

Definition. *The* **Weierstrass $\wp$-function** *for a lattice Λ is defined by*

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \smallsetminus \{0\}} \left[\frac{1}{(z-w)^2} - \frac{1}{\omega^2} \right].$$

Theorem 11.1.11. *For any lattice Λ, $\wp(z)$ is an elliptic function with poles of order 2 at the points of Λ and no other poles. Moreover, $\wp(-z) = \wp(z)$ and $\wp'(z) = -2F_3(z)$.*

Proof. (Sketch) To show $\wp(z)$ is meromorphic, one estimates the summands by

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| \leq \frac{D}{|\omega|^3}$$

for some constant D and all $z \in B_r, \omega \in \Lambda \smallsetminus \Lambda_r$ as in the previous proof.

Next, $\wp(z)$ can be differentiated term-by-term to obtain the expression $\wp'(z) = -2F_3(z)$. And proving that $\wp(z)$ is odd is straightforward:

$$\begin{aligned} \wp(-z) &= \frac{1}{(-z)^2} + \sum_{\omega \in \Lambda \smallsetminus \{0\}} \left[\frac{1}{(-z-\omega)^2} - \frac{1}{\omega^2} \right] \\ &= \frac{1}{z^2} + \sum_{-\omega \in \Lambda \smallsetminus \{0\}} \left[\frac{1}{(z-(-\omega))^2} - \frac{1}{(-\omega)^2} \right] = \wp(z) \end{aligned}$$

after switching the order of summation.

Finally, proving $\wp(z)$ is doubly periodic is difficult since we don't necessarily have absolute convergence. However, one can reduce to proving $\wp(z+\omega_1) = \wp(z) = \wp(z+\omega_2)$. Then using the formula for $\wp'(z)$, we have

$$\begin{aligned} \frac{d}{dz}[\wp(z+\omega_1) - \wp(z)] &= -2F_3(z+\omega_1) + 2F_3(z) \\ &= -2F_3(z) + 2F_3(z) = 0 \end{aligned}$$

since $F_3(z)$ is elliptic by Proposition 11.1.10. Hence $\wp(z+\omega_1) - \wp(z) = c$ is constant. Evaluating at $z = -\frac{\omega_1}{2}$, we see that $c = \wp\left(\frac{\omega_1}{2}\right) - \wp\left(-\frac{\omega_1}{2}\right) = 0$ since $\wp(z)$ is odd. Hence $c = 0$, so it follows that $\wp(z)$ is doubly periodic and therefore elliptic. $\square$

Lemma 11.1.12. *Let $\wp(z)$ be the Weierstrass $\wp$-function for a lattice $\Lambda \subseteq \mathbb{C}$ and let Π be the fundamental domain of Λ. Then*

(1) For any $u \in \mathbb{C}$, the function $\wp(z) - u$ has either two simple roots or one double root in Π.

(2) The zeroes of $\wp'(z)$ in Π are simple and they only occur at $\frac{\omega_1}{2}, \frac{\omega_2}{2}$ and $\frac{\omega_1+\omega_2}{2}$.

(3) *The numbers* $u_1 = \wp\left(\frac{\omega_1}{2}\right), u_2 = \wp\left(\frac{\omega_2}{2}\right)$ *and* $u_3 = \wp\left(\frac{\omega_1+\omega_2}{2}\right)$ *are precisely those* u *for which* $\wp(z) - u$ *has a double root.*

Proof. (1) follows from Corollary 11.1.6.

(2) By Theorem 11.1.11, $\deg \wp'(z) = 3$ so it suffices to show that $\frac{\omega_1}{2}, \frac{\omega_2}{2}$ and $\frac{\omega_1+\omega_2}{2}$ are all roots. For $z = \frac{\omega_1}{2}$, we have

$$\wp'\left(\frac{\omega_1}{2}\right) = -\wp'\left(-\frac{\omega_1}{2}\right) = -\wp'\left(\frac{\omega_1}{2} - \omega_1\right) = -\wp'\left(\frac{\omega_1}{2}\right)$$

since $\wp'(z)$ is elliptic. Thus $\wp'\left(\frac{\omega_1}{2}\right) = 0$. The others are similar.

(3) The double roots occur exactly when $\wp'(u) = 0$, so use (2). $\square$

We now prove that any elliptic function can be written in terms of $\wp(z)$ and $\wp'(z)$.

Theorem 11.1.13. *Fix a lattice* $\Lambda \subseteq \mathbb{C}$ *and let* $\mathcal{E}(\Lambda)$ *be the field of all elliptic functions with lattice of periods* Λ*. Then* $\mathcal{E}(\Lambda) = \mathbb{C}(\wp, \wp')$*.*

Proof. Take $f(z) \in \mathcal{E}(\Lambda)$. Then $f(-z) \in \mathcal{E}(\Lambda)$ as well and thus we can write $f(z)$ as the sum of an even and an odd elliptic function:

$$f(z) = f_{even}(z) + f_{odd}(z) = \frac{f(z) + f(-z)}{2} + \frac{f(z) - f(-z)}{2}.$$

We will prove that every even elliptic function is rational in $\wp(z)$, but this will imply the theorem, since then $f_{even}(z) = \varphi(\wp(z))$ and $\frac{f_{odd}(z)}{\wp'(z)} = \psi(\wp(z))$ for some $\varphi, \psi \in \mathbb{C}(\wp(z))$ and we can then write $f(z) = \varphi(\wp(z)) + \wp'(z)\psi(\wp(z))$.

Assume $f(z)$ is an even elliptic function. It's enough to construct $\varphi(\wp(z))$ such that $\frac{f(z)}{\varphi(\wp(z))}$ only has (potential) zeroes and poles at $z = 0$ in the fundamental parallelogram for Λ, since then by Corollary 11.1.6, $\frac{f(z)}{\varphi(\wp(z))}$ is holomorphic and then by Proposition 11.1.4 it is constant. Suppose $f(a) = 0$ for a some zero of order m. Consider $\wp(z) = u$. If $u \neq \wp\left(\frac{\omega_1}{2}\right), \wp\left(\frac{\omega_2}{2}\right), \wp\left(\frac{\omega_1+\omega_2}{2}\right)$ then $\wp(z) = u$ has precisely two solutions in the fundamental parallelogram, $z = a$ and $z = a^*$ where

$$a^* = \begin{cases} \omega_1 + \omega_2 - a & \text{if } a \in \mathrm{Int}(\Pi) \\ \omega_1 - a & \text{if } a \text{ is parallel to } \omega_1 \\ \omega_2 - a & \text{if } a \text{ is parallel to } \omega_2. \end{cases}$$

(Notice that since f is even, $f(a) = 0$ implies $f(a^*) = 0$ as well.) Moreover, if $\mathrm{ord}_a f = 0$ then $\mathrm{ord}_{a^*} f = m$. Note that $a = a^*$ holds precisely when a is in the set $\Theta := \left\{0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}\right\}$.

Let Z (resp. P) be the set of zeroes (resp. poles) of $f(z)$ in Π. Then the assignment $a \mapsto a^*$ is in fact an involution on Z and P, so we can write

$$Z = Z_1' \cup \cdots \cup Z_r' \cup Z_1'' \cup \cdots \cup Z_s''$$
$$P = P_1' \cup \cdots \cup P_u' \cup P_1'' \cup \cdots \cup P_v''$$

where the Z_i' and P_i' are the 2-element orbits of the involution and the Z_j'' and P_j'' are the 1-element orbits. Of course then $s, v \leq 3$. For $a_i' \in Z_i'$, set $\mathrm{ord}_{a_i'} f = m_i'$ and for $a_j'' \in Z_j''$, set $\operatorname{ord}_{a_i''} f = m_i''$, which is even. Likewise, for $b_i' \in P_i'$, set $\operatorname{ord}_{b_i'} f = n_i'$ and for $b_j'' \in P_j''$, set $\operatorname{ord}_{b_i''} f = n_i''$ which is even. Then we define $\varphi(\wp(z))$ by

$$\varphi(\wp(z)) = \frac{\prod_{i=1}^{r}(\wp(z) - \wp(a_i'))^{m_i'} \prod_{j=1}^{s}(\wp(z) - \wp(a_j''))^{m_j''/2}}{\prod_{i=1}^{u}(\wp(z) - \wp(b_i'))^{n_i'} \prod_{j=1}^{v}(\wp(z) - \wp(b_j''))^{n_j}}.$$

Then $\varphi(\wp(z))$ has only potential zeroes/poles at $z = 0$ in the fundamental parallelogram, so we are done. $\square$

11.2 Elliptic Curves

Let $\Lambda \subseteq \mathbb{C}$ be a lattice. There is a canonical way to associate to the complex torus $\mathbb{C}/\Lambda$ an elliptic curve E such that $\mathbb{C}/\Lambda \cong E(\mathbb{C})$. We would also like to reverse this process, i.e. given an elliptic curve E, define a lattice $\Lambda \subseteq \mathbb{C}$ such that $\mathbb{C}/\Lambda \cong E(\mathbb{C})$. This procedure generalizes for a curve C of genus $g > 1$ and produces its Jacobian, $C \hookrightarrow \mathbb{C}^g/\Lambda = J(C)$.

We need the following lemma.

Lemma 11.2.1. *Suppose $f_0, f_1, f_2, \ldots$ is a sequence of analytic functions on the ball $B_r(z_0)$ with Taylor expansions*

$$f_n(z) = \sum_{k=0}^{\infty} a_k^{(n)} (z - z_0)^k.$$

Then if $F(z) = \sum_{n=0}^{\infty} f_n(z)$ converges uniformly on $B_\rho(z_0)$ for all $\rho < r$, each series $A_k = \sum_{n=0}^{\infty} a_k^{(n)}$ converges and $F(z)$ has Taylor expansion

$$F(z) = \sum_{k=0}^{\infty} A_k (z - z_0^k).$$

Let $\wp(z)$ be the Weierstrass $\wp$-function for Λ. Then $\wp'(z)^2$ is an even elliptic function, so by Theorem 11.1.13, $\wp'(z)^2 \in \mathbb{C}(\wp)$. On a small enough neighborhood around $z_0 = 0$,

$$\wp(z) - \frac{1}{z^2} = \sum_{\omega \in \Lambda \setminus \{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right]$$

is analytic. Moreover, for each $\omega \in \Lambda \setminus \{0\}$ we have

$$\begin{aligned} \frac{1}{(z-\omega)^2} &= \frac{1}{\omega^2} + \frac{2z}{\omega^3} + \frac{3z^2}{\omega^4} + \ldots \\ \implies \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} &= \frac{2z}{\omega^2} + \frac{3z^2}{\omega^4} + \ldots \end{aligned}$$

which is uniformly convergent. Hence Lemma 11.2.1 shows that

$$\wp(z) - \frac{1}{z^2} = \sum_{\omega \in \Lambda \setminus \{0\}} \sum_{k=1}^{\infty} \frac{k+1}{\omega^{k+2}} z^k = \sum_{k=1}^{\infty} (k+1) G_{k+2} z^k$$

where $G_m = G_m(\Lambda) := \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^m}$. These G_m are examples of *modular forms*.

Definition. *The series $G_m(\Lambda) = \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^m}$ is called the* **Eisenstein series** *for Λ of weight m.*

From the above work, we obtain the following formulas:

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + 3G_4z^2 + 5G_6z^4 + 7G_8z^6 + \dots \\
\wp(z)^2 &= \frac{1}{z^4} + 6G_4 + \dots \\
\wp(z)^3 &= \frac{1}{z^6} + \frac{9G_4}{z^2} + 15G_6 + \dots \\
\wp'(z) &= -\frac{2}{z^3} + 6G_4z + \dots \\
\wp'(z)^2 &= \frac{4}{z^6} - \frac{24G_4}{z^2} - 80G_6 - \dots
\end{aligned}$$

This implies:

Proposition 11.2.2. *The functions $\wp$ and $\wp'$ satisfy the following relation:*

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

where $g_2 = 60G_4$ and $g_3 = 140G_6$.

Consider the polynomial $p(x) = 4x^3 - g_2x - g_3$, where the g_n are defined for the lattice $\Lambda \subseteq \mathbb{C}$.

Proposition 11.2.3. *$p(x) = 4(x-u_1)(x-u_2)(x-u_3)$ where $u_1 = \wp\left(\frac{\omega_1}{2}\right), u_2 = \wp\left(\frac{\omega_2}{2}\right)$ and $u_3 = \wp\left(\frac{\omega_1+\omega_2}{2}\right)$ are distinct roots.*

Thus $(x,y) = (\wp(z), \wp'(z))$ determine an equation $y^2 = 4x^3 - g_2x - g_3$ which is the defining equation for an elliptic curve E_0 over $\mathbb{C}$. Let $E = E_0 \cup \{[0,1,0]\} \subseteq \mathbb{P}^2$ be the projective closure of E_0. The point $[0,1,0]$ is sometimes denoted ∞.

Theorem 11.2.4. *The map*

$$\begin{aligned}
\varphi : \mathbb{C}/\Lambda &\longrightarrow E(\mathbb{C}) \\
z + \Lambda &\longmapsto \varphi(z+\Lambda) = \begin{cases} [\wp(z), \wp'(z), 1], & z \notin \Lambda \\ [0,1,0], & z \in \Lambda \end{cases}
\end{aligned}$$

is a bijective, biholomorphic map.

Proof. Assume $z_1, z_2 \in \mathbb{C}$ are such that $z_1 + \Lambda \neq z_2 + \Lambda$. Without loss of generality we may assume $z_1, z_2 \in \Pi$, the fundamental domain of Λ (otherwise, translate). If $\wp(z_1) = \wp(z_2)$ and $\wp'(z_1) = \wp'(z_2)$, then with the notation of Theorem 11.1.13, we must have $z_2 = z_1^* \neq z_1$ and thus $z_1, z_2 \notin \Theta = \left\{0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}\right\}$. Since $\wp'(z)$ is odd, we get $\wp'(z_1) = \wp'(z_2) = -\wp'(-z_2) = -\wp'(z_1)$, but this implies $\wp(z_1) = 0$, contradicting $z_1 \notin \Theta$. Therefore φ is one-to-one.

Next, we must show that for any $(x_0, y_0) \in E(\mathbb{C})$, $x_0 = \wp(z)$ and $y_0 = \wp'(z)$ for some $z \in \mathbb{C}$. If $\wp(z_1) = x_0$, then it's clear that $\wp'(z_1) = y_0$ or $-y_0$. Now one shows as in the previous paragraph that we must have $\wp'(z_1) = y_0$.

Now consider $F(x,y) = y^2 - p(x)$, where $p(x) = 4x^3 - g_2x - g_3$. If (x_0, y_0) satisfies $F(x_0, y_0) = 0$ and $y_0 \neq 0$, then $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$ and thus the assignment $(x,y) \mapsto x$ is a local chart about (x_0, y_0). Likewise, $(x,y) \mapsto y$ defines a local chart about (x_0, y_0) when $x_0 \neq 0$. Finally, we conclude by observing that a locally biholomorphic map is biholomorphic. $\square$

In general, an elliptic curve can be defined by a *Weierstrass equation*

$$E : y^2 = f(x) = ax^3 + bx^2 + cx + d.$$

This embeds into projective space via $(x, y) \mapsto [x, y, 1]$. Setting $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$, we also obtain a homogeneous equation for the curve:

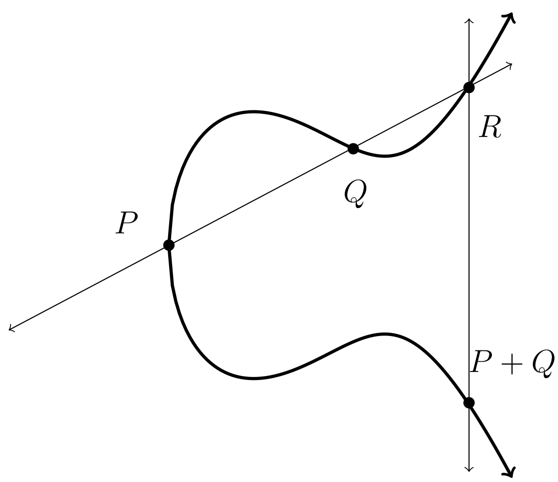
$$E : ZY^2 = aX^3 + bX^2Z + cXZ^2 + dZ^3.$$

The single point at infinity, $[0, 1, 0]$, can be studied by dehomogenizing via the coordinates $\tilde{z} = \frac{Z}{Y}$ and $\tilde{x} = \frac{X}{Y}$, which yield

$$E : \tilde{z} = a\tilde{x}^3 + b\tilde{x}^2\tilde{z} + a\tilde{x}\tilde{z}^2 + d\tilde{z}^3.$$

We have shown that a lattice $\Lambda \subseteq \mathbb{C}$ determines elliptic functions $\wp(z)$ and $\wp'(z)$ that satisfy $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ and that this polynomial expression has no multiple roots. Therefore the mapping $z \mapsto (\wp(z), \wp'(z))$ determines a bijective correspondence $\mathbb{C}/\Lambda \smallsetminus \{0\} \to E(\mathbb{C}) \smallsetminus \{\infty\}$ which can be extended to all of $\mathbb{C}/\Lambda \to E(\mathbb{C})$ (this is Theorem 11.2.4). There is a natural group structure on $\mathbb{C}/\Lambda$ induced from $\mathbb{C}$, but what is not so obvious is that $E(\mathbb{C})$ also possesses a group structure, the so-called "chord-and-tangent method".

Let E be an elliptic curve over an arbitrary field k, let $O \in E(k)$ be the point at infinity and fix two points $P, Q \in E(k)$. In the plane $\mathbb{P}^2_k$, there is a unique line containing P and Q; call it L. (If $P = Q$, then take L to be the tangent line to E at P.) By Bézout's theorem, $E \cap L = \{P, Q, R\}$ for some third point $R \in E(k)$, which may not be distinct from P and Q if multiplicity is counted. Let L' be the line through R and O and call its third point R'.



Addition of two points $P, Q \in E(k)$ is defined by $P + Q = R'$, where R' is the unique point lying on the line through R and O. If $R = O$, we set $R' = O$.

Proposition 11.2.5. *Let E be an elliptic curve with $O \in E(k)$. Then*

(a) If L is a line in $\mathbb{P}^2$ such that $E \cap L = \{P, Q, R\}$, then $(P + Q) + R = O$.

(b) For all $P \in E(k)$, $P + O = P$.

(c) For all $P, Q \in E(k)$, $P + Q = Q + P$.

(d) For all $P \in E(k)$, there exists a point $-P \in E(k)$ satisfying $P + (-P) = O$.

(e) For all $P, Q, R \in E(k)$, $(P + Q) + R = P + (Q + R)$.

Together, (b) – (e) say that chord-and-tangent addition of points defines an associative, commutative group law on $E(k)$. The proofs of (a) – (d) are rather routine using the definition of this addition law, whereas verifying associativity is notoriously difficult. There are formulas for the coordinates of $P + Q$ that make this possible though (see Silverman).

Theorem 11.2.6. *The map $\varphi : \mathbb{C}/\Lambda \to E(\mathbb{C})$ is an isomorphism of abelian groups.*

Proof. Consider the diagram

$$\begin{tikzcd} \mathbb{C}/\Lambda \times \mathbb{C}/\Lambda \arrow[r, "\varphi \times \varphi"] \arrow[d, "\alpha"'] & E(\mathbb{C}) \times E(\mathbb{C}) \arrow[d, "\beta"] \\ \mathbb{C}/\Lambda \arrow[r, "\varphi"'] & E(\mathbb{C}) \end{tikzcd}$$

where α and β are the respective group operations. Since $\mathbb{C}/\Lambda \times \mathbb{C}/\Lambda$ is a topological group, it's enough to show the diagram commutes on a dense subset of $\mathbb{C}/\Lambda \times \mathbb{C}/\Lambda$. Consider

$$\widetilde{X} = \{(u_1, u_2) \in \mathbb{C}^2 \mid u_1, u_2, u_1 \pm u_2, 2u_1 + u_2, u_1 + 2u_2 \notin \Lambda\}.$$

Then $\widetilde{X} \cong \mathbb{C}^2$ so $X = \widetilde{X} \mod \Lambda \times \Lambda$ is dense in $\mathbb{C}/\Lambda \times \mathbb{C}/\Lambda$. Take $(u_1 + \Lambda, u_2 + \Lambda) \in X$ and set $u_3 = -(u_1 + u_2)$. Then $u_1 + u_2 + u_3 = 0$ in $\mathbb{C}/\Lambda$. Set $P = \varphi(u_1), Q = \varphi(u_2)$ and $R = \varphi(u_3) \in E(\mathbb{C})$. By the assumptions on X, the points P, Q, R are distinct. We want to show $\varphi(u_1 + u_2) = \varphi(u_1) + \varphi(u_2) = P + Q$. Since $\wp(z)$ is even and $\wp'(z)$ is odd, we see that $\varphi(-z) = -\varphi(z)$ for all $z \in \mathbb{C}/\Lambda$. Thus $\varphi(u_1 + u_2) = -\varphi(-(u_1 + u_2)) = -R$ so we need to show $P + Q + R = O$, i.e. P, Q, R are colinear. Since $u_1 \neq u_2$, the line $\overline{PQ}$ is not vertical, so there exist a, b such that $\wp'(u_i) = a\wp(u_i) + b$ for $i = 1, 2$. Consider the elliptic function

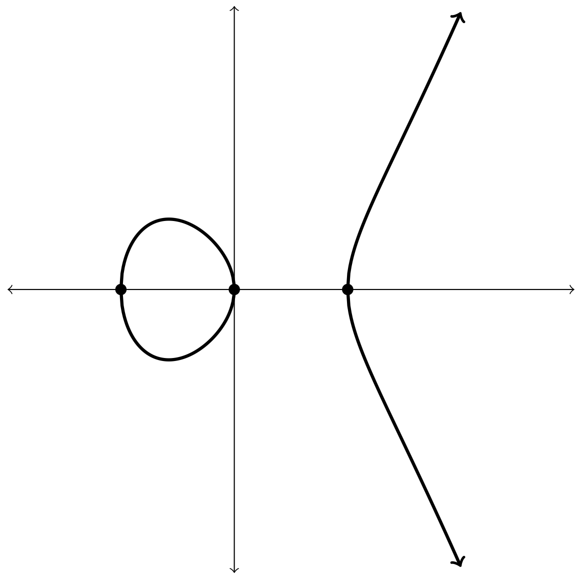
$$f(z) = \wp'(z) - (a\wp(z) + b).$$

Then on the fundamental domain Π, f only has a pole at 0, so $\mathrm{ord}_0 f = -3$. Also, u_1 and u_2 are distinct zeroes of f, so there is a third point $\omega \in \Pi$ such that $\deg(f) = u_1 + u_2 + \omega - 3 \cdot 0 = 0$, i.e. $u_1 + u_2 + \omega = 0$. Solving for ω, we get $\omega = -(u_1 + u_2) = u_3$. It follows that $R = \varphi(u_3)$ is on the same line as P and Q, so we are done. $\square$

The compatibility of the group operations of $\mathbb{C}/\Lambda$ and $E(\mathbb{C})$ is highly useful. For example, fix $N \in \mathbb{N}$ and let

$$E[N] = \{P \in E(\mathbb{C}) \mid [N]P = O\},$$

where $[N]P = \underbrace{P + \ldots + P}_{N}$. The points of $E[N]$ are called the N-torsion points of E. For $N = 2$, the points P such that $P = -P$ are exactly the intersection points of E with the x-axis along with $O = [0, 1, 0]$:



In general, one can show that $\#E[N] = N^2$. This is hard to see from the geometric picture, but working with the isomorphism $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$ from Theorem 11.2.6, we see that since $\mathbb{C}/\Lambda = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ as an abelian group, the N-torsion is given by $(\mathbb{C}/\Lambda)[N] = \frac{1}{N}\mathbb{Z}/\mathbb{Z} \times \frac{1}{N}\mathbb{Z}/\mathbb{Z}$. This is a group of order N^2.

A morphism in the category of elliptic curves is called an *isogeny*. Explicitly, $\varphi : E_1 \to E_2$ is an isogeny between two elliptic curves if it is a (nonconstant) morphism of schemes that takes the basepoint $O_1 \in E_1$ to the basepoint $O_2 \in E_2$.

Proposition 11.2.7. *Suppose $\Lambda_1, \Lambda_2 \subseteq \mathbb{C}$ are lattices and $f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ is a holomorphic map. Then there exist $a, b \in \mathbb{C}$ such that $a\Lambda_1 \subseteq \Lambda_2$ and*

$$f(z \mod \Lambda_1) = az + b \mod \Lambda_2.$$

Proof. As topological spaces, $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are complex tori with the same universal covering space $\mathbb{C}$, so any $f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ lifts to $F : \mathbb{C} \to \mathbb{C}$ making the diagram commute:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "F"] \arrow[d, "\pi_1"'] & \mathbb{C} \arrow[d, "\pi_2"] \\
\mathbb{C}/\Lambda_1 \arrow[r, "f"'] & \mathbb{C}/\Lambda_2
\end{tikzcd}$$

Since covers are local homeomorphisms, it follows that F is holomorphic as well. Thus for any $z \in \mathbb{C}, \ell \in \Lambda_1$,

$$\pi_2(F(z+\ell) - F(z)) = f(\pi_1(z+\ell) - \pi_1(z)) = f(\pi_1(z) - \pi_1(z)) = f(0) = 0.$$

So $F(z+\ell) - F(z) \in \Lambda_1$ for any $\ell \in \Lambda_1$ and the function $L(z) = F(z+\ell) - F(z)$ is constant. It follows that $F'(z+\ell) = F'(z)$, so F' is holomorphic and elliptic, but this means by Proposition 11.1.4 that $F'(z) = a$ for some constant a. Hence $F(z) = az + b$ as claimed. $\square$

Corollary 11.2.8. *Any holomorphic map $f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ is, up to translation, a group homomorphism. In particular, if $f(0) = 0$ then f is a homomorphism.*

Corollary 11.2.9. *For any elliptic curve E, the group of endomorphisms* $\mathrm{End}(E)$ *has rank at most* 2.

Proof. Viewing $E(\mathbb{C}) = \mathbb{C}/\Lambda$ for some $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, we get

$$\begin{aligned}
\mathrm{End}(E) &= \{f : E \to E \mid f \text{ is an isogeny}\} \\
&= \{f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda \mid f \text{ is holomorphic and } f(0) = 0\} \quad \text{by Corollary 11.2.8} \\
&= \{z \in \mathbb{C} \mid z\Lambda \subseteq \Lambda\} \\
&= \{z \in \mathbb{C} \mid z(\mathbb{Z} + \mathbb{Z}\tau) \subseteq (\mathbb{Z} + \mathbb{Z}\tau)\} \\
&\subseteq \mathbb{Z} + \mathbb{Z}\tau.
\end{aligned}$$

Hence $\operatorname{rank} \mathrm{End}(E) \leq 2$. $\square$

It turns out that there are two possible cases for the rank of $\mathrm{End}(E)$, breaking down as follows:

- $\mathrm{End}(E) = \mathbb{Z}$.
- $\mathrm{End}(E)$ is an *order* $\mathcal{O}$ in some imaginary quadratic number field $K/\mathbb{Q}$. In this case, E is said to have *complex multiplication*.

11.3 The Classical Jacobian

For the isomorphism $\varphi : \mathbb{C}/\Lambda \to E(\mathbb{C})$ in Theorem 11.2.6, let $\psi = \varphi^{-1} : E(\mathbb{C}) \to \mathbb{C}/\Lambda$ be the inverse map. To understand this map explicitly, we will show how to construct a torus for every elliptic curve, i.e. find a lattice $\Lambda \subseteq \mathbb{C}$ such that $\mathbb{C}/\Lambda \cong E(\mathbb{C})$.

Lemma 11.3.1. *Any lattice $\Lambda \subseteq \mathbb{C}$ can be written*

$$\Lambda = \left\{ \int_0^P dz : P \in \Lambda \right\}.$$

Notice that each differential form dz on $\mathbb{C}$ satisfies $d(z + \ell) = dz$ for all $\ell \in \Lambda$ by Lemma 11.3.1. Thus dz descends to a differential form on $\mathbb{C}/\Lambda$, which by abuse of notation we will also denote by dz. Formally, this is the pushforward of dz along the quotient $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$. This implies:

Lemma 11.3.2. *Any lattice $\Lambda \subseteq \mathbb{C}$ can be written*

$$\Lambda = \left\{ \int_\gamma dz : \gamma \text{ is a closed curve in } \mathbb{C}/\Lambda \text{ passing through } 0 \right\}.$$

For an elliptic curve E defined by the equation $y^2 = f(x)$, fix a holomorphic differential form ω on $E(\mathbb{C})$. (In general, the space of holomorphic differential forms on a curve has dimension equal to the genus of the curve, so in the elliptic curve case, there is exactly one such ω, up to scaling.)

Definition. *The* **lattice of periods** *for an elliptic curve E is*

$$\Lambda = \left\{ \int_\gamma \omega : \gamma \text{ is a closed curve in } E \text{ passing through } P \right\}$$

where $P \in E(\mathbb{C})$ is fixed.

Example 11.3.3. Under the map $\varphi : \mathbb{C}/\Lambda \to E(\mathbb{C}), z \mapsto (x, y) = (\wp(z), \wp'(z))$, we see that

$$dx = \wp'(z)\, dz = y\, dz$$

so $\omega = \frac{dx}{y}$ is a differential form on $E(\mathbb{C})$. In fact, $\omega = \frac{dx}{f'(x)}$, where E is defined by $y^2 = f(x)$, is holomorphic because $f'(x) \not\equiv 0$. This differential form is also holomorphic at $O = [0, 1, 0]$, so up to scaling, this is the unique holomorphic form on E.

Historically, mathematicians were interested in studying solutions to *elliptic integrals*, or integrals of the form

$$\int \frac{dx}{\sqrt{ax^3 + bx + c}}.$$

When $f(x) = ax^3 + bx + c$, the expression $\omega = \frac{dx}{\sqrt{ax^3+bx+c}}$ is precisely the holomorphic differential form defining the lattice of periods of the elliptic curve $E : y^2 = f(x)$.

For a more functorial description, let $V_E = \Gamma(E, \Omega_E)$ be the space of all holomorphic differential forms on E. If γ is a curve in $E(\mathbb{C})$, there is an associated linear functional $\varphi_\gamma \in V_E^*$ defined by

$$\begin{aligned} \varphi_\gamma : V_E &\longrightarrow \mathbb{C} \\ \omega &\longmapsto \int_\gamma \omega. \end{aligned}$$

Fixing the basepoint $O \in E(\mathbb{C})$, the lattice of periods for E can be written

$$\Lambda = \{\varphi_\gamma : \gamma \in \pi_1(E(\mathbb{C}), O)\}.$$

In other words, this defines a map $\pi_1(E(\mathbb{C}), O) \to V_E^*, \gamma \mapsto \varphi_\gamma$.

Definition. *The* **Jacobian** *of an elliptic curve E is the quotient $J(E) = V_E^*/\Lambda$.*

For each point $P \in E(\mathbb{C})$, the coset $\varphi_\gamma + \Lambda$ is an element of the Jacobian, where γ is a path from O to P. This defines an injective map $i : E \hookrightarrow J(E)$.

Proposition 11.3.4. *Suppose $\sigma : E_1 \to E_2$ is an isogeny between elliptic curves, so that $\sigma(O_1) = O_2$. Then there is a map $\tau : J(E_1) \to J(E_2)$ making the following diagram commute:*

$$\begin{tikzcd} E_1 \arrow[r, "\sigma"] \arrow[d, "i_1"'] & E_2 \arrow[d, "i_2"] \\ J(E_1) \arrow[r, "\tau"'] & J(E_2) \end{tikzcd}$$

Proof. The pullback gives a contravariant map $\sigma^* : V_{E_2} \to V_{E_1}, \omega \mapsto \sigma^*\omega = \omega \circ \sigma$. Taking the dual of this gives a linear map $\sigma^{**} : V_{E_1}^* \to V_{E_2}^*$ defined by $(\sigma^{**}\rho)(\omega) = \rho(\sigma^*\omega)$ for any $\rho \in V_{E_1}^*$ and $\omega \in V_{E_2}$. Taking $\rho = \varphi_{\gamma_1}$ for a path γ_1 in E_1 gives

$$\rho(\sigma^*\omega) = \varphi_{\gamma_1}(\sigma^*\omega) = \int_{\gamma_1} \sigma^*\omega = \int_{\sigma(\gamma_1)} \omega = \varphi_{\sigma(\gamma_1)}\omega.$$

Thus $\sigma^{**}\varphi_{\gamma_1} = \varphi_{\sigma(\gamma_1)}$. If γ_1 is a closed curve through O_1, then $\sigma(\gamma_1)$ is a closed curve passing through $O_2 = \sigma(O_1)$. Hence if $\Lambda_{E_1}, \Lambda_{E_2}$ are the lattices of periods for E_1, E_2, respectively, we have $\sigma^{**}(\lambda_{E_1}) \subseteq \Lambda_{E_2}$. So σ^{**} factors through the quotients, defining τ:

$$\tau = \overline{\sigma^{**}} : V_{E_1}^*/\Lambda_{E_1} \longrightarrow V_{E_2}^*/\Lambda_{E_2}.$$

It is immediate the diagram commutes. $\square$

Lemma 11.3.5. *For any elliptic curve E, the inclusion $i : E \hookrightarrow J(E)$ induces an isomorphism*

$$i^* : \pi_1(E, O) \longrightarrow \pi_1(J(E), i(O)).$$

Unfortunately, the construction of the Jacobian given so far is not algebraic so it would be hard to carry over to curves over an arbitrary ground field. To construct Jacobians algebraically, we will prove Abel's theorem:

Theorem 11.3.6 (Abel)**.** *Suppose $\Lambda \subseteq \mathbb{C}$ is a lattice with fundamental domain Π and take any set $\{a_i\} \subset \Pi$ such that there are integers $m_i \in \mathbb{Z}$ satisfying $\sum m_i = 0$ and $\sum m_i a_i \in \Lambda$. Then there exists an elliptic function $f(z)$ whose set of zeroes and poles is $\{a_i\}$ and whose orders of vanishing/poles are $\operatorname{ord}_{a_i} f = m_i$.*

Given a lattice $\Lambda \subseteq \mathbb{C}$, we may assume $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ for some $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$.

Definition. *The* **theta function** *for a lattice Λ is*

$$\theta(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i(n^2\tau + 2nz)}.$$

One has $|e^{\pi i(n^2\tau+2nz)}| = e^{-\pi(n^2 \operatorname{Im}\tau + 2n \operatorname{Im} z)}$ for any $z \in \mathbb{C}$, which implies that the above series converges absolutely.

Proposition 11.3.7. *Fix a theta function $\theta(z) = \theta(z,\tau)$. Then*

(1) $\theta(z) = \theta(-z)$.

(2) $\theta(z+1) = \theta(z)$.

(3) $\theta(z+\tau) = e^{-\pi i(\tau+2z)}\theta(z)$.

Properties (2) and (3) together say that $\theta(z)$ is what's known as a *semielliptic function.* For our purposes, this will be good enough. Notice that for $z = \frac{1+\tau}{2}$, we have

$$\begin{aligned}\theta\left(\frac{1+\tau}{2}\right) &= \theta\left(-\frac{1+\tau}{2} + (1+\tau)\right)\\ &= e^{\pi i\left(\tau + 2\left(-\frac{1+\tau}{2}\right)\right)}\theta\left(-\frac{1+\tau}{2}\right)\\ &= e^{\pi i}\theta\left(-\frac{1+\tau}{2}\right) = -\theta\left(\frac{1+\tau}{2}\right).\end{aligned}$$

Thus $z = \frac{1+\tau}{2}$ is a zero of $\theta(z)$.

Lemma 11.3.8. *All zeroes of $\theta(z,t)$ are simple and are of the form $\frac{1+\tau}{2} + \ell$ for $\ell \in \Lambda$.*

Lemma 11.3.9. *For $x \in \mathbb{C}$, set $\theta^{(x)}(z,\tau) = \theta\left(z - \frac{1+\tau}{2} - x\right)$. Then $\theta^{(x)}(z) = \theta^{(x)}(z,\tau)$ satisfies:*

(1) $\theta^{(x)}(z+1) = \theta^{(x)}(z)$.

(2) $\theta^{(x)}(z+\tau) = e^{-\pi i(2(z-x)-1)}\theta^{(x)}(z)$.

We now prove Abel's theorem (11.3.6).

Proof. Given such a set $\{a_i\} \subset \Pi$, let $x_1, \ldots, x_n$ be the list of all a_i with $m_i > 0$, listed with repetitions corresponding to the number m_i. For example, if $m_1 = 2$ then $x_1 = x_2 = a_1$. Likewise, let $y_1, \ldots, y_n$ be the list of all a_i with $m_i < 0$, once again with repetitions. By the hypothesis $\sum m_i = 0$, there are indeed an equal number of each. Set

$$f(z) = \frac{\prod_{i=1}^n \theta^{(x_i)}(z)}{\prod_{i=1}^n \theta^{(y_i)}(z)}.$$

Then by Lemma 11.3.9, $f(z+1) = f(z)$. On the other hand, the lemma also gives

$$\begin{aligned} f(z+\tau) &= \frac{\prod_{i=1}^n \theta^{(x_i)}(z+\tau)}{\prod_{i=1}^n \theta^{(y_i)}(z)} \\ &= e^{2\pi i\left(\sum_{i=1}^n x_i - \sum_{i=1}^n y_i\right)} f(z) \\ &= e^{2\pi i \sum m_i a_i} f(z) \\ &= f(z) \quad \text{since } \sum m_i a_i = 0. \end{aligned}$$

Therefore $f(z)$ is elliptic. □

Note that $\theta(z)$ is a meromorphic function, so the integral

$$\frac{1}{2\pi i} \int_{\partial \Pi} \frac{\theta'(z)}{\theta(z)}\, dz$$

counts the number of zeroes of $\theta(z)$ in the fundamental domain Π, up to multiplicity. To ensure no zeroes lying on $\partial\Pi$ are missed, we may shift $\Pi \to \Pi_\alpha$ for an appropriate $\alpha \in \mathbb{C}$. Parametrize $\partial\Pi$ as in Proposition 11.1.5. Then once again the integrals along γ_2 and γ_4 cancel since $\theta(z+1) = \theta(z)$. On the other hand,

$$\begin{aligned} &\theta(z+\tau) = e^{-\pi i(\tau+2z)}\theta(z) \\ \Longrightarrow\ &\theta'(z+\tau) = e^{-\pi i(\tau+2z)}(-2\pi i\theta(z) + \theta'(z)) \\ \Longrightarrow\ &\frac{\theta'(z+\tau)}{\theta(z+\tau)} = -2\pi i + \frac{\theta'(z)}{\theta(z)}. \end{aligned}$$

This implies

$$\begin{aligned} \int_{\partial\Pi} \frac{\theta'(z)}{\theta(z)}\, dz &= \int_{\gamma_1} \frac{\theta'(z)}{\theta(z)}\, dz + \int_{\gamma_2} \frac{\theta'(z)}{\theta(z)}\, dz + \int_{\gamma_3} \frac{\theta'(z)}{\theta(z)}\, dz + \int_{\gamma_4} \frac{\theta'(z)}{\theta(z)}\, dz \\ &= \left(\int_{\gamma_1} \frac{\theta'(z)}{\theta(z)}\, dz + \int_{\gamma_3} \frac{\theta'(z)}{\theta(z)}\, dz\right) + \left(\int_{\gamma_2} \frac{\theta'(z)}{\theta(z)}\, dz + \int_{\gamma_4} \frac{\theta'(z)}{\theta(z)}\, dz\right) \\ &= \left(\int_{\gamma_1} \frac{\theta'(z)}{\theta(z)}\, dz - \int_{\gamma_1} \frac{\theta'(z)}{\theta(z)}\, dz + 2\pi i\right) + 0 \\ &= 2\pi i. \end{aligned}$$

It follows that $\theta(z)$ has exactly one zero in Π, and it must be $z = \frac{1+\tau}{2}$.

Definition. *For a curve E (need not be elliptic), define:*

- *A* **divisor** *on E is a formal sum $D = \sum n_P P$ over the points $P \in E$, with $n_P \in \mathbb{Z}$. The abelian group of all divisors is denoted* $\mathrm{Div}(E)$.
- *The* **degree** *of a divisor $D = \sum n_P P \in \mathrm{Div}(E)$ is $\deg(D) = \sum n_P$. The set of all* **degree 0 divisors** *is denoted* $\mathrm{Div}^0(E)$.
- *For a meromorphic function f on $E(\mathbb{C}) = \mathbb{C}/\Lambda$, the* **principal divisor** *associated to f is $(f) = \sum \deg_P P$ where $n_P = \mathrm{ord}_P f$. The group of all principal divisors is denoted* $\mathrm{PDiv}(E)$.
- *The* **Picard group** *of E is the quotient group* $\mathrm{Pic}(E) = \mathrm{Div}(E)/\mathrm{PDiv}(E)$. *The degree zero part of the Picard group is written* $\mathrm{Pic}^0(E) = \mathrm{Div}^0(E)/\mathrm{PDiv}(E)$.

The inverse map $\psi : E \to \mathbb{C}/\Lambda$ extends to the group of divisors on E:

$$\begin{aligned} \Psi : \mathrm{Div}(E) &\longrightarrow \mathbb{C}/\Lambda \\ \sum n_P P &\longmapsto \sum n_P \psi(P). \end{aligned}$$

Definition. *The map $\Psi : \mathrm{Div}(E) \to \mathbb{C}/\Lambda$ is called the* **Abel-Jacobi map**.

Recall that $\psi : P \mapsto \int_{\gamma_P} \omega + \Lambda \in \mathbb{C}/\Lambda$ where ω is a fixed holomorphic differential form on E and γ_P is a path connecting $O \in E(\mathbb{C})$ to P. If O' is another basepoint and ψ' is the corresponding map, we have $\psi(P) = \psi(O') + \psi'(P)$ for all $P \in E$. So it appears that Ψ is not well-defined. However, this issue vanishes when we restrict Ψ to $\mathrm{Div}^0(E)$: if $D = \sum n_P P$ is a degree 0 divisor, then

$$\begin{aligned} \Psi(D) &= \sum n_P \psi(P) \\ &= \sum n_P(\psi(O') + \psi'(P)) \\ &= \psi(O') \sum n_P + \sum n_P \psi'(P) \\ &= 0 + \sum n_P \psi'(P) = \Psi'(D). \end{aligned}$$

Corollary 11.3.10. *The map $\Psi : \mathrm{Div}^0(E) \to \mathbb{C}/\Lambda$ induces an isomorphism* $\mathrm{Pic}^0(E) \cong \mathbb{C}/\Lambda$.

Proof. One can prove that Ψ is a surjective group homomorphism. Moreover, Abel's theorem (11.3.6) implies that $\ker \Psi = \mathrm{PDiv}(E)$. $\square$

Consider the map $i_O : E \to \mathrm{Div}^0(E)$ that sends $P \mapsto P - O$. This fits into a commutative diagram:

$$\begin{tikzcd}
\mathrm{Div}^0(E) \arrow[dr, "\Psi"] & \\
& \mathbb{C}/\Lambda \\
E \arrow[uu, "i_O"] \arrow[ur, "\psi_O"'] &
\end{tikzcd}$$

On the level of the Picard group, this diagram looks like

$$\begin{tikzcd}
\mathrm{Pic}^0(E) \arrow[dr, "\overline{\Psi}"] & \\
& \mathbb{C}/\Lambda \\
E \arrow[uu, "\overline{i_O}"] \arrow[ur, "\psi_O"'] &
\end{tikzcd}$$

and every arrow is a bijection.

11.4 Jacobians of Higher Genus Curves

Let C be a complex curve of genus $g \geq 2$ and let $V = \Gamma(C, \Omega_C)$ be the vector space of holomorphic differential forms on C. Then $\dim_{\mathbb{C}} V = g$, so $V^* \cong \mathbb{C}^g$. As in the previous section, for any path ω in C the assignment $\varphi_\gamma : \omega \mapsto \int_\gamma \omega$ defines a functional $\varphi_\gamma \in V^*$. As for elliptic curves, we define:

Definition. *The* **lattice of periods** *for C is*

$$\Lambda = \{\varphi_\gamma \in V^* \mid \gamma \text{ is a closed curve in } C\}.$$

Lemma 11.4.1. *Λ is a lattice in V^*.*

Definition. *The* **Jacobian** *of C is the quotient space $J(C) = V^*/\Lambda$.*

As with elliptic curves, we have a map $\psi : C \to J(C)$ called the *Abel-Jacobi map*, which sends $P \mapsto \varphi_{\gamma_P} + \Lambda$, where γ_P is a curve through P. Also, ψ extends to the divisor group of C as a map

$$\Psi : \mathrm{Div}(C) \longrightarrow J(C)$$

which is canonical when restricted to $\mathrm{Div}^0(C)$. The Abel-Jacobi theorem generalizes Theorem 11.3.6 and Corollary 11.3.10.

Theorem 11.4.2. *Let C be a curve of genus $g > 0$ and let $\Psi : \mathrm{Div}^0(C) \to J(C)$ be the Abel-Jacobi map. Then*

(1) (Abel) $\ker \Psi = \mathrm{PDiv}(C)$.

(2) (Jacobi) Ψ is surjective.

Therefore Ψ induces an isomorphism $\mathrm{Pic}^0(C) \cong J(C)$.

Just as with elliptic curves, if we fix a basepoint $O \in C$, the map $i_O : C \to \mathrm{Div}^0(C), P \mapsto P - O$ determines a commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}^0(C) \arrow[dr, "\overline{\Psi}"] & \\
& J(C) \\
C \arrow[uu, "\overline{i_O}"] \arrow[ur, "\psi_O"'] &
\end{tikzcd}$$

However, this time not every map is a bijection. In particular, $\dim C = 1 < g = \dim J(C)$. To remedy this, let C^g be the g-fold product of C and consider the map

$$\begin{aligned}
\psi^g : C^g &\longrightarrow J(C) \\
(P_1, \ldots, P_g) &\longmapsto \psi(P_1) + \ldots + \psi(P_g)
\end{aligned}$$

where $+$ denotes the group law on $J(C)$.

Theorem 11.4.3 (Jacobi). *$\psi^g : C^g \longrightarrow J(C)$ is surjective.*

There is still work to do to show that the natural map $C^g \to \mathrm{Pic}^0(C)$ is surjective. It turns out that $J(C)$ is birationally equivalent to the *symmetric power* $C^{(g)} = C^g/\sim$, where $(P_1, \ldots, P_g) \sim (P_{\sigma(1)}, \ldots, P_{\sigma(g)})$ for any permutation $\sigma \in S_g$. Jacobi proved that this birational equivalence is enough to endow $\mathrm{Pic}^0(C) \cong J(C)$ with the structure of an algebraic group.

Theorem 11.4.4. *$J(C)$ is an abelian variety.*

Part III

Functional Analysis

Chapter 12

Introduction

Part III follows a course on Banach spaces taught by Dr. Stephen Robinson at Wake Forest University during the spring of 2014. The primary reference for the semester was a set of course notes compiled by Dr. Robinson.

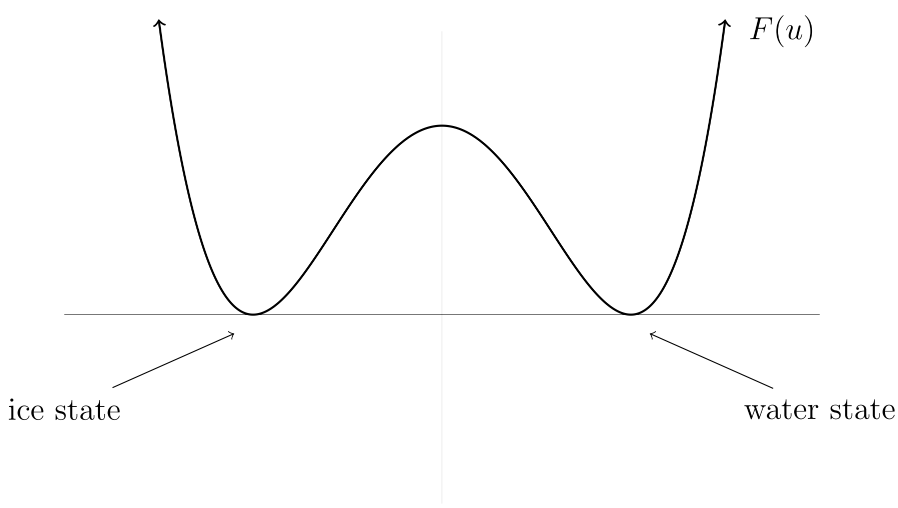
Our motivation for studying functional analysis is the following question which describes phase transitions in physics (e.g. water $\leftrightarrow$ ice).

Question. *Given a system whose energy is described by the integral equation*

$$E(u) = \int_0^1 |u'|^2 \, du + \int_0^1 F(u) \, du$$

for some function F, what are the values for which the system's energy is minimized?

In general, this can be thought of as a critical point search. For example, consider



The trick is the domain of $E(u)$ is some set of functions u, whereby E is a function from this set of functions to the real numbers $\mathbb{R}$. For this reason we want to be able to perform calculus on such sets of functions, called function spaces. They are a special type of vector space called a normed linear space.

The main topics covered in these notes are

- Normed linear spaces, Banach spaces and their properties
- The Arzela-Ascoli Theorem
- Polynomial approximation and the Weierstrass Approximation Theorem
- Contraction mapping
- Completion of a metric space
- Calculus on normed linear spaces, including the Fréchet derivative, the Mean Value Theorem, Sard's Theorem and the Inverse Function Theorem

Chapter 13

Normed Linear Spaces and Banach Spaces

13.1 Normed Linear Spaces

We begin with the fundamental object of study in analysis: a metric space.

Definition. *A space X is a* **metric space** *if there is a function $d : X \times X \to \mathbb{R}$ satisfying*

1) *(Positivity) For all $x, y \in X$, $d(x,y) \geq 0$ and equality holds if and only if $x = y$.*

2) *(Symmetry) For all $x, y \in X$, $d(x,y) = d(y,x)$.*

3) *For all $x, y, z \in X$, $d(x,z) \leq d(x,y) + d(y,z)$.*

Examples.

① $\mathbb{R}$ is a metric space under the usual distance function $d(x,y) = |x - y|$.

② For any $n \geq 2$, $\mathbb{R}^n$ is a metric space under the Euclidean norm

$$d(x,y) = ||x - y|| := \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2}.$$

③ There is a generalization of Euclidean n-space called ℓ^2 which will be important in subsequent sections. One can think of ℓ^2 as the space of sequences of real numbers $(x_1, x_2, x_3, \ldots)$. Then ℓ^2 is a metric space under the 2-norm:

$$d(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^2 \right)^{1/2}$$

④ Let $C[0,1]$ denote the space of continuous functions on the interval $[0,1]$. Then $C[0,1]$ is a metric space with several compatible metrics. The two most important are:

$$d(f,g) := \sup_{x \in [0,1]} |f(x) - g(x)|$$

$$\text{and} \qquad d_2(f,g) := \left(\int_0^1 |f(x) - g(x)|^2 \, dx \right)^{1/2}.$$

All of the spaces in these examples have a vector space structure. In addition, the notion of distance is derived from the distance between vectors: $d(x,y) = ||x - y||$. In the study of functional analysis, we want to restrict our consideration to a certain type of metric space.

Definition. *Suppose that X is a real vector space and there is a function $|| \cdot || : X \to \mathbb{R}$ satisfying*

(1) *$||x|| \geq 0$ for all $x \in X$, and $||x|| = 0$ if and only if x is the zero vector.*

(2) *$||x + y|| \leq ||x|| + ||y||$ for all $x, y \in X$.*

(3) *$||\alpha x|| = |\alpha| \, ||x||$ for any $\alpha \in \mathbb{R}$.*

Then X is a **normed linear space***. If only (2) and (3) hold, then $||\cdot||$ is called a* **seminorm***.*

Example 13.1.1. Not every metric space is a normed linear space. For instance, let $X = \mathbb{R}$ and define

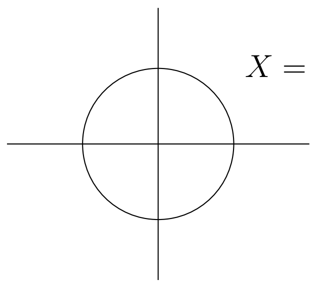
$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

which is called the *discrete metric*. By virtue of d being a metric, (1) and (2) above are satisfied. However, if (3) held we would have

$$\begin{aligned} 1 = d(2, 0) &= ||2 - 0|| = ||2|| = 2||1|| = 2||1 - 0|| \\ &= 2 \cdot d(1, 0) = 2 \cdot 1 = 2 \end{aligned}$$

clearly a contradiction. Therefore $(\mathbb{R}, d)$ is not a normed linear space.

Example 13.1.2. Consider the unit circle in $\mathbb{R}^2$:



$$X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

X is not a vector space, so it is certainly not a normed linear space.

While not every metric space is a normed linear space, the reverse implication is true.

Theorem 13.1.3. *If* $(X, ||\cdot||)$ *is a normed linear space, then* (X, d) *is a metric space under the function* $d(x, y) := ||x - y||$.

Proof. Norms are nonnegative, so $d \geq 0$. Likewise, $d(x, y) = 0 \iff x = y$ follows directly from (1) in the definition of a normed linear space. For the triangle inequality (2), consider

$$d(x, z) = ||x - z|| = ||x - y + y - z|| \leq ||x - y|| + ||y - z|| = d(x, y) + d(y, z).$$

Finally, to show symmetry consider

$$d(y, x) = ||y - x|| = || - (x - y)|| = | - 1| \, ||x - y|| = ||x - y|| = d(x, y).$$

Hence d is a metric so (X, d) is a metric space. $\square$

13.2 Generalizing the Reals

In this section we will focus on generalizing the following properties of the real numbers to Euclidean n-space $\mathbb{R}^n$ for finite n:

- $\mathbb{R}$ is a normed linear space (easy).
- $\mathbb{R}$ is complete, that is, every Cauchy sequence converges (Theorem 2.3.8).
- A set $D \subset \mathbb{R}$ is compact if and only if D is closed and bounded (the Heine-Borel theorem).
- The set of rationals $\mathbb{Q}$ is dense in $\mathbb{R}$ (Theorem 1.3.6).

For notational convenience, we will actually restrict our focus to the proofs in $\mathbb{R}^2$, but moving up in dimension doesn't alter the proofs much. Consider $\mathbb{R}^2$ with $||x|| = \sqrt{x_1^2 + x_2^2}$ for $x = (x_1, x_2) \in \mathbb{R}^2$. We first prove

Proposition 13.2.1. $\mathbb{R}^2$ *is a normed linear space.*

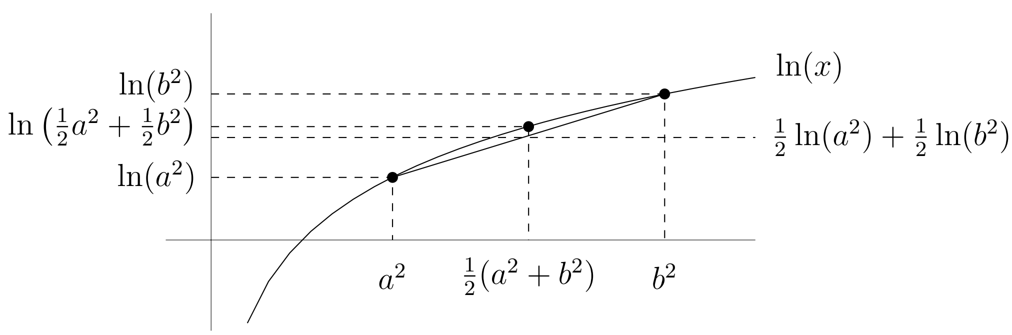
The main ingredients in this proof are three inequalities, which we state and prove separately before returning to the main proof.

Lemma 13.2.2 (Young's Inequality)**.** *For any nonnegative* $a, b \in \mathbb{R}$, $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$.

Proof. Consider

$$\ln(ab) = \ln(a) + \ln(b) = \tfrac{1}{2}\ln(a^2) + \tfrac{1}{2}\ln(b^2) \leq \ln\left(\tfrac{1}{2}a^2 + \tfrac{1}{2}b^2\right)$$

where the inequality comes from the concavity of $\ln(x)$ – see the diagram below.



Taking the exponential of both sides of the inequality $\ln(ab) \leq \ln\left(\frac{1}{2}a^b + \frac{1}{2}b^2\right)$ proves Young's Inequality. $\square$

Lemma 13.2.3 (Hölder's Inequality)**.** $\displaystyle\sum_{i=1}^{2} a_i b_i \leq \left(\sum_{i=1}^{2} a_i^2\right)^{1/2} \left(\sum_{i=1}^{2} b_i^2\right)^{1/2}.$

Proof. By Young's Inequality (Lemma 13.2.2),

$$\sum_{i=1}^{2} a_i b_i \leq \sum_{i=1}^{2} |a_i b_i| \leq \sum_{i=1}^{2} \left(\tfrac{1}{2}a_i^2 + \tfrac{1}{2}b_i^2\right).$$

If we suppose that $||a|| = 1 = ||b||$ then the inequality becomes

$$\frac{1}{2}\sum_{i=1}^{2} a_i^2 + \frac{1}{2}\sum_{i=1}^{2} b_i^2 = \frac{1}{2} + \frac{1}{2} = 1 = ||a||\,||b||$$

which proves the desired inequality. Now if a and b are vectors of any length, then the normal vectors in the directions of a and b satisfy Hölder's Inequality:

$$\sum_{i=1}^{2}\left(\frac{a_i}{||a||}\right)\left(\frac{b_i}{||b||}\right) \leq 1 \quad \Longrightarrow \quad \sum_{i=1}^{2} a_i b_i \leq ||a||\,||b||.$$

□

We will see more general versions of Young's and Hölder's inequalities in Section 20.3.

Lemma 13.2.4 (Minkowski's Inequality). $\left(\sum_{i=1}^{2}(a_i+b_i)^2\right)^{1/2} \leq \left(\sum_{i=1}^{2} a_i^2\right)^{1/2} + \left(\sum_{i=1}^{2} b_i^2\right)^{1/2}.$

Proof. Consider

$$\begin{aligned}
\sum_{i=1}^{2}(a_i+b_i)^2 &= \sum_{i=1}^{2} |a_i+b_i|^2 \\
&= \sum_{i=1}^{2} |a_i+b_i|\,|a_i+b_i| \\
&\leq \sum_{i=1}^{2}(|a_i|+|b_i|)\,|a_i+b_i| \quad \text{by the triangle inequality for } \mathbb{R} \\
&= \sum_{i=1}^{2} |a_i|\,|a_i+b_i| + \sum_{i=1}^{2} |b_i|\,|a_i+b_i| \\
&\leq \left(\sum_{i=1}^{2} a_i^2\right)^{1/2}\left(\sum_{i=1}^{2} |a_i+b_i|^2\right)^{1/2} + \left(\sum_{i=1}^{2} b_i^2\right)^{1/2}\left(\sum_{i=1}^{2} |a_i+b_i|^2\right)^{1/2} \\
&\quad \text{by Hölder's Inequality (Lemma 20.3.4).}
\end{aligned}$$

Dividing through by $\left(\sum_{i=1}^{2}(a_i+b_i)^2\right)^{1/2}$ then yields

$$\left(\sum_{i=1}^{2}(a_i+b_i)^2\right)^{1/2} \leq \left(\sum_{i=1}^{2} a_i^2\right)^{1/2} + \left(\sum_{i=1}^{2} b_i^2\right)^{1/2}.$$

□

Young's, Hölder's and Minkowski's inequalities generalize to finite n with only minor changes in each proof. These allow us to prove

Proposition 13.2.5. $(\mathbb{R}^2, ||\cdot||)$ *is a normed linear space.*

Proof. First, $\mathbb{R}^2$ is a vector space. By properties of square roots, $||x|| \geq 0$ and $||x|| = 0 \iff x = 0$. To show the linear condition, let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} ||\alpha x|| &= \left(\sum_{i=1}^{2} |\alpha x_i|^2\right)^{1/2} = \left(\sum_{i=1}^{2} |\alpha|^2 |x_i|^2\right)^{1/2} \\ &= \left(|\alpha|^2 \sum_{i=1}^{2} |x_i|^2\right)^{1/2} = |\alpha| \left(\sum_{i=1}^{2} |x_i|^2\right)^{1/2} = |\alpha|\, ||x||. \end{aligned}$$

Finally, Minkowski's Inequality (Lemma 13.2.4) gives us $||x+y|| \leq ||x|| + ||y||$, the triangle inequality. Hence $\mathbb{R}^2$ is a normed linear space. $\square$

Again, note that all proofs generalize to $\mathbb{R}^n$, so that we may conclude that $\mathbb{R}^n$ is a normed linear space for finite n.

Lemma 13.2.6. *A sequence (x_n) in $\mathbb{R}^2$ converges if and only if the component sequences of (x_n) converge in $\mathbb{R}$.*

Proof. Suppose that $(x_n) \subset \mathbb{R}^2$ converges to x. Then given any $\varepsilon > 0$, there is some $N > 0$ such that $||x_n - x|| < \varepsilon$ for all $n \geq N$. Note that

$$\begin{aligned} |x_{n1} - x_1| &\leq \sqrt{|x_{n1} - x_1|^2 + |x_{n2} - x_2|^2} = ||x_n - x|| \\ \text{and likewise} \quad |x_{n2} - x_2| &\leq \sqrt{|x_{n1} - x_1|^2 + |x_{n2} - x_2|^2} = ||x_n - x||. \end{aligned}$$

Hence $|x_{n1} - x_1|, |x_{n2} - x_2| < \varepsilon$ for all $n \geq N$, so $(x_{n1}) \to x_1$ and $(x_{n2}) \to x_2$ in $\mathbb{R}$.

Conversely, suppose $(x_{n1}) \to x_1$ and $(x_{n2}) \to x_2$ for some $x_1, x_2 \in \mathbb{R}$. Let $\varepsilon > 0$. By convergence, there exist natural numbers N_1, N_2 such that

$$\begin{aligned} |x_{n1} - x_1| &< \tfrac{\varepsilon}{\sqrt{2}} \quad \text{for } n > N_1 \\ \text{and} \quad |x_{n2} - x_2| &< \tfrac{\varepsilon}{\sqrt{2}} \quad \text{for } n > N_2. \end{aligned}$$

Let $N = \max\{N_1, N_2\}$ and set $x = (x_1, x_2)$. Then

$$||x_n - x|| = \sqrt{|x_{n1} - x_1|^2 + |x_{n2} - x_2|^2} < \sqrt{\tfrac{\varepsilon^2}{2} + \tfrac{\varepsilon^2}{2}} = \varepsilon.$$

Hence (x_n) converges to $x \in \mathbb{R}^2$. $\square$

Definition. *The* p**-norm** *on $\mathbb{R}^2$ is defined for all $x \in \mathbb{R}^2$ by*

$$||x||_p = (|x_1|^p + |x_2|^p)^{1/p}$$

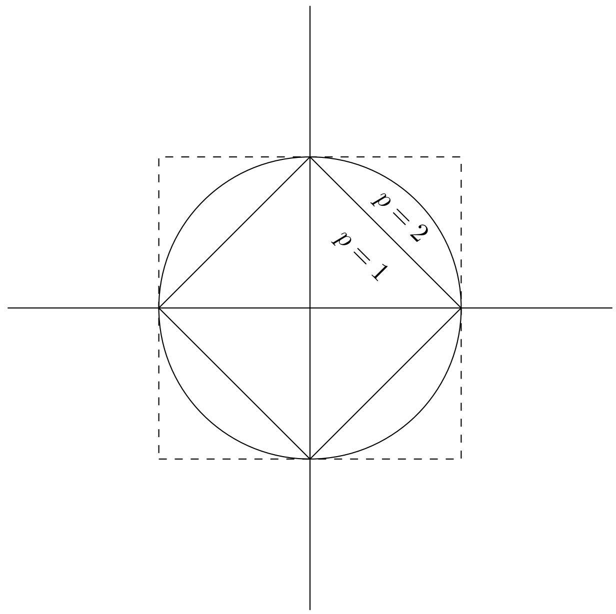
for any $1 \leq p < \infty$.

Example 13.2.7. The Euclidean norm $||\cdot||$ is simply the $p = 2$ case of this definition.

Proposition 13.2.8. *$||\cdot||_p$ is a norm on $\mathbb{R}^2$.*

Proof. Omitted. This requires a generalization of Minkowski's Inequality (Lemma 13.2.4). $\square$

Example 13.2.9. A unit circle with respect to a p-metric is the set of points $\{x \in \mathbb{R}^2 : ||x||_p = 1\}$. Some examples of unit circles for various values of p are shown below.



As $p \to \infty$, the circles converge to the unit square.

Definition. *The* ∞-**norm***, sometimes called the* **sup norm** *or* **max norm***, is defined for all $x \in \mathbb{R}^2$ by*

$$||x||_\infty = \max\{|x_1|, |x_2|\}.$$

Note that the unit circle under $||\cdot||_\infty$ is precisely the unit square, shown with dashed lines above. When consider any of these norms, we will often write $||\cdot||_p$ for $1 \le p \le \infty$.

Definition. *A normed linear space which is complete is called a* **Banach space**.

Theorem 13.2.10. *$(\mathbb{R}^2, ||\cdot||_2)$ is complete, and therefore a Banach space.*

Proof. Let $(x_n) \subset \mathbb{R}^2$ be a Cauchy sequence. Consider $(x_{n1}) \subset \mathbb{R}$, the sequence of first components of (x_n). Then for any m, n,

$$|x_{n1} - x_{m1}| \le \sqrt{|x_{n1} - x_{m1}|^2 + |x_{n2} - x_{m2}|^2} = ||x_n - x_m||_2$$

and this implies that (x_{n1}) is Cauchy since (x_n) is assumed to be. Now $\mathbb{R}$ is complete so (x_{n1}) converges. By a similar argument, $(x_{n2}) \subset \mathbb{R}$ converges as well. Hence by Lemma 13.2.6, (x_n) converges in $\mathbb{R}^2$ and we conclude $\mathbb{R}^2$ is complete. $\square$

Theorem 13.2.11. *If (x_n) is a bounded sequence in $\mathbb{R}^2$, then (x_n) has a convergent subsequence.*

Proof. Let (x_n) be a bounded sequence in $\mathbb{R}^2$. Consider (x_{n1}), the sequence of first components in $\mathbb{R}$. We know $|x_{n1}| \leq ||x_n||$ so (x_{n1}) is bounded as well. By the Bolzano-Weierstrass Theorem, (x_{n1}) has a convergent subsequence, say (x_{n_k1}). Then $(x_{n_k}) \subset \mathbb{R}^2$ has converging first components.

Now consider (x_{n_k2}), the sequence of second components of (x_{n_k}). By Bolzano-Weierstrass this has a convergent subsequence, $(x_{n_{k_j}2}) \subset \mathbb{R}$. Therefore $(x_{n_{k_j}})$ has both component sequences converging, and thus this subsequence converges by Lemma 13.2.6. □

This has the following consequence for characterizing compact sets in $\mathbb{R}^2$.

Corollary 13.2.12. *A subset $K \subset \mathbb{R}^2$ that is closed and bounded is compact.*

This generalizes one direction of the Heine-Borel Theorem. It turns out that the other direction holds as well (we won't prove this), so that we have the following characterization of compact sets in $\mathbb{R}^n$.

Theorem 13.2.13 (Heine-Borel)**.** *For every $n \geq 1$, $K \subset \mathbb{R}^n$ is compact if and only if K is closed and bounded.*

Next we prove the two-dimensional analog (the case for $\mathbb{R}^n$ is similar) of the density of the rationals in the reals.

Theorem 13.2.14. *$\mathbb{Q}^2$ is dense in $\mathbb{R}^2$.*

Proof. Let $x \in \mathbb{R}^2$ and suppose $\varepsilon > 0$ is given. Let $r_1 \in \mathbb{Q}$ such that $|x_1 - r_1| < \frac{\varepsilon}{\sqrt{2}}$ and let $r_2 \in \mathbb{Q}$ such that $|x_2 - r_2| < \frac{\varepsilon}{\sqrt{2}}$; these choices are possible by the density of $\mathbb{Q} \subset \mathbb{R}$. As a result, if $r = (r_1, r_2) \in \mathbb{Q}^2$ we have

$$||x - r|| = \sqrt{|x_1 - r_1|^2 + |x_2 - r_2|^2} < \sqrt{\left(\frac{\varepsilon}{\sqrt{2}}\right)^2 + \left(\frac{\varepsilon}{\sqrt{2}}\right)^2} = \varepsilon.$$

Therefore $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$. □

Definition. *Two norms $|| \cdot ||_1$ and $|| \cdot ||_2$ on $\mathbb{R}^n$ are said to be* **equivalent** *if there exist $a, b > 0$ such that $b||x||_2 \leq ||x||_1 \leq a||x||_2$ for all $x \in \mathbb{R}^2$.*

Proposition 13.2.15. *The p-norms $|| \cdot ||_p$ for $1 \leq p \leq \infty$ are equivalent on $\mathbb{R}^2$.*

Proof. Let $x = (x_1, x_2) \in \mathbb{R}^2$ and first consider

$$\begin{aligned} ||x||_1 = |x_1| + |x_2| &= \sum_{i=1}^{2} |x_i| \cdot 1 \\ &\leq \left(\sum_{i=1}^{2} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{2} 1^q\right)^{1/q} \qquad \text{by Hölder's Inequality (Lemma 20.3.4)} \\ &= 2^{1/q}||x||_p \end{aligned}$$

for any p, q such that $\frac{1}{p} + \frac{1}{q} = 1$. In particular, this shows that $||x||_1 \leq 2||x||_p$ for any p. Now compare $||x||_p$ and $||x||_\infty$:

$$\begin{aligned} ||x||_p &= \left(\sum_{i=1}^{2} |x_i|^p \right)^{1/p} \\ &\leq \left(\sum_{i=1}^{2} ||x||_\infty^p \right)^{1/p} \\ &= ||x||_\infty \left(\sum_{i=1}^{2} 1^p \right)^{1/p} = 2^{1/p} ||x||_\infty. \end{aligned}$$

So $||x||_p \leq 2||x||_\infty$. Finally, we close the circle:

$$||x||_\infty \max\{|x_1|, |x_2|\} \leq \sum_{i=1}^{2} |x_i| = ||x||_1.$$

Hence all the p-norms for $1 \leq p \leq \infty$ are equivalent. $\square$

Notice again that this proof is easily modified for $\mathbb{R}^n$, $n > 2$. An important consequence of the above fact is that convergence in $\mathbb{R}^2$ (or $\mathbb{R}^n$) is the same with respect to *any* p-norm.

13.3 Sequences Spaces

In the last section we saw how to generalize the intrinsic properties of $\mathbb{R}$ to any finite-dimensional Euclidean space $\mathbb{R}^n$. The natural extensions of such spaces are called ℓ^p spaces.

Definition. *A* **sequence space***, or ℓ^p space, is defined for each*

$$\ell^p = \left\{ (x_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

Example 13.3.1. The most important of the ℓ^p spaces is ℓ^2, which is the space of sequences of real numbers that converge with respect to the 2-norm. ℓ^2 is notable for being the only sequence space which is also a *Hilbert space*, that is, ℓ^2 is complete with respect to the inner product induced by $||\cdot||_2$. (We will study these spaces in Chapter 20.)

An example of a 'point' in ℓ^2 is the sequence $x_n = \frac{1}{n}$, which converges with respect to the ℓ^2-norm since

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty.$$

On the other hand, the sequence $x_n = \frac{1}{\sqrt{n}}$ is *not* an element of ℓ^2 since

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} \right)^2 = \sum_{n=1}^{\infty} \frac{1}{n},$$

the harmonic series, diverges.

It may be useful after reading the first two sections to think of ℓ^2 as $\mathbb{R}^\infty$ and indeed it exhibits many of the same characteristics. However, the convergent sum requirement makes things 'nice' in ℓ^2, whereas in an infinite dimensional vector space of real numbers we may not be so lucky.

We are about to prove that ℓ^2 is a normed linear space – as with the different p-norms in the previous section, slight modifications will make the proof run for any ℓ^p as well. First, note that Young's Inequality (Lemma 13.2.2) doesn't change when moving to the infinite dimensional case. Then Hölder's and Minkowski's inequalities follow in much the same way.

Lemma 13.3.2 (Hölder's Inequality)**.** $\lim\limits_{N\to\infty} \sum_{i=1}^{N} |a_i b_i| \leq lim_{N\to\infty} \left(\sum_{i=1}^{N} |a_i|^2 \right)^{1/2} \left(\sum_{i=1}^{N} |b_i|^2 \right)^{1/2}.$

Proof. Take a limit of Hölder's Inequality (Lemma 20.3.4) in the finite case. □

Lemma 13.3.3 (Minkowski's Inequality)**.** $\left(\sum_{i=1}^{\infty} (x_i + y_i)^2 \right)^{1/2} \leq \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2} + \left(\sum_{i=1}^{\infty} |y_i|^2 \right)^{1/2}.$

Proof. Take a limit of the n-dimensional Minkowski's Inequality; limits preserve inequality. □

Theorem 13.3.4. *ℓ^2 is a normed linear space.*

Proof. Since $||x||_2 = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2}$ is a convergent sum of nonnegative terms, we have $||x||_2 \geq 0$, and clearly $||x||_2 = 0 \iff x_i = 0$ for all i. For linearity, let $\alpha \in \mathbb{R}$ and consider

$$\begin{aligned} ||\alpha x||_2 &= \left(\sum_{i=1}^{\infty} |\alpha x_i|^2\right)^{1/2} = \left(\sum_{i=1}^{\infty} |\alpha|^2 |x_i|^2\right)^{1/2} \\ &= \left(|\alpha|^2 \sum_{i=1}^{\infty} |x_i|^2\right)^{1/2} = |\alpha|\, ||x||_2. \end{aligned}$$

Finally, the generalized Minkowski's Inequality above directly implies the triangle inequality. Hence ℓ^2 is a normed linear space. □

Example 13.3.5. Consider the sequence of sequences (x_n) in ℓ^2 defined by

$$\begin{aligned} x_1 &= (1, 0, 0, \ldots) \\ x_2 &= (0, 1, 0, \ldots) \\ x_3 &= (0, 0, 1, \ldots) \\ &\vdots \\ x_n &= (\ldots, \overset{n-1}{0}, \overset{n}{1}, \overset{n+1}{0}, \ldots) \end{aligned}$$

We calculate that

$$\begin{aligned} ||x_2 - x_1||_2 &= \left(\sum_{i=1}^{\infty} |x_{2i} - x_{1i}|^2\right)^{1/2} \\ &= (1^2 + 1^2 + 0^2 + 0^2 + 0^2 + \ldots)^{1/2} = \sqrt{2}. \end{aligned}$$

and in fact this is the difference between any pair of distinct terms in (x_n). Thus (x_n) has the following surprising properties.

- (x_n) is bounded.
- (x_n) is *not* Cauchy, so the sequence is not convergent in ℓ^2.
- The sequence does have componentwise convergence, as for each n, x_n is a sequence in $\mathbb{R}$ converging to 0.
- (x_n) has no converging subsequences.

Theorem 13.3.6. *ℓ^2 is a Banach space.*

Proof. We proved ℓ^2 is a normed linear space, so it remains to show that ℓ^2 is complete with respect to the ℓ^2 norm. Let (x_n) be a Cauchy sequence in ℓ^2. Consider the sequence of kth components (x_{nk}), which is itself a sequence of real numbers. Then we see that for any n, m,

$$|x_{nk} - x_{mk}| = \sqrt{|x_{nk} - x_{mk}|} \leq \left(\sum_{j=1}^{\infty} |x_{nj} - x_{mj}|^2\right)^{1/2} = ||x_n - x_m||_2.$$

This implies that since (x_n) is Cauchy, so is (x_{nk}). By completeness of $\mathbb{R}$, (x_{nk}) converges to some real number $\bar{x}_k$. Since k was arbitrary, we have componentwise convergence for the original sequence (x_n). Let $\bar{x}$ be the sequence of these limits, i.e. $\bar{x} = (\bar{x}_k)$. To show $\bar{x} \in \ell^2$, note that there is some M such that

$$\sum_{k=1}^{\infty} |x_{nk}|^2 \leq M$$

for all n, since Cauchy sequences are bounded. Look at the partial sum:

$$\sum_{k=1}^{N} |\bar{x}_k|^2 = \lim_{n\to\infty} |x_{nk}|^2 \leq M.$$

It follows that for all N, $\displaystyle\sum_{k=1}^{N} |\bar{x}_k|^2 \leq M$, so taking the limit as $N \to \infty$ will preserve this bound. Hence $\displaystyle\sum_{k=1}^{\infty} |\bar{x}_k|^2$ is finite and we conclude that $\bar{x} \in \ell^2$.

Now we will show that $||x_n - \bar{x}||_2 \longrightarrow 0$. Consider

$$\left(\sum_{k=1}^{N} |x_{nk} - \bar{x}_k|^2\right)^{1/2} \leq \left(\sum_{k=1}^{N} |x_{nk} - x_{mk}|^2\right)^{1/2} + \left(\sum_{k=1}^{N} |x_{mk} - \bar{x}_k|^2\right)^{1/2}$$

by Minkowski's Inequality (Lemma 13.3.3)

$$\leq ||x_n - x_m||_2 + \left(\sum_{k=1}^{N} |x_{mk} - \bar{x}_k|^2\right)^{1/2}.$$

For a given $\varepsilon > 0$, we can choose N_0 such that $n, m > N_0$ implies

$$\left(\sum_{k=1}^{N} |x_{nk} - \bar{x}_k|^2\right)^{1/2} < \varepsilon + \left(\sum_{k=1}^{N} |x_{mk} - \bar{x}_k|^2\right)^{1/2}.$$

Letting $m \to \infty$, the left side doesn't change and $(x_{mk}) \to \bar{x}_k$ by componentwise convergence, so we have

$$\left(\sum_{k=1}^{N} |x_{nk} - \bar{x}_k|^2\right)^{1/2} < \varepsilon + 0 = \varepsilon$$

for all $n > N_0$. Finally, taking the limit as $N \to \infty$ gives us $||x_n - \bar{x}||_2 < \varepsilon$ so (x_n) converges. Hence ℓ^2 is complete. $\square$

Example 13.3.7. In this example we illustrate some differences between ℓ^2 and ℓ^3. Recall that $\left(\frac{1}{\sqrt{n}}\right) \notin \ell^2$. However, in the ℓ^3 norm we see that

$$\sum_{n=1}^{\infty}\left|\frac{1}{n^{1/2}}\right|^3 = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$$

which converges. Thus $\ell^3 \not\subset \ell^2$. On the other hand, suppose $x \in \ell^2$, i.e. $x = (x_n)$ is a sequence such that $\sum |x_n|^2 < \infty$. Then there must be some $N > 0$ such that $|x_n| < 1$ for all $n > N$. This implies

$$\sum_{n=1}^{\infty}|x_n|^3 = \sum_{n=1}^{N}|x_n|^3 + \sum_{n=N+1}^{\infty}|x_n|^3 \le \sum_{n=1}^{N}|x_n|^3 + \sum_{n=N+1}^{\infty}|x_n|^2 < \infty.$$

Hence $\ell^2 \subset \ell^3$ and this containment is proper.

Proposition 13.3.8. *In ℓ^2, any sequence (x_n) that is bounded has a subsequence that converges componentwise.*

Proof. Suppose $(x_n) \subset \ell^2$ is a bounded sequence. Consider the kth components $x_{1k}, x_{2k}, x_{3k}, \ldots$ and observe that they are bounded for each $k \in \mathbb{N}$. First choose a subsequence (x_{1i}) of (x_n) such that the first components of (x_{1i}) converges; this is possible by the Heine-Borel Theorem on $\mathbb{R}$. Next choose a subsequence (x_{2i}) of the first subsequence (x_{1i}) such that the second components of (x_{2i}) converge, again using Heine-Borel. Note that $(x_{2i}) \subset (x_{1i})$ implies the first components of (x_{2i}) also converge. In the inductive step, choose (x_{ji}) to be a subsequence of $(x_{j-1,i})$ such that the jth components converge. Since the chosen subsequences are nested, (x_{ji}) will have convergence in its first j components.

We want to find a subsequence of (x_n) that converges in every component. To do this, we select the 'diagonalization' of the nested subsequences above. Consider (x_{ii}); this is a subsequence of (x_n) that converges componentwise, since for any $k \ge 1$ and $j \ge k$, (x_{ji}) is a subsequence of (x_{ki}) so its kth components converge. $\square$

Note that in most cases componentwise convergence does not give us ℓ^2 convergence. Hence boundedness does *not* imply compactness in ℓ^2. However, we next introduce an object called the Hilbert cube which is closed, bounded and compact in ℓ^2.

Definition. *The* **Hilbert cube** *is the subset $K \subset \ell^2$ defined by*

$$K = \left\{x \in \ell^2 : |x_k| \le \tfrac{1}{k}\right\}.$$

Notice that $\displaystyle\sum_{k=1}^{\infty}|x_k|^2 \le \sum_{k=1}^{\infty}\frac{1}{k^2} < \infty$ so K does indeed lie in ℓ^2. Also, by construction K is bounded. We will show that K is also closed and compact.

Lemma 13.3.9. *K is closed.*

Proof. Let $(x_n) \subset K$ be a convergent sequence with respect to the ℓ^2 norm and denote its limit of convergence by $x \in \ell^2$. It follows that (x_n) is componentwise convergent to the components of x. In other words, $\lim_{n\to\infty} x_{nk} = x_k$ for all k. Thus

$$|x_k| = \lim n \to \infty |x_{nk}| \leq \frac{1}{k}$$

and this shows $x \in K$ by definition, so K is closed. □

Lemma 13.3.10. *K is compact.*

Proof. Let $(x_n) \subset K$ be a bounded sequence. By Proposition 13.3.8, we may assume without loss of generality that (x_n) converges componentwise. For each k, let $x_k = \lim_{n\to\infty} x_{nk}$. By the proof of Lemma 13.3.9, $x = (x_k) \in K$.

We want to show $||x_n - x||_2 \longrightarrow 0$. Let $\varepsilon > 0$ be given and consider

$$\begin{aligned}(||x_n - x||_2)^2 &= \sum_{k=1}^{\infty} |x_{nk} - x_k|^2 \\ &= \sum_{k=1}^{N} |x_{nk} - x_k|^2 + \sum_{k=N+1}^{\infty} |x_{nk} - x_k|^2\end{aligned}$$

for any fixed N. Notice that $|x_{nk} - x_k| \leq \frac{2}{k}$ for each k, by definition of the Hilbert cube. So it follows that

$$\sum_{k=N+1}^{\infty} |x_{nk} - x_k|^2 \leq \sum_{k=N+1}^{\infty} \frac{4}{k^2}$$

but the series on the right converges, so we may choose N large enough so that

$$\sum_{k=N+1}^{\infty} \frac{4}{k^2} < \frac{\varepsilon}{2}.$$

Now consider the other term. We know by componentwise convergence that

$$\lim_{n\to\infty} \sum_{k=1}^{N} |x_{nk} - x_k|^2 = 0.$$

In other words, there is an M such that for every $n > M$,

$$\sum_{k=1}^{N} |x_{nk} - x_k|^2 < \frac{\varepsilon}{2}.$$

Putting these together, for all $n > \max\{N, M\}$, $(||x_n - x||_2)^2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence (x_n) converges. □

In $\mathbb{R}^n$ we saw that $\mathbb{Q}^n$ is an example of a dense subset. Since ℓ^2 consists of sequences of real numbers that are eventually small, we must adjust things slightly to obtain a dense subset of rational sequences. In fact, consider the set Q of all $x \in \ell^2$ such that $x = (r_1, r_2, \ldots, r_k, 0, 0, 0, \ldots)$ where $r_1, \ldots, r_k \in \mathbb{Q}$ and k is finite. Then Q is dense in ℓ^2.

Chapter 14

Function Spaces

In Section 13.3 we defined a sequence space as a vector space whose elements (vectors) are infinite sequences of real numbers. Alternatively, a sequence can be thought of as a function $\mathbb{N} \to \mathbb{R}$, so that each point in ℓ^2 is such a function. In this chapter we generalize the notion of a function space and explore four important examples in functional analysis:

- $C[0,1]$ is the set of continuous functions on the interval $[0,1]$.
- $C^1[0,1]$ is the set of differentiable functions on $[0,1]$ that have continuous derivatives.
- $L^2[0,1]$ is the *completion* of $C[0,1]$ as a metric space.
- $W^{1,2}[0,1]$, the so-called Sobolev space, is a completion of $C^1[0,1]$ with respect to a certain norm.

14.1 Norms on Function Spaces

The most important space in functional analysis is $C[0,1]$:

Definition. *The* **space of continuous functions** *on the interval* $[0,1]$ *is denoted*

$$C[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}.$$

Definition. *On* $C[0,1]$*, we define the* **sup norm** *by* $||f||_\infty = \max\limits_{x\in[0,1]} |f(x)|$.

From calculus, we know that a continuous function on a closed interval achieves its maximum, so the sup and max norms are interchangeable on $C[0,1]$. Another norm of interest is a generalization of the 2-norm from Chapter 13.

Definition. *On* $C[0,1]$*, we define the* **2-norm** *by*

$$||f||_2 = \left(\int_0^1 |f(x)|^2\,dx\right)^{1/2}.$$

Proposition 14.1.1. $(C[0,1], ||\cdot||_\infty)$ *is a normed linear space.*

Proof. We will establish the triangle inequality and note that the proof of the other properties is routine. Let $x \in [0,1]$ and $f,g \in C[0,1]$. Then

$$|f(x)+g(x)| \le |f(x)| + |g(x)| \le ||f||_\infty + ||g||_\infty.$$

Notice that the right side no longer depends on x, so taking the sup of both sides preserves the inequality, giving us

$$||f+g||_\infty \le ||f||_\infty + ||g||_\infty.$$

It follows that $||\cdot||_\infty$ is a norm on $C[0,1]$. □

In order to prove that $C[0,1]$ is a normed linear space with the 2-norm, we need Hölder's inequality:

Lemma 14.1.2 (Hölder's Inequality for Continuous Functions)**.** *For any functions* $f,g \in C[0,1]$*,*

$$\int_0^1 |fg| \le \left(\int_0^1 |f|^2\right)^{1/2}\left(\int_0^1 |g|^2\right)^{1/2}.$$

Proof. First suppose $||f||_2 = ||g||_2 = 1$. Then

$$\begin{aligned}\int_0^1 |fg| &\le \int_0^1 \left(\tfrac{1}{2}|f|^2 + \tfrac{1}{2}|g|^2\right) \quad \text{by Young's inequality (Lemma 13.2.2)}\\ &= \frac{1}{2}\int_0^1 |f|^2 + \frac{1}{2}\int_0^1 |g|^2\\ &= \tfrac{1}{2}(1) + \tfrac{1}{2}(1) = 1 = ||f||_2\,||g||_2.\end{aligned}$$

This proves Hölder's inequality for the unit norm case. In general this may applied to $\frac{f}{||f||_2}$ and $\frac{g}{||g||_2}$ to produce the inequality for all f,g. □

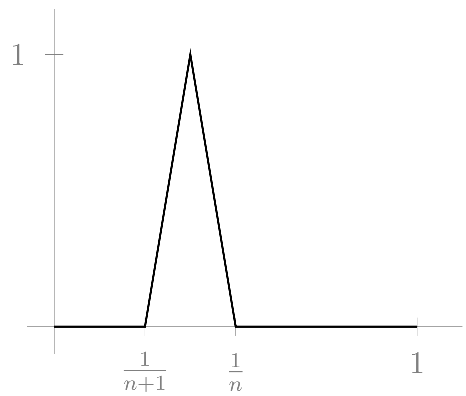
Proposition 14.1.3. $(C[0,1], ||\cdot||_2)$ *is a normed linear space.*

Proof. We will check the triangle inequality; the rest of the properties are routine to verify. For any $f, g \in C[0,1]$, consider

$$\begin{aligned}
(||f+g||_2)^2 &= \int_0^1 |f+g|^2 \\
&= \int_0^1 |f+g|\,|f+g| \\
&\leq \int_0^1 |f|\,|f+g| + \int_0^1 |g|\,|f+g| \quad \text{by the regular triangle inequality} \\
&\leq ||f||_2\,||f+g||_2 + ||g||_2\,||f+g||_2 \quad \text{by Hölder's inequality (Lemma 14.1.2).}
\end{aligned}$$

Dividing through by $||f+g||_2$ gives us $||f+g||_2 \leq ||f||_2 + ||g||_2$. Hence we can conclude that $(C[0,1], ||\cdot||_2)$ is a normed linear space. □

Example 14.1.4. To illustrate the differences between $||\cdot||_\infty$ and $||\cdot||_2$, consider the family of functions $\{f_n\}$ described by the picture below.



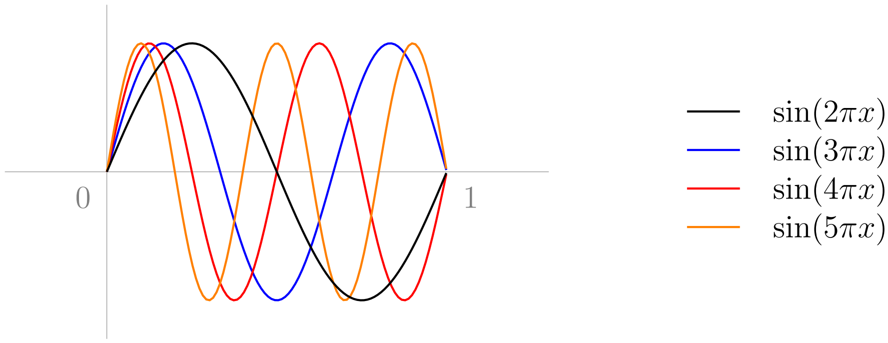
By construction, $||f_n||_\infty = 1$ for each n, and for any $m \neq n$, $||f_n - f_m||_\infty = 1$. Hence the sequence (f_n) is *not* Cauchy in the function space $(C[0,1], ||\cdot||_\infty)$. However, it is easy to see that (f_n) is pointwise convergent to 0, since for any x, $f_n(x) \to 0$.

Now consider (f_n) in $C[0,1]$ with the $||\cdot||_2$ norm. For each n,

$$\begin{aligned}
||f_n||_2 &= \left(\int_0^1 |f(x)|^2\,dx\right)^{1/2} = \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} |f(x)|^2\right)^{1/2} \leq \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} 1\,dx\right)^{1/2} \\
&= \left(\frac{1}{n} - \frac{1}{n+1}\right)^{1/2} = \sqrt{\frac{1}{n(n+1)}}.
\end{aligned}$$

Thus we can see that $||f_n - 0||_2 \longrightarrow 0$ as $n \to \infty$, so the sequence (f_n) converges to $f(x) = 0$ in $(C[0,1], ||\cdot||_2)$. In fact, we could even let the peaks of $\{f_n\}$ tend towards ∞ at a certain rate and still have $||f_n||_2 \longrightarrow 0$, while in that case we would see $||f_n||_\infty \longrightarrow \infty$.

Example 14.1.5. Let $\{f_n\}$ be the family of functions $f_n(x) = \sin(n\pi x)$ for $n = 2, 3, 4, \ldots$.



Note that

$$\int_0^1 \sin^2(n\pi x)\,dx = \int_0^1 \cos^2(n\pi x)\,dx$$

$$\text{and} \quad \int_0^1 \left(\sin^2(n\pi x) + \cos^2(n\pi x)\right)\,dx = \int_0^1 1\,dx = 1.$$

This shows that

$$(||f_n||_2)^2 = \int_0^1 \sin^2(n\pi x)\,dx = \frac{1}{2}$$

which can also be verified using integration by parts. Now consider for some $m \neq n$,

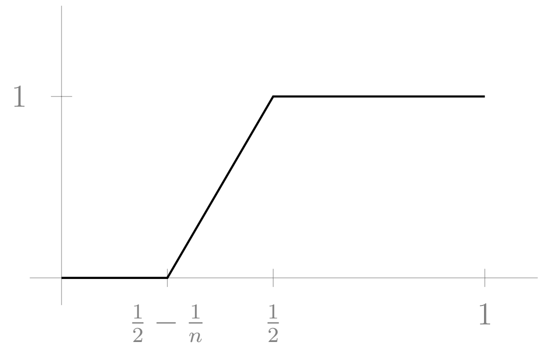
$$\begin{aligned}(||f_n - f_m||_2)^2 &= \int_0^1 (\sin(n\pi x) - \sin(m\pi x))^2\,dx \\ &= \int_0^1 (\sin^2(n\pi x) + \sin^2(m\pi x) - 2\sin(n\pi x)\sin(m\pi x))\,dx \\ &= \frac{1}{2} + \frac{1}{2} - 2\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx \\ &= 1 - 2(0) = 1.\end{aligned}$$

Therefore (f_n) is not Cauchy. Note the similarities between this example and the ℓ^2 sequence

$$\begin{aligned} x_1 &= (1, 0, 0, \ldots) \\ x_2 &= (0, 1, 0, \ldots) \\ x_3 &= (0, 0, 1, \ldots) \\ &\vdots \end{aligned}$$

It turns out that the family of functions $\{f_n\}$ in $C[0,1]$ is isomorphic to (x_n) in ℓ^2. These are examples of orthonormal bases for the vector spaces $C[0,1]$ and ℓ^2.

Example 14.1.6. Define the family of functions $\{f_n\}$ for $n \geq 2$ described by the picture below.



Then (f_n) converges pointwise to the step function

$$f(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Consider these functions with the sup norm (we must use the sup norm instead of the max norm since $f(x)$ is discontinuous):

$$||f_n - f||_\infty = \sup_{x \in [0,1]} |f_n(x) - f(x)| = 1 \text{ for all } n \geq 2.$$

It looks like (f_n) does not converge with respect to $||\cdot||_\infty$. For any $m \neq n$, we have

$$||f_n - f_m||_\infty = \sup_{x \in [0,1]} |f_n(x) - f_m(x)|.$$

If we choose any $n \geq 2$, then as $m \to \infty$, $||f_n - f_m||_\infty \longrightarrow 1$ by the picture. Hence the sequence is not Cauchy in $(C[0,1], ||\cdot||_\infty)$. Note that convergence with respect to $||\cdot||_\infty$ is equivalent to uniform convergence, so clearly a sequence cannot converge to a discontinuous function such as the step function.

Now consider (f_n) with the 2-norm:

$$(||f_n - f||_2)^2 = \int_0^1 |f_n(x) - f(x)|^2 \, dx = \int_{\frac{1}{2} - \frac{1}{n}}^{\frac{1}{2}} 1^2 \, dx = \frac{1}{n}.$$

So $||f_n - f||_2 \longrightarrow 0$, but since f is not continuous, (f_n) still doesn't converge in $(C[0,1], ||\cdot||_2)$. By a similar estimate, (f_n) is Cauchy in $(C[0,1], ||\cdot||_2)$. In particular, this suggests that $(C[0,1], ||\cdot||_2)$ is *not complete*. We prove this next.

Proposition 14.1.7. *$C[0,1]$ is not complete with respect to $||\cdot||_2$ and hence not a Banach space.*

Proof. We proved that the sequence (f_n) defined above is Cauchy with respect to $||\cdot||_2$ and showed that the sequence converges to the step function $f(x)$ which is not in $C[0,1]$. Now we must verify that (f_n) doesn't have *any* limit in $C[0,1]$.

Suppose (f_n) converges to some $f(x)$ in $C[0,1]$. Take $x_0 \in \left(0, \frac{1}{2}\right)$ such that $f(x_0) \neq 0$. Then since f is continuous, there must be some neighborhood of x_0 on which f is nonzero. This means there is a $\delta > 0$ such that

$$|f(x)| \geq \tfrac{1}{2}|f(x_0)| \quad \text{for } x \in (x_0 - \delta, x_0 + \delta) \subset \left(0, \tfrac{1}{2}\right).$$

Choose N such that $x_0 + \delta < \frac{1}{2} - \frac{1}{n}$ for all $n > N$ and consider

$$\begin{aligned} \int_0^1 |f_n(x) - f(x)|^2\, dx &\geq \int_{x_0-\delta}^{x_0+\delta} |f_n(x) - f(x)|^2\, dx \\ &= \int_{x_0-\delta}^{x_0+\delta} |f(x)|^2\, dx \quad \text{for all } n > N \\ &\geq \tfrac{1}{4}|f(x_0)|^2\, 2\delta \quad \text{by the work above.} \end{aligned}$$

Taking the square root of both sides, we have

$$||f_n - f||_2 \geq \frac{\sqrt{2\delta}}{2}|f(x_0)| \quad \text{for all } n > N.$$

Hence $f_n \not\to f$ so we must have $f(x) = 0$ for all $x \in \left(0, \frac{1}{2}\right)$. A similar proof shows that $f(x) = 1$ for all $x \in \left(\frac{1}{2}, 1\right)$. Now these show that

$$\lim_{x\to\frac{1}{2}^-} f(x) = 0 \quad \text{and} \quad \lim_{x\to\frac{1}{2}^+} f(x) = 1.$$

Thus $\lim\limits_{x\to\frac{1}{2}} f(x)$ does not exist, so in fact f is not continuous on $[0, 1]$, contradicting our choice of f. Therefore we conclude that $(C[0, 1], ||\cdot||_2)$ is not complete. □

The last few examples and propositions illustrate an important point about function spaces: there is a difference between pointwise and uniform convergence, and a sequence may exhibit one with respect to a particular norm but not the other. It turns out that convergence with respect to $||\cdot||_\infty$ does guarantee convergence in the space $C[0, 1]$, as the next lemma shows.

Lemma 14.1.8. *If* $(f_n) \subset C[0, 1]$ *and* $f : [0, 1] \to \mathbb{R}$ *such that* $||f_n - f||_\infty \longrightarrow 0$ *then* $f \in C[0, 1]$.

Proof. This is just a restatement of Theorem 3.6.1. □

Theorem 14.1.9. $(C[0, 1], ||\cdot||_\infty)$ *is a Banach space.*

Proof. We proved that $C[0, 1]$ is a normed linear space with respect to the sup norm so it remains to check this space is complete. Let (f_n) be a Cauchy sequence in $C[0, 1]$. Then for a fixed $x \in [0, 1]$ and $m \neq n$,

$$|f_n(x) - f_m(x)| \leq ||f_n - f_m||_\infty$$

so it follows that $(f_n(x)) \subset \mathbb{R}$ is Cauchy. By completeness of $\mathbb{R}$, $(f_n(x))$ converges, say to $f(x)$. Let f be the function whose values are these convergent limits of $(f_n(x))$ for each $x \in [0, 1]$. Consider

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)|$$

$$leq||f_n - f_m||_\infty + |f_m(x) - f(x)|.$$

Then given $\varepsilon > 0$ we can choose N such that $n, m > N$ imply $||f_n - f_m||_\infty < \varepsilon$ by the Cauchy property of (f_n). Now taking $m \to \infty$, $|f_m(x) - f(x)| \longrightarrow 0$ by pointwise convergence. Hence for all x and $n > N$,

$$|f_n(x) - f(x)| \leq ||f_n - f_m||_\infty + |f_m(x) - f(x)| < \varepsilon + 0 = \varepsilon.$$

Taking the supremum of both sides respects the inequality, so $||f_n - f||_\infty < \varepsilon$. By Lemma 14.1.8, f is continuous so (f_n) converges in $C[0, 1]$ and therefore $(C[0, 1]|| \cdot ||_\infty)$ is complete. $\square$

14.2 The Arzela-Ascoli Theorem

In this section we explore compactness in $C[0,1]$. Recall that in ℓ^2, a bounded sequence contained a subsequence which converged componentwise. We might expect a bounded sequence in $C[0,1]$ to have a subsequence with pointwise convergence. It turns out that the conditions required for compactness are

- The slopes may be controlled via the Lipschitz condition:

$$|f_n(x) - f_n(y)| \leq k|x-y| \quad \text{for all } n, x, y \text{ and for some } k.$$

- Equicontinuity (to be defined).

Assume $(f_n) \subset C[0,1]$ is a sequence satisfying $||f_n||_\infty \leq M$ for all n. We will not be able to show pointwise convergence for every point in $[0,1]$ because the unit interval is uncountable. Instead, we will take a countable subset of $[0,1]$ (hint: what's a nice, countable subset of any real interval?) and show pointwise convergence on this subset.

Let $x_1 \in [0,1]$ and consider $(f_n(x_1)) \subset \mathbb{R}$. Note that $|f_n(x_1)| \leq ||f_n||_\infty \leq M$ for all n, so $(f_n(x_1))$ is a bounded sequence in $\mathbb{R}$. By the Bolzano-Weierstrass Theorem (2.4.5), $(f_n(x_1))$ has a converging subsequence $(f_{1n}(x_1))$, and thus (f_{1n}) is a subsequence of (f_n) that converges when evaluated at x_1. Next, let $x_2 \in [0,1]$ such that $x_2 \neq x_1$ and consider $(f_{1n}(x_2))$. Notice that $|f_{1n}(x_2)| \leq ||f_{1n}||_\infty \leq M$ for all n, so $(f_{1n}(x_2))$ is another bounded sequence in $\mathbb{R}$ with a converging subsequence $(f_{2n}(x_2))$. As above, (f_{2n}) is a subsequence of (f_n) with pointwise convergence at both x_1 and x_2.

Continue in this way to obtain nested subsequences such that $(f_{kn}(x_i))$ converges for all $1 \leq i \leq k$ and $x_i \neq x_j$ for all $1 \leq i < j \leq k$. As we did in Section 13.3, consider the 'diagonal' sequence (f_{nn}). Choose $x_k \in [0,1]$. Then for $n \geq k$, (f_{nn}) is a subsequence of (f_{kn}). Hence (f_{nn}) converges. In practice we let (x_n) be an enumeration of $\mathbb{Q} \cap [0,1]$ but it can be any sequence that enumerates a countable, dense subset of $[0,1]$. In the proof below, we will show pointwise convergence at every point in (x_n).

The last ingredient, as mentioned above, is the following condition:

Definition. *A sequence $(f_n) \subset C[0,1]$ is* **equicontinuous at a point** $x_0 \in [0,1]$ *if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f_n(x) - f_n(x_0)| < \varepsilon$ for all n. The sequence is* **equicontinuous** *if it is equicontinuous at every point in $[0,1]$, and* **uniformly equicontinuous** *if δ does not depend on x_0.*

Now we can prove the main theorem characterizing compactness in $C[0,1]$.

Theorem 14.2.1 (Arzela-Ascoli)**.** *If $(f_n) \subset C[0,1]$ is bounded and uniformly equicontinuous, then (f_n) has a converging subsequence.*

Proof. By the arguments above, we may assume (f_n) is a bounded sequence in $C[0,1]$ such that $(f_n(r))$ converges for all $r \in \mathbb{Q} \cap [0,1]$. By uniform equicontinuity, for every $\varepsilon > 0$ there is a single $\delta > 0$ such that

$$|x-y| < \delta \implies |f_n(x) - f_n(y)| < \tfrac{\varepsilon}{3} \text{ for all } n.$$

Let $0 = r_0 < r_1 < r_2 < \cdots < r_k = 1$ such that $r_i \in \mathbb{Q}$ and $|r_{i+1} - r_i| < \delta$; this is possible by the density of the rationals in any interval of real numbers. For each r_i, choose N_i such that $|f_n(r_i) - f_m(r_i)| < \frac{\varepsilon}{3}$ for all $n, m > N_i$. Then set $N = \max\limits_{0 \leq i \leq k}\{N_i\}$ and consider

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq |f_n(x) - f_n(r)| + |f_n(r) - f_m(r)| + |f_m(r) - f_m(x)| \\
&\qquad \text{for some } r = r_i \text{ such that } |x - r_i| < \delta \\
&< \tfrac{\varepsilon}{3} + |f_n(r) - f_m(r)| + \tfrac{\varepsilon}{3} \quad \text{by equicontinuity} \\
&< \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon \quad \text{for all } m, n \geq N \geq N_i.
\end{aligned}$$

This string of inequalities is valid for all $x \in [0, 1]$ by our choice of the r_i. Therefore $||f_n - f_m||_\infty < \varepsilon$ for all $n, m \geq N$, and since $(C[0, 1]|| \cdot ||_\infty)$ is complete, this shows (f_n) converges. □

We immediately obtain the following corollary.

Corollary 14.2.2. *If $K \subset C[0, 1]$ is closed, bounded and equicontinuous then K is compact.*

A partial converse is also true.

Proposition 14.2.3. *If $K \subset C[0, 1]$ is compact then K is equicontinuous.*

Proof. Suppose to the contrary that K is compact but not equicontinuous. Then there is an $\varepsilon > 0$ and sequences $(x_n), (y_n) \subset [0, 1]$ and $(f_n) \subset K$ such that $|x_n - y_n| \longrightarrow 0$ but $|f_n(x_n) - f_n(y_n)| \geq \varepsilon$ for all n. Without loss of generality, we may assume since $[0, 1]$ and K are compact that $(x_n) \to x$ and $(y_n) \to y$ in $[0, 1]$ and $(f_n) \to f$ in K. Note that $|x_n - y_n| \longrightarrow 0$ implies $x = y$. Then

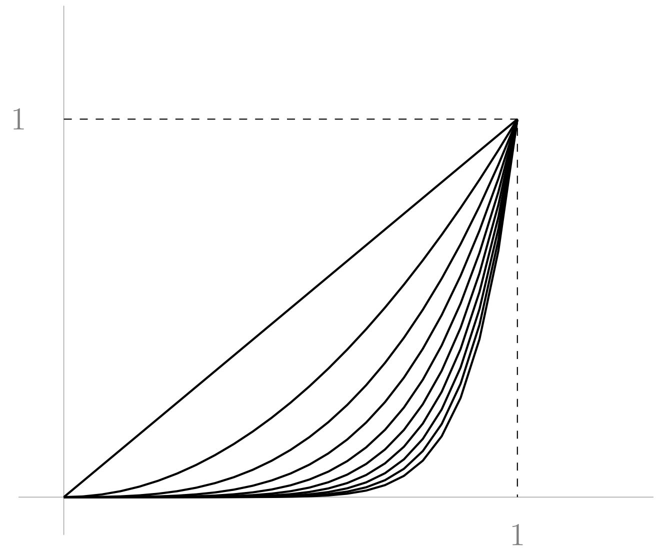
$$\begin{aligned}
|f_n(x_n) - f(x)| &\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\
&\leq ||f_n - f||_\infty + |f(x_n) - f(x)|,
\end{aligned}$$

but $\lim\limits_{n\to\infty} ||f_n - f||_\infty = 0$, and moreover $\lim\limits_{n\to\infty} |f(x_n) - f(x)|$ because $(f_n) \to f$, $(x_n) \to x$ and f is a continuous function. This shows that $\lim\limits_{n\to\infty} f_n(x_n) = f(x)$; similarly, $\lim\limits_{n\to\infty} f_n(y_n) = f(y) = f(x)$. Finally, this shows that

$$\lim_{n\to\infty} |f_n(x_n) - f_n(y_n)| = 0,$$

a contradiction. Therefore compactness implies equicontinuity in $C[0, 1]$. □

Example 14.2.4. Consider the sequence of continuous functions $f_n(x) = x^n$. The first 10 elements of the sequence are shown below.



It can be shown that any subsequence of (f_n) must converge to

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

but $f \notin C[0,1]$ so by the Arzela-Ascoli Theorem, (f_n) must not be equicontinuous. In fact, let $\varepsilon = \frac{1}{2}$ and consider the interval $[1-\delta, 1]$ for sufficiently small δ. Let $x = 1$ and $y = 1 - \delta$, by which $|x - y| \leq \delta$ and

$$|f_n(x) - f_n(y)| = |1^n - (1-\delta)^n| = |1 - (1-\delta)^n|.$$

Note that since $1 - \delta < 1$, $\lim\limits_{n\to\infty} (1-\delta)^n = 0$ so $(1-\delta)^n \leq \frac{1}{4}$ for large enough n. Thus

$$|f_n(x) - f_n(y)| \geq \left|1 - \tfrac{1}{4}\right| = \tfrac{3}{4} > \tfrac{1}{2} = \varepsilon.$$

Hence (f_n) is not equicontinuous.

The next lemma gives us a nice criterion for checking that a sequence is equicontinuous.

Lemma 14.2.5. *Let $(f_n) \subset C[0,1]$ where each f_n is differentiable and suppose there is some $M > 0$ such that $||f_n'||_\infty \leq M$ for all n. Then (f_n) is equicontinuous.*

Proof. Given any $x, y \in [0,1]$, the Mean Value Theorem (4.3.3) says that there is some $c \in [0,1]$ such that $|f_n(x) - f_n(y)| = |f_n'(c)|\,|x-y| \leq M|x-y|$ for all n. So given $\varepsilon > 0$, setting $\delta = \frac{\varepsilon}{M}$ shows that (f_n) is equicontinuous. $\square$

There is an alternate proof of this lemma that uses the fundamental theorem of calculus.

The Arzela-Ascoli Theorem has an important application in the study of initial value problems. Suppose we have a first order differential equation

$$y' = f(y)$$
$$y(0) = y_0.$$

Assume $f : \mathbb{R} \to \mathbb{R}$ is bounded and continuous. Then Peano's Theorem says the IVP has a solution.

Theorem 14.2.6 (Peano's Theorem)**.** *Given a bounded, continuous function $f : \mathbb{R} \to \mathbb{R}$ and a first order differential equation $y' = f(y)$, then every initial value problem $y(0) = y_0$ has a solution.*

Proof sketch. The first step is to change the IVP into a fixed-point problem. We do this via the fundamental theorem of calculus:

$$\begin{aligned} y(t) - y(0) &= \int_0^t f(y(s))\,ds \\ \implies y(t) &= y_0 + \int_0^t f(y(s))\,ds. \end{aligned}$$

Next we set up an iteration scheme:

$$\begin{aligned} y_0(t) &= y_0 \\ y_1(t) &= y_0 + \int_0^t f(y_0(s))\,ds \\ &\vdots \\ y_{n+1}(t) &= y_0 + \int_0^t f(y_n(s))\,ds. \end{aligned}$$

If the sequence (y_n) converges to some function $y = y(t)$ then y is a solution to the initial value problem. To show convergence, we prove that (y_n) is bounded and equicontinuous on $[0, 1]$ and apply the Arzela-Ascoli Theorem. Consider

$$\begin{aligned} |y_n(t)| &= \left| y_0 + \int_0^t f(y_{n-1}(s))\,ds \right| \\ &\le |y_0| + \int_0^t |f(y_{n-1}(s))|\,ds \\ &\le |y_0| + tM \quad \text{since } f \text{ is bounded} \\ &\le |y_0| + M \quad \text{for all } t \in [0, 1]. \end{aligned}$$

Hence $||y_n||_\infty \le |y_0| + M$. Now use the fundamental theorem of calculus to write

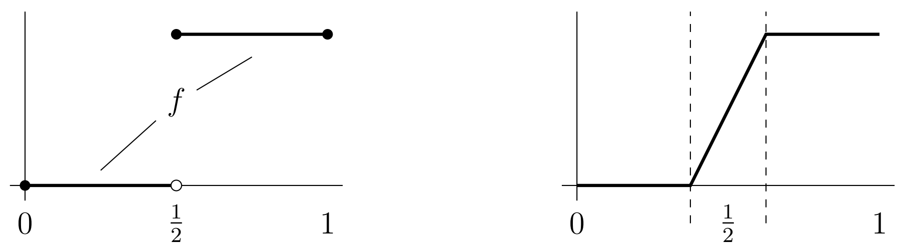
$$|y_n'(t)| = |f(y_{n-1}(t))| \le M$$

for all $t \in [0, 1]$. This shows that $||y_n'||_\infty \le M$, so by Lemma 14.2.5 (y_n) is equicontinuous. Hence by Arzela-Ascoli, (y_n) has a converging subsequence and the limit of such a subsequence represents a solution to the initial value problem. □

14.3 Approximation

In many different branches of mathematics, it is useful to be able to use 'nice' objects to approximate not-so-nice objects. In the world of functional analysis, this takes the form of polynomial approximation (there are other types of functional approximations but we study the most common form here). We will prove the Weierstrass Approximation Theorem, which says that every function in $C[0,1]$ can be approximated to an arbitrary degree of accuracy by a polynomial.

Suppose we are trying to approximate a discontinuous function such as the step function shown below (left).



The idea is to average over small intervals. Let $f_n(x)$ be the average of $f(x)$ on $\left[x-\frac{1}{n}, x+\frac{1}{n}\right]$. Explicitly, this can be written as an integral:

$$f_n(x) = \frac{n}{2}\int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(t)\,dt.$$

At times we may need to extend the function outside $[0,1]$, which is typically done by letting $f(x)=0$ for $x \notin [0,1]$. Next, we change the limits of integration by setting

$$f_n(x) = \int_{-\infty}^{\infty} \frac{n}{2}\chi_{\left[x-\frac{1}{n}, x+\frac{1}{n}\right]}(t) f(t)\,dt,$$

where χ_I denotes the **characteristic function** of an interval I, equal to 1 on the defining interval and 0 elsewhere. The component $\frac{n}{2}\chi_{\left[x-\frac{1}{n}, x+\frac{1}{n}\right]}$ is typically referred to as the **averaging kernel** and it may be written more compactly as

$$g_n(t-x) \quad \text{where} \quad g_n(t) = \frac{n}{2}\chi_{\left[-\frac{1}{n},\frac{1}{n}\right]}(t).$$

This family of functions g_n has the following properties

(1) $g_n \geq 0$ for all n.

(2) $\displaystyle\int_{-\infty}^{\infty} g_n(t)\,dt = 1.$

(3) The g_n concentrate at 0, i.e. as $n \to \infty$, there is a greater emphasis on values near 0. Rigorously, the concentration property says that for all $\delta > 0$,

$$\lim_{n\to\infty}\int_{-\delta}^{\delta} g_n(t)\,dt = 1, \quad \text{or equivalently,} \quad \lim_{n\to\infty}\int_{[-\delta,\delta]^C} g_n(t)\,dt = 0.$$

The reason for using these g_n is that

$$f_n(x) = \int_{-\infty}^{\infty} g_n(t-x) f(t)\, dt$$

inherits nice properties from g_n. This description of f_n is a special case of

Definition. *For two functions $f, g : \mathbb{R} \to \mathbb{R}$, the* **convolution** *of f by g is*

$$f * g(x) = \int_{-\infty}^{\infty} g(x-t) f(t)\, dt.$$

Note that since our g_n are all even functions, $g_n(x-t) = g_n(t-x)$ so that in the language of convolutions, $f_n = f * g_n$.

Proposition 14.3.1. *Convolution is commutative, i.e. for any $f, g : \mathbb{R} \to \mathbb{R}$, $f * g(x) = g * f(x)$.*

Proof. Start with $f * g(x) = \displaystyle\int_{-\infty}^{\infty} g(x-t) f(t)\, dt$. Set $u = x - t$ so that $du = -dt$. Then

$$f * g(x) = -\int_{\infty}^{-\infty} g(u) f(x-u)\, du = \int_{-\infty}^{\infty} g(u) f(x-u)\, du = g * f(x).$$

□

Our strategy for approximating $C[0,1]$ functions by polynomials is as follows:

(1) Start with a function $f \in C[0,1]$.

(2) Extend f by $\bar{f}$ on $(-\infty, \infty)$ so that

   (a) $\bar{f}$ is uniformly continuous.

   (b) $\bar{f}$ is bounded.

(3) Define $f_n(x) := \bar{f} * g_n(x) = \displaystyle\int_{-\infty}^{\infty} g_n(x-t)\bar{f}(t)\, dt$.

Lemma 14.3.2. *(f_n) converges to $\bar{f}$ uniformly.*

Proof. Let $\varepsilon > 0$ and choose $\delta > 0$ such that $|\bar{f}(x) - \bar{f}(y)| < \frac{\varepsilon}{2}$ whenever $|x-y| < \delta$, and choose N such that for every $n \geq N$, $\displaystyle\int_{[-\delta,\delta]^C} g_n(t)\, dt < \frac{\varepsilon}{4M}$ for some M, by the concentration property (3) of g_n. Then

$$\begin{aligned}
|f_n(x) - \bar{f}(x)| &= \left| \int_{-\infty}^{\infty} \bar{f}(x-t) g_n(t)\, dt - \bar{f}(x) \cdot 1 \right| \\
&= \left| \int_{-\infty}^{\infty} \bar{f}(x-t) g_n(t)\, dt - \bar{f}(x) \int_{-\infty}^{\infty} g_n(t)\, dt \right| \quad \text{by property (2)} \\
&= \left| \int_{-\infty}^{\infty} (\bar{f}(x-t) - \bar{f}(x)) g_n(t)\, dt \right| \\
&\le \int_{-\infty}^{\infty} |\bar{f}(x-t) - \bar{f}(x)| g_n(t)\, dt \quad \text{by property (1)} \\
&= \int_{[-\delta,\delta]} |\bar{f}(x-t) - \bar{f}(x)| g_n(t)\, dt + \int_{[-\delta,\delta]^C} |\bar{f}(x-t) - \bar{f}(x)| g_n(t)\, dt \\
&\le \int_{[-\delta,\delta]} \frac{\varepsilon}{2} g_n(t)\, dt + \int_{[-\delta,\delta]^C} 2M g_n(t)\, dt \\
&\qquad \text{by uniform convergence and boundedness of } \bar{f} \\
&\le \frac{\varepsilon}{2} \cdot 1 + 2M \cdot \frac{\varepsilon}{4M} \quad \text{for large enough } n \text{ by property (3)} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence $(f_n) \to \bar{f}$ and since δ didn't depend on x or y, the convergence is uniform. □

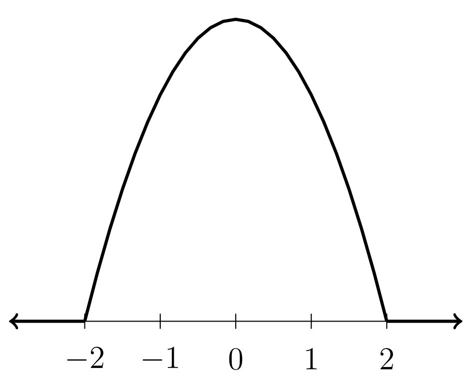
What if $g_n(x)$ is a polynomial? What can we say about f_n? For example, if $g_n(t) = c_0 + c_1 t + c_2 t^2$ then f_n inherits the property of being a polynomial, at least on $[0,1]$:

$$\begin{aligned}
\int_{-\infty}^{\infty} g_n(x-t) \bar{f}(t)\, dt &= \int_{-\infty}^{\infty} (c_0 + c_1(x-t) + c_2(x-t)^2) \bar{f}(t)\, dt \\
&= c_0 \int_{-\infty}^{\infty} \bar{f}(t)\, dt + c_1 x \int_{-\infty}^{\infty} \bar{f}(t)\, dt - c_1 \int_{-\infty}^{\infty} t \bar{f}(t)\, dt \\
&\quad + c_2 x^2 \int_{-\infty}^{\infty} \bar{f}(t)\, dt - 2c_2 x \int_{-\infty}^{\infty} t \bar{f}(t)\, dt + c_2 \int_{-\infty}^{\infty} t^2 \bar{f}(t)\, dt.
\end{aligned}$$

We can collect these since the integrals are just constant, yielding

$$f_n(x) = d_0 + d_1 x + d_2 x^2,$$

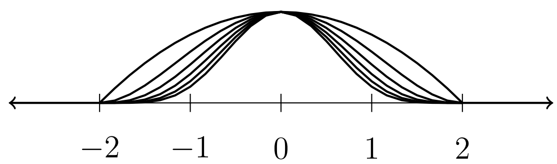
a polynomial.



A technical issue is that our choice of g_n must satisfy the given properties, so we will choose a quadratic-type polynomial on $[-2,2]$. We make the following observations about our chosen family of functions g_n:

- $\displaystyle\int_{-\infty}^{\infty} g_n(x-t)\bar{f}(t)\,dt = \int_{-1}^{2} g_n(x-t)\bar{f}(t)\,dt.$
- We are primarily interested in $x \in [0,1]$ to study $C[0,1]$ functions, so we only need $g_n(x-t)$ to be a polynomial for $0 \le x \le 1$ and $-1 \le t \le 2$. Added together, these give us $-2 \le x-t \le 2$ which is why we choose g_n to be polynomials on $[-2,2]$.

To construct g_n explicitly, first consider the following family of functions.



$$p_n(x) = \begin{cases} \left(1-\frac{x^2}{4}\right)^n & -2 \le x \le 2 \\ 0 & \text{elsewhere.} \end{cases}$$

We can see that $p_n \ge 0$, that is, p_n satisfy nonnegativity. Let

$$c_n = \int_{-\infty}^{\infty} p_n(x)\,dx = \int_{-2}^{2} \left(1-\frac{x^2}{4}\right)^n dx.$$

Now we can define $g_n(x) = \frac{1}{c_n}p_n(x)$. By construction, the g_n satisfy the nonnegativity and unit area properties. Note that as $n \to \infty$, c_n and p_n both tend to 0 so we will estimate c_n to control this term in the formula for g_n. Differentiating p_n gives us

$$p_n'(x) = n\left(1-\frac{x^2}{4}\right)^{n-1}\left(-\frac{2x}{4}\right) = -\frac{nx}{2}\left(1-\frac{x^2}{4}\right)^{n-1}.$$

It's also helpful to know concavity of g_n so we differentiate again:

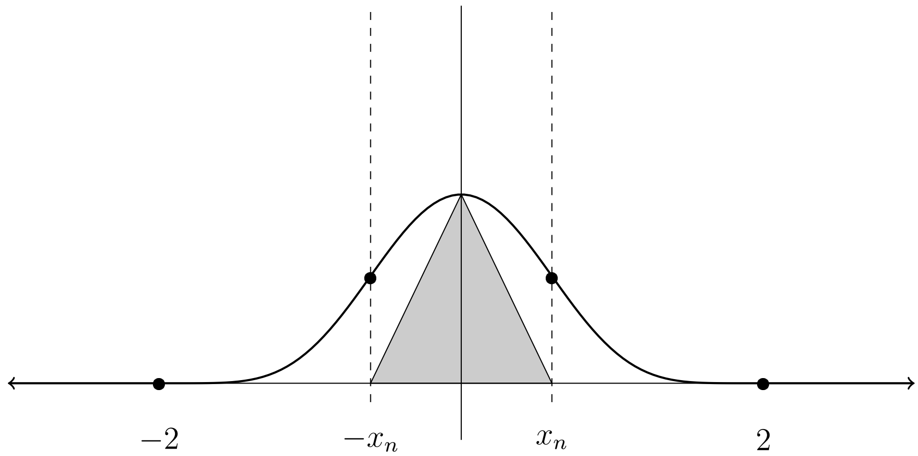
$$\begin{aligned} p_n''(x) &= -\frac{n}{2}\left(1-\frac{x^2}{4}\right)^{n-1} - \frac{nx}{2}(n-1)\left(1-\frac{x^2}{4}\right)^{n-2}\left(-\frac{2x}{4}\right) \\ &= -\frac{n}{2}\left(1-\frac{x^2}{4}\right)^{n-1} + n(n-1)\frac{x^2}{4}\left(1-\frac{x^2}{4}\right)^{n-2} \\ &= n\left(1-\frac{x^2}{4}\right)^{n-2}\left[-\frac{1}{2}\left(1-\frac{x^2}{4}\right) + (n-1)\frac{x^2}{4}\right]. \end{aligned}$$

The inflection points are found by solving the following equations:

$$0 = \left(1-\frac{x^2}{4}\right)^{n-2} \implies x = \pm 2$$

$$\begin{aligned} 0 &= -\frac{1}{2}\left(1-\frac{x^2}{4}\right) + (n-1)\frac{x^2}{4} \\ 0 &= -\frac{1}{2} + \frac{1}{2}\cdot\frac{x^2}{4} + (n-1)\frac{x^2}{4} \implies x_n = \pm\frac{2}{\sqrt{2n-1}}. \end{aligned}$$

The latter set of inflection points are the ones we are interested in; these are depicted on the graph below.



Looking at the inscribed triangle, we see its area is $\frac{1}{2} \cdot (2x_n) \cdot 1 = x_n$ and this is a lower bound for the total area under the curve, which is c_n. Hence $c_n \geq \frac{2}{\sqrt{2n-1}}$.

Now we can prove the concentration property for g_n by taking an arbitrary $\delta > 0$ and computing the integral

$$\begin{aligned}
\int_{\delta}^{2} g_n(x)\,dx \leq (2-\delta)g_n(\delta) &= \frac{1}{c_n} p_n(\delta)(2-\delta) \\
&\leq \frac{\sqrt{2n-1}}{2}\left(1-\frac{\delta^2}{4}\right)^n (2-\delta) \\
&= \frac{\sqrt{2n-1}}{2} r^n (2-\delta)
\end{aligned}$$

for some $0 < r < 1$. This term goes to 0 as n gets large and therefore as $n \to \infty$, $\displaystyle\int_{[-\delta,\delta]^C} g_n(x)\,dx \longrightarrow 0$ as $n \to \infty$, showing g_n satisfies the concentration property.

With our g_n in hand that satisfies the important properties of an approximating polynomial, we next prove that $f_n(x) = \bar{f} * g_n(x)$ is a polynomial for all $x \in [0,1]$. For $x \in [0,1]$ and $t \in [-1,2]$, we showed that $-2 \leq x - t \leq 2$. Thus

$$g_n(x-t) = \frac{1}{c_n}\left(1 - \frac{(x-t)^2}{4}\right)^n = \sum_{k=0}^{2n} a_k t^k x^{2n-k} + b.$$

This allows us to compute the convolution:

$$\begin{aligned}
\bar{f} * g_n(x) &= \int_{-\infty}^{\infty} g_n(x-t)\bar{f}(t)\,dt \\
&= \int_{-1}^{2}\left(\sum_{k=0}^{2n} a_k t^k x^{2n-k} + b\right)\bar{f}(t)\,dt \\
&= \sum_{k=0}^{2n}\left(a_k x^{2n-k}\int_{-1}^{2} t^k \bar{f}(t)\,dt\right) + b\int_{-1}^{2}\bar{f}(t)\,dt,
\end{aligned}$$

which is a polynomial in x.

We are now ready to prove the Weierstrass Approximation Theorem, which in general terms says that every $f \in C[0,1]$ can be approximated to any level of accuracy by a polynomial on $[0,1]$. In analytic terms,

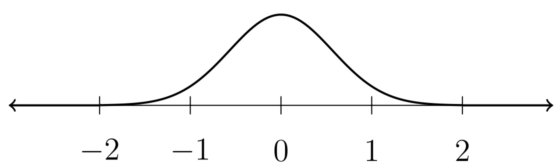
Theorem 14.3.3 (Weierstrass Approximation)**.** *The set of polynomials is dense in $C[0,1]$.*

Proof. We have defined a sequence (f_n) defined appropriately for any $f(x) \in C[0,1]$ such that each f_n is a polynomial on $[0,1]$. Moreover, we proved that (f_n) converges to f uniformly on $[0,1]$. This shows the set of polynomials is dense in $C[0,1]$. $\square$

Another approximation technique is to let

$$c_\varepsilon = \int_{-\infty}^{\infty} e^{-(t/\varepsilon)^2}\,dt \quad \text{and} \quad g_\varepsilon(x) = \frac{1}{c_\varepsilon} e^{-(x/\varepsilon)^2}$$

for any $\varepsilon > 0$. Then each g_ε condenses around 0 and decays rapidly outside $[0,1]$.



To compute numerical approximations for $f(x)$ (e.g. with *Mathematica*), we can define a family $g_\varepsilon(x)$ that approximate f and then further approximate the g_ε with polynomials.

14.4 Contraction Mapping

Definition. *Let (X, d) be a complete metric space and suppose $F : X \to X$ is a function on this metric space. If there is a number $\alpha \in (0, 1)$ such that $d(F(x), F(y)) \le \alpha\, d(x, y)$ for all $x, y \in X$, then F is called a* **contraction**.

Lemma 14.4.1. *Contractions are continuous.*

Proof. Let $F : X \to X$ be a contraction. Then given $\varepsilon > 0$, choose $\delta = \varepsilon$ and note that if $d(x, y) < \delta$, we have

$$d(F(x), F(y)) \le \alpha\, d(x, y) < \alpha\delta < \delta = \varepsilon.$$

Hence F is continuous. □

The main reason we study contraction maps is to determine *fixed points*. The following theorem, known as the contraction mapping theorem, asserts that a contraction on a complete metric space has a unique fixed point.

Theorem 14.4.2 (Contraction Mapping Theorem)**.** *Let X be a complete metric space and $F : X \to X$ be a contraction. Then F has a unique fixed point $x \in X$ such that $F(x) = x$.*

Proof. Let $x_1 \in X$ and for each $n \ge 2$, define $x_{n+1} = F(x_n)$. Although we have no target for convergence, X is complete so we will show the sequence (x_n) is Cauchy. Take $n, m \in \mathbb{N}$ – without loss of generality, assume $n > m$. Then consider

$$\begin{aligned} d(x_n, x_m) &= d(F(x_{n-1}), F(x_{m-1})) \\ &\le \alpha\, d(x_{n-1}, x_{m-1}) \\ &\le \alpha^{m-1} d(x_{n-m+1}, x_1) \quad \text{after } m-1 \text{ iterations.} \end{aligned}$$

Now we can write

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \ldots + d(x_{m+1}, x_m) \quad \text{by the triangle inequality} \\ &\le \alpha^{n-2} d(x_2, x_1) + \alpha^{n-3} d(x_2, x_1) + \ldots + \alpha^{m-1} d(x_2, x_1) \quad \text{by the above} \\ &= d(x_2, x_1)(\alpha^{m-1} + \ldots + \alpha^{n-2}) \\ &\le d(x_2, x_1)(\alpha^{m-1} + \ldots + \alpha^{n-2} + \alpha^{n-1} + \alpha^n + \ldots) \\ &= d(x_2, x_1)\frac{\alpha^{m-1}}{1-\alpha} \quad \text{by geometric series.} \end{aligned}$$

Now, given $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ such that $d(x_2, x_1)\frac{\alpha^{N-1}}{1-\alpha} < \varepsilon$. Then by the above estimate, $d(x_n, x_m) < \varepsilon$ for all $n, m > N$. Hence (x_n) is Cauchy and by completeness, $(x_n) \to x$ for some $x \in X$.

We will show that x is the unique fixed point of F. First, since F is continuous by Lemma 14.4.1, we can take the limit of our iterative definition:

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} x_n \implies F(x) = x.$$

Hence x is a fixed point of F. Suppoes y is another fixed point of F, that is, $F(y) = y$. If x and y are distinct,

$$0 < d(x, y) = d(F(x), F(y)) \leq \alpha\, d(x, y)$$

and dividing through by $d(x, y)$ (which is assumed to be nonzero) shows that $1 \leq \alpha$, contradicting $0 < \alpha < 1$. Therefore the fixed point x is unique. □

The search for fixed points, and in particular *unique* fixed points, is critical in the study of differential equations (recall Peano's theorem, Section 14.2). The contraction mapping theorem may be used to proved the so-called Existence and Uniqueness Theorem for ODEs.

Theorem 14.4.3. *Suppose we have an initial value problem $y'(t) = f(y(t), t)$, $y(0) = y_0$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying the Lipschitz condition. Then there exists an $\varepsilon > 0$ such that the IVP has a unique solution on the interval $[0, \varepsilon]$.*

Proof sketch. The Lipschitz condition says that there is some $k > 0$ such that

$$|f(t, y_2) - f(t, y_1)| \leq k\, |y_2, y_1| \quad \text{for all } y_1, y_2 \in \mathbb{R}.$$

As in Peano's theorem, we change the IVP into an integral equation:

$$y(t) = y_0 + \int_0^t f(y(s), s)\, ds.$$

To prove Theorem 14.4.3, we will find a continuous function y satisfying this equation. Let $X = \{y \in C[0, \varepsilon] : |y(t) - y_0| \leq \delta \text{ for all } t \in [0, \varepsilon]\}$. One can show that X is a complete metric space. For each $y \in X$, define

$$F(y) = y_0 + \int_0^t f(y(s), s)\, ds.$$

Now we have a fixed point problem, so we will show F is a contraction. First, $F(y)$ is a continuous function on $[0, \varepsilon]$ for each $y \in X$ because $y_0 + \int_0^t f(y(s), s)\, ds$ is differentiable on $[0, \varepsilon]$ (apply the fundamental theorem of calculus). Next, the extreme value theorem says that f is bounded on the closed interval $[0, \varepsilon]$. Then

$$\begin{aligned}
|F(y)(t) - y_0| &= \left| \int_0^t f(y(s), s)\, ds \right| \\
&\leq \int_0^t |f(y(s), s)|\, ds \\
&\leq Mt \quad \text{since } f \text{ is bounded on } [0, \varepsilon] \\
&\leq M\varepsilon.
\end{aligned}$$

This shows that the appropriate restriction on ε is $\varepsilon \leq \frac{\delta}{M}$ and shows that F is well-defined.

To show F is a contraction, let $y_1, y_2 \in X$ and consider

$$\begin{aligned}
|F(y_2)(t) - F(y_1)(t)| &= \left| \int_0^t (f(y_2(s), s) - f(y_1(s), s))\, ds \right| \\
&\leq \int_0^t |f(y_2(s), s) - f(y_1(s), s)|\, ds \\
&\leq k \int_0^t |y_2(s) - y_1(s)|\, ds \quad \text{by the Lipschitz condition} \\
&\leq k||y_2 - y_1||_\infty \int_0^t 1\, ds = kt||y_2 - y_1||_\infty \\
&\leq k\varepsilon||y_2 - y_1||_\infty.
\end{aligned}$$

This shows that if we choose $\varepsilon \leq \frac{1}{2k}$, then $|F(y_2)(t) - F(y_1)(t)| \leq \frac{1}{2}||y_2 - y_1||_\infty$ and taking the sup of both sides yields

$$||F(y_2) - F(y_1)||_\infty \leq \tfrac{1}{2}||y_2 - y_1||_\infty.$$

Hence F is a contraction, so by the contraction mapping theorem, F has a *unique* fixed point y satisfying

$$y(t) = F(y)(t) = y_0 + \int_0^t f(y(s), s)\, ds.$$

Therefore $y(t)$ is the unique solution to our original IVP. □

To recap a bit, the necessary assumptions to make a formal proof of the existence and uniqueness theorem work are

- Assume $k > 0$ is a value satisfying the Lipschitz condition for f.
- Pick $\delta > 0$ and notice that f is continuous on the closed rectangle $0 \leq t \leq 1$, $y_0 - \delta \leq y \leq y_0 + \delta$, so there exists an $M > 0$ such that $|f(y, t)| \leq M$ on this rectangle.
- Assume $\varepsilon \leq 1$, $\varepsilon \leq \frac{\delta}{M}$ and $\varepsilon \leq \frac{1}{2k}$.

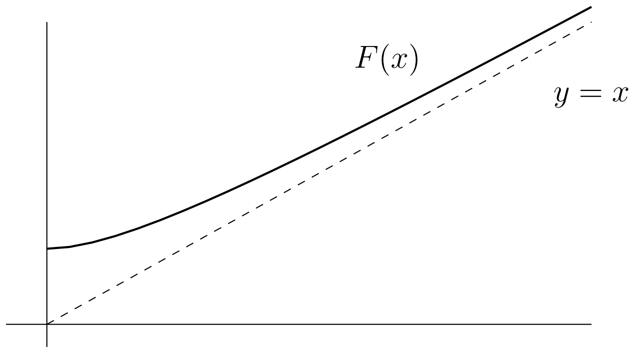
Example 14.4.4. Consider the IVP $y' = y^2$, $y(0) = 1$. In the study of ODEs, this is called a *separable differential equation* because it can be written

$$\frac{dy}{y^2} = dt.$$

We can integrate, yielding $-y^{-1} = t + c$ for a constant c. By the initial condition, $c = -1$ so we have $y = \frac{1}{1-t}$. Notice that y^2 fails the Lipschitz condition on $\mathbb{R}$, but the proof above can be adapted to show this IVP has a unique solution.

The above example shows that even without the Lipschitz condition, we can prove existence and uniqueness of solutions to initial value problems. This brings up a related question: can fixed points be found only with the Lipschitz condition? The following example shows that the answer is no in general.

Example 14.4.5. Consider the metric space $(C[0,\infty), |\cdot|)$ and the function $F(x) = x + \frac{1}{1+x}$.



Note that $|F(x) - F(y)| < |x - y|$ for all $x, y \in [0, \infty)$ so F *almost* satisfies the hypotheses of the contraction mapping theorem. However, no fixed point exists – graphically, a fixed point is represented by a point on the line $y = x$, but as we can see from the picture, $F(x)$ never intersects this line.

14.5 Differentiable Function Spaces

Recall the space $C^1[0,1] = \{f : [0,1] \to \mathbb{R} : f \text{ is differentiable}\}$. We turn $C^1[0,1]$ into a normed linear space by defining

$$||f||_{C^1} = ||f||_\infty + ||f'||_\infty.$$

It turns out that $(C^1[0,1], ||\cdot||_{C^1})$ is complete and therefore a Banach space. The Weierstrass Approximation Theorem (14.3.3) has a nice generalization to $C^1[0,1]$:

Theorem 14.5.1. *The polynomials are dense in* $C^1[0,1]$.

Proof. Let $f \in C^1[0,1]$. Since f is continuously differentiable, $f' \in C[0,1]$. By the Weierstrass Approximation Theorem (14.3.3), there exists a polynomial p such that $||f' - p||_\infty < \frac{\varepsilon}{2}$ for any given $\varepsilon > 0$. Let

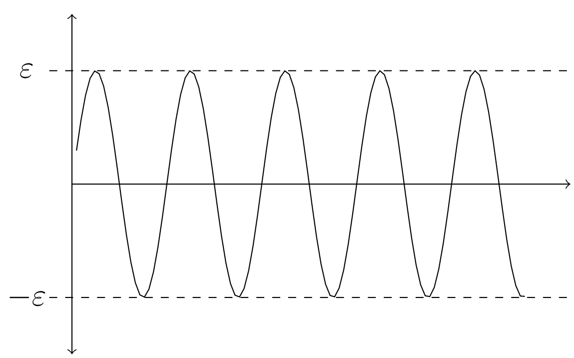
$$P(x) = f(0) = \int_0^x p(t)\,dt.$$

By the fundamental theorem of calculus, $f(x) = f(0) + \int_0^x f'(t)\,dt$ so we have

$$\begin{aligned} |f(x) - P(x)| &= \left| \int_0^x f'(t)\,dt - \int_0^x p(t)\,dt \right| \\ &\leq \int_0^x |f'(t) - p(t)|\,dt \\ &< \frac{\varepsilon x}{2} \leq \frac{\varepsilon}{2} \quad \text{for all } x \in [0,1]. \end{aligned}$$

This shows $||f - P||_\infty < \frac{\varepsilon}{2}$ which implies $||f - P||_{C^1} = ||f - P||_\infty + ||f' - p|| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence the polynomials are dense in $C^1[0,1]$. □

Example 14.5.2. Consider an oscillating function such as $f(x) = \varepsilon \sin(Mx)$, $\varepsilon > 0$, $M \geq 1$.



Clearly $||f||_\infty \leq \varepsilon$. However, $||f'||_\infty$ may be large, depending on how we choose M. Thus $||f||_{C^1}$ is not bounded by any constant multiple of $||f||_\infty$. By definition, $||g||_\infty \leq ||g||_{C^1}$ for any $g \in C^1[0,1]$, so this shows that the C^1 norm is finer than the sup norm.

14.6 Completing a Metric Space

Recall that $C[0,1]$ is complete with respect to $||\cdot||_\infty$, but *not complete* with respect to $||\cdot||_2$. In functional analysis we still want to be able to use $||\cdot||_2$ since it is useful in other settings. For example, calculating angles between "vectors" (functions) and defining an inner product (creating a Hilbert space; see Section 20.2) are possible in $(C[0,1], ||\cdot||_2)$. Therefore, our motivation in this section is to develop a *completion* of $(C[0,1], ||\cdot||_2)$.

The most useful analogy is the completion of $\mathbb{Q}$ into the real numbers $\mathbb{R}$. We want to reconstruct the aspects of this construction as closely as possible. Let (X,d) be a metric space. We define a relation $\sim$ on the set of Cauchy sequences in X by $(x_n) \sim (y_n)$ if and only if $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 14.6.1. *$\sim$ is an equivalence relation.*

Proof. For all $n \in \mathbb{N}$, $d(x_n, x_n) = 0$ so $(x_n) \sim (x_n)$. If $(x_n) \sim (y_n)$, then $\lim\limits_{n\to\infty} d(x_n, y_n) = \lim\limits_{n\to\infty} d(y_n, x_n)$ shows that $(y_n) \sim (x_n)$. Lastly, suppose $(x_n) \sim (y_n)$ and $(y_n) \sim (z_n)$. Then the triangle inequality gives us

$$d(x_n, z_n) \le d(x_n, y_n) + d(y_n, z_n)$$

and both of the terms on the right can be made small with sufficiently large n, so $d(x_n, z_n) \to 0$. Hence $(x_n) \sim (z_n)$ so $\sim$ is an equivalence relation. $\square$

This allows us to partition the set of Cauchy sequences in (X,d) via $\sim$. For any Cauchy sequence $(x_n) \subset X$, denote the equivalence class of (x_n) by

$$[(x_n)] = \{(y_n) \subset X \mid (y_n) \text{ is Cauchy and } (y_n) \sim (x_n)\}.$$

Lemma 14.6.2. *For any Cauchy sequences $(x_n), (y_n) \subset X$, $\lim\limits_{n\to\infty} d(x_n, y_n)$ exists.*

Proof. We show that $d(x_n, y_n)$ is a Cauchy sequence of real numbers and use the completeness of $\mathbb{R}$. First, for any $n, m \in \mathbb{N}$,

$$\begin{aligned} & d(x_n, y_n) \le d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \\ \Longrightarrow\ & d(x_n, y_n) - d(x_m, y_m) \le d(x_n, x_m) + d(y_m, y_n). \end{aligned}$$

Similarly, $d(x_m, y_m) - d(x_n, y_n) \le d(x_n, x_m) + d(y_n, y_m)$. Thus

$$|d(x_n, y_n) - d(x_m, y_m)| \le d(x_n, x_m) + d(y_n, y_m)$$

and given any $\varepsilon > 0$ we can choose N large enough so that for all $n, m > N$, $d(x_n, x_m) < \frac{\varepsilon}{2}$ and $d(y_n, y_m) < \frac{\varepsilon}{2}$ by Cauchy. Hence

$$|d(x_n, y_n) - d(x_m, y_m)| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon \quad \text{for all } n, m > N.$$

So $d(x_n, y_n)$ is a Cauchy sequence in $\mathbb{R}$ and therefore $\lim\limits_{n\to\infty} d(x_n, y_n)$ exists. $\square$

Definition. *For a metric space (X, d), define the* **completion** *of X with respect to d by $\overline{X} = \{[(x_n)] : (x_n) \subset X \text{ is Cauchy}\}$. Elements of $\overline{X}$ are equivalence classes of Cauchy sequences, which are written $\bar{x} = [(x_n)]$. A metric on $\overline{X}$ may be defined for any $\bar{x}, \bar{y} \in \overline{X}$ by*

$$\bar{d}(\bar{x}, \bar{y}) = \lim_{n\to\infty} d(x_n, y_n).$$

Lemma 14.6.2 shows that $\bar{d}(\bar{x}, \bar{y})$ exists for all $\bar{x}, \bar{y} \in \overline{X}$ but to shows $\bar{d}$ is well-defined, we need to verify that $d(x_n, y_n)$ does not depend on which Cauchy sequences we pick from each equivalence class.

Lemma 14.6.3. *Let $(x_n), (x'_n)$ and (y_n) be Cauchy sequences in X and suppose $(x_n) \sim (x'_n)$. Then*

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y_n).$$

Proof. By the triangle inequality,

$$\begin{aligned} d(x_n, y_n) &\le d(x_n, x'_n) + d(x'_n, y_n) \\ \implies d(x_n, y_n) - d(x'_n, y_n) &\le d(x_n, x'_n). \end{aligned}$$

Similarly, $d(x'_n, y_n) - d(x_n, y_n) \le d(x_n, x'_n)$, so

$$|d(x_n, y_n) - d(x'_n, y_n)| \le d(x_n, x'_n).$$

The right side can be made small for sufficiently large n since $(x_n) \sim (x'_n)$, so we conclude

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y_n).$$

□

The properties of a metric are now easy to verify for $\bar{d}$, which shows

Proposition 14.6.4. *$(\overline{X}, \bar{d})$ is a metric space.*

Recall that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$. In the same way, we want to first think of X as a subset of $\overline{X}$ – formally, it is not defined this way – and then show X is dense in $\overline{X}$. In other words, we want to define an injective function $\varphi : X \to \overline{X}$, i.e. a function such that $\varphi(x) = [(x_n)]$ for some Cauchy sequence $(x_n) \subset X$.

Definition. *Let $\varphi : X \to \overline{X}$ be the map $\varphi(x) = [(x_n)]$ where $x_n = x$ for all n.*

The constant sequence $(x_n) = (x, x, x, x, \dots)$ is clearly Cauchy, so this gives us $\varphi(X)$ as a subset of $\overline{X}$, i.e. φ is an embedding.

Lemma 14.6.5. *φ is an isometry.*

Proof. Let $x, y \in X$ and define the constant sequences $(x_n) = (x, x, x, \dots)$ and $(y_n) = (y, y, y, \dots)$. Then $\bar{d}(\varphi(x), \varphi(y)) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x, y) = d(x, y)$. Hence φ preserves distances, that is, φ is an isometry. □

Lemma 14.6.6. *$\varphi(X)$ is dense in $\overline{X}$.*

Proof. Let $\bar{x} \in \overline{X}$, where $\bar{x} = [(x_n)]$ for some representative Cauchy sequence $(x_n) \subset X$. Consider the sequence $(\varphi(x_k)) \subset \overline{X}$. Then $\bar{d}(\varphi(x_k), \bar{x}) = \lim_{n\to\infty} d(x_k, x_n)$. For a given $\varepsilon > 0$, let $N > 0$ such that for all $k, n > N$, $d(x_k, x_n) < \varepsilon$. This is possible since (x_n) is Cauchy. Then $\bar{d}(\varphi(x_k), \bar{x}) < \varepsilon$ for all $k > N$. Hence $\varphi(X)$ is a dense subset of $\overline{X}$. □

Our main goal is to show that $\overline{X}$ is a complete metric space. Our strategy has two parts:

(1) Show that if $(\bar{x}_n) \subset \varphi(X)$ is Cauchy, then $(\bar{x}_n)$ converges in $\overline{X}$.

(2) Show that any arbitrary Cauchy sequence $(\bar{x}_n) \subset \overline{X}$ converges in $\overline{X}$.

Lemma 14.6.7. *If $(\bar{x}_k) \subset \varphi(X)$ is Cauchy then $(\bar{x}_k)$ converges in $\overline{X}$.*

Proof. Let $\bar{x}_k = \varphi(x_k)$, i.e. $\bar{x}_k = [(x_{kn})]$ for the sequence (x_{kn}), $x_{kn} = x_k$ for all n. Since $(\varphi(x_k))$ is Cauchy in $\overline{X}$ and φ is an isometry, (x_k) is Cauchy in X and therefore we can define $\bar{x} = [(x_n)]$. Consider

$$\bar{d}(\bar{x}_k, \bar{x}) = \bar{d}(\varphi(x_k), \bar{x}) = \lim_{n\to\infty} d(x_{kn}, x_n) = \lim_{n\to\infty} d(x_k, x_n).$$

Then as in the last proof, for a given $\varepsilon > 0$ we can choose $N > 0$ such that for all $k, n > N$, $d(x_k, x_n) < \varepsilon$ by the Cauchy condition. Thus $\bar{d}(\bar{x}_k, \bar{x}) < \varepsilon$ for all $k > N$, so $(\bar{x}_k)$ converges to $\bar{x}$. □

Theorem 14.6.8. *$\overline{X}$ is a complete metric space.*

Proof. We have proven that $\overline{X}$ is a metric space, so it remains to show that if $(\bar{x}_n) \subset \overline{X}$ is a Cauchy sequence then $(\bar{x}_n)$ converges in $\overline{X}$. Given such a Cauchy sequence, choose $(\bar{y}_n) \subset \varphi(X)$ such that $\bar{d}(\bar{x}_n, \bar{y}_n) < \frac{1}{n}$ for all n, which is possible since $\varphi(X) \subset \overline{X}$ is dense. Consider the following inequality:

$$\bar{d}(\bar{y}_n, \bar{y}_m) \leq \bar{d}(\bar{y}_n, \bar{x}_n) + \bar{d}(\bar{x}_n, \bar{x}_m) + \bar{d}(\bar{x}_m, \bar{y}_m) < \tfrac{1}{n} + \bar{d}(\bar{x}_n, \bar{x}_m) + \tfrac{1}{m}.$$

Given $\varepsilon > 0$, choose $N > 0$ such that for all $n, m > N$, $\frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2}$ and $\bar{d}(\bar{x}_n, \bar{x}_m) < \frac{\varepsilon}{2}$, using the Archimedean Principle and the fact that $(\bar{x}_n)$ is Cauchy. Then for all $n, m > N$,

$$\bar{d}(\bar{y}_n, \bar{y}_m) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon,$$

which shows $(\bar{y}_n)$ is Cauchy. By Lemma 14.6.7, $(\bar{y}_n)$ converges to some $\bar{y} \in \overline{X}$. We will show that $(\bar{x}_n)$ also converges to $\bar{y}$ to complete the proof. Consider

$$\bar{d}(\bar{x}_n, \bar{y}) \leq \bar{d}(\bar{x}_n, \bar{y}_n) + \bar{d}(\bar{y}_n, \bar{y}) < \tfrac{1}{n} + \bar{d}(\bar{y}_n, \bar{y}).$$

Taking the limit as $n \to \infty$ makes the right side small since $(\bar{y}_n) \to \bar{y}$, so this shows $\bar{d}(\bar{x}_n, \bar{y}) \to 0$. Hence $(\bar{x}_n)$ converges to $\bar{y}$. □

Remark. The technique used in the proof above is easily modified for any metric space with a dense subset.

Definition. *The* **Lebesgue space** $L^2[0,1]$ *is defined as the completion of* $C[0,1]$ *with respect to the norm* $||\cdot||_2$.

Example 14.6.9. Consider the step function

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Recall that we found a sequence $(f_n) \subset C[0,1]$ that is Cauchy and "converges" to $f(x)$ *outside* $C[0,1]$. Technically, $f(x)$ is not an element of $L^2[0,1]$ because objects in the completion $L^2[0,1]$ are equivalence classes of Cauchy sequences in $C[0,1]$. However, it's common to denote $[(f_n)]$ by $f(x) \in L^2[0,1]$ and proceed with further computation.

14.7 Sobolev Space

For the space of differentiable functions $C^1[0,1]$ we can equip an alternate norm $||\cdot||_{1,2}$ defined by

$$||f||_{1,2} = ||f||_2 + ||f'||_2 = \left(\int_0^1 f^2\right)^{1/2} + \left(\int_0^1 (f')^2\right)^{1/2}.$$

Unlike with the $||\cdot||_{C^1}$ norm, the normed linear space with respect to $||\cdot||_{1,2}$ is not complete.

Definition. *The* **Sobolev space** $W^{1,2}[0,1]$ *is the completion of* $(C^1[0,1], ||\cdot||_{1,2})$.

This space has turned out to be of vital importance in modern analysis for finding solutions to differential equations, since $W^{1,2}[0,1]$ has some notion of differentiability.

There is even a limited notion of differentiability for $L^2[0,1]$ functions. Let $\bar{f} \in L^2[0,1]$, so $\bar{f} = [(f_n)]$ for a Cauchy sequence $(f_n) \subset C[0,1]$. Then for any $g \in C[0,1]$ and $0 \le a < b \le 1$,

$$\begin{aligned} \left|\int_a^b g(t)\,dt\right| &\le \int_a^b |g(t)|\,dt \\ &\le \left(\int_a^b 1^2\,dt\right)^{1/2}\left(\int_a^b g(t)^2\,dt\right)^{1/2} \quad \text{by Hölder's inequality (Lemma 14.1.2)} \\ &= (b-a)^{1/2}||g||_2. \end{aligned}$$

So $\left|\int_a^b g\right| \le ||g||_2$. Applied to the sequence (f_n), this means

$$\left|\int_a^b (f_n - f_m)\right| \le ||f_n - f_m||_2$$

which implies $\left(\int_a^b f_n\right)$ is a Cauchy sequence in $\mathbb{R}$ and hence converges. To show that this integral doesn't depend on our choice of (f_n), suppose $(h_n) \sim (f_n)$. Then by the above work,

$$\left|\int_a^b f_n - \int_a^b h_n\right| = \left|\int_a^b (f_n - h_n)\right| \le ||f_n - h_n||_2,$$

but since the sequences are in the same equivalence class, $||f_n - h_n||_2 \longrightarrow 0$ as n gets large. Hence the integrals are the same and we can state the following definition.

Definition. *For any* $\bar{f} = [(f_n)] \in L^2[0,1]$*, we define the* $\mathbf{L^2}$ **integral**

$$\int_a^b \bar{f} = \lim_{n\to\infty}\left(\int_a^b f_n\right).$$

It can be verified that $\int_a^b \bar{f}$ has all the nice properties of the ordinary integral from single-variable calculus. The story doesn't end there, however. Objects in $W^{1,2}[0,1]$ are even nicer for integration. Take $\bar{f} \in W^{1,2}[0,1]$, with $\bar{f} = [(f_n)]$ where (f_n) is now Cauchy with respect to $||\cdot||_{1,2}$. For any $g \in C^1[0,1]$, differentiability allows us to write

$$g(x) = g(0) + \int_0^x g'(t)\,dt$$

for any $x \in [0,1]$, which means

$$\begin{aligned}
|g(0)| &\le |g(x)| + \left|\int_0^x g'(t)\,dt\right| \\
&\le |g(x)| + \left(\int_0^x 1^2\,dt\right)^{1/2}\left(\int_0^x g'(t)^2\,dt\right)^{1/2} \quad \text{by Hölder's inequality (Lemma 14.1.2)} \\
&\le |g(x)| + ||g'||_2.
\end{aligned}$$

If we integrate both sides over the whole interval, this becomes

$$\begin{aligned}
\int_0^1 |g(0)|\,dt &\le \int_0^1 |g(x)|\,dx + \int_0^1 ||g'||_2\,dx \\
|g(0)| &\le ||g||_2 + ||g'||_2 \quad \text{using Hölder's inequality again} \\
&= ||g||_{1,2}.
\end{aligned}$$

Now for any $x \in [0,1]$, we have

$$\begin{aligned}
|g(x)| &\le |g(0)| + \int_0^x |g'(t)|\,dt \\
&\le ||g||_{1,2} + ||g'||_2 \quad \text{using the above manipulations} \\
&\le ||g||_{1,2} + ||g||_{1,2} = 2||g||_{1,2}
\end{aligned}$$

and taking the max gives us the following norm comparison: $||g||_\infty \le 2||g||_{1,2}$. Hence $||\cdot||_\infty$ is bounded by the $||\cdot||_{1,2}$ norm.

Returning to the sequence (f_n), our work so far shows that $||f_n - f_m||_\infty \le 2||f_n - f_m||_{1,2}$ so since (f_n) is Cauchy with respect to $||\cdot||_{1,2}$, this implies it is also Cauchy with respect to $||\cdot||_\infty$. We proved $(C[0,1], ||\cdot||_\infty)$ is complete, so (f_n) converges to some $f \in C[0,1]$. Suppose we also have $(g_n) \sim (f_n)$. Then the inequalities above give us $||g_n - f_n||_\infty \le 2||g_n - f_n||_{1,2}$ but (f_n) and (g_n) are in the same equivalence class, so the right side goes to 0 as $n \to \infty$. Therefore there is a well-defined choice of limit $f \in C[0,1]$ which satisfies $\lim(f_n) = f$ for $\bar{f} = [(f_n)] \in W^{1,2}[0,1]$. In fact, this gives us an embedding $W^{1,2}[0,1] \hookrightarrow C[0,1]$.

The most important theorem for Sobolev spaces is

Theorem 14.7.1 (Rellich-Kondrakov). *Let $(\bar{f}_n) \subset W^{1,2}[0,1]$ be a bounded sequence and let (f_n) be a representative sequence of $(\bar{f}_n)$ in $C[0,1]$. Then (f_n) has a subsequence which converges in $C[0,1]$.*

Proof. If $(\bar{f}_n)$ is bounded then there is an $M > 0$ such that $||f_n||_{1,2} \le ||\bar{f}_n|| \le M$ for all n. We showed that $||\cdot||_\infty$ is bounded by $||\cdot||_{1,2}$ so this implies the $||f_n||_\infty$ are bounded as well.

Write $f_n(x) = f_n(0) + \int_0^x f_n'(t)\,dt$ using the fundamental theorem of calculus. Then for all $x, y \in [0,1]$ and $n \in \mathbb{N}$,

$$\begin{aligned}
|f_n(y) - f_n(x)| &= \left| \int_x^y f_n'(t)\,dt \right| \\
&\leq \int_x^y |f_n'(t)|\,dt \\
&\leq \left(\int_x^y 1^2\,dt \right)^{1/2} \left(\int_x^y |f_n'(t)|^2\,dt \right)^{1/2} \qquad \text{by Hölder's inequality (Lemma 14.1.2)} \\
&= |y - x|^{1/2} ||f_n||_{1,2} \\
&\leq M|y - x|^{1/2}.
\end{aligned}$$

Thus (f_n) is both bounded and equicontinuous in $C[0,1]$ so the Arzela-Ascoli Theorem (14.2.1) says there is a subsequence which converges in $C[0,1]$. $\square$

Corollary 14.7.2. *There is a compact embedding of* $W^{1,2}[0,1]$ *into* $L^2[0,1]$.

Chapter 15

Calculus on Normed Linear Spaces

15.1 Differentiability

In single-variable calculus (see Section 4.2), if $\lim_{h \to 0} \frac{f(x_0+h) - f(x)}{h}$ exists then we say f is differentiable at x_0. As a consequence we can form linear approximations $f(x_0 + h) \approx f(x_0) + f'(x_0)h$, or alternatively $f(x) = f(x_0) + f'(x_0)(x - x_0) + error$. We can solve for the error term, $error = f(x) - (f(x_0) + f'(x_0)(x - x_0))$, and divide out by $x - x_0$:

$$\frac{error}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)$$

to see that $\lim_{x \to x_0} \frac{error}{x - x_0} = 0$. This is expressed more compactly in 'little o' notation.

Definition. *Let f be a continuous function. We say $f(x) = o(h)$ if*

$$\lim_{h \to 0} \frac{f(x)}{h} = 0.$$

In this notation, $error = o(x - x_0)$ in a linear approximation.

To generalize the notion of differentiability, let $f : X \to Y$ where X and Y are normed linear spaces. We want to associate the differentiability of f with an expression of the form

$$f(x) = f(x_0) + L(x - x_0) + o(x - x_0)$$

where L is a particularly 'nice' function, acting in a similar fashion as the single-variable derivative f'.

Definition. *A function $L : X \to Y$ is called a* **linear operator** *if*

(1) $L(cx) = cL(x)$ for all $c \in \mathbb{R}$ and $x \in X$.

(2) $L(x_1 + x_2) = L(x_1) + L(x_2)$ for all $x_1, x_2 \in X$.

The first two facts that one often verifies in single-variable calculus is that the derivative satisfies (1) and (2), so linear operators do indeed generalize the derivative. This allows us to define

Definition. *For normed linear spaces X and Y and a continuous function $f : X \to Y$, we say f is* **differentiable** *at $x_0 \in X$ if there is a continuous linear operator $L : X \to Y$ such that $f(x) = f(x_0) + L(x - x_0) + o(x - x_0)$. If such an L exists, it is called the* **Fréchet derivative** *of f at x_0, denoted $Df(x_0)$.*

Examples.

1. Consider $f(x) = x^2$ at $x_0 = 1$. The Fréchet derivative of f is simply the first derivative $f'(x) = 2x$ so the linear approximation at x_0 is
$$x^2 = 1 + 2(x-1) + error.$$
Solving for error, we have $error = x^2 - 2x + 1 = (x-1)^2$, and we see that
$$\lim_{x\to 1} \frac{(x-1)^2}{|x-1|} = 0$$
so $error = o(x-1)$. Hence $f(x) = x^2$ is differentiable.

2. Consider $f(x,y) = x^2 + y^2$ at $(x_0, y_0) = (1,2)$. The partial derivatives of f are $f_x = 2x$ and $f_y = 2y$ so at (x_0, y_0), the linear approximation is described by
$$x^2 + y^2 = 5 + 2(x-1) + 4(y-2) + error.$$
Solving for error, we have $error = x^2 - 2x + y^2 - 4y + 5 = (x-1)^2 + (y-2)^2$, so
$$\lim_{(x,y)\to(1,2)} \frac{(x-1)^2 + (y-2)^2}{||(x,y) - (1,2)||} = \frac{(x-1)^2 + (y-2)^2}{\sqrt{(x-1)^2 + (y-2)^2}} = 0.$$
Hence $error = o(\bar{x} - \bar{x}_0)$ so f is differentiable at $\bar{x}_0 = (1,2)$.

3. Consider the function $f(x,y) = (x_2 + y^2, x^2 - y^2) = (f_1(x,y), f_2(x,y))$ at $(x_0, y_0) = (1,2)$. There is a linear approximation of each component function:
$$f_1(x,y) = 5 + 2(x-1) + 4(y-2) + error_1 \quad \text{and} \quad f_2(x,y) = -3 + 2(x-1) - 4(y-2) + error_2,$$
which can be written together in matrix-vector form:
$$f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} + \begin{bmatrix} 2 & 4 \\ 2 & -4 \end{bmatrix} \begin{pmatrix} x-1 \\ y-2 \end{pmatrix} + \overrightarrow{error}.$$
Notice that the matrix $Df(x_0, y_0)$ is the Jacobian of f, i.e. the matrix of partials:
$$Df(x_0, y_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}.$$

4. Is the function $f(x,y) = |x|^{1/2}|y|^{1/2}$ differentiable at $(0,0)$? Although derivatives may be computed at $(0,0)$, e.g.
$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0,$$
an approximation would look like
$$|x|^{1/2}|y|^{1/2} = 0 + 0(x-0) + 0(y-0) + error \implies error = |x|^{1/2}|y|^{1/2}.$$

Then along the path $y = x$, the little o limit is

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{|x|^{1/2}|y|^{1/2}}{\sqrt{x^2+y^2}} = \frac{|x|}{|x|\sqrt{2}} = \frac{1}{\sqrt{2}} \neq 0.$$

Therefore f is not differentiable at $(0,0)$. This shows the importance of the tangent approximation structure in defining differentiability in higher-dimension spaces, rather than just the existence of derivatives. However, the next theorem shows that in the two-dimensional case it suffices to show that the partial derivatives exist on a *neighborhood* of (x_0, y_0) – this generalizes well to any finite n.

Theorem 15.1.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on a neighborhood of some (x_0, y_0). Then f is differentiable at (x_0, y_0).*

Proof. Let $error = f(x,y) - f(x_0,y_0) - f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0)$. By the Mean Value Theorem (Section 15.4), this can be written

$$\begin{aligned} error &= f(x,y) - f(x,y_0) + f(x,y_0) - f(x_0,y_0) - f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0) \\ &= f_y(x,y^*)(y-y_0) + f_x(x^*,y_0)(x-x_0) - f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0) \end{aligned}$$

for some y^* between y and y_0, and x^* between x and x_0. Then taking the little o limit,

$$\lim_{\bar{x}\to\bar{x}_0} \frac{error}{||\bar{x}-\bar{x}_0||} = \lim_{\bar{x}\to\bar{x}_0} \left[(f_x(x^*,y_0) - f_x(x_0,y_0)) \frac{x-x_0}{||\bar{x}-\bar{x}_0||} + (f_y(x,y^*) - f_y(x_0,y_0)) \frac{y-y_0}{||\bar{x}-\bar{x}_0||} \right].$$

Notice that

$$\frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \leq \frac{x-x_0}{\sqrt{(x-x_0)^2}} = 1$$

$$\text{and likewise} \quad \frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \leq \frac{y-y_0}{\sqrt{(y-y_0)^2}} = 1,$$

so these two partials are bounded. Also, since the x^* and y^* terms are squeezed between x, x_0 and y, y_0, respectively, the continuity of f_x and f_y on the neighborhood of (x_0, y_0) implies that $f_x(x^*, y_0) - f_x(x_0, y_0) \longrightarrow 0$ and $f_y(x, y^*) - f_y(x_0, y_0) \longrightarrow 0$. Hence

$$\lim_{\bar{x}\to\bar{x}_0} \frac{error}{||\bar{x}-\bar{x}_0||} = 0$$

so f is differentiable at (x_0, y_0). ☐

15.2 Linear Operators

In this section we further study the properties of linear operators introduced in the last section. Since Fréchet derivatives are linear operators, this will have implications in the study of differentiable functions on normed linear spaces.

Definition. *A linear operator $L : X \to Y$ is* **bounded** *if there is a $c > 0$ such that $||Lx||_Y \leq c||x||_X$ for all $x \in X$.*

Intuitively, c is a bound on how far a vector $x \in X$ may be "stretched" by L.

Lemma 15.2.1. *If X and Y are normed linear spaces and X is finite dimensional then any linear operator $L : X \to Y$ is bounded.*

Proof. We may assume $X = \mathbb{R}^n$ for some $n < \infty$. Let $\{e_1, \dots, e_n\}$ be the standard basis. Then for any $x \in \mathbb{R}^n$,

$$\begin{aligned}
||L(x)|| &= ||L(x_1e_1 + \ldots + x_ne_n)|| \\
&\leq \sum_{i=1}^{n} |x_i|\,||L(e_i)|| \quad \text{by the triangle inequality} \\
&\leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} ||L(e_i)||^2\right)^{1/2} \qquad \text{by Hölder's inequality (Lemma 14.1.2)} \\
&= c||x||
\end{aligned}$$

where $c = \left(\sum_{i=1}^{n} ||L(e_i)||^2\right)^{1/2}$. Hence L is bounded. ☐

Examples.

1. Define $D : (C^1[0,1], ||\cdot||_\infty) \to (C[0,1], ||\cdot||_\infty)$ by $Df = f'$. Then for any $f, g \in C^1[0,1]$ and $\alpha \in \mathbb{R}$,

$$D(f+g) = (f+g)' = f' + g' = Df + Dg$$
$$\text{and} \quad D(\alpha f) = (\alpha f)' = \alpha f' = \alpha Df,$$

so D is a linear operator. However, D is unbounded since, for example, $D\sin(nx) = n\cos x$ for any n and this sequence is unbounded as $n \to \infty$.

2. Define the integral operator $L : (C[0,1], ||\cdot||_\infty) \to (C^1[0,1], ||\cdot||_{C^1})$ by

$$L(f) = \int_0^x f(t)\,dt.$$

Then since f is continuous on $[0,1]$, f is Riemann integrable so L is well-defined. Let $F(x) = Lf(x)$ so that by the fundamental theorem of calculus, $F'(x) = f(x)$. Then

$$||L(f)||_{C^1} = ||F||_{C^1} = ||F||_\infty + ||F'||_\infty = ||F||_\infty + ||f||_\infty.$$

For any $x \in [0,1]$, consider

$$|F(x)| \leq \int_0^x |f(t)|\,dt \leq \int_0^x ||f||_\infty\,dt \leq ||f||_\infty.$$

Then taking the sup on both sides gives us $||F||_\infty \leq ||f||_\infty$, which further implies $||L(f)||_{C^1} \leq 2||f||_\infty$ so the integral operator is bounded.

3. We saw in example 3 in Section 15.1 that many times a derivative is a matrix operator. Consider for example

$$L = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2.$$

Notice that L is a symmetric matrix so it is (1) diagonalizable, (2) has real eigenvalues and (3) its eigenvalues are orthogonal. To explicitly compute the eigenvalues, we set the determinant $\det(L-\lambda)$ equal to 0:

$$\begin{vmatrix} -1-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 0,$$

which produces the equation $0 = \lambda^2 - \lambda - 6 = (\lambda-3)(\lambda+2)$. So the eigenvalues for L are $\lambda_1 = 3$ and $\lambda_2 = -2$. Denote unit eigenvectors for λ_1, λ_2 by $\vec{x}_1$ and $\vec{x}_2$, respectively. Let $\vec{x} \in \mathbb{R}^2$. Then since $\vec{x}_1, \vec{x}_2$ are a basis for $\mathbb{R}^2$ we can write $\vec{x} = \alpha\vec{x}_1 + \beta\vec{x}_2$ for some $\alpha, \beta \in \mathbb{R}$. By orthonormality of $\vec{x}_1, \vec{x}_2$, we have

$$||\vec{x}||_2 = ||\alpha\vec{x}_1 + \beta\vec{x}_2||_2 = \sqrt{\alpha^2+\beta^2}.$$

Moreover, since $\vec{x}_1, \vec{x}_2$ are eigenvectors, $L\vec{x} = \alpha L\vec{x}_1 + \beta L\vec{x}_2 = 3\alpha\vec{x}_1 - 2\beta\vec{x}_2$. Then

$$\begin{aligned} ||L\vec{x}||_2 &= ||3\alpha\vec{x}_1 - 2\beta\vec{x}_2||_2 = \sqrt{9\alpha^2+4\beta^2} \\ &\leq \sqrt{9\alpha^2+9\beta^2} = 3\sqrt{\alpha^2+\beta^2} = 3||\vec{x}||_2. \end{aligned}$$

Hence L is a bounded linear operator and it is easy to show that the best bound will always be the largest (in magnitude) of the eigenvectors; this is sometimes referred to as the *principal eigenvalue* of the operator.

4. The Laplacian $L = -\Delta$ is a linear operator on $C^2[0,1]$: $Lu = -(u_{xx}+u_{yy})$.

Remark. To show L is a bounded linear operator, it suffices to show that $||Lu|| \leq c||u||$ for all *unit vectors u*.

Definition. *Let X, Y be normed linear spaces. Define the vector space*

$$\mathcal{L}(X,Y) = \{L : X \to Y \mid L \textit{ is a bounded linear operator}\}$$

and the **operator norm** $||L||_{op} = \inf\{c > 0 : ||Lx||_Y \leq c||x||_X$ *for all* $x \in X\}$.

Lemma 15.2.2. $||\cdot||_{op}$ *is a norm on* $\mathcal{L}(X,Y)$.

Proof. Positivity follows from the definition. Suppose $||L||_{op} = 0$. Then for any $x \in X$, $||Lx|| \leq c||x||$ is true for all $c > 0$. This implies $||Lx|| = 0$ for all x, so L must be the zero operator. To show the triangle inequality, take $x \in X$ and consider

$$\begin{aligned} ||(L_1 + L_2)x||_Y &= ||L_1x + L_2x||_Y \\ &\leq ||L_1x||_Y + ||L_2x||_Y \quad \text{by triangle inequality on } Y \\ &\leq ||L_1||_{op}||x||_X + ||L_2||_{op}||x||_X \\ &= (|L_1||_{op} + ||L_2||_{op})||x||_X. \end{aligned}$$

It follows that $||L_1 + L_2||_{op} \leq ||L_1||_{op} + ||L_2||_{op}$. Scalars are proven similarly. Hence $(\mathcal{L}(X,Y), ||\cdot||_{op})$ is a normed linear space. ☐

Lemma 15.2.3. *Suppose $L : X \to Y$ is a linear operator. Then L is bounded if and only if L is continuous.*

Proof. ($\Longrightarrow$) If L is bounded then $||Lx_2 - Lx_1||_Y = ||L(x_2 - x_1)||_Y \leq ||L||_{op}||x_2 - x_1||_X$. Given $\varepsilon > 0$, choose $\delta = \dfrac{\varepsilon}{||L||_{op}}$ so that

$$||x_2 - x_1||_X < \delta \implies ||Lx_2 - Lx_1||_Y < \varepsilon.$$

This shows L is continuous.

($\Longleftarrow$) Conversely, suppose L is continuous. We will use continuity at $x_0 = 0$. Let $\varepsilon = 1$ and choose $\delta > 0$ such that $||Lx||_Y \leq 1$ for all $||x||_X \leq \delta$. In particular, consider $x \in X$ for which $||x||_X = \delta$. Then

$$\left|\left|L\left(\tfrac{x}{||x||}\right)\right|\right|_Y \leq \tfrac{1}{\delta}.$$

This shows L is bounded for all unit vectors, so by the remark above, L is bounded for all $x \in X$. ☐

We make some observations about $||\cdot||_{op}$. For any linear operator L, if $x = 0$ then $Lx = 0$ so we need only check Lx for $x \neq 0$. Also, by an earlier remark, $||Lx|| \leq c||x||$ for all $x \neq 0 \iff ||Lx|| \leq c$ for all $x \in X$ with $||x|| = 1$. Therefore an equivalent definition of the operator norm is

$$||L||_{op} = \sup\{||Lx||_Y : x \in X, ||x||_X = 1\}.$$

With this in mind, it is useful to think of $||L||_{op}$ as the 'maximum stretch' of the unit vectors in X by L.

Lemma 15.2.4. *For any $T \in \mathcal{L}(X,Y)$,*

$$||T||_{op} = \sup_{\substack{x \in X \\ ||x||_X = 1}} ||Tx||_Y = \inf\{C \geq 0 : ||Tx||_Y \leq C||x||_X \text{ for all } x \in X\}.$$

Remark. If $X \xrightarrow{T} Y \xrightarrow{S} Z$ are bounded operators, then for any $x \in X$,

$$||STx||_Z \leq ||S||_{op}||T||_{op}||x||_X.$$

In particular $||ST||_{op} \leq ||S||_{op}||T||_{op} < \infty$, so $ST \in \mathcal{L}(X,Z)$. A special case of this shows that $\mathcal{L}(X,X)$ is an algebra.

Theorem 15.2.5. *If Y is a Banach space then $\mathcal{L}(X,Y)$ is a Banach space.*

Proof. What this statement boils down to is that $\mathcal{L}(X,Y)$ is complete if Y is complete. Let $(L_n) \subset \mathcal{L}(X,Y)$ be a Cauchy sequence of bounded linear operators $L_n : X \to Y$. The first step as with $C[0,1]$ is to show pointwise convergence of this sequence. Take $x \in X$ and consider the sequence $(L_n(x))$ in Y. Then

$$||L_n(x) - L_m(x)||_Y = ||(L_n - L_m)x||_Y \le ||L_n - L_m||_{op}||x||_X.$$

Since (L_n) is Cauchy, for any $\varepsilon > 0$ we may choose an $N \in \mathbb{N}$ such that $||L_n - L_m||_{op} < \frac{\varepsilon}{||x||_X}$ for all $n, m > N$, which by the above gives us $||L_n(x) - L_m(x)||_Y < \varepsilon$ for all $n, m > N$. Hence $(L_n(x)) \subset Y$ is Cauchy. Since Y is complete, this sequence converges to some element y in Y. Define a function $L : X \to Y$ by $L(x) = y$ for this element y.

We will next show that $L \in \mathcal{L}(X,Y)$. For any $x_1, x_2 \in X$, consider

$$\begin{aligned}
L(x_1 + x_2) &= \lim_{n\to\infty} L_n(x_1 + x_2) \\
&= \lim_{n\to\infty} L_n(x_1) + L_n(x_2) \quad \text{by linearity of the } L_n \\
&= \lim_{n\to\infty} L_n(x_1) + \lim_{n\to\infty} L_n(x_2) \quad \text{by pointwise convergence} \\
&= L(x_1) + L(x_2).
\end{aligned}$$

A similar proof shows $L(\alpha x) = \alpha L(x)$ for all $x \in X, \alpha \in \mathbb{R}$, so L is linear. To show L is bounded, note that Cauchy sequences are bounded in any space, so there exists an $M > 0$ such that $||L_n||_{op} \le M$. Then

$$||Lx||_Y = \lim_{n\to\infty} ||L_n x||_Y \le \lim_{n\to\infty} ||L_n||_{op}||x||_X \le \lim_{n\to\infty} M||x||_X = M||x||_X.$$

Hence L is a bounded linear operator.

Finally, we prove that (L_n) converges to L. Using the triangle inequality, we can write

$$\begin{aligned}
||L_n x - Lx||_Y &= ||L_n x - L_m x + L_m x - Lx||_Y \\
&\le ||L_n x - L_m x||_Y + ||L_m x - Lx||_Y \\
&\le ||L_n - L_m||_{op}||x||_X + ||L_m x - Lx||_Y.
\end{aligned}$$

For a given $\varepsilon > 0$, we can choose N large enough so that $||L_n - L_m||_{op} < \varepsilon$ for all $n, m > N$ by the Cauchy property. This gives us

$$||L_n x - Lx||_Y < \varepsilon||x||_X + ||L_m x - Lx||_Y$$

and as $m \to \infty$, $||L_m x - Lx||_Y \longrightarrow 0$ by pointwise convergence, so $||L_n x - Lx||_Y < \varepsilon||x||_X$ for all $x \in X$ and $n > N$. Hence $||L_n - L||_{op} < \varepsilon$ for all $n > N$, so $(L_n) \to L$ in $\mathcal{L}(X,Y)$. We therefore conclude that $\mathcal{L}(X,Y)$ is a complete normed linear space. □

Proposition 15.2.6. *Define the set $D = \{L \in \mathcal{L}(X,X) : L \text{ is invertible with bounded inverse}\}$ for a Banach space X.*

(1) D is open.

(2) The function $\varphi : D \to D$ defined by $\varphi(L) = L^{-1}$ is continuous.

Proof omitted. □

15.3 Rules of Differentiation

In this section we generalize the usual properties of derivatives in a single variable (Section 4.2) to Fréchet derivatives. These are

1. (Additivity) Suppose $f : X \to Y$ and $g : X \to Y$ are differentiable at x_0. Then $f + g$ is differentiable at x_0 with $D(f + g)(x_0) = Df(x_0) + Dg(x_0)$.

2. (Product Rule) Let $f : X \to \mathbb{R}$ and $g : X \to Y$ be differentiable at x_0. Then $fg : X \to Y$ is differentiable at x_0 with $D(fg)(x_0) = Df(x_0) \cdot g(x_0) + f(x_0) \cdot Dg(x_0)$.

3. (Chain Rule) Suppose $f : X \to Y$ is differentiable at x_0 and $g : Y \to Z$ is differentiable at $f(x_0)$. Then the composition $g \circ f : X \to Z$ is differentiable at x_0 with $D(g \circ f)(x_0) = Dg(f(x_0)) \cdot Df(x_0)$.

Notice that in 2 and 3 we must be careful with how we multiply and compose linear operators. The product rule does not hold, for example, for two differentiable functions $f, g : X \to Y$ since it is not clear how elements of Y should be multiplied.

Proof of 1. Since f and g are differentiable, we have

$$\begin{aligned}(f+g)(x) &= f(x) + g(x) \\ &= [f(x_0) + Df(x_0)(x - x_0) + o_f(x - x_0)] + [g(x_0) + Dg(x_0)(x - x_0) + o_g(x - x_0)] \\ &= (f(x_0) + g(x_0)) + (Df(x_0) + Dg(x_0))(x - x_0) + o_f(x - x_0) + o_g(x - x_0).\end{aligned}$$

Observe that $Df(x_0) + Dg(x_0)$ is a linear operator, and

$$\lim_{x \to x_0} \frac{o_f(x - x_0) + o_g(x - x_0)}{||x - x_0||} = \lim_{x \to x_0} \frac{o_f(x - x_0)}{||x - x_0||} + \lim_{x \to x_0} \frac{o_g(x - x_0)}{||x - x_0||} = 0 + 0 = 0.$$

So $o_f(x - x_0) + o_g(x - x_0) = o(x - x_0)$. Hence $f + g$ is differentiable. □

Proof of 2. Again, differentiability allows us to write

$$\begin{aligned}(fg)(x) &= f(x)g(x) \\ &= [f(x_0) + Df(x_0)(x - x_0) + o_f(x - x_0)][g(x_0) + Dg(x_0)(x - x_0) + o_g(x - x_0)] \\ &= f(x_0)g(x_0) + [g(x_0)Df(x_0) + f(x_0)Dg(x_0)](x - x_0) + f(x_0)o(x - x_0) \\ &\quad + Df(x_0)(x - x_0)Dg(x_0)(x - x_0) + Df(x_0)(x - x_0)o(x - x_0) \\ &\quad + o(x - x_0)g(x_0) + o(x - x_0)Dg(x_0)(x - x_0) + o(x - x_0)o(x - x_0).\end{aligned}$$

We deal with the end terms one at a time to show they are each little o; by the proof of 1, the sum of little o terms is again little o. Consider

$$\lim_{x \to x_0} \frac{f(x_0)o(x - x_0)}{||x - x_0||} = f(x_0) \lim_{x \to x_0} \frac{o(x - x_0)}{||x - x_0||} = 0.$$

Similarly, $o(x-x_0)g(x_0) = o(x-x_0)$ — moreover, this shows that a constant times a little o term is little o. Next,

$$\lim_{x\to x_0} \frac{o(x-x_0)o(x-x_0)}{||x-x_0||} = \lim_{x\to x_0} o(x-x_0) \lim_{x\to x_0} \frac{o(x-x_0)}{||x-x_0||} = 0\cdot 0 = 0,$$

and this also shows products of little o terms are little o. For the rest of the terms, it's helpful to observe how fast their norms shrink:

$$\begin{aligned} ||Df(x_0)(x-x_0)o(x-x_0)|| &\le |Df(x_0)(x-x_0)| \cdot ||o(x-x_0)|| \\ &\le ||Df||_{op}||x-x_0|| \cdot ||o(x-x_0)||. \end{aligned}$$

So we have

$$\lim_{x\to x_0} \left|\left| \frac{Df(x_0)(x-x_0)o(x-x_0)}{||x-x_0||} \right|\right| \le \lim_{x\to x_0} ||Df||_{op}||x-x_0|| \cdot ||o(x-x_0)|| = 0$$

and therefore $Df(x_0)(x-x_0)o(x-x_0) = o(x-x_0)$. Similarly $o(x-x_0)Dg(x_0)(x-x_0) = o(x-x_0)$. Lastly,

$$\begin{aligned} \lim_{x\to x_0} \left|\left| \frac{Df(x_0)(x-x_0)Dg(x_0)(x-x_0)}{||x-x_0||} \right|\right| &\le \lim_{x\to x_0} ||Df||_{op}||Dg||_{op} \frac{||x-x_0||^2}{||x-x_0||} \\ &= \lim_{x\to x_0} ||Df||_{op}||Dg||_{op}||x-x_0|| = 0. \end{aligned}$$

Thus most of the expression for $(fg)(x)$ is little o, so we have

$$(fg)(x) = (fg)(x_0) + [g(x_0)Df(x_0) + f(x_0)Dg(x_0)](x-x_0) + o(x-x_0).$$

Hence fg is differentiable. □

Proof of 3. For notational convenience, let $y = f(x)$ and $y_0 = f(x_0)$. Then differentiability allows us to write

$$\begin{aligned} g(f(x)) &= g(y_0) + Dg(y_0)(y-y_0) + o(y-y_0) \\ &= g(y_0) + Dg(y_0)(f(x) - f(x_0)) + o(y-y_0) \\ &= g(y_0) + Dg(y_0)[Df(x_0)(x-x_0) + o(x-x_0)] + o(y-y_0) \\ &= g(y_0) + Dg(y_0)Df(x_0)(x-x_0) + Dg(y_0)o(x-x_0) + o(y-y_0). \end{aligned}$$

Consider $Dg(y_0)o(x-x_0)$; the first part is a linear operator in $\mathcal{L}(X,Y)$ and the little o term is an element of Y, so this whole term lies in Z. Then

$$||Dg(y_0)o(x-x_0)||_Z \le ||Dg||_{op}||o(x-x_0)||_Y$$

so in the limit this becomes

$$\lim_{x\to x_0} \frac{||Dg(y_0)o(x-x_0)||_Z}{||x-x_0||_X} \le \lim_{x\to x_0} ||Dg||_{op} \frac{||o(x-x_0)||_Y}{||x-x_0||_X} = 0.$$

Next, consider

$$\begin{aligned}||o(y-y_0)|| &= \frac{||o(y-y_0)||}{||y-y_0||}||y-y_0|| = \frac{||o(y-y_0)||}{||y-y_0||}||f(x)-f(x_0)|| \\ &= \frac{||o(y-y_0)||}{||y-y_0||}||Df(x_0)(x-x_0)+o(x-x_0)||.\end{aligned}$$

Then taking norms and applying the triangle inequality produces

$$\begin{aligned}||o(y-y_0)|| &\leq \frac{||o(y-y_0)||}{||y-y_0||}(||Df(x_0)(x-x_0)||+||o(x-x_0)||) \\ &= \frac{||o(y-y_0)||}{||y-y_0||}(||Df||_{op}||x-x_0||+||o(x-x_0)||) \\ \implies \lim_{x\to x_0}\frac{||o(y-y_0)||}{||x-x_0||} &\leq \lim_{x\to x_0}\frac{||o(y-y_0)||}{||y-y_0||}\left(||Df||_{op}+\frac{||o(x-x_0)||}{||x-x_0||}\right) \\ &= \lim_{x\to x_0}\frac{||o(y-y_0)||}{||y-y_0||}(||Df||_{op}+0).\end{aligned}$$

By continuity, $y=f(x)\to f(x_0)=y_0$ as $x\to x_0$, so the entire limit goes to 0. Condensing all the little o terms, we have

$$g(f(x)) = g(f(x_0)) + Dg(f(x_0))Df(x_0)(x-x_0) + o(x-x_0).$$

Hence $g\circ f$ is differentiable. □

15.4 The Mean Value Theorem

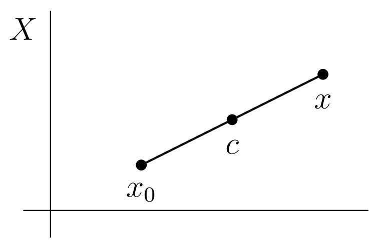
Recall the mean value theorem from single-variable calculus (Theorem 4.3.3):

Theorem. *For any function $f \in C^1[a,b]$, there exists a number c between a and b such that $f(b) - f(a) = f'(c)(b-a)$.*

The natural question is: Does the mean value theorem generalize to continuous functions on normed linear spaces? For this to make sense, the equation would have to be

$$f(x) - f(x_0) = Df(c)(x - x_0)$$

for some element $c \in X$ lying on a 'segment' between x and x_0:



Example 15.4.1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (x^2, y^3)$. The Fréchet derivative of this map is the linear operator

$$Df(x,y) = \begin{bmatrix} 2x & 0 \\ 0 & 3y^2 \end{bmatrix}.$$

Let $x_0 = (0,0)$ and $x = (1,1)$. If there were a point on the line segment between x_0 and x, it would be of the form (t,t), $t \in (0,1)$ and we would then have

$$\begin{aligned} f(1,1) - f(0,0) &= Df(t,t)((1,1) - (0,0)) \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{bmatrix} 2t & 0 \\ 0 & 3t^2 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix}. \end{aligned}$$

However there is no such t satisfying $1 = 2t$ and $1 = 3t^2$ simultaneously, so a generalization of the mean value theorem fails in this scenario.

Luckily, we can generalize the mean value theorem to continuous, *real-valued* functions.

Theorem 15.4.2 (Mean Value Theorem)**.** *Suppose $f : X \to \mathbb{R}$ is continuously differentiable and $x_0, x_1 \in X$. Then there is some $c \in (0,1)$ such that*

$$f(x_1) - f(x_0) = Df(x_0 + c(x_1 - x_0))(x_1 - x_0).$$

Proof. Let $g(t) = f(x_0 + t(x_1 - x_0))$. Then $g \in C^1[0,1]$ with derivative vector $x_1 - x_0$. By the single-variable mean value theorem (above), there exists some $c \in (0,1)$ such that $g'(c) = (g(1) - g(0)) \cdot 1$. Using the chain rule (3 from Section 15.3),

$$f(x_1) - f(x_0) = g(1) - g(0) = g'(c) = Df(x_0 + c(x_1 - x_0)) \cdot (x_1 - x_0).$$

$\square$

For the general case $f : X \to Y$, the previous example shows that we can't hope for a full mean value theorem analog. However, we still develop a slightly weaker mean value inequality:

Theorem 15.4.3 (Mean Value Inequality). *If $f : X \to Y$ is continuous and Fréchet differentiable with bounded Fréchet derivative, then for any $x_0, x_1 \in X$,*

$$||f(x_1) - f(x_0)||_Y \leq M||x_1 - x_0||_X \quad \text{where } M = \sup ||DF(x)||_{op}.$$

Proof. Suppose $f : X \to Y$ is differentiable and $\sup ||Df(x)||_{op} = M > 0$. As above, let $g(t) = f(x_0 + t(x_1 - x_0))$. We want to show taht $||f(x_1) - f(x_0)||_Y \leq M||x_1 - x_0||_X$ so take norms of the linear approximation:

$$\begin{aligned} f(x) - f(x_0) &= Df(x_0)(x - x_0) + o(x - x_0) \\ ||f(x) - f(x_0)||_Y &= ||Df(x_0)(x - x_0) + o(x - x_0)||_Y \\ &\leq ||Df(x_0)(x - x_0)||_Y + ||o(x - x_0)||_Y \quad \text{by triangle inequality} \\ &\leq ||Df(x_0)||_{op}||x - x_0||_X + \frac{||o(x - x_0)||_Y}{||x - x_0||_X}||x - x_0||_X \\ &\leq M||x - x_0||_X + \frac{||o(x - x_0)||_Y}{||x - x_0||_X}||x - x_0||_X. \end{aligned}$$

Given $\varepsilon > 0$, we can choose $\delta > 0$ such that $\dfrac{||o(x - x_0)||_Y}{||x - x_0||_X} < \varepsilon$ whenever $||x - x_0||_X < \delta$ by the little o property. Thus for all $t \leq \dfrac{\delta}{||x_1 - x_0||}$,

$$||f(x_0 + t(x_1 - x_0)) - f(x_0)|| \leq (M + \varepsilon)t||x_1 - x_0||.$$

Define $T = \sup\{s > 0 : ||f(x_0 + t(x_1 - x_0)) - f(x_0)|| \leq (M + \varepsilon)t||x_1 - x_0|| \text{ for all } t \in [0, s]\}$. We know that $T \geq \dfrac{\delta}{||x_1 - x_0||}$ and by continuity,

$$||f(x_0 + t(x_1 - x_0)) - f(x_0)|| \leq (M + \varepsilon)t||x_1 - x_0||$$

for all $0 \leq t \leq T$. We want to show that $T = 1$. Suppose to the contrary that $T < 1$. By a similar argument as above, there exists a $\delta_T > 0$ such that

$$||f(x) - f(x_0 + T(x_1 - x_0))|| \leq (M + \varepsilon)||x - (x_0 + T(x_1 - x_0))||$$

for all x satisfying $||x - (x_0 + T(x_1 - x_0))|| < \delta_T$. In particular, this inequality works for $x = x_0 + (T + \mu)(x_1 - x_0)$ for any arbitrarily small $\mu > 0$. This implies

$$||(x_0 + (T + \mu)(x_1 - x_0)) - (x_0 + T(x_1 - x_0))|| = \mu||x_1 - x_0|| < \delta_T.$$

It follows that $||f(x_0 + t(x_1 - x_0)) - f(x_0)|| \leq (M + \varepsilon)t||x_1 - x_0||$ for all $0 \leq t \leq \dfrac{\delta_T}{||x_1 - x_0||}$, but this contradicts the definition of T as a supremum. Therefore $T = 1$ so $||f(x_1) - f(x_0)|| \leq (M + \varepsilon)||x_1 - x_0||$. Finally, $\varepsilon > 0$ was arbitrary, so we conclude that

$$||f(x_1) - f(x_0)|| \leq M||x_1 - x_0||.$$

□

15.5 Mixed Partials

In this section we use the mean value theorem to prove the property from multivariable calculus that for a twice-continuously differentiable function $f(x, y)$, the mixed partial derivatives f_{xy} and f_{yx} are equal.

Theorem 15.5.1. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a twice-continuously differentiable function on a neighborhood of the point (x_0, y_0) then $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.*

Proof. In terms of linear approximations, we can write the first and second partials as

$$\begin{aligned}
f_x(x_0, y_0) &= \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \\
f_x(x_0, y_0 + k) &= \lim_{h\to 0} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)}{h} \\
\text{and} \quad f_{xy}(x_0, y_0) &= \lim_{k\to 0} \frac{f_x(x_0, y_0 + k) - f_x(x_0, y_0)}{k} \\
&= \lim_{h,k\to 0} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0 + k) - f(x_0 + h, y_0) + f(x_0, y_0)}{hk}.
\end{aligned}$$

Set $\Delta = f(x_0 + h, y_0 + k) - f(x_0 + h, y_0) - f(x_0, y_0 + k) + f(x_0, y_0)$. For a fixed k, define the function $g(t) = f(x_0 + t, y_0 + k) - f(x_0 + t, y_0)$ and note that

$$\begin{aligned}
g(h) &= f(x_0 + h, y_0 + k) - f(x_0 + h, y_0) \\
\text{and} \quad g(0) &= f(x_0, y_0 + k) - f(x_0, y_0)
\end{aligned}$$

so $\Delta = g(h) - g(0)$. By the mean value theorem, there's some $\bar{h}$ between 0 and h such that $\Delta = g(h) - g(0) = g'(\bar{h})(h - 0) = hg'(\bar{h})$. Differentiating g, we have

$$g'(t) = f_x(x_0 + t, y_0 + k) - f_x(x_0 + t, y_0)$$

so $\Delta = h(f_x(x_0 + \bar{h}, y_0 + k) - f_x(x_0 + \bar{h}, y_0))$. Now let $p(t) = f_x(x_0 + \bar{h}, y_0 + t)$ so we can vary k. Then by the mean value theorem (15.4.2), for some $\bar{k}$ between 0 and k,

$$\Delta = h(p(k) - p(0)) = hkp'(\bar{k}).$$

Consider $\frac{\Delta}{hk} = f_{xy}(x_0 + \bar{h}, y_0 + \bar{k})$. By continuity of f_{xy},

$$\lim_{h,k\to 0} \frac{\Delta}{hk} = f_{xy}(x_0, y_0).$$

Now return to where we defined $g(t)$ and vary k first and then h to show

$$\lim_{h,k\to 0} \frac{\Delta}{hk} = f_{yx}(x_0, y_0).$$

Hence the mixed partials are equal. □

15.6 Directional Derivatives

Since a normed linear space X is a vector space, we should be able to generalize the notion of a derivative in *any direction*, along the lines of multivariable calculus. We define

Definition. *Let $f : X \to Y$ be differentiable at u_0 and let $v \in X$. The* **directional derivative** *at u_0 in the direction of v is*

$$D_v f(u_0) = \lim_{t\to 0} \frac{f(u_0 + tv) - f(u_0)}{t}.$$

Using the fact that f is differentiable, we can actually evaluate this further:

$$\begin{aligned} D_v f(u_0) &= \lim_{t\to 0} \frac{f(u_0 + tv) - f(u_0)}{t} \\ &= \lim_{t\to 0} \left(\frac{t\, Df(u_0)(v)}{t} + \frac{o(t)}{t||v||}||v|| \right) \\ &= Df(u_0)(v) + 0 = Df(u_0)(v). \end{aligned}$$

Example 15.6.1. Consider the function $F : C[0,1] \to C^1[0,1]$ defined for all $u \in C[0,1]$ by

$$F(u) = \int_0^1 u^2(x)\, dx.$$

If F is differentiable at u_0 then for any $v \in C[0,1]$, the directional derivative in the direction of v is computed as

$$\begin{aligned} D_v F(u_0) &= \lim_{t\to 0} \frac{F(u_0 + tv) - F(u_0)}{t} \\ &= \lim_{t\to 0} \frac{1}{t} \left[\int_0^1 (u_0 + tv)^2 - \int_0^1 u_0^2 \right] \\ &= \lim_{t\to 0} \frac{1}{t} \int_0^1 2tu_0 v + \lim_{t\to 0} \frac{1}{t} \int_0^1 t^2 v^2 \\ &= \lim_{t\to 0} 2 \int_0^1 u_0 v + \lim_{t\to 0} t \int_0^1 v^2 \\ &= 2 \int_0^1 u_0 v + 0 = 2 \int_0^1 u_0 v. \end{aligned}$$

It's easy to check that $D_v F(u_0)$ is a linear operator. Moreover, we can write $F(u) = F(u_0) + DF(u_0)(u - u_0) + error$ and solving for error produces

$$\begin{aligned} error &= F(u) - f(u_0) - DF(u_0)(u - u_0) \\ &= \int_0^1 u^2 - \int_0^1 u_0^2 - 2 \int_0^1 u_0(u - u_0) \\ &= \int_0^1 (u^2 - 2uu_0 + u_0^2) = \int_0^1 (u - u_0)^2. \end{aligned}$$

We can estimate the last expression by $||F(u-u_0)|| \le ||u-u_0||_\infty^2$ which shows that

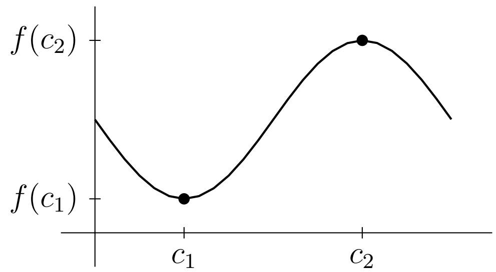
$$\lim_{u\to u_0} \frac{||error||}{||u-u_0||} \le \lim_{u\to u_0} \frac{||u-u_0||^2}{||u-u_0||} = 0.$$

Hence $error = o(u-u_0)$ so F is indeed differentiable and the directional derivative is well-defined.

Example 15.6.2. Along similar lines, a formula for the directional derivatives of the integral operator $F(u) = \int_0^1 u(x)\,dx$ is $D_v F(u_0) = \int_0^1 f'(u_0)v$.

15.7 Sard's Theorem

A common problem in calculus is identifying critical points of a function on an interval/domain. For example, the function below has two critical points, $\mathcal{C} = \{c_1, c_2\}$, and two critical values, $f(\mathcal{C}) = \{f(c_1), f(c_2)\}$.



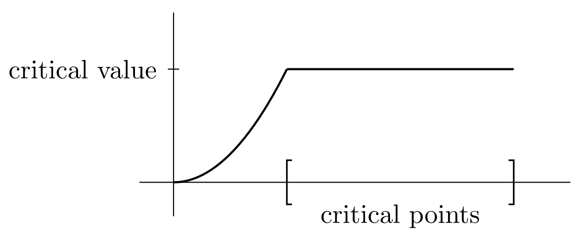
In loose terms, Sard's Theorem says that the set $f(\mathcal{C})$ of critical values of a function is small. What does it mean to be small? There are many notions for measuring a set's size in mathematics. The 'size' used in Sard's Theorem is *measure* – to state Sard's Theorem, it will suffice to define a measure zero set.

Definition. *A set $A \subset \mathbb{R}$ has* **measure zero***, denoted $|A| = 0$, if given any $\varepsilon > 0$, there is a collection of intervals $I_1, \ldots, I_n$ such that $A \subseteq \bigcup_{k=1}^{n} I_k$ and $\sum_{k=1}^{n} |I_k| < \varepsilon$.*

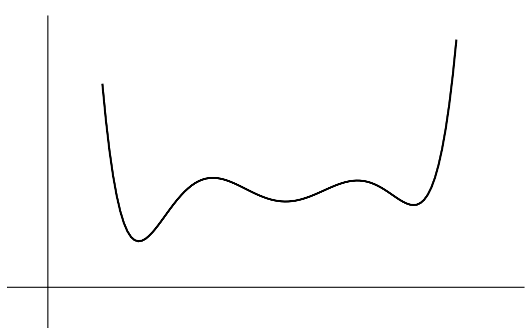
The length notation for an interval $I = [a, b]$ is standard: $|I| = b - a$. Sard's Theorem states that $|f(\mathcal{C})| = 0$, that is, the set of critical values of a function has measure zero.

Examples.

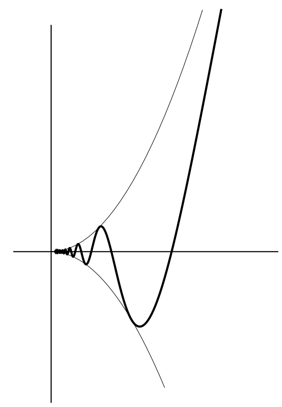
1. A function may have infinitely many critical *points*, as in the function below. However, the set of critical values is still small – in this case, there is a unique critical value.



2. A finite number of critical values are easy to capture with small intervals, as in the function below.



③ Consider the function $f(x) = x^2 \sin\left(\frac{1}{x}\right)$.



Here, $f(\mathcal{C})$ is countably infinite. However, we can still capture some of the critical values with an interval of length $\frac{\varepsilon}{2}$ about the origin and a finite number of intervals adding up to length $\frac{\varepsilon}{2}$ to capture the rest.

Theorem 15.7.1 (Sard's Theorem)**.** *Let $f \in C^1[0,1]$ and let $\mathcal{C} = \{x \in [0,1] : Df(x) = 0\}$. Then the set of critical values $f(\mathcal{C})$ has measure zero.*

Proof. Let $\varepsilon > 0$. Since f is differentiable, we can write $f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0)$ for any $x_0 \in [0,1]$, or even better, $f(x) = f(x_0) + o(x - x_0)$. Then for $x \neq x_0$,

$$f(x) - f(x_0) = o(x - x_0) = \frac{o(x - x_0)}{|x - x_0|}\,|x - x_0|,$$

which is small. By the mean value theorem, there is some c between x and x_0 such that we can write

$$\begin{aligned} o(x - x_0) &= f(x) - f(x_0) - f'(x_0)(x - x_0) \\ &= f'(c)(x - x_0) - f'(x_0)(x - x_0) \\ &= (f'(c) - f'(x_0))(x - x_0). \end{aligned}$$

So $\frac{|o(x-x_0)|}{|x-x_0|} = |f'(c) - f(x_0)|$ for such a number c. Recall that since f' is continuous on a closed interval, it is uniformly continuous on that interval. Then since c is between x and x_0, $|c - x_0| \leq |x - x_0|$. By uniform continuity, we can choose $\delta > 0$ such that if $|x_1 - x_2| < \delta$, then $|f'(x_1) - f'(x_2)| < \frac{\varepsilon}{2}$. Thus $|f'(x) - f'(c)| < \frac{\varepsilon}{2}$ for $|x - x_0| < \delta$, so there is a single $\delta > 0$ such that

$$|x - x_0| < \delta \quad \text{implies} \quad \frac{|o(x - x_0)|}{|x - x_0|} < \frac{\varepsilon}{2}.$$

Now choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$ and partition the interval $[0,1]$ into intervals $I_1, \ldots, I_n$ of length $\frac{1}{n}$. If I_k contains a critical point $x_k \in \mathcal{C}$, then for all $x \in I_k$, we have $f(x) - f(x_k) = o(x - x_k)$, so by our previous estimate,

$$|f(x) - f(x_k)| = |o(x - x_k)| < \frac{\varepsilon}{2} \cdot |x - x_k| < \frac{\varepsilon}{2} \cdot \frac{1}{n}.$$

Then $f(I_k) \subset J_k := \left[f(x_k) - \frac{\varepsilon}{2n}, f(x_k) + \frac{\varepsilon}{2n}\right]$ and $f(\mathcal{C}) \subset \bigcup_{\substack{1\le k\le n\\ I_k\cap\mathcal{C}\neq\varnothing}} J_k$. Additionally,

$$\sum_{\substack{1\le k\le n\\ I_k\cap\mathcal{C}\neq\varnothing}} |J_k| \le \sum_{k=1}^{n} \frac{\varepsilon}{n} = n\cdot\frac{\varepsilon}{n} = \varepsilon.$$

Therefore $f(\mathcal{C})$ has measure zero. $\square$

Sard's Theorem can be generalized to differentiable functions on n-dimensional spaces. First we need to define critical points and measure zero sets for $\mathbb{R}^n$.

Definition. *A differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$ has a* **critical point** *at x_0 if the differential $Df(x_0)$ is not invertible. In this case the point $f(x_0)$ is called a* **critical value** *of the function.*

Definition. *A set $A \subset \mathbb{R}^n$ is said to have* **measure zero** *if given any $\varepsilon > 0$, there is a countable collection of rectangles $\{R_1, R_2, \ldots\}$ such that $K \subset \bigcup_{n=1}^{\infty} R_n$ and $\sum_{n=1}^{\infty} |R_n| < \varepsilon$.*

Here the notation $|R|$ denotes the volume of a rectangle R in n-dimensional Euclidean space; if $R = [a_1, b_1] \times [a_2, b_2] \cdots \times [a_n, b_n]$ then its volume is

$$|R| = (b_1 - a_1)(b_2 - a_2)\cdots(b_n - a_n) = \prod_{k=1}^{n}(b_k - a_k).$$

Sard's Theorem for functions of n-dimensional space then reads:

Theorem 15.7.2 (Sard's Theorem)**.** *Let $f \in C^1([0,1]^n, \mathbb{R}^n)$ and let $\mathcal{C} = \{x \in [0,1]^n : Df(x) = 0\}$. Then the set $f(\mathcal{C})$ of critical values of f has measure zero.*

Proof. We prove the case when $n = 2$ and remark that things generalize to higher dimensions, although the notation is cumbersome.

Suppose $f = (f_1, f_2) \in C^1([0,1]^2, \mathbb{R}^2)$. Then $x_0 \in [0,1]^2$ is a critical point of the function if $\det Df(x_0) = 0$. This happens precisely when the columns of the Jacobian are linearly dependent. In particular, there is a vector v such that the columns are multiples of v and for any w, $Df(x_0)\cdot w = cv$ for some c. As we did in the one-dimensional case, we first need to estimate $o(x - x_0)$:

$$\begin{aligned}
o(x - x_0) &= f(x) - f(x_0) - Df(x_0)(x - x_0)\\
&= \begin{bmatrix} f_1(x) - f_1(x_0)\\ f_2(x) - f_2(x_0)\end{bmatrix} - \begin{bmatrix}\nabla f_1(x_0)(x - x_0)\\ \nabla f_2(x_0)(x - x_0)\end{bmatrix}\\
&= \begin{bmatrix} f_1(x) - f_1(x_0) - \nabla f_1(x_0)(x - x_0)\\ f_2(x) - f_2(x_0) - \nabla f_2(x_0)(x - x_0)\end{bmatrix}.
\end{aligned}$$

Now f_1 and f_2 are each real-valued functions so by the mean value theorem, there are numbers c_1 and c_2 which let us write

$$\begin{aligned}
o(x - x_0) &= \begin{bmatrix}\nabla f_1(c_1)(x - x_0) - \nabla f_1(x_0)(x - x_0)\\ \nabla f_2(c_2)(x - x_0) - \nabla f_2(x_0)(x - x_0)\end{bmatrix}\\
&= \begin{bmatrix}\nabla f_1(c_1) - \nabla f_1(x_0)\\ \nabla f_2(c_2) - \nabla f_2(x_0)\end{bmatrix}(x - x_0).
\end{aligned}$$

Taking norms,

$$||o(x-x_0)|| = \left|\left| \begin{bmatrix} \nabla f_1(c_1) - \nabla f_1(x_0) \\ \nabla f_2(c_2) - \nabla f_2(x_0) \end{bmatrix} (x-x_0) \right|\right|.$$

For the first component, the Cauchy-Schwartz inequality gives us

$$\begin{aligned} ||(\nabla f_1(c_1) - \nabla f_1(x_0)) \cdot (x-x_0)|| &\leq ||\nabla f_1(c_1) - \nabla f_1(x_0)|| \, ||x-x_0|| \\ &< \varepsilon \, ||x-x_0|| \quad \text{when } ||x-x_0|| < \delta_1 \end{aligned}$$

for a well-chosen $\delta_1 > 0$; this is possible, as in the proof before, by the uniform continuity of ∇f_1. Likewise we can choose a $\delta_2 > 0$ such that

$$||(\nabla f_2(c_2) - \nabla f_2(x_0)) \cdot (x-x_0)|| < \varepsilon \, ||x-x_0|| \quad \text{when } ||x-x_0|| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then if $||x-x_0|| < \delta$, $||o(x-x_0)|| < \varepsilon \, ||x-x_0||$ by the above work.

Partition $[0,1]^2$ into small squares such that no two points inside the same square are further than δ apart. We do this by choosing $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\delta}{\sqrt{2}}$ and setting

$$S_{ij} = \left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right].$$

The squares $\{S_{ij}\}_{i,j=0}^{n-1}$ form our partition of $[0,1]^2$. Let $S = S_{ij}$ for some i,j and suppose $x_0 \in S$ is a critical point of f. Then for any $x \in S$, $f(x) = f(x_0) + Df(x_0)(x-x_0) + o(x-x_0)$. Let v be the unit vector such that $Df(x_0) = \alpha v$ for some $\alpha \in \mathbb{R}$. Then $f(x) = f(x_0) + \alpha v + o(x-x_0)$ for any $x \in S$. By Lemma 15.2.3, $Df(x_0)$ is bounded, i.e. $||Df(x_0)|| \leq M$ for some $M > 0$. Then for any $x \in S$,

$$||Df(x_0)(x-x_0)|| \leq ||Df(x_0)|| \, ||x-x_0|| \leq M||x-x_0|| < M\frac{\sqrt{2}}{n}.$$

So $|\alpha| < M\frac{\sqrt{2}}{n}$. Now we turn our attention to the error term $o(x-x_0)$. Since $||x-x_0|| < \frac{\sqrt{2}}{n} < \delta$, we know $||o(x-x_0)|| < \varepsilon$. Let $v^\perp$ be a unit vector perpendicular to v. Then $o(x-x_0) = \beta_1 v + \beta_2 v^\perp$ for some β_1, β_2 since $\{v, v^\perp\}$ is an orthonormal basis for $\mathbb{R}^2$. Thus

$$||o(x-x_0)|| = \sqrt{\beta_1^2 + \beta_2^2} < \varepsilon \, ||x-x_0|| \leq \varepsilon \, \frac{\sqrt{2}}{n}$$

which shows that $|\beta_i| < \varepsilon \, \frac{\sqrt{2}}{n}$ for each $i = 1,2$. Now $f(x) = f(x_0) + (\alpha + \beta_1)v + \beta_2 v^\perp$ with $|\alpha| < M\frac{\sqrt{2}}{n}$, $|\beta_1| < \varepsilon\frac{\sqrt{2}}{n}$ and $|\beta_2| < \varepsilon\frac{\sqrt{2}}{n}$. Define a rectangle

$$R = \left\{ f(x_0) + sv + tv^\perp : s \in \left[-(M+\varepsilon)\tfrac{\sqrt{2}}{n}, (M+\varepsilon)\tfrac{\sqrt{2}}{n}\right], t \in \left[-\varepsilon\tfrac{\sqrt{2}}{n}, \varepsilon\tfrac{\sqrt{2}}{n}\right] \right\}.$$

By construction, $f(S) \subseteq R$ and the volume of R is

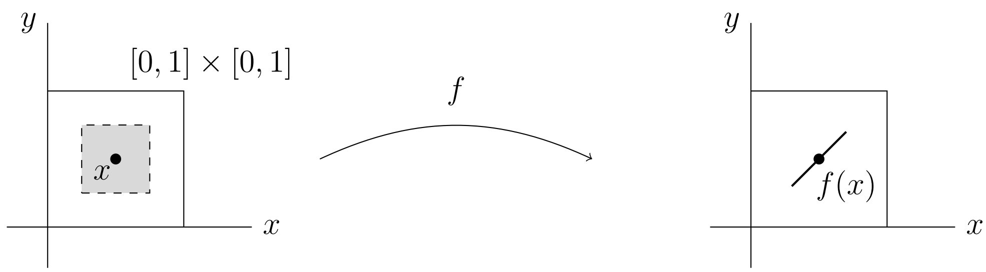
$$|R| = \left(2(M+\varepsilon)\tfrac{\sqrt{2}}{n}\right)\left(2\varepsilon\tfrac{\sqrt{2}}{n}\right) = 8\varepsilon(M+\varepsilon)\frac{1}{n^2}.$$

Let $S_1, \dots, S_N$ be all of the squares in the collection $\{S_{ij}\}$ that contain critical points of f. Let R_k be the rectangle constructed for S_k by the above procedure. Then $f(\mathcal{C})$ is contained in $\bigcup_{k=1}^N R_k$ and the total area of the rectangles is

$$\sum_{k=1}^{N} |R_k| = 8\varepsilon(M+\varepsilon)\frac{N}{n^2} \leq 8\varepsilon(M+\varepsilon).$$

Since $\varepsilon > 0$ was arbitrarily small, $8\varepsilon(M+\varepsilon)$ is also arbitrarily small so this proves that $f(\mathcal{C})$ has measure zero. $\square$

Intuitively, the 2-dimensional case says that under the function f, the neighborhood around each critical point gets 'squished' to a line segment with zero area.



Generalizing the proof above to a function $f : \mathbb{R}^n \to \mathbb{R}^n$ requires finding an orthonormal basis of $n-1$ vectors, which is difficult but always possible by the Gram-Schmidt algorithm.

Corollary 15.7.3. *For a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$, the set of critical values $f(\mathcal{C})$ has measure zero.*

Proof. We prove the case when $n = 2$ and remark that the proof generalizes to $n \geq 3$.

Let $\{R_{ij}\}_{(i,j)\in\mathbb{N}^2}$ be a countable collection of rectangles in $\mathbb{R}^2$ that contain all critical points of f. If need be, let $R_{ij} = [i, i+1] \times [j, j+1]$ so that $\{R_{ij}\}_{(i,j)\in\mathbb{N}^2}$ covers $\mathbb{R}^2$. For each pair (i, j), consider the restriction $f_{ij} = f|_{R_{ij}} : R_{ij} \to \mathbb{R}^2$. Then Sard's Theorem says that the set $\mathcal{C}_{ij}$ of critical values of f_{ij} has measure zero. In particular, for any $\varepsilon > 0$ there is a collection of squares $\{S_k^{ij}\}$ such that $f_{ij}(\mathcal{C}_{ij}) \subset \bigcup\limits_{k=1}^{\infty} S_k^{ij}$ and $\sum\limits_{k=1}^{\infty} |S_k^{ij}| \leq \frac{\varepsilon}{2^{i+j}}$. Then the set of critical values of f is contained in the union of all the $f(\mathcal{C}_{ij})$, which is in turn contained in $\bigcup\limits_{(i,j,k)\in\mathbb{N}^3} S_k^{ij}$. Moreover,

$$\begin{aligned}
\sum_{(i,j,k)\in\mathbb{N}^3} |S_k^{ij}| = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |S_k^{ij}| \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\varepsilon}{2^{i+j}} = \varepsilon \sum_{i=1}^{\infty} \frac{1}{2^i} \sum_{j=1}^{\infty} \frac{1}{2^j} \\
&= \varepsilon \sum_{i=1}^{\infty} \frac{1}{2^i} \left(\frac{\frac{1}{2}}{1-\frac{1}{2}} \right) \quad \text{by geometric series} \\
&= \varepsilon \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot 1 = \varepsilon \left(\frac{\frac{1}{2}}{1-\frac{1}{2}} \right) \quad \text{by geometric series again} \\
&= \varepsilon \cdot 1 = \varepsilon.
\end{aligned}$$

Therefore the set of critical values of f has measure zero. $\square$

Another useful consequence is that whenever the codomain of f has higher dimension than the domain, the *entire image* of f has measure zero.

Corollary 15.7.4. *Suppose $f : [0,1]^n \subset \mathbb{R}^n \to \mathbb{R}^m$ is a continuously differentiable function and $m > n$. Then $f(D)$ has measure zero.*

Proof. We prove the case where $n = 2$ and $m = 3$. Suppose $f : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ is continuously differentiable, with

$$f(x, y) = (f_1(x, y), f_2(x, y), f_3(x, y)).$$

We can view D as a subset of $\mathbb{R}^3$ via the inclusion map $\iota : \mathbb{R}^2 \hookrightarrow \mathbb{R}^3$, and redefine f as $F : D \subset \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$F(x, y, z) = f(x, y, 0) = (f_1(x, y, 0), f_2(x, y, 0), f_3(x, y, 0)).$$

F naturally inherits continuous differentiability from f, so by Sard's Theorem, the set of critical values of F has measure zero. We know that a point $\bar{x}_0 \in D$ is a critical point of the function if the Jacobian matrix is not invertible. But for any $\bar{x}_0 \in D$,

$$DF(\bar{x}_0) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{bmatrix} \cdot \bar{x}_0 = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & 0 \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & 0 \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & 0 \end{bmatrix} \cdot \bar{x}_0$$

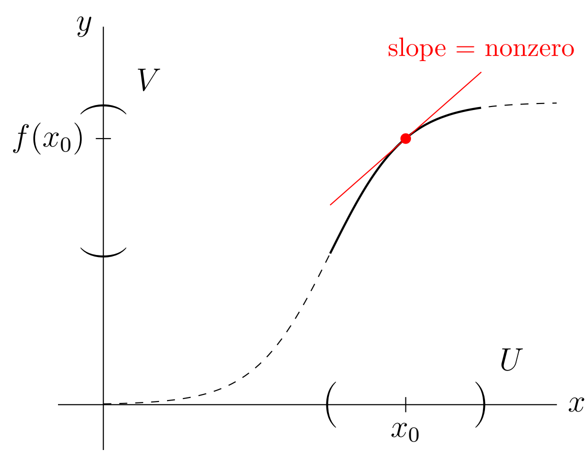
which is clearly not invertible. Hence *every* point in D is a critical point of F. Then by the above, $f(D)$ has measure zero in $\mathbb{R}^3$. $\square$

15.8 Inverse Function Theorem

One of the most important tools in modern analysis is the following generalization of Theorem 3.2.5.

Theorem 15.8.1 (Inverse Function Theorem)**.** *Let X and Y by Banach spaces and let $f : X \to Y$ be a continuously differentiable function. Suppose there is some point $x_0 \in X$ such that $Df(x_0)$ has a bounded inverse, $(Df(x_0))^{-1}$. Then there are neighborhoods U and V of x and $f(x_0)$, respectively, such that $f : U \to V$ is invertible with a continuously differentiable inverse $f^{-1} : V \to U$ whose differential satisfies $Df^{-1}(f(x_0)) = (Df(x_0))^{-1}$.*

In the one-dimensional case, the bounded inverse condition means that the slope of f at x_0 is nonzero, so there's an interval U so that $f'(x) \geq \varepsilon > 0$ on the whole interval.



We prove the theorem in several steps. First we prove the special case when $X = Y$, $x_0 = 0$, $f(x_0) = 0$ and $Df(x_0)$ is the identity operator. Although these conditions seem restrictive, we will see that there is only a small adjustment needed to make the proof run in the general case.

Suppose $X = Y, x_0 = 0, f(x_0) = 0$ and $Df(x_0) = I$. Observe that

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + o(x - x_0) = x + o(x).$$

Let $F(x) = y - o(x)$ for a fixed $y \in X$. Then the problem comes down to finding a fixed point of F, so that $F(x) = x = y - o(x)$. To do this we will employ the contraction mapping theorem (14.4.2).

Lemma 15.8.2. *There is a $\delta > 0$ such that $||o(x_2) - o(x_1)|| \leq \frac{1}{2}||x_2 - x_1||$ for all $x_1, x_2 \in \overline{B}_\delta(0)$.*

Proof. Observe that $o(x) = f(x) - x$ so $o(x)$ is continuously differentiable with $Do(x) = Df(x) - I$. In particular,

$$Do(0) = Df(0) - I = I - I = 0,$$

the zero operator. By continuity, there is some $\delta > 0$ such that $||Do(x)|| \leq \frac{1}{2}$ on $\overline{B}_\delta(0)$. Then by the mean value inequality (Theorem 15.4.3), $||o(x_2) - o(x_1)|| \leq \frac{1}{2}||x_2 - x_1||$ for all $x_1, x_2 \in \overline{B}_\delta(0)$. □

Lemma 15.8.3. *There is some neighborhood of* $0 \in X$ *such that* $Df(x_0)$ *has a bounded inverse for all* x_0 *in the neighborhood.*

Proof. Consider the neighborhood $B_\delta(0)$. Let $x_0 \in B_\delta(0)$; our goal is to show $Df(x_0)$ has a bounded inverse L which is a linear operator. Let $K = Df(x_0)$ and notice that $K = I-(I-K)$ where I is the identity operator. We claim that $K^{-1} = I+(I-K)+(I-K)^2+\ldots$ (the idea here is that $\frac{1}{1-x} = 1+x+x^2+\ldots$ for small x in the real number case). Consider the partial sum $S_n = I+(I-K)+(I-K)^2+\ldots+(I-K)^n$ and let $J = I-K$ to compress notation. Then if $n > m$,

$$\begin{aligned}
||S_n - S_m||_{op} &= ||J^n + \ldots + J^{m+1}||_{op} \\
&\leq ||J^n||_{op} + \ldots + ||J^{m+1}||_{op} \quad \text{by the triangle inequality} \\
&\leq ||J||_{op}^n + \ldots + ||J||_{op}^{m+1} \quad \text{by induction on } J^k \\
&\leq \frac{||J||_{op}^{m+1}}{1-||J||_{op}}.
\end{aligned}$$

By Lemma 15.8.2, $||Do(x)||_{op} \leq \frac{1}{2}$ and since $J = K - I = Df(x_0) - I = Do(x_0)$, it follows that S_n is a Cauchy sequence. Now since X is a Banach space, $\mathcal{L}(X,X)$ is complete so S_n converges to some $M \in \mathcal{L}(X,X)$. Consider

$$\begin{aligned}
KM = (I-J)M &= \lim_{n\to\infty}(I-J)S^n \\
&= \lim_{n\to\infty}[(I-J)(I+J+J^2+\ldots+J^n)] \\
&= \lim_{n\to\infty}(I-J^{n+1}) = I.
\end{aligned}$$

This shows that $M = K^{-1}$ and by construction, $M = \lim S_n = I+(I-K)+(I-K)^2+\ldots$ as claimed. □

Returning to the function $F(x) = y - o(x)$, we want to apply the contraction mapping theorem (14.4.2) to find a fixed point of F. First, note that for all $x \in B_\delta(0)$ and $y \in \overline{B}_{\delta/2}(0)$,

$$\begin{aligned}
||F(x)|| &= ||y - o(x)|| \\
&\leq ||y|| + ||o(x)|| \quad \text{by the triangle inequality} \\
&\leq ||y| + \tfrac{1}{2}||x|| \quad \text{by Lemma 15.8.2} \\
&\leq ||y|| + \tfrac{\delta}{2} \\
&\leq \tfrac{\delta}{2} + \tfrac{\delta}{2} = \delta.
\end{aligned}$$

Additionally, for all $x_1, x_2 \in B_\delta(0)$, we have

$$||F(x_2) - F(x_1)|| = ||o(x_2) - o(x_1)|| \leq \tfrac{1}{2}||x_2 - x_1||$$

So if $y \in \overline{B}_{\delta/2}(0)$ then $F : \overline{B}_\delta(0) \to \overline{B}_\delta(0)$ is a contraction. It is a fact that any closed subset of a complete space is itself complete, so $\overline{B}_\delta(0)$ is complete. Therefore by the contraction mapping theorem, there is a unique $x \in \overline{B}_\delta(0)$ such that $F(x) = x$, i.e. $f(x) = y$. Call this $x = f^{-1}(y)$. Then we have constructed an inverse function $f^{-1} : \overline{B}_{\delta/2}(0) \to \overline{B}_\delta(0)$.

Notice that this restricts to $f^{-1} : B_{\delta/2}(0) \to B_\delta(0)$ so define $U = f^{-1}(B_{\delta/2}(0)) \cap B_\delta(0)$ and $V = B_{\delta/2}(0)$. Then U and V are neighborhoods of 0 and $f(0)$ that satisfy the inverse function theorem *as long as* f^{-1} is continuously differentiable. Thus we have a bijection $f : U \to V$ so it remains to check the differentiability of f^{-1}.

Lemma 15.8.4. *$f^{-1} : V \to U$ is Lipschitz continuous.*

Proof. Let $y_1, y_2 \in V = B_{\delta/2}(0)$ and let $x_1, x_2 \in U$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then $x_1 + o(x_1) = y_1$ and $x_2 + o(x_2) = y_2$, so $f^{-1}(y_1) + o(f^{-1}(y_1)) = y_1$ and $f^{-1}(y_2) + o(f^{-1}(y_2)) = y_2$. Consider

$$\begin{aligned} ||f^{-1}(y_2) - f^{-1}(y_1)|| &= ||(y_2 - o(f^{-1}(y_2))) - (y_1 - o(f^{-1}(y_1)))|| \\ &\leq ||y_2 - y_1|| + ||o(f^{-1}(y_2)) - o(f^{-1}(y_1))|| \\ &\leq ||y_2 - y_1|| + \tfrac{1}{2}||f^{-1}(y_2) - f^{-1}(y_1)||. \end{aligned}$$

So $\frac{1}{2}||f^{-1}(y_2) - f^{-1}(y_1)|| \leq ||y_2 - y_1||$, or $||f^{-1}(y_2) - f^{-1}(y_1)|| \leq 2||y_2 - y_1||$. Hence f^{-1} is Lipschitz continuous with Lipschitz constant $k = 2$. □

Since Lipschitz continuity implies regular continuity, we get as a consequence that f^{-1} is continuous. Next we verify differentiability.

Lemma 15.8.5. *f^{-1} is differentiable at 0 with $Df^{-1}(0) = (Df(0))^{-1} = I$.*

Proof. Recall that $f(x) = x + o(x)$, so $f^{-1}(y) + o(f^{-1}(y)) = y$ for any $y \in V$. That is, $f^{-1}(y) = y - o(f^{-1}(y)) = f^{-1}(0) + I(y - 0) - o(f^{-1}(y))$. It therefore suffices to prove $o(f^{-1}(y)) = o(y)$. Consider

$$\begin{aligned} \frac{||o(f^{-1}(y))||}{||y||} &= \frac{||o(f^{-1}(y))||}{||f^{-1}(y)||} \cdot \frac{||f^{-1}(y)||}{||y||} \\ &\leq \frac{||o(f^{-1}(y))||}{||f^{-1}(y)||} \cdot \frac{2||y||}{||y||} \quad \text{by Lemma 15.8.4.} \end{aligned}$$

Now by continuity of f^{-1}, as $y \to 0$, $f^{-1}(y) \to f^{-1}(0) = 0$ so we have

$$\lim_{y \to 0} \frac{||o(f^{-1}(y))||}{||y||} = 2 \lim_{y \to 0} \frac{||o(f^{-1}(y))||}{||f^{-1}(y)||} = 0.$$

Therefore $o(f^{-1}(y)) = o(y)$ so we conclude that f^{-1} is differentiable at 0. □

We have proven all of the properties of f^{-1} in the special case, so now it remains to generalize and finish the proof of the inverse function theorem.

Proof. Assume $f : X \to Y$ is continuously differentiable with an invertible derivative, with bounded inverse, at $x_0 \in X$. We have a sequence of invertible maps:

$$\begin{tikzcd} X \arrow[r, "x \mapsto x + x_0"] & X \arrow[l, bend left] \arrow[r, "f"] & Y \arrow[r] & Y \arrow[l, bend left, "y \mapsto y + y_0"] \arrow[r, "(Df(x_0))^{-1}"] & X \end{tikzcd}$$

Define $\bar{f} : X \to X$ by the composition along these maps:

$$\bar{f}(x) = (Df(x_0))^{-1}(f(x + x_0) - f(x_0)).$$

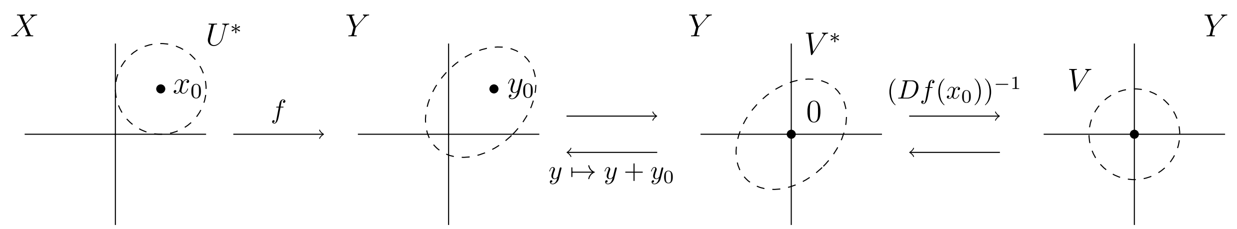
Notice that $\bar{f}(0) = 0$ and by the chain rule (Section 15.3),

$$D\bar{f}(0) = (Df(x_0))^{-1})(Df(x_0) - 0) = (Df(x_0))^{-1}Df(x_0) = I.$$

So the hypotheses of the special case are satisfied for $\bar{f}$. Consider

$$\begin{aligned}\bar{f}(x) &= (Df(x_0))^{-1}(f(x + x_0) - f(x_0)) \\ &= (Df(x_0))^{-1}(Df(x_0)(x) + o(x) - f(x_0)) \\ &= x + (Df(x_0))^{-1}(o(x)).\end{aligned}$$

By Lemmas 15.8.4 and 15.8.5, there exist open neighborhoods U and V of 0 such that $\bar{f} : U \to V$ is invertible with all of the previously stated properties. Now we have

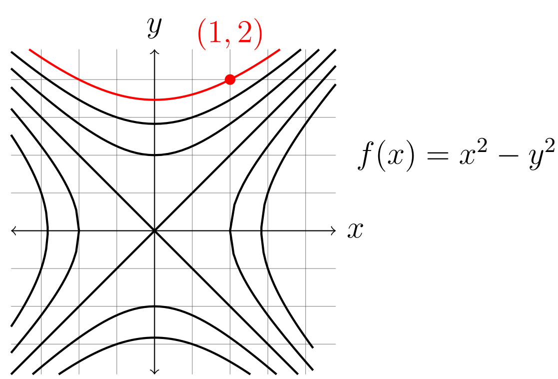


Consider $f^{-1}(y) = x_0 + \bar{f}^{-1}((Df(x_0))^{-1}(y - y_0))$ as illustrated above. Then $f^{-1} : V^* + y_0 \to U^*$ is defined and the chain rule shows that it is invertible with $Df^{-1}(x_0) = (Df(x_0))^{-1}$. This completes the proof. $\square$

Example 15.8.6. Consider the function $f(x, y) = x^2 - y^2$ at $(1, 2)$, where $f(1, 2) = -3$. In multivarible calculus, we would draw level curves to understand the graph of this function. In doing so, we are using the inverse function theorem to say that there is a neighborhood of $(1, 2)$ on which $f(x, y) \approx -3$ holds. Consider $F(x, y) = (f(x, y), y)$. This map is differentiable with

$$DF(x, y) = \begin{bmatrix} f_x & f_y \\ 0 & 1 \end{bmatrix}.$$

In particular, $DF(1, 2) = \begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix}$ which is invertible. We can even compute its inverse quickly using linear algebra. By the inverse function theorem, there are neighborhood; g is a smooth curve in $\mathbb{R}^2$ that is invertible within U. This is how we construct level curves and graph the function locally.



Part IV

Measure Theory

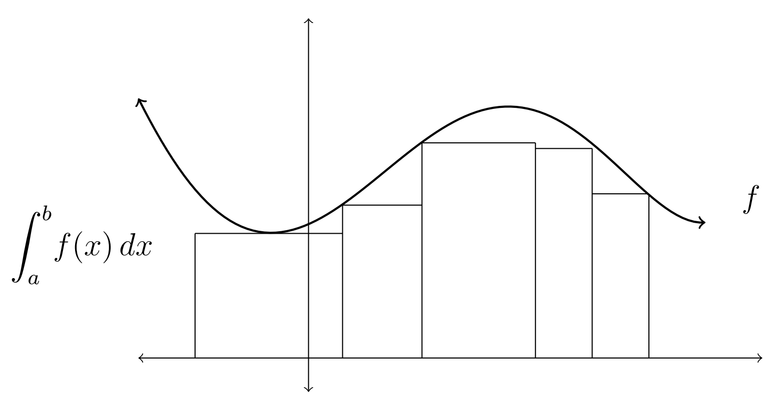
Chapter 16

Introduction

Part IV is comprised by notes from a graduate real analysis course taught by Dr. Tai Melcher at the University of Virginia in Spring 2016. The companion text for the course is Folland's *Real Analysis: Modern Techniques and Their Applications, 2nd ed.* The main topics covered are:

- A review of some concepts in set theory
- Measure theory
- Integration theory
- Differentiation
- Some functional analysis, including normed linear spaces
- The theory of L^p spaces

By far the most fundamental subject in real analysis is measure theory. In a general sense, measure theory gives us a way of extending the concrete notions of length, area, volume, etc. and also of extending the theory of Riemann integration from calculus. Recall that the Riemann integral of a real-valued function $f(x)$ on an interval $[a, b]$ is defined as the limit of the areas of a sequence of increasingly thinner rectangles that approximate the area under the curve of f:



If we are to use this limiting process on say a function like the characteristic function

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

then what should the value of $\int_0^1 \chi_{\mathbb{Q}}(x)\, dx$ be? Riemann integration fails to give a value to this integral, but since the "rectangles" summed up by such an integral would all have zero area ($\mathbb{Q}$ is disconnected), we should expect the integral to equal 0.

We seek to develop an abstract notion of "length" to replace the length of the subintervals in Riemann integration that allows for integration over a broader class of sets.

Two important set theoretic constructions are:

Definition. *Given a sequence of sets $\{A_n\}_{n=1}^{\infty}$ in a space X, the* **outer limit** *of the A_n is the set of $x \in X$ such that $x \in A_n$ for infinitely many n:*

$$\limsup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n.$$

This may also be written as the set $\{x \in X \mid x \in A_n \text{ i.o.}\}$, where i.o. stands for "infinitely often".

Definition. *Given a sequence of sets $\{A_n\}_{n=1}^{\infty}$ in X, the* **inner limit** *of the A_n is the set of $x \in X$ such that $x \in A_n$ for all but a finite number of n:*

$$\liminf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n.$$

This is also written $\{x \in X \mid x \in A_n \text{ e.a.}\}$, where a.a. stands for "eventually always".

These may be viewed as generalizations of lim sup and lim inf from real analysis.

16.1 The Discrete Sum

A concrete case of integration is found in the notion of a discrete sum:

Definition. *Let X be a (not necessarily countable) set and let $f : X \to [0, \infty]$ be a function. The* **discrete sum** *of f over X is defined as*

$$\sum_X f := \sup \left\{ \sum_{x \in A} f(x) : A \subset X \text{ is finite} \right\}.$$

We will develop the theory of the discrete sum as a case study for what is to come later in the general theory of integration. The main theorems (Monotone Convergence Theorem, Fatou's Lemma, Dominated Convergence Theorem, etc.) will be proven again later with the generalized notion of integration with respect to a measure.

Example 16.1.1. If $X = \mathbb{N}$, the discrete sum is just the regular counting sum:

$$\sum_{\mathbb{N}} f = \sum_{n=1}^{\infty} f(n) = \sup_N \sum_{n=1}^{N} f(n).$$

One of the most important theorems in the theory of sequences is the Monotone Convergence Theorem. This theorem is often proven in an introductory real analysis course. The version stated here is slightly more abstract.

Theorem 16.1.2 (Monotone Convergence for Sums)**.** *For a sequence of nonnegative functions $f_n : X \to [0, \infty]$, if $f_n(x) \leq f_{n+1}(x)$ for all $x \in X$ and for all $n \in \mathbb{N}$, then*

$$\lim_{n \to \infty} \sum_{x \in X} f_n(x) = \sum_{x \in X} \lim_{n \to \infty} f_n(x).$$

Proof. First observe that since $f_n(x)$ is an increasing sequence for all $x \in X$, $\sum_X f_n(x)$ is also an increasing sequence. Therefore $\lim_{n\to\infty} \sum_X f_n(x)$ exists in $[0, \infty]$. Fix $n \in \mathbb{N}$. By hypothesis, for all $x \in X$, $f(x) = \sup f_n(x) \geq f_n(x)$. Thus for any finite subset $A \subset X$,

$$\sum_X f(x) \geq \sum_A f(x) \geq \sum_A f_n(x).$$

Taking the sup on the right over all finite $A \subset X$ preserves the inequality, so we have

$$\sum_X f(x) \geq \sum_X f_n(x).$$

Since this holds for all $n \in \mathbb{N}$, taking the limit as $n \to \infty$ again preserves the limit, giving us one of the desired inequalities:

$$\sum_X f(x) \geq \lim_{n \to \infty} \sum_X f_n(x).$$

Now fix a finite set $A \subset X$. Then for all $n \in \mathbb{N}$, we have

$$\sum_A f_n(x) \le \sum_X f_n(x) \le \lim_{n\to\infty} \sum_X f_n(x).$$

(The last inequality uses the fact that $\sum_X f_n(x)$ is an increasing sequence.) Taking the limit on the left as $n \to \infty$ gives us

$$\sum_A \lim_{n\to\infty} f_n(x) \le \lim_{n\to\infty} \sum_X f_n(x).$$

Finally, taking the sup on the left over all finite $A \subset X$ preserves the inequality, producing

$$\sum_X \lim_{n\to\infty} f_n(x) \le \lim_{n\to\infty} \sum_X f_n(x).$$

Since $\lim f_n(x) = f(x)$, we have proven both inequalities required. □

Lemma 16.1.3. *If $\sum_{x\in X} f(x) < \infty$ then the set $\{x \in X \mid f(x) > 0\}$ is at most countable.*

Proof. We can write

$$\{x \in X \mid f(x) > 0\} = \bigcup_{n=1}^{\infty} \left\{x \mid f(x) > \tfrac{1}{n}\right\}.$$

Now each set $\left\{x \mid f(x) > \frac{1}{n}\right\}$ is finite since the discrete sum is assumed to converge. Then the countable union of these finite sets is countable, so the set in question is countable. □

Theorem 16.1.4 (Tonelli's Theorem for Sums)**.** *If X and Y are sets and $f : X \times Y \to [0,\infty]$ is a function, then*

$$\sum_{(x,y)\in X\times Y} f(x,y) = \sum_{x\in X} \sum_{y\in Y} f(x,y) = \sum_{y\in Y} \sum_{x\in X} f(x,y).$$

Proof. By symmetry, it is sufficient to show the first equality. On one hand, let $\Lambda \subset X \times Y$ be a finite subset. Choose finite subsets $\alpha \subset X$ and $\beta \subset Y$ such that $\Lambda \subset \alpha \times \beta$. For any such α, β, we have

$$\sum_\Lambda f(x,y) \le \sum_{\alpha\times\beta} f(x,y) = \sum_\alpha \sum_\beta f(x,y) \le \sum_\alpha \sum_Y f(x,y) \le \sum_X \sum_Y f(x,y).$$

Taking the supremum over all such Λ proves the inequality $\sum_{X\times Y} f(x,y) \le \sum_X \sum_Y f(x,y)$.

Going the other way, for each $x \in X$ choose a sequence of finite subsets $\beta_n^x \subset Y$ such that $\beta_n^x \nearrow Y$ and

$$\sum_Y f(x,y) = \lim_{n\to\infty} \sum_{y\in\beta_n^x} f(x,y).$$

If $\alpha \subset X$ is finite, then set $\beta_n = \bigcup_{x\in\alpha} \beta_n^x$ and observe that because β_n are finite sets,

$$\sum_Y f(x,y) = \lim_{n\to\infty} \sum_{y\in\beta_n} f(x,y)$$

holds for all $x \in \alpha$. Hence

$$\begin{aligned}\sum_{x\in\alpha}\sum_{y\in Y} f(x,y) &= \sum_{x\in\alpha} \lim_{n\to\infty} \sum_{y\in\beta_n} f(x,y) \\ &= \lim_{n\to\infty} \sum_{x\in\alpha}\sum_{y\in\beta_n} f(x,y) \text{ since } \alpha \text{ is finite} \\ &= \lim_{n\to\infty} \sum_{\alpha\times\beta_n} f(x,y) \\ &\le \sum_{X\times Y} f(x,y).\end{aligned}$$

Taking the supremum over all such $\alpha \subset X$, we get the other inequality: $\sum_X \sum_Y f(x,y) \le \sum_{X\times Y} f(x,y)$. Therefore equality holds. ☐

Theorem 16.1.5 (Fatou's Lemma for Sums)**.** *If $f_n : X \to [0,\infty]$ is a sequence of nonnegative functions then*

$$\sum_X \liminf f_n \le \liminf \sum_X f_n.$$

Proof. Define $g_k = \inf\{f_n \mid n \ge k\}$ so that the sequence (g_k) increases from below to $\liminf f_n$. In particular, $g_k \le f_n$ for all $n \ge k$, so

$$\sum_X g_k \le \sum_X f_n \text{ for all } n \ge k.$$

By the Monotone Convergence Theorem for discrete sums (Theorem 16.1.2),

$$\sum_X \liminf f_n = \sum_X \lim_{k\to\infty} g_k = \lim_{k\to\infty} \sum_X g_k \le \liminf \sum_X f_n.$$

This proves Fatou's Lemma for discrete sums. ☐

Remark. If $A = \sum_X f(x)$ then for every $\varepsilon > 0$, there exists a finite subset $\alpha_\varepsilon \subset X$ such that $A - \varepsilon \le \sum_{\alpha_\varepsilon} f \le A$. Furthermore, these inequalities hold for any set α containing α_ε.

Next we extend the discrete sum to complex-valued functions.

Definition. *If $f : X \to \mathbb{C}$ is a function, we say the sum $\sum_X f = \sum_{x\in X} f(x)$ exists and equals $A \in \mathbb{C}$ if for every $\varepsilon > 0$, there exists a finite subset $\alpha_\varepsilon \subset X$ such that for all finite sets $\alpha \supseteq \alpha_\varepsilon$,*

$$\left|\sum_\alpha f - A\right| < \varepsilon.$$

Notice that unlike before, this does not allow for infinite sums. In order to access our work with discrete sums of nonnegative functions, we will write a real-valued function $f : X \to \mathbb{R}$ as $f = f^+ - f^-$, where $f^+(x) := \max\{f(x), 0\}$ and $f^-(x) := \min\{-f(x), 0\}$ are called the *positive* and *negative parts* of f, respectively. To describe all complex-valued functions, recall that a function $g : X \to \mathbb{C}$ may be divided into its real and imaginary parts, $g = \operatorname{Re} g + i \operatorname{im} g$.

Definition. *We say a function $f : X \to \mathbb{C}$ is* **summable** *if $\sum_X |f| < \infty$.*

Proposition 16.1.6. *The sum $\sum_X f$ exists if and only if f is summable. In this case,*

$$\left|\sum_X f\right| \leq \sum_X |f|.$$

Proof. ($\Longleftarrow$) Suppose $\sum_X |f| < \infty$. Then $\sum_X (\operatorname{Re} f)^{\pm}$ and $\sum_X (\operatorname{im} f)^{\pm}$ are all finite. Consider

$$\sum_X f = \sum_X (\operatorname{Re} f)^+ - \sum_X (\operatorname{Re} f)^- + i\left[\sum_X (\operatorname{im} f)^+ - \sum_X (\operatorname{im} f)^-\right].$$

Now each piece on the right is finite, so the whole expression exists.

($\Longrightarrow$) Conversely, suppose $\sum_X |f| = \infty$. By the triangle inequality, $|f| \leq |\operatorname{Re} f| + |\operatorname{im} f|$. Then either $\sum_X |\operatorname{Re} f| = \infty$ or $\sum_X |\operatorname{im} f| = \infty$. Without loss of generality, let us suppose that f is a real-valued function. Then $|f| = f^+ + f^-$ and $\sum_X |f| = \sum_X f^+ + \sum_X f^- = \infty$. Thus at least one of $\sum_X f^{\pm}$ is infinite. Without loss of generality, assume the discrete sum of the positive part sums to infinity. Let $X' = \{x \in X \mid f(x) \geq 0\}$ so that $\sum_{X'} f = \infty$. Then there is a sequence of finite subsets $\alpha_n \subset X'$ such that $\sum_{x \in \alpha_n} f(x) \geq n$ for each $n \in \mathbb{N}$. Thus for any finite subset $\alpha \subset X$, we have

$$\lim_{n \to \infty} \sum_{\alpha \cup \alpha_n} f = \infty.$$

It follows that $\sum_X f$ cannot exist in $\mathbb{R}$. The last statement of the proposition follows from the definition of the discrete sum. $\square$

Theorem 16.1.7 (Dominated Convergence Theorem for Sums)**.** *Suppose $f_n : X \to \mathbb{C}$ is a sequence of complex functions satisfying*

(1) $\lim_{n \to \infty} f_n(x) = f(x)$ exists for every $x \in X$; and

(2) There exists a summable function $g : X \to [0, \infty]$ such that $|f_n(x)| \leq g(x)$ for every $x \in X$ and $n \in \mathbb{N}$.

Then $\displaystyle\lim_{n \to \infty} \sum_{x \in X} f_n(x) = \sum_{x \in X} f(x).$

Proof. This will follow from the general Dominated Convergence Theorem in Section 18.2. $\square$

From the Dominated Convergence Theorem (DCT), we obtain some useful results about continuity and differentiability in sums.

Corollary 16.1.8. *Suppose X is a set, $V \subset \mathbb{R}^n$ is an open set and $f : V \times X \to \mathbb{C}$ is a function satisfying*

(a) for each $x \in X$, the function $t \mapsto f(t, x)$ is continuous on V, and

(b) *there is a summable function $g : X \to [0, \infty)$ such that $|f(t, x)| \le g(x)$ for all $t \in V$ and $x \in X$.*

Then the function $F : V \to \mathbb{C}$ given by $F(t) := \sum_{x\in X} f(t, x)$ is a continuous function.

Proof. First, $|f(t, x)| \le g(x)$ for all $x \in X$ implies $\sum_{x\in X} |f(t, x)| \le \sum_{x\in X} g(x) < \infty$ so $f(t, x)$ is summable for each $t \in V$. Thus the sum $F(t) = \sum_{x\in X} f(t, x)$ is defined. Fix $t_0 \in V$. It suffices to show that for any sequence $(t_n) \subset V$ converging to t_0, we have $\lim_{n\to\infty} F(t_n) = F(t_0)$. Let $(t_n) \subset V$ be such a sequence. This defines a sequence of functions $f_n := f(t_n, x) : X \to \mathbb{C}$ such that $|f_n(x)| \le g(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Further, since $t \mapsto f(t, x)$ is continuous, $\lim_{n\to\infty} f_n(x) = f(t_0, x)$ for all $x \in X$. By DCT, we have

$$\lim_{n\to\infty} F(t_n) = \lim_{n\to\infty} \sum_{x\in X} f_n(x) = \sum_{x\in X} f(t_0, x) = F(t_0).$$

This proves $(F(t_n)) \to F(t_0)$ so F is continuous at any $t_0 \in V$. □

Corollary 16.1.9. *Suppose that X is a set, (a, b) is an open interval in $\mathbb{R}$ and $f : (a, b) \times X \to \mathbb{C}$ is a function satisfying*

(a) *for each $x \in X$, the function $t \mapsto f(t, x)$ is differentiable on (a, b);*

(b) *there is a summable function $g : X \to [0, \infty)$ such that*

$$|f'(t, x)| := \left|\frac{\partial}{\partial t} f(t, x)\right| \le g(x) \text{ for all } t \in (a, b) \text{ and } x \in X; \text{ and}$$

(c) *there is a $t_0 \in (a, b)$ such that $\sum_{x\in X} |f(t_0, x)| < \infty$.*

Then for $F : (a, b) \to \mathbb{C}$ given by $F(t) = \sum_{x\in X} f(t, x)$, F is differentiable and

$$F'(t) = \sum_{x\in X} f'(t, x).$$

Proof. We first show that for all $t \in (a, b)$, $\sum_{x\in X} |f(t, x)| < \infty$. Let $t_0 \in (a, b)$ be as in condition (c) and take any $t \in (a, b), t \ne t_0$. By (a), we can apply the Mean Value Theorem to produce c between t and t_0 such that

$$f(t, x) = f(t_0, x) + \frac{\partial f(c, x)}{\partial t}(t - t_0).$$

This implies

$$\begin{aligned} |f(t, x) - f(t_0, x)| &= \frac{df(c, x)}{dt}|t - t_0| \le g(x)|t - t_0| \\ \implies |f(t, x)| &\le |f(t_0, x)| + g(x)|t - t_0| \quad \text{by (b).} \end{aligned}$$

Now sum to obtain

$$\sum_{x\in X} |f(t, x)| \le \sum_{x\in X} \Big(|f(t_0, x)| + g(x)|t - t_0|\Big) = \sum_{x\in X} |f(t_0, x)| + |t - t_0| \sum_{x\in X} g(x) < \infty.$$

Therefore $\sum_{x\in X} |f(t,x)| < \infty$ as claimed.

Now fix $t \in (a,b)$ and let (t_n) be a sequence in $(a,b) \smallsetminus \{t\}$ such that $(t_n) \to t$. For each $n \in \mathbb{N}$, set $f_n = \frac{f(t_n)-f(t,x)}{t_n-t}$ so that

$$\frac{F(t_n)-F(t)}{t_n-t} = \sum_{x\in X} \frac{f(t_n,x)-f(t,x)}{t_n-t} = \sum_{x\in X} f_n(x).$$

Then by (a), we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{f(t_n,x)-f(t,x)}{t_n-t} = \frac{\partial f}{\partial t}(t,x).$$

With this and (b), the conditions of the DCT are satisfied, so we conclude that

$$\lim_{n\to\infty} \frac{F(t_n)-F(t)}{t_n-t} = \lim_{n\to\infty} \sum_{x\in X} f_n(x) = \sum_{x\in X} \frac{\partial f}{\partial t}(t,x) = \sum_{x\in X} f'(t,x).$$

Moreover, $|f'(t,x)| \le g(x)$ for all x, so $\sum_{x\in X} |f'(t,x)| \le \sum_{x\in X} g(x) < \infty$, proving $f'(t,x)$ is summable. Thus the above proves that F is differentiable with derivative

$$F'(t) = \sum_{x\in X} f'(t,x).$$

□

The Dominated Convergence Theorem is equivalent to the Monotone Convergence Theorem and Fatou's Lemma. We will prove all three in full generality in Chapter 18. DCT (due to Lebesgue) provides another way to switch a limit with a discrete sum, as we saw in the Monotone Convergence Theorem (16.1.2) and Fatou's Lemma (16.1.5) for nonnegative functions. Here, we must be able to control the sequence (f_n) with a summable function g in order to switch the limit and sum.

Theorem 16.1.10 (Fubini's Theorem for Sums)**.** *Suppose $f : X \times Y \to \mathbb{C}$ is summable, i.e. $\sum_{X\times Y} |f| < \infty$. Then*

$$\sum_{X\times Y} f(x,y) = \sum_X \sum_Y f(x,y) = \sum_Y \sum_X f(x,y).$$

This generalizes Tonelli's Theorem (16.1.4) for complex-valued functions. When we develop the theory of integration with respect to a general measure, the approach will be the same: prove Tonelli's Theorem for nonnegative functions and then generalize with Fubini's Theorem.

Chapter 17

Measure Theory

Assigning length, area, volume, hypervolume, etc. to a set is a natural action on many spaces. However, there are sets (e.g. the Cantor set) for which such a notion of measurement makes no sense. The primary goal of measure theory is to generalize the idea of length/area/volume to more abstract spaces and sets.

Let's begin by discussing $\mathbb{R}^n$. We want to define a function $\mu : \mathbb{P}(\mathbb{R}^n) \to [0, \infty]$ on the power set of $\mathbb{R}^n$ which satisfies a few key properties:

(1) For every countable collection $\{E_i\}_{i=1}^{\infty}$ of disjoint subsets of $\mathbb{R}^n$, $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$. This property is called *(disjoint) countable additivity.*

(2) If two subsets $E, F \subseteq \mathbb{R}^n$ are equivalent under a rigid motion (a rotation, translation, reflection, or composition of any of these), then $\mu(E) = \mu(F)$.

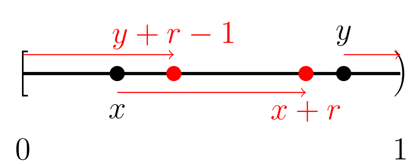
(3) If Q is the unit cube in $\mathbb{R}^n$, then $\mu(Q) = 1$.

The next example shows that it is impossible to meet all three criteria when constructing such a function on all of $\mathbb{P}(\mathbb{R}^n)$.

Example 17.0.1. Consider $\mathbb{R}$ and define an equivalence relation $\sim$ by $x \sim y$ if $x - y \in \mathbb{Q}$. This allows us to partition the unit interval $[0, 1)$ into its equivalence classes under $\sim$. Let N be a set containing one element from each equivalence class; such a set is possible to construct using the Axiom of Choice. Set $R = \mathbb{Q} \cap [0, 1)$. For each $r \in R$, define the set

$$N_r = \{x + r \mid x \in N, r \in N \cap [0, 1 - r)\} \cup \{x + r - 1 \mid r \in N \cap [1, r - 1)\}.$$

In nontechnical terms, N_r is a translation of N by r modulo 1, meaning if adding r to x takes us out of the interval $[0, 1)$ then we subtract 1, or "wrap back around" to add the rest on after 0. This is illustrated below.



Claim. *The N_r for $r \in R$ partition* $[0, 1)$.

Proof. Take $x \in [0, 1)$. Then there is an element $n \in N$ such that $x \sim n$, since we chose N to contain an element from every equivalence class of $\sim$. This means $x - n \in \mathbb{Q}$, so set $r = n - x$. Since x and n both lie in $[0, 1)$, $|n - x| < 1$ which means that either $n - x \in [0, 1)$ or $x - n \in [0, 1)$. Accordingly, either $r \in R$ or $-r \in R$. Without loss of generality suppose $r \in R$. Then by definition $x \in N_r$ because $x + r = x + (n - x) = n \in N$. Hence the N_r cover $[0, 1)$.

Now suppose $N_r \cap N_s$ is nonempty for some $r, s \in R$. Then for some $x \in N$ and $r \in R$, $x + r \in N_s$ or $x + r - 1 \in N_s$. Say $x + r \in N_s$; the proof will be similar in the other case. Then there is some $y \in N$ such that $x + r = y + s$ or $x + r = y + s - 1$. In either case, y differs from x by a rational, $r - s$ or $r - s + 1$. Hence $x \sim y$, but then $y = x$ since N contains exactly one element from each equivalence class of $\sim$. This implies $x + r = x + s \iff r = s$ or $x + r = x + s - 1 \iff r = s - 1$ but since $r, s \in R \subset [0, 1)$, the latter is impossible. Therefore $r = s$ and hence $N_r = N_s$. Finally, we conclude that because the N_r are a disjoint cover of $[0, 1)$, they form a partition of $[0, 1)$. □

Now suppose $\mu : \mathbb{P}(\mathbb{R}) \to [0, \infty]$ is a function satisfying conditions (1) – (3). Then for all $r \in R$, we must have

$$\mu(N) = \mu(N \cap [0, 1 - r)) + \mu(N \cap [1 - r, 1)) = \mu(N_r),$$

using disjoint additivity (1) and translation-invariance (2). Moreover, by (1) and (3) we should have

$$1 = \mu([0, 1)) = \mu\left(\bigcup_{r \in R} N_r\right) = \sum_{r \in R} \mu(N_r).$$

Since each $\mu(N_r)$ is equal, they can either add up to 0 or ∞, but nothing else in between. This contradicts condition (3).

The above example shows that no matter how we define a measure, if we want it to satisfy conditions (1) – (3) then it cannot possibly be defined on all of the power set $\mathbb{P}(\mathbb{R})$.

One may think, 'Hey, wait! What if the problem was with *countable* additivity?!?' Unfortunately, this doesn't resolve the issue. There is a famous theorem called the Banach-Tarski Paradox that allows one to transform a sphere of the size of a pea, for example, into another sphere the size of the Earth (or the Sun, or whatever size you want!) using only rigid motions. The catch is that the procedure only breaks down the pea into a *finite* number of sets, so countable additivity isn't the issue here. In fact the sets used in the Banach-Tarski Paradox are nonmeasurable, a concept we will define in the next section or two.

17.1 σ-Algebras

To get around the issues presented in the introduction, we will define our measures μ on certain restricted subsets of the power set $\mathbb{P}(X)$.

Definition. *Given a set X, an* **algebra** *on X is a subset $\mathcal{A}$ of the power set $\mathbb{P}(X)$ such that*

(1) $\varnothing, X \in \mathcal{A}$.

(2) $\mathcal{A}$ is closed under taking complements: if $A \in \mathcal{A}$ then $A^C \in \mathcal{A}$.

(3) $\mathcal{A}$ is closed under finite intersections: if $\{A_i\}_{i=1}^n \subset \mathcal{A}$ then $\bigcap_{i=1}^n A_i \in \mathcal{A}$.

Notice that (2), (3) and deMorgan's Laws imply that an algebra $\mathcal{A}$ is also closed under finite unions. We sometimes say an algebra is "closed under finite set operations" to capture all of this at once.

Definition. *An algebra $\mathcal{A}$ on X is a* **σ-algebra** *if $\mathcal{A}$ is also closed under countable intersections: if $\{A_i\}_{i=1}^\infty \subset \mathcal{A}$ then $\bigcap_{i=1}^\infty A_i \in \mathcal{A}$. The pair $(X, \mathcal{A})$ is called a* **measurable space***.*

Compare this to the definition of a topology on X:

Definition. *Given a set X, a* **topology** *on X is a subset $\mathcal{T}$ of the power set $\mathbb{P}(X)$ that satisfies*

(1) $\varnothing, X \in \mathcal{T}$.

(2) $\mathcal{T}$ is closed under arbitrary unions: if $\{A_\alpha\}_{\alpha\in I} \subset \mathcal{T}$ then $\bigcup_{\alpha\in I} A_\alpha \in \mathcal{T}$.

(3) $\mathcal{T}$ is closed under finite intersections: if $\{A_i\}_{i=1}^n \subset \mathcal{T}$ then $\bigcap_{i=1}^n A_i \in \mathcal{T}$.

Sets $A \in \mathcal{T}$ are called **open** *in X. If $A \in \mathcal{T}$ then its complement A^C is called* **closed** *in X. The pair $(X, \mathcal{T})$ is called a* **topological space***.*

Proposition 17.1.1. *If $\mathcal{E} \subseteq \mathbb{P}(X)$ is a collection of subsets of X, then there exist a unique topology $\mathcal{T}(\mathcal{E})$, a unique algebra $\mathcal{A}(\mathcal{E})$ and a unique σ-algebra $\sigma(\mathcal{E})$ which are respectively the smallest topology, algebra and σ-algebra containing $\mathcal{E}$.*

Proof. Define

$$\mathcal{T}(\mathcal{E}) = \bigcap_{\substack{\text{topologies}\\ \mathcal{T}\supseteq\mathcal{E}}} \mathcal{T}, \qquad \mathcal{A}(\mathcal{E}) = \bigcap_{\substack{\text{algebras}\\ \mathcal{A}\supseteq\mathcal{E}}} \mathcal{A} \quad \text{and} \quad \sigma(\mathcal{E}) = \bigcap_{\substack{\sigma\text{-algebras}\\ \mathcal{A}\supseteq\mathcal{E}}} \mathcal{A}.$$

In each case, the object $\square(\mathcal{E})$ is the intersection of all $\square$'s containing $\mathcal{E}$. Beginning with $\mathcal{T}(\mathcal{E})$, note that this is nonempty since $\varnothing, X \in \mathcal{T}$ for any topology $\mathcal{T}$ on X, and therefore $\varnothing, X$ are contained in the intersection. If $\{A_\alpha\}_{\alpha\in I}$ lie in $\mathcal{T}(\mathcal{E})$ then they all lie in any topology $\mathcal{T} \supseteq \mathcal{E}$ and so $\bigcup_{\alpha\in I} A_\alpha \in \mathcal{T}$ for every such $\mathcal{T}$. Therefore the union lies in $\mathcal{T}(\mathcal{E})$. Likewise, if $\{A_i\}_{i=1}^n \subset \mathcal{T}(\mathcal{E})$ then the A_i lie in every topology $\mathcal{T} \supseteq \mathcal{E}$ so $\bigcap_{i=1}^n A_i \in \mathcal{T}$ and therefore $\bigcap_{i=1}^n A_i \in \mathcal{T}(\mathcal{E})$. This shows $\mathcal{T}(\mathcal{E})$ is a topology.

The proofs for $\mathcal{A}(\mathcal{E})$ and $\sigma(\mathcal{E})$ are similar. We will show the proof for $\mathcal{A}(\mathcal{E})$. Since each $\mathcal{A}$ in the intersection is an algebra, $\varnothing, X \in \mathcal{A}$ and so $\varnothing, X \in \bigcap_{\mathcal{A} \supseteq \mathcal{E}} \mathcal{A} = \mathcal{A}(\mathcal{E})$. Now if $A \in \mathcal{A}(\mathcal{E})$ then A lies in each algebra $\mathcal{A} \supseteq \mathcal{E}$. So $A^C \in \mathcal{A}$ and therefore $A^C \in \mathcal{A}(\mathcal{E})$. Similarly, if $\{A_i\}_{i=1}^n \subset \mathcal{A}(\mathcal{E})$ then $A_i \in \mathcal{A}$ for all i and for all algebras $\mathcal{A} \supseteq \mathcal{E}$. Thus $\bigcap_{i=1}^n A_i \in \mathcal{A}$ for every such $\mathcal{A}$, and so $\bigcap_{i=1}^n A_i \in \mathcal{A}(\mathcal{E})$. This proves that $\mathcal{A}(\mathcal{E})$ is an algebra. $\square$

Example 17.1.2. On the set $X = \{1, 2\}$, we can define various topologies. The *trivial topology* $\mathcal{T}_0 = \{\varnothing, X\}$ is a topology on any set X. The power set $\mathbb{P}(X)$ is also always a topology. The only nontrivial topologies on this space are

$$\mathcal{T}_1 = \{\varnothing, \{1\}, X\} = \mathcal{T}(\{1\}) \quad \text{and} \quad \mathcal{T}_2 = \{\varnothing, \{2\}, X\}.$$

As with topologies, the trivial collection $\{\varnothing, X\}$ and the power set $\mathbb{P}(X)$ are always (σ-)algebras on any X. An example of a different algebra here is

$$\sigma_1 = \{\varnothing, \{1\}, \{2\}, X\} = \sigma(\{1\}) = \sigma(\{2\}).$$

Example 17.1.3. Let $X = \{1, 2, 3\}$. Then some topologies and algebras on X include:

$$\begin{aligned}
&\mathcal{T}_0 = \{\varnothing, X\}, \text{ the trivial topology/algebra} \\
&\mathcal{T}_1 = \mathcal{T}(\{1\}) = \{\varnothing, \{1\}, X\} \\
&\mathcal{T}_2 = \mathcal{T}(\{1\}, \{2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, X\} \\
&\mathcal{T}_3 = \mathcal{T}(\{1, 2\}, \{2, 3\}) = \{\varnothing, \{1, 2\}, \{2, 3\}, \{2\}, X\} \\
&\sigma_1 = \sigma(\{2\}) = \{\varnothing, \{2\}, \{1, 3\}, X\} \\
&\sigma_2 = \sigma(\{1\}, \{2\}) = \mathbb{P}(X).
\end{aligned}$$

Definition. *The* **Borel σ-algebra** *on a topological space $(X, \mathcal{T})$ is the σ-algebra generated by $\mathcal{T}$, written $\mathcal{B}_X = \sigma(\mathcal{T})$.*

Example 17.1.4. The most important Borel σ-algebra is that of the real numbers, $\mathcal{B} = \mathcal{B}_{\mathbb{R}}$, which contains all of the subsets of $\mathbb{R}$ for which it makes sense to define "length" (plus their complements, intersections, etc.) Consider the following σ-algebras generated by four different sets:

(i) σ(standard open sets)

(ii) $\sigma(\{(a, \infty) : a \in \mathbb{R}\})$

(iii) $\sigma(\{(a, \infty) : a \in \mathbb{Q}\})$

(iv) $\sigma(\{[a, \infty) : a \in \mathbb{Q}\})$.

We claim that all four are in fact the same σ-algebra, which is the Bore σ-algebra on $\mathbb{R}$. Let $\mathcal{M}_1$ be the σ-algebra in (i); $\mathcal{M}_2$ in (ii); $\mathcal{M}_3$ in (iii); and $\mathcal{M}_4$ in (iv). We will show $\mathcal{M}_4 \subseteq \mathcal{M}_3 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_4$ in order to prove the claim. For each step, it will suffice to show the sets generating $\mathcal{M}_i$ are all contained in $\mathcal{M}_{i-1}$ in order to prove $\mathcal{M}_i \subseteq \mathcal{M}_{i-1}$ by definition of a σ-algebra generated by a set.

To show $\mathcal{M}_4 \subseteq \mathcal{M}_3$, take an interval $[a, \infty)$, $a \in \mathbb{Q}$. This can be written

$$[a, \infty) = \bigcup_{n=1}^{\infty} \left(a + \frac{1}{n}, \infty \right)$$

and clearly each $\left(a + \frac{1}{n}, \infty\right)$ lies in $\mathcal{M}_3$, so because σ-algebras are closed under countable unions, we see that $[a, \infty) \in \mathcal{M}_3$. Hence $\mathcal{M}_4 \subseteq \mathcal{M}_3$.

The next inclusion, $\mathcal{M}_3 \subseteq \mathcal{M}_2$, is trivial since the generators of $\mathcal{M}_3$ are already among the generators of $\mathcal{M}_2$, and hence lie in $\mathcal{M}_2$ itself.

Next, take (a, ∞) for $a \in \mathbb{R}$ and write this as

$$(a, \infty) = \bigcup_{n=1}^{\infty} (a, a+n).$$

Then each $(a, a+n)$ is a standard open set on $\mathbb{R}$ and thus lies in $\mathcal{M}_1$. Since σ-algebras are closed under countable unions, we have $(a, \infty) \in \mathcal{M}_1$ which implies as before that $\mathcal{M}_2 \subseteq \mathcal{M}_1$.

Finally, take a standard open set $(a, b) \subseteq \mathbb{R}$ where $-\infty < a < b < \infty$. First, we can approximate the left endpoint using the density of $\mathbb{Q}$ in any nonempty subset of $\mathbb{R}$:

$$(a, b) = \bigcup_{r \in (a,b) \cap \mathbb{Q}} [r, b).$$

This is of course a countable union since $\mathbb{Q}$ is countable. Now for each of these intervals $[r, b)$, we can write

$$[r, b) = [r, \infty) \cap \bigcap_{n=N}^{\infty} \left[b - \frac{1}{n}, \infty \right)$$

where N is the first natural number such that $a < b - \frac{1}{N}$, which exists since $b - a > 0$ by assumption. Thus each $[r, b)$ is a countable intersection of sets of the form $[x, \infty)$ so we have shown that (a, b) is a countable union of a countable intersection of sets of the form $[x, \infty)$. Thus $(a, b) \in \mathcal{M}_4$, so $\mathcal{M}_1 \subseteq \mathcal{M}_4$. This completes the proof that $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}_3 = \mathcal{M}_4$.

Let X be a set and $\mathcal{E} \subseteq \mathbb{P}(X)$. Denote by $\mathcal{E}_f$ the collection of all finite intersections of sets in $\mathcal{E} \cup \{\varnothing, X\}$.

Lemma 17.1.5. *$\mathcal{T}(\mathcal{E})$ is equal to $\mathcal{T}$, the collection of all arbitrary unions of sets in $\mathcal{E}_f$.*

Proof. Notice that $\mathcal{E}$ is trivially contained in both $\mathcal{E}_f$ and $\mathcal{T}$. Let $A, C \in \mathcal{T}$ and set $\mathcal{E}' = \mathcal{E} \cup \{\varnothing, X\}$. Then

$$A = \bigcup_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} \bigcap_{B \in \mathcal{F}_\alpha} B \quad \text{and} \quad C = \bigcup_{\beta \in J} C_\beta = \bigcup_{\beta \in J} \bigcap_{D \in \mathcal{G}_\beta} D$$

where each $\mathcal{F}_\alpha, \mathcal{G}_\beta \subset \mathcal{E}'$ is a finite subcollection. Consider

$$A \cap C = \left(\bigcup_\alpha A_\alpha \right) \cap \left(\bigcup_\beta C_\beta \right) = \bigcup_{\alpha, \beta} (A_\alpha \cap C_\beta).$$

This lies in $\mathcal{T}$ since each $A_\alpha \cap C_\beta \in \mathcal{E}_f$. Thus $\mathcal{T}$ is closed under finite intersections. Clearly $\mathcal{T}$ is also closed under arbitrary unions by construction, so $\mathcal{T}$ is a topology containing $\mathcal{E}$ and hence $\mathcal{T}(\mathcal{E}) \subseteq \mathcal{T}$.

Conversely, if $A \in \mathcal{T}$ then we want to show A lies in every topology containing $\mathcal{E}$. Write

$$A = \bigcup_\alpha A_\alpha = \bigcup_\alpha \bigcap_{\beta \in \mathcal{F}_\alpha} B \quad \text{for finite } \mathcal{F}_\alpha \subset \mathcal{E}'.$$

Then each A_α is a finite intersection of sets in $\mathcal{E}'$ and so it lies in any topology $\mathcal{T}' \supseteq \mathcal{E}$. Taking unions keeps us in $\mathcal{T}'$, so we see that $A \in \mathcal{T}'$ for all such $\mathcal{T}'$. In particular $A \in \mathcal{T}(\mathcal{E})$. This proves $\mathcal{T}(\mathcal{E}) = \mathcal{T}$ as claimed. □

Now let $\mathcal{E}_c = \mathcal{E} \cup \{\varnothing, X\} \cup \mathcal{E}^C$, where $\mathcal{E}^C = \{A^C \mid A \in \mathcal{E}\}$.

Lemma 17.1.6. *$\mathcal{A}(\mathcal{E})$ is equal to the collection of finite unions of finite intersections of sets in $\mathcal{E}_c$.*

Proof. Similar. □

Example 17.1.7. On $\mathbb{R}$, consider the set $\mathcal{E} = \{(a, b] : -\infty \le a < b\} \cup \{\varnothing, \mathbb{R}\}$. Then $\mathcal{T}(\mathcal{E})$ consists of all arbitrary unions of sets from $\mathcal{E}$. In general, it's not guaranteed that $\sigma(\mathcal{E}) = \sigma(\mathcal{T}(\mathcal{E}))$ but when $\mathcal{E}$ contains enough sets, such as the example above, this does hold. In this case, $\mathcal{E}_c = \mathcal{E} \cup \{(-\infty, a] \cup (b, \infty) : a < b\}$ and $\mathcal{A}(\mathcal{E})$ is the collection of finite unions of sets in $\mathcal{E}_c$. Notice that

$$(a, b) = \bigcup_{n=1}^{\infty} (a, b_n] \in \sigma(\mathcal{E}), \text{ where } b_n \nearrow b$$

$$\{b\} = (a, b] \setminus (a, b) \in \sigma(\mathcal{E}) \quad \text{and} \quad [a, b] = \{a\} \cup (a, b] \in \sigma(\mathcal{E})$$

so all "reasonable" sets, or sets that we would normally want to measure, are in $\sigma(\mathcal{E})$. In fact, $\sigma(\mathcal{E}) = \mathcal{B}$, the Borel σ-algebra on $\mathbb{R}$.

We next prove the suprising fact that every countable σ-algebra (on an arbitrary space X) must be finite. To do so, we need a lemma:

Lemma 17.1.8. *Suppose that $\mathcal{M} \subset \mathbb{P}(X)$ is a σ-algebra and that $\mathcal{M}$ is countable. Then there is a unique partition $\mathbb{F} \subset \mathcal{M}$ such that for each $A \in \mathcal{M}$,*

$$A = \bigcup_{\alpha \in \mathbb{F}, \alpha \subset A} \alpha.$$

Proof. For each $x \in X$, set $A_x = \bigcap\{A \in \mathcal{M} : x \in A\}$. Since $\mathcal{M}$ is a σ-algebra and therefore closed under countable intersections, and this is a countable intersection since $\mathcal{M}$ is countable by hypothesis, A_x is the smallest element of $\mathcal{M}$ containing x. Suppose $x, y \in X$ such that $A_x \cap A_y \neq \varnothing$. Then neither A_x nor A_y is empty, so $x \in A_x \cap A_y$ and $y \in A_x \cap A_y$. Since $A_x, A_y \in \mathcal{M}$ and $\mathcal{M}$ is closed under intersections, we have $A_x \cap A_y \in \mathcal{M}$. But as shown above, A_x is the smallest element of $\mathcal{M}$ containing x, so it must be that $A_x \subseteq A_x \cap A_y$, which implies $A_x = A_x \cap A_y$. Likewise, A_y is the smallest element of $\mathcal{M}$ containing y, so we have

$A_y \subseteq A_x \cap A_y$, implying $A_y = A_x \cap A_y = A_x$. Hence $A_x = A_y$, so $\mathbb{F} = \{A_x \in \mathcal{M} : x \in X\}$ is a partition.

Now fix $A \in \mathcal{M}$. Then clearly $\displaystyle\bigcup_{\alpha\in\mathbb{F},\alpha\subset A} \alpha \subseteq A$ since each α in the union is a subset of A. On the other hand, for each $x \in A$, $x \in A_x \subset A$ so A_x is one of the α in the union, which means $x \in A_x \subset \displaystyle\bigcup_{\alpha\in\mathbb{F},\alpha\subset A} \alpha$. Since this holds for each $x \in A$, we have $A \subseteq \displaystyle\bigcup_{\alpha\in\mathbb{F},\alpha\subset A} \alpha$ and therefore $A = \displaystyle\bigcup_{\alpha\in\mathbb{F},\alpha\subset A} \alpha$.

To show uniqueness, suppose $\mathcal{G}$ is another partition of $\mathcal{M}$ such that for any $A \in \mathcal{M}$, we have

$$A = \bigcup_{\alpha\in\mathbb{F},\alpha\subset A} \alpha = \bigcup_{\beta\in\mathcal{G},\beta\subset A} \beta.$$

Pick out one of these $\beta \in \mathcal{G}, \beta \subset A$. Then $\beta \in \mathcal{M}$ so we can write it as a disjoint countable union of sets in $\mathbb{F}$ (since $\mathbb{F} \subset \mathcal{M}$ is countable):

$$\beta = \coprod_{k=1}^{\infty} \alpha_k \quad \text{where each } \alpha_k \subset \beta.$$

In turn, each $\alpha_k \in \mathbb{F} \subset \mathcal{M}$ so we can write them as disjoint countable unions of sets in $\mathcal{G}$: $\alpha_k = \coprod_{i=1}^{\infty} \beta_{ik}$ where $\beta_{ik} \in \mathcal{G}, \beta_{ik} \subset \alpha_k$ for all $i \in \mathbb{N}$. Now we have

$$\beta = \coprod_{k=1}^{\infty}\coprod_{i=1}^{\infty} \beta_{ik}$$

which is a disjoint union of sets in $\mathcal{G}$. Since $\mathcal{G}$ is a partition, it must be that $\beta = \beta_{ik}$ for *unique* choices of $i, k \in \mathbb{N}$. This shows $\beta = \beta_{ik} \subset \alpha_k$ for this particular k, but recall that $\alpha_k \subset \beta$ originally. Therefore $\beta = \alpha_k \in \mathbb{F}$. Since $\beta \in \mathcal{G}$ was arbitrary, we have shown that $\mathcal{G} \subseteq \mathbb{F}$. This argument is symmetric in $\mathbb{F}$ and $\mathcal{G}$, so these partitions are equal. Hence $\mathbb{F}$ is the unique partition of $\mathcal{M}$ with the prescribed property. $\square$

Theorem 17.1.9. *Suppose $\mathcal{M} \subset \mathbb{P}(X)$ is a σ-algebra such that $\mathcal{M}$ is countable. Then $\mathcal{M}$ is finite.*

Proof. Suppose to the contrary that $\mathcal{M}$ is infinite and let $\mathbb{F}$ be the partition from Lemma 17.1.8. This implies two things: (1) X must be infinite, since if X were finite, $|\mathcal{M}| \leq |\mathbb{P}(X)| = 2^{|X|} < \infty$; and (2) $\mathbb{F}$ is infinite, since if it were a finite partition of $\mathcal{M}$, $\mathcal{M}$ would be finite as well. By hypothesis, $\mathcal{M}$ is countable so $\mathbb{F} \subset \mathcal{M}$ is countable is as well. Write $\mathbb{F} = \{\alpha_m\}_{m=1}^{N}$, where N is possibly equal to ∞. Define a set function

$$\begin{aligned} f : \mathcal{M} &\longrightarrow Z = \{(z_1, z_2, \ldots,) : z_i \in \{0, 1\} \text{ for each } i = 1, \ldots, N\} \\ A &\longmapsto (z_i) \text{ where } z_i = \begin{cases} 0 & \text{if } \alpha_i \not\subset A \\ 1 & \text{if } \alpha_i \subset A. \end{cases} \end{aligned}$$

We claim f is bijective. Note that since $\mathcal{M}$ is a σ-algebra, it is closed under countable unions so any countable union of sets in $\mathbb{F}$ lie in $\mathcal{M}$. Take an arbitrary sequence $(z_1, z_2, \ldots) \in Z$ and let m_1 be the first index such that $z_{m_1} = 1$; m_2 the second occurrence of 1; m_3 the third; and so on. Then $A = \alpha_{m_1} \cup \alpha_{m_2} \cup \cdots$ is an element of $\mathcal{M}$ by the comment above, and we have $f(A) = (z_i)$ by construction, so f is surjective. Now suppose $f(A) = f(B) = (z_1, z_2, \dots)$ for sets $A, B \in \mathcal{M}$. Let z_{i_1} be the first occurrence of 1 in the sequence; z_{i_2} be the second occurrence; and so on. Then A by Lemma 17.1.8, A can be written *uniquely* as the union

$$A = \bigcup_{\alpha_i \in \mathbb{F}, \alpha_i \subset A} \alpha_i \supseteq \bigcup_r \alpha_{i_r}$$

where r ranges over all indices such that $z_{i_r} = 1$. The containment above is clear by the definition of the the function f. On the other hand, if some $\alpha_\ell \in \mathbb{F}$ is not a subset of A then it does not show up in the partition $A = \bigcup \alpha_i$. Therefore $A = \bigcup_r \alpha_{i_r}$ and the same argument holds for B, showing $A = B$. Hence f is a bijection, meaning $|\mathcal{M}| = |Z| = 2^{|\mathbb{F}|}$. Now either $|\mathbb{F}| < \infty$, in which case $2^{|\mathbb{F}|} < \infty$ and $\mathcal{M}$ is not infinite, or $|\mathbb{F}| = |\mathbb{N}|$, in which case $|\mathcal{M}| = 2^{|\mathbb{N}|} > |\mathbb{N}|$ and $\mathcal{M}$ is not countable. Both are contradictions, so $\mathcal{M}$ must have been finite to begin with. $\square$

17.2 Measures

Let X be a set and consider a function $\mu : \mathbb{F} \to [0, \infty]$ on some subset $\mathbb{F} \subset \mathbb{P}(X)$.

Definition. *We say μ is* **additive** *if for any disjoint sets $\{A_i\}_{i=1}^n \subset \mathbb{F}$ such that their union $A = \coprod_{i=1}^n A_i$ also lies in $\mathbb{F}$, we have*

$$\mu(A) = \sum_{i=1}^n \mu(A_i).$$

Further, μ is **σ-additive** *if the property holds for any countable collection of disjoint sets whose union is in $\mathbb{F}$.*

Definition. *We say μ is* **subadditive** *if for any sets $A_i \in \mathbb{F}$ such that their (not necessarily disjoint) union $A = \bigcup_{i=1}^\infty A_i$ lies in $\mathbb{F}$, we have*

$$\mu(A) \leq \sum_{i=1}^\infty \mu(A_i).$$

Definition. *The function $\mu : \mathbb{F} \to [0, \infty]$ is* **σ-finite** *if there exists a countable collection $\{A_n\}_{n=1}^\infty \subset \mathbb{F}$ such that $X = \bigcup_{n=1}^\infty A_n$ and $\mu(A_n) < \infty$ for all n.*

This is not to be confused with:

Definition. *μ is* **finite** *if $\mu(X) < \infty$.*

The most important definition in measure theory is the notion of a measure:

Definition. *Let $(X, \mathcal{M})$ be a measurable space. A* **measure** *on $\mathcal{M}$ is a function $\mu : \mathcal{M} \to [0, \infty]$ which is σ-additive and satisfies $\mu(\varnothing) = 0$. We call the triple $(X, \mathcal{M}, \mu)$ a* **measure space***, and sets $E \in \mathcal{M}$ are* **measurable sets***.*

If $\mathcal{A}$ is an algebra and $\mu : \mathcal{A} \to [0, \infty]$ is additive and satisfies $\mu(\varnothing) = 0$, then μ is sometimes called a *finitely additive measure*.

A measure satisfies some important properties:

Proposition 17.2.1. *Let $\mu : \mathcal{M} \to [0, \infty]$ be a measure on a measurable space $(X, \mathcal{M})$. Then*

(1) (Monotonicity) If $E, F \in \mathcal{M}$ and $E \subseteq F$ then $\mu(E) \leq \mu(F)$.

(2) (Subadditivity) If $\{A_n\}_{n=1}^\infty \subset \mathcal{M}$ then $\mu\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \mu(A_n)$.

(3) (Continuity from below) If $\{E_n\}_{n=1}^\infty \subset \mathcal{M}$ converges to E from below, then $\mu(E_n)$ converges to $\mu(E)$ from below.

(4) (Continuity from above) If $\{E_n\}_{n=1}^\infty \subset \mathcal{M}$ converges to E from above and $\mu(E_1) < \infty$ then $\mu(E_n)$ converges to $\mu(E)$ from above.

Proof. (1) Write $F = E \cup (F \setminus E) \in \mathcal{M}$ by closure under unions and complements. Then by (disjoint) additivity, we have

$$\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E)$$

since μ is nonnegative.

(2) Define a new sequence $\{A_n'\}_{n=1}^\infty$ by $A_1' = A$ and for each $n \geq 2$, $A_n' = A_n \setminus \bigcup_{j=1}^{n-1} A_j$. By construction the A_n' are disjoint and $\bigcup_{n=1}^\infty A_n' = \bigcup_{n=1}^\infty A_n$. Then $A_n' \subseteq A_n$ for each n, so by (1),

$$\mu\left(\bigcup_{n=1}^\infty A_n\right) = \mu\left(\bigcup_{n=1}^\infty A_n'\right) = \sum_{n=1}^\infty \mu(A_n') \leq \sum_{n=1}^\infty \mu(A_n).$$

This proves subadditivity.

(3) Define $E_n' = E_n \setminus \bigcup_{j=1}^{n-1} E_j = E_n \setminus E_{n-1}$ for each n. Then for every $N \in \mathbb{N}$, $E_N = \bigcup_{n=1}^N E_n'$ and the E_n' are disjoint. By additivity, we hvae

$$\begin{aligned} \mu(E_N) &= \mu\left(\bigcup_{n=1}^N E_n'\right) = \sum_{n=1}^N \mu(E_n') \\ \implies \mu(E) &= \mu\left(\bigcup_{n=1}^\infty E_n'\right) = \sum_{n=1}^\infty \mu(E_n') = \lim_{N\to\infty} \mu(E_N). \end{aligned}$$

Finally, the convergence is from below by monotonicity.

(4) By assumption $E = \bigcap_{n=1}^\infty E_n$. Consider $E_1 \setminus E = \bigcup_{n=1}^\infty (E_1 \setminus E_n)$. For each n, set $A_n = E_1 \setminus E_n$. Then by construction, $A_n \nearrow (E_1 \setminus E)$. By (3),

$$\begin{aligned} \lim_{n\to\infty} \mu(A_n) &= \lim_{n\to\infty} \mu(E_1 \setminus E_n) = \mu(E_1 \setminus E) \\ &= \mu(E_1) - \mu(E) = \lim_{n\to\infty} (\mu(E_1) - \mu(E_n)) \\ &= \mu(E_1) - \lim_{n\to\infty} \mu(E_n). \end{aligned}$$

Since $\mu(E_1) < \infty$, we can subtract it from the two expressions above in which it appears, leaving us with $\lim_{n\to\infty} \mu(E_n) = \mu(E)$. $\square$

Note that if μ is a finite measure, that is $\mu(X) < \infty$, then continuity from above holds for all converging sequences $\{E_n\} \searrow E$.

Definition. *A collection $\mathcal{E} \subseteq \mathbb{P}(X)$ is an* **elementary class** *if*

(1) $\varnothing \in \mathcal{E}$.

(2) $\mathcal{E}$ is closed under finite intersections.

(3) If $E \in \mathcal{E}$ then $E^C = \bigcup_{i=1}^n F_i$ for finitely many disjoint sets $F_i \in \mathcal{E}$.

Note that $\mathcal{E}$ is an elementary class then $\mathcal{A}(\mathcal{E})$ consists of disjoint unions of sets in $\mathcal{E}$.

Example 17.2.2. The collection $\mathcal{E} = \{(a, b] \cap \mathbb{R} : a < b \text{ and } a, b \in \mathbb{R} \cup \{-\infty, \infty\}\}$ is an elementary class on $\mathbb{R}$.

Proposition 17.2.3. *Let $\mathcal{A} \subset \mathbb{P}(X)$ and $\mathcal{B} \subset \mathbb{P}(X)$ be elementary classes. Then*

$$\mathcal{E} = \mathcal{A} \times \mathcal{B} = \{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$$

is an elementary class on $X \times X$.

Proof. First, $\varnothing \in \mathcal{A}$ and $\varnothing \in \mathcal{B}$ so $\varnothing \times \varnothing \in \mathcal{A} \times \mathcal{B}$. Next, if $\{C_i\}_{i=1}^n \subset \mathcal{A} \times \mathcal{B}$ then each C_i is of the form $C_i = A_i \times B_i$ for $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$. We claim $\bigcap_{i=1}^n (A_i \times B_i) = (\bigcap_{i=1}^n A_i) \times (\bigcap_{i=1}^n B_i)$. For any $(a, b) \in X \times X$, consider

$$\begin{aligned} (a,b) \in \bigcap_{i=1}^n (A_i \times B_i) &\iff (a,b) \in A_i \times B_i \text{ for all } i \\ &\iff a \in A_i \text{ and } b \in B_i \text{ for all } i \\ &\iff a \in \bigcap_{i=1}^n A_i \text{ and } b \in \bigcap_{i=1}^n B_i \\ &\iff (a,b) \in \left(\bigcap_{i=1}^n A_i\right) \times \left(\bigcap_{i=1}^n B_i\right) \end{aligned}$$

The claim holds. Now since $\mathcal{A}$ and $\mathcal{B}$ are elementary classes, $\bigcap_{i=1}^n A_i \in \mathcal{A}$ and $\bigcap_{i=1}^n B_i \in \mathcal{B}$. Therefore $\bigcap_{i=1}^n C_i = \bigcap_{i=1}^n (A_i \times B_i) \in \mathcal{A} \times \mathcal{B}$.

Finally, take $E \in \mathcal{E}$. Then $E = A \times B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We claim the complement of E is precisely $(A \times B^C) \cup (A^C \times B) \cup (A^C \times B^C)$. Indeed, for any $(a, b) \in X \times X$,

$$\begin{aligned} (a,b) \in (A \times B)^C &\iff (a,b) \notin A \times B \iff a \notin A \text{ or } b \notin B \\ &\iff (a,b) \in (A \times B^C) \cup (A^C \times B) \cup (A^C \times B^C). \end{aligned}$$

Observe that this union is disjoint. Now since $A \in \mathcal{A}$, there are disjoint sets $\{F_i\}_{i=1}^n \subset \mathcal{A}$ such that $A^C = \coprod_{i=1}^n F_i$. Likewise, since $B \in \mathcal{B}$, there are disjoint sets $\{G_j\}_{j=1}^m \subset \mathcal{B}$ such that $B^C = \coprod_{j=1}^m G_j$. Then we have

$$\begin{aligned} E^C &= (A \times B^C) \cup (A^C \times B) \cup (A^C \times B^C) \\ &= \left(A \times \left(\coprod_{j=1}^m G_j\right)\right) \cup \left(\left(\coprod_{i=1}^n F_i\right) \times B\right) \cup \left(\left(\coprod_{i=1}^n F_i\right) \times \left(\coprod_{j=1}^m G_j\right)\right) \\ &= \left(\coprod_{j=1}^m (A \times G_j)\right) \cup \left(\coprod_{i=1}^n (F_i \times B)\right) \cup \left(\coprod_{i=1}^n \coprod_{j=1}^m (F_i \times G_j)\right) \end{aligned}$$

where in the last step we use the same distributive law of unions over products as we proved for intersections earlier in the proof. This exhibits E^C as a finite disjoint union of sets of the form

$$(A \times G_j), \qquad (F_i \times B) \quad \text{or} \quad (F_i \times G_j),$$

each of which lies in $\mathcal{A} \times \mathcal{B}$. Therefore $\mathcal{E} = \mathcal{A} \times \mathcal{B}$ is an elementary class. $\square$

An important question in measure theory is, given an algebra $\mathcal{A}$ on a set X and a measure $\mu : \sigma(\mathcal{A}) \to [0, \infty]$, is μ completely determined by its values on $\mathcal{A}$? The answer in general is no, but with an additional criterion we can study *extensions* of finitely additive measures on algebras to their σ-algebras.

Some notation: if $\mathcal{E} \subset \mathbb{P}(X)$ is a collection of sets in X then we let $\mathcal{E}_\sigma$ denote the collection of countable unions of sets in $\mathcal{E}$, and $\mathcal{E}_\delta$ denote the collection of countable intersections of sets in $\mathcal{E}$.

Definition. *A* **monotone class** *on a set X is a collection $\mathbb{F} \subset \mathbb{P}(X)$ with the property that if $\{B_n\}_{n=1}^\infty \subset \mathbb{F}$ and $B_n \nearrow B$ then $B \in \mathbb{F}$, and if $B_n \searrow B$ then $B \in \mathbb{F}$. In other words, $\mathbb{F}$ is closed under monotone sequences of subsets.*

Example 17.2.4. Every σ-algebra is a monotone class.

The following theorem, due to Halmos, is the most important result regarding monotone classes.

Theorem 17.2.5 (Monotone Class Theorem)**.** *If $\mathcal{A}$ is an algebra on X and $\mathcal{A} \subseteq \mathbb{F} \subseteq \mathbb{P}(X)$ for a monotone class $\mathbb{F}$, then $\sigma(\mathcal{A}) \subseteq \mathbb{F}$.*

To prove the theorem, we need the following lemma.

Lemma 17.2.6. *A monotone class which is an algebra is a σ-algebra.*

Proof. We just need to check that if $\{A_n\}_{n=1}^\infty \subset \mathcal{M}$ then $\bigcup_{n=1}^\infty A_n \in \mathcal{M}$. Define $B_n = \bigcup_{k=1}^n A_k$ for each $n \in \mathbb{N}$. Then $B_1 \subset B_2 \subset \cdots$ and $\bigcup_{n=1}^\infty B_n = \bigcup_{n=1}^\infty A_n$. Since $\mathcal{M}$ is a monotone class, $\bigcup_{n=1}^\infty B_n \in \mathcal{M}$. □

Now for the proof of Theorem 17.2.5:

Proof. First define $m(\mathbb{F}_0)$ to be the monotone class generated by $\mathbb{F}_0$, that is the smallest monotone class containing $\mathbb{F}_0$. It suffices to show $m(\mathbb{F}_0)$ is an algebra and then apply Lemma 17.2.6.

(1) $\Omega \in m(\mathbb{F}_0)$ since $\Omega \in \mathbb{F}_0 \subset m(\mathbb{F}_0)$ by definition of an algebra.

(2) Suppose $\mathcal{G} = \{A \mid A^C \in m(\mathbb{F}_0)\}$. Since the definition of monotone class is symmetric with respect to complements, we see $\mathcal{G}$ is a monotone class. Moreover, since $\mathbb{F}_0$ is an algebra, $\mathcal{G} \supset \mathbb{F}_0$ so by minimality of $m(\mathbb{F}_0)$, $\mathcal{G} \supset m(\mathbb{F}_0)$. Thus $A^C \in m(\mathbb{F}_0)$.

(3) Define $\mathcal{G}_1 = \{A \in m(\mathbb{F}_0) \mid A \cup B \in m(\mathbb{F}_0) \text{ for all } B \in \mathbb{F}_0\}$. If $\{A_i\}_{i=1}^\infty \subset \mathcal{G}_1$ with $A_i \subset A_{i+1}$ for all i, then for any $B \in \mathbb{F}_0$,

$$\left(\bigcup_{i=1}^\infty A_i\right) \cup B = \bigcup_{i=1}^\infty (A_i \cup B)$$

and each of these pieces is in $\mathcal{G}_1$. Hence $\mathcal{G}_1$ is a monotone class. Now $\mathcal{G}_1 \supset \mathbb{F}_0$ since $\mathbb{F}_0$ is a field, so by Lemma 17.2.6 $\mathcal{G}_1$ must contain $\sigma(\mathbb{F}_0)$. Set

$$\mathcal{G}_2 = \{B \in m(\mathbb{F}_0) \mid A \cup B \in m(\mathbb{F}_0) \text{ for all } A \in m(\mathbb{F}_0)\}.$$

For the same reason as above, $\mathcal{G}_2$ is a monotone class, and $\mathcal{G}_2 \supset \mathbb{F}_0$ since $\mathcal{G}_1 \supset m(\mathbb{F}_0)$. This shows that $\mathcal{G}_2 \supset m(\mathbb{F}_0)$ so for every $A, B \in m(\mathbb{F}_0)$, $A \cup B$ is also in $m(\mathbb{F}_0)$.

Hence we conclude that $m(\mathbb{F}_0)$ is an algebra, and applying Lemma 17.2.6 shows that $m(\mathbb{F}_0)$ is a σ-algebra. By definition $\sigma(\mathbb{F}_0)$ is the smallest σ-algebra containing $\mathbb{F}_0$), so this proves finally that $\mathcal{M} \supset \sigma(\mathbb{F}_0)$. □

Theorem 17.2.7 (Regularity)**.** *Let $(X, \mathcal{M})$ be a measurable space and take $\mathcal{A}$ to be an algebra on X such that $\mathcal{M} = \sigma(\mathcal{A})$. Suppose $\mu, \nu : \mathcal{M} \to [0, \infty]$ are measures defined on a measurable space $(X, \mathcal{M})$ such that $\mu = \nu$ on $\mathcal{A}$, and there exists a sequence $\{X_n\} \subset \mathcal{A}$ such that $X_n \nearrow X$ and $\mu(X_n) = \nu(X_n) < \infty$ for all n. Then*

(1) $\mu = \nu$ on all of $\mathcal{M}$.

(2) For any $A \in \mathcal{M}$, $\mu(A) = \inf\{\mu(B) \mid B \in \mathcal{A}_\sigma, A \subseteq B\}$.

Proof. (1) Assume $\mu(X) = \nu(X) < \infty$. Let $\mathbb{F} = \{A \in \mathcal{M} = \sigma(\mathcal{A}) \mid \mu(A) = \nu(A)\}$. By assumption, $\mathcal{A} \subseteq \mathbb{F}$. We claim that $\mathbb{F}$ is a monotone class. Let $A_n \in \mathbb{F}$ be a sequence of sets converging to A from below. Then since $\mu(A_n) = \nu(A_n)$ for all n,

$$\mu(A) = \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \nu(A_n) = \nu(A)$$

by continuity from below. Therefore $A \in \mathbb{F}$. The proof for a descending sequence $A_n \searrow A$ is similar, using continuity from above. Hence $\mathbb{F}$ is a monotone class, so the Monotone Class Theorem implies $\sigma(\mathcal{A}) \subseteq \mathbb{F}$, that is, $\mu = \nu$ on $\sigma(\mathcal{A})$.

(2) First suppose μ is finite, that is, $\mu(X) < \infty$. Set $\mu^*(A) = \inf\{\mu(B) \mid B \in \mathcal{A}_\sigma, A \subseteq B\}$ for all $A \in \mathcal{M}$ and define the collection

$$\mathbb{F} = \{A \in \mathcal{M} \mid \mu^*(A) = \mu(A)\}.$$

Note that $\mathcal{A} \subseteq \mathcal{A}_\sigma \subseteq \mathbb{F}$ so we're done once we show $\mathbb{F}$ is a monotone class. Let $\{A_n\}_{n=1}^\infty \subset \mathbb{F}$ so that $A_n \nearrow A$ and let $\varepsilon > 0$. For each $n \in \mathbb{N}$, there exists a $B_n \in \mathcal{A}_\sigma$ such that $A_n \subseteq B_n$ and $\mu(B_n \smallsetminus A_n) = \mu(B_n) - \mu(A_n) = \frac{\varepsilon}{2^n}$. (Note that the subtraction property is only possible in a finite measure space.) Then if $B = \bigcup_{n=1}^\infty B_n$, we have $A \subseteq B$ and

$$\begin{aligned} \mu(B \smallsetminus A) = \mu(B) - \mu(A) &= \mu\left(\left(\bigcup_{n=1}^\infty B_n\right) \smallsetminus A\right) \\ &= \mu\left(\bigcup_{n=1}^\infty (B_n \smallsetminus A)\right) \\ &\le \sum_{n=1}^\infty \mu(B_n \smallsetminus A) \quad \text{by subadditivity} \\ &\le \sum_{n=1}^\infty \mu(B_n \smallsetminus A_n) < \sum_{n=1}^\infty \frac{\varepsilon}{2^n} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have $\mu(A) = \inf\{\mu(B) \mid B \in \mathcal{A}_\sigma, A \subseteq B\} = \mu^*(A)$. So $\mathbb{F}$ is closed under ascending sequences. On the other hand, if $\{A_n\}_{n=1}^\infty \subset \mathbb{F}$ so that $A_n \searrow A$ and let $\varepsilon > 0$. Then there exist $B_n \in \mathcal{A}_\sigma$ such that $A_n \subseteq B_n$ and $\mu(B_n \smallsetminus A_n) < \frac{\varepsilon}{2}$ for each n. Then $A \subseteq A_n \subseteq B_n$ implies that

$$\mu(B_n \smallsetminus A) = \mu(B_n \smallsetminus A_n) + \mu(A_n \smallsetminus A) < \frac{\varepsilon}{2} + \mu(A_n) - \mu(A).$$

Since μ is finite, continuity from above gives us $\mu(A_n) \searrow \mu(A)$ as $n \to \infty$. Thus for sufficiently large n, $\mu(B_n \smallsetminus A) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Therefore $\mu(A) = \inf\{\mu(B) \mid B \in \mathcal{A}_\sigma, A \subseteq B\} = \mu^*(A)$ once again, so $\mathbb{F}$ is closed under descending sequences. Hence $\mathbb{F}$ is a monotone class, so by the Monotone Class Theorem we have $\mathcal{M} = \sigma(\mathcal{A}) \subseteq \mathbb{F}$, i.e. $\mu^* = \mu$ on $\mathcal{M}$.

Now in the infinite case, the σ-finite property allows us to define set functions $\mu_n : \mathcal{M} \to [0, \infty]$ by $\mu_n(A) = \mu(A \cap X_n)$ for all $n \in \mathbb{N}$ and $A \in \mathcal{M}$, where the X_n are as in the statement of the theorem. Then each μ_n is a finite measure since $\mu_n(X) = \mu(X_n) < \infty$ by assumption. Thus for any $\varepsilon > 0$, there are sets $B_n \in \mathcal{A}_\sigma$ such that $A \subseteq B_n$ and

$$\mu_n(B_n \smallsetminus A) < \frac{\varepsilon}{2^n}.$$

Let $B = \bigcup_{n=1}^\infty (B_n \cap X_n)$. Then $B \in \mathcal{A}_\sigma$, $A \subseteq B$, and we have

$$\begin{aligned}
\mu(B \smallsetminus A) &= \mu\left(\left(\bigcup_{n=1}^\infty (B_n \cap X_n)\right) \smallsetminus A\right) \\
&= \mu\left(\bigcup_{n=1}^\infty (B_n \cap X_n) \smallsetminus A\right) \\
&\le \sum_{n=1}^\infty \mu((B_n \cap X_n) \smallsetminus A) \quad \text{by subadditivity} \\
&\le \sum_{n=1}^\infty \mu(X_n \cap (B_n \smallsetminus A)) \\
&= \sum_{n=1}^\infty \mu_n(B_n \smallsetminus A) < \sum_{n=1}^\infty \frac{\varepsilon}{2^n} = \varepsilon.
\end{aligned}$$

As before, this proves $\mu(B) = \mu^*(B)$ on all of $\mathcal{M}$, proving (2). □

Corollary 17.2.8 (Extension Theorem)**.** *Suppose $\mu_0 : \mathcal{A} \to [0, \infty]$ is an additive measure on an algebra $\mathcal{A}$ that is σ-finite and σ-additive on $\mathcal{A}$. Then there is a unique measure μ on $\mathcal{M} = \sigma(\mathcal{A})$ such that $\mu|_{\mathcal{A}} = \mu_0$ and for all $A \in \mathcal{M}$,*

$$\mu(A) = \inf\left\{\sum_{n=1}^\infty \mu_0(A_n) : A_n \in \mathcal{A}, A \subseteq \bigcup_{n=1}^\infty A_n\right\}.$$

The infimum definition above is an example of an *outer measure*, defined in general below.

Definition. *A set function $\mu^* : \mathbb{P}(X) \to [0, \infty]$ is called an* **outer measure** *on X if*

(1) $\mu^*(\varnothing) = 0$.

(2) (Monotonicity) For all $A \subseteq B$, $\mu^*(A) \le \mu^*(B)$.

(3) (Subadditivity) For any collection $\{A_i\}_{i=1}^\infty$ of sets in X, $\mu^*\left(\bigcup_{i=1}^\infty A_i\right) \le \sum_{i=1}^\infty \mu^*(A_i)$.

Example 17.2.9. The function $\mu^*(A) = \inf\{\mu(B) : B \in \mathcal{A}_\sigma, A \subseteq B\}$ is an example of an outer measure on $\mathcal{A}_\sigma$. The extension theorem then says that if μ_0 is σ-additive and σ-finite on $\mathcal{A}$, then μ_0 extends uniquely to the outer measure μ^* on $\sigma(\mathcal{A})$.

Definition. *Given an outer measure $\mu^* : \mathbb{P}(X) \to [0,\infty]$, a set $A \subseteq X$ is* **μ^*-measurable** *if for every $E \subseteq X$, $\mu^*(E) = \mu^*(A \cap E) + \mu^*(A^C \cap E)$. This is also called the* **Carathéodory condition**.

Definition. *A measure space $(X, \mathcal{M}, \mu)$ is* **complete** *if for all $A \in \mathcal{M}$ with $\mu(A) = 0$ and for all $B \subseteq A$, $B \in \mathcal{M}$ and $\mu(B) = 0$.*

Theorem 17.2.10. *Given an outer measure $\mu^* : \mathbb{P}(X) \to [0,\infty]$, the collection*

$$\mathcal{M} = \{A \in \mathbb{P}(X) \mid A \text{ is } \mu^*\text{-measurable}\}$$

is a σ-algebra on X and $(X, \mathcal{M}, \mu^*)$ is complete.

There is an analagous notion of inner measure:

Definition. *A set function $\mu_* : \mathbb{P}(X) \to [0,\infty]$ is an* **inner measure** *on X if*

(1) $\mu_*(\varnothing) = 0$.

(2) For any disjoint sets $A, B \in \mathbb{P}(X)$, $\mu^*(A \cup B) \ge \mu_*(A) + \mu_*(B)$.

(3) For any descending sequence $A_n \searrow A$ such that $\mu_*(A_1) < \infty$,

$$\mu_*\left(\bigcap_{n=1}^\infty A_n\right) = \lim_{n\to\infty} \mu_*(A_n).$$

(4) If $\mu_*(A) = \infty$ then for all $M > 0$, there is a set $B \subseteq A$ such that $M \le \mu_*(B) < \infty$.

One can define an inner measure from a σ-additive set function $\mu_0 : \mathcal{A} \to [0,\infty]$ by

$$\mu_*(A) = \sup\{\mu(B) : B \in \mathcal{A}_\sigma, A \supseteq B\}.$$

Then $A \in \mathcal{A}$ is measurable (with respect to μ_0) if and only if $\mu_*(A) = \mu^*(A)$.

The uniqueness statement in Theorem 17.2.7 can also be proven using π- and λ-systems.

Definition. *A* **π-system** *is a collection $\mathbb{P} \subset \mathbb{P}(X)$ for which $A_1, A_2 \in \mathbb{P} \implies A_1 \cap A_2 \in \mathbb{P}$.*

Definition. *A* **λ-system** *is a collection $\mathcal{L} \subset \mathbb{P}(X)$ satisfying*

(1) $X \in \mathcal{L}$.

(2) $A \in \mathcal{L} \implies A^C \in \mathcal{L}$.

(3) *If* $\{A_i\}_{i=1}^{\infty}$ *are disjoint sets in* $\mathcal{L}$ *then* $\displaystyle\bigcup_{i=1}^{\infty} A_i \in \mathcal{L}$.

Notice that a λ-system is *almost* a σ-algebra – the disjoint requirement in (3) is a key difference. Further, a collection that is both a λ-system and a π-system is automatically a σ-algebra. To see this, let $\{A_i\}_{i=1}^{\infty}$ be a collection of not necessarily disjoint sets in such a system. Then by setting $B_i = A_i \setminus \bigcup_{j=1}^{n-1} A_j$ we see that the B_i have the same union as the A_i and are disjoint.

Remark. Notice that an equivalent condition to (2), given that (1) and (3) are true, is $A_1, A_2 \in \mathcal{L}$ and $A_1 \subset A_2 \implies A_2 \setminus A_1 \in \mathcal{L}$.

Theorem 17.2.11 (Dynkin's π-λ Theorem)**.** *Suppose* $\mathbb{P}$ *is a* π*-system and* $\mathcal{L}$ *is a* λ*-system with* $\mathbb{P} \subset \mathcal{L}$. *Then* $\sigma(\mathbb{P}) \subset \mathcal{L}$.

Proof. Define $\mathcal{L}_0$ to be the λ-system generated by $\mathbb{P}$, i.e. the smallest λ-system containing $\mathbb{P}$. Then $\mathbb{P} \subset \mathcal{L}_0 \subset \mathcal{L}$. We will show that $\sigma(\mathbb{P}) \subset \mathcal{L}_0$ which implies the result. We do this by showing $\mathcal{L}_0$ is a π-system and by the comments above this will mean $\mathcal{L}_0$ is a σ-algebra.

Define $\mathcal{L}_A = \{B \subset \mathcal{L}_0 \mid B \cap A \in \mathcal{L}_0\}$ for a set $A \subset X$. First assume that $A \in \mathcal{L}_0$. Under this hypothesis we can show that $\mathcal{L}_A$ is a λ-system:

(1) $X \cap A = A \in \mathcal{L}_0$ so $X \in \mathcal{L}_A$.

(2) Suppose $B_1, B_2 \in \mathcal{L}_A$ and $B_1 \subset B_2$. Then $A \cap (B_2 \setminus B_1) = (A \cap B_2) \setminus (A \cap B_1)$ and we notice that $A \cap B_2$ and $A \cap B_1$ are both in $\mathcal{L}_0$. Since $\mathcal{L}_0$ is a λ-system, the whole expression above is in $\mathcal{L}_0$. Hence $B_2 \setminus B_1 \in \mathcal{L}_A$ so (2) holds by the remark.

(3) Suppose $\{B_i\}_{i=1}^{\infty}$ are disjoint elements of $\mathcal{L}A$. Then $A \cap (\bigcup_{i=1}^{\infty} B_i) = \bigcup_{i=1}^{\infty}(A \cap B_i)$ and $A \cap B_i \in \mathcal{L}_0$ for each i, so because $\mathcal{L}_0$ is a λ-system, $\bigcup_{i=1}^{\infty}(A \cap B_i) \in \mathcal{L}_0$. Thus $\bigcup_{i=1}^{\infty} B_i \in \mathcal{L}_A$.

Now suppose $A, B \in \mathbb{P}$, which implies $A \cap B \in \mathbb{P}$ since $\mathbb{P}$ is a π-system. That means that if $A \in \mathbb{P}$ then $\mathcal{L}_A \supset \mathbb{P}$ and moreover $\mathcal{L}_A$ is a λ-system containing $\mathbb{P}$, so $\mathcal{L}_A \supset \mathcal{L}_0$. Therefore if $A \in \mathbb{P}$ and $B \in \mathcal{L}_0$ then $A \cap B \in \mathcal{L}_0$. Switching the roles of A and B, we can see that $\mathcal{L}_B \supset \mathbb{P}$ if $B \in \mathcal{L}_0$. Thus for every $B \in \mathcal{L}_0$, $\mathcal{L}_B$ is a λ-system containing P, which implies $\mathcal{L}_B \supset \mathcal{L}_0$. Finally, for all $A, B \in \mathcal{L}_0$, $A \cap B \in \mathcal{L}_0$ which shows $\mathcal{L}_0$ is a π-system. Hence $\mathcal{L}_0$ is a σ-algebra and the theorem is proved. □

The following gives an alternate proof of the uniqueness statement of Theorem 17.2.7.

Theorem 17.2.12. *Let* $\mathbb{P}$ *be a* π*-system and suppose* μ *and* ν *are measures on* $\sigma(\mathbb{P})$ *satisfying* $\mu|_{\mathbb{P}} = \nu|_{\mathbb{P}}$. *Then* $\mu = \nu$.

Proof. Let $\mathcal{L} = \{A \in \sigma(\mathbb{P}) \mid P_1(A) = P_2(A)\}$. We will prove that $\mathcal{L}$ is a λ-system, which will imply $\mathcal{L} \supset \sigma(\mathbb{P})$ by the π-λ Theorem.

(1) Clearly $X \in \mathcal{L}$.

(2) If $A \in \mathcal{L}$ then $P_1(A^C) = 1 - P_1(A) = 1 - P_2(A) = P_2(A^C)$ so $A^C \in \mathcal{L}$.

(3) If $\{A_n\}_{n=1}^{\infty}$ are disjoint sets in $\mathcal{L}$ then

$$\begin{aligned} P_1\left(\bigcup_{n=1}^{\infty} A_n\right) &= \sum_{n=1}^{\infty} P_1(A_n) \\ &= \sum_{n=1}^{\infty} P_2(A_n) \\ &= P_2\left(\bigcup_{n=1}^{\infty} A_n\right) \end{aligned}$$

because P_1 and P_2 are both countably additive.

Hence $\mathcal{L}$ is a λ-system and the result follows. □

The regularity theorem (17.2.7) generalizes in the following way.

Theorem 17.2.13. *Suppose $\mathcal{T}$ is a topology on X with the property that, for every closed $C \subset X$, there exists $\{V_n\}_{n=1}^{\infty} \subset \mathcal{T}$ such that $V_n \searrow C$. Let $\mathcal{M} = \sigma(\mathcal{T})$ and $\mu : \mathcal{M} \to [0, \infty]$ be a measure which is σ-finite on $\mathcal{T}$.*

(i) For all $\varepsilon > 0$ and $A \in \mathcal{M}$, there exists $V \in \mathcal{T}$ and a closed set F such that $F \subset A \subset V$ and $\mu(V \setminus F) < \varepsilon$.

(ii) For all $B \in \mathcal{M}$, there exists $A \in F_\sigma$ and $C \in G_\delta$ such that $A \subset B \subset C$ and $\mu(C \setminus A) = 0$.

Proof. (i) Let $\mathcal{G}$ be the collection of all $A \in \mathcal{M}$ such that for every $\varepsilon > 0$, there exist an open set V and a closed set F such that $F \subset A \subset V$ and $\mu(V \setminus F) < \varepsilon$. We will prove $\mathcal{G}$ is a σ-algebra containing $\mathcal{T}$. The second part is easy: If U is open, U^C is closed so there is an open sequence $V_n \searrow U^C$. Taking complements, we get a closed sequence $V_n^C \nearrow U$, so for any $\varepsilon > 0$, there is a closed set V_n^C such that $\mu(U \setminus V_n^C) < \varepsilon$. Therefore every open set U lies in $\mathcal{G}$, and hence $\mathcal{T} \subset \mathcal{G}$.

To prove $\mathcal{G}$ is a σ-algebra, first observe that $\varnothing \in \mathcal{G}$ since $\varnothing$ is both open and closed, and $\mu(\varnothing) = 0$. For the rest of the proof, let $\varepsilon > 0$ be arbitrary. Take any $A \in \mathcal{G}$ and let V be open and F be closed such that $F \subset A \subset V$ and $\mu(V \setminus F) < \varepsilon$. Then taking complements gives us $V^C \subset A^C \subset F^C$ where F^C is now open and V^C is closed. Moreover, $\mu(F^C \setminus V^C) = \mu(F^C \cap V) = \mu(V \cap F^C) = \mu(V \setminus F) < \varepsilon$. So we see that $A^C \in \mathcal{G}$. Finally, take a countable collection $\{A_n\}_{n=1}^{\infty} \subset \mathcal{G}$ and set $A = \bigcup_{n=1}^{\infty} A_n$. Consider the sets $B_m := \bigcup_{n=1}^{m} A_n$. Then $B_m \nearrow A$ as $m \to \infty$. By continuity from below, we have

$$\lim_{m\to\infty} \mu(B_m) = \mu(A).$$

Since the $B_m \in \mathcal{M}$, for every $m \in \mathbb{N}$ there is a closed set F_m and an open set V_m such that $F_m \subset B_m \subset V_m$ and $\mu(V_m \setminus F_m) < \frac{1}{m}$. Set $V = \bigcup_{m=1}^{\infty} V_m$ and $F = \bigcap_{m=1}^{\infty} F_m$, so that V is open, F is closed and $(V_m \setminus F_m) \nearrow (V/F)$ by construction. For each $m \in \mathbb{N}$,

$A_m \subseteq B_m \subseteq V_m \subseteq V$ so in particular $A \subset V$. Similarly, for each $m \in \mathbb{N}$, $A \supseteq B_m \supset F_m \supset F$, so in particular $A \supset F$. Then we have $F \subset A \subset V$. Finally,

$$\begin{aligned}\mu(V \smallsetminus F) &= \lim_{m\to\infty} \mu(V_m \smallsetminus F_m) \quad \text{by continuity from below} \\ &= \lim_{m\to\infty} \frac{1}{m} = 0.\end{aligned}$$

Therefore $A = \bigcup_{n=1}^{\infty} A_n$ lies in $\mathcal{G}$, so $\mathcal{G}$ is a σ-algebra. We proved that $\mathcal{T} \subset \mathcal{G}$, so we conclude that $\mathcal{M} \subset \mathcal{G}$.

(ii) Let $B \in \mathcal{M}$. Then by (i), for every $n \in \mathbb{N}$ there is a closed set A_n and an open set C_n such that $A_n \subset B \subset C_n$ and $\mu(C_n \smallsetminus A_n) \leq \frac{1}{n}$. Setting $A = \bigcup_{n=1}^{\infty} A_n$ and $C = \bigcap_{n=1}^{\infty} C_n$, we see that $A \in F_\sigma$ and $C \in G_\delta$. Moreover, let X have a cover $\bigcup_{k=1}^{\infty} X_k = X$ such that $\mu(X_k) < \infty$ for each $k \in \mathbb{N}$. Then, intersecting the C_n with X_k's if necessary, we may assume each C_n has finite measure. Therefore we may apply continuity from above to the convergent sequence $(C_n \smallsetminus A_n) \searrow (C \smallsetminus A)$ to obtain

$$\mu(C \smallsetminus A) = \lim_{n\to\infty} \mu(C_n \smallsetminus A_n) \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

So $\mu(C \smallsetminus A) = 0$ as required. □

Lemma 17.2.14. *Suppose (X, ρ) is a metric space and $\mathcal{T} = \mathcal{T}_\rho$ is the topology generated by ρ. Then for every closed set F, there exists a sequence of open sets $F_\varepsilon \searrow F$ as $\varepsilon \searrow 0$.*

Proof. Given a set $F \subset X$ and $\varepsilon > 0$, let F_ε be the open set

$$F_\varepsilon = \bigcup_{x\in F} B(x, \varepsilon)$$

where $B(x, \varepsilon)$ is the ρ-ball centered on x of radius ε. First, we show that if F is closed, then $F_\varepsilon \searrow F$ as $\varepsilon \searrow 0$. Suppose $0 < \varepsilon_1 \leq \varepsilon_0$. Then for each $y \in F_{\varepsilon_1}$, $y \in B(x, \varepsilon_1)$ for some $x \in F$. Since $\varepsilon_1 \leq \varepsilon_0$, we have $B(x, \varepsilon_1) \subseteq B(x, \varepsilon_0)$ so $y \in B(x, \varepsilon_0)$ as well. Thus $y \in B(x, \varepsilon_0)$ as well, so $F_{\varepsilon_1} \subseteq F_{\varepsilon_0}$. This proves the sequence F_ε is descending. We finish by proving $\bigcap_{\varepsilon>0} F_\varepsilon = F$. On one hand, every $x \in F$ lies in $B(x, \varepsilon)$ for every $\varepsilon > 0$, so $x \in F_\varepsilon$ for all $\varepsilon > 0$, and thus $x \in \bigcap_{\varepsilon>0} F_\varepsilon$. This proves $F \subseteq \bigcap_{\varepsilon>0} F_\varepsilon$. On the other hand, suppose $y \in \bigcap_{\varepsilon>0} F_\varepsilon$. Then for every $n \in \mathbb{N}$, there is some $x_n \in F$ such that $y \in B\left(x_n, \frac{1}{n}\right)$. Then as $n \to \infty$, $\rho(x_n, y) \to 0$ so the sequence (x_n) converges to y. Thus y is a limit point of F, but as F is closed, this means $y \in F$. Hence $F = \bigcap_{\varepsilon>0} F_\varepsilon$. We conclude that $F_\varepsilon \searrow F$ as $\varepsilon \searrow 0$. □

Corollary 17.2.15 (Regularity on $\mathbb{R}^n$)**.** *Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}^n$ and suppose that $\mu : \mathcal{B} \to [0, \infty]$ is a measure such that $\mu(A) < \infty$ for all bounded sets $A \in \mathcal{B}$. Then*

(i) For all $A \in \mathcal{B}$ and $\varepsilon > 0$, there exist a closed set F and an open set V such that $F \subset A \subset V$ and $\mu(V \smallsetminus F) < \varepsilon$.

(ii) If $\mu(A) < \infty$, then the set F in (i) may be chosen to be compact.

(iii) For all $A \subset \mathcal{B}$, we have

$$\mu(A) = \inf\{\mu(V) \mid A \subset V \text{ and } V \text{ is open}\} = \sup\{\mu(K) \mid A \supset K \text{ and } K \text{ is compact}\}.$$

Proof. (i) Lemma 17.2.14 shows that the hypotheses of Theorem 17.2.13 are satisfied, so (i) follows immediately.

(ii) If $\mu(A) < \infty$, by (i) there is a closed set F and open set V such that $F \subset A \subset V$ and $\mu(V \setminus F) < \frac{\varepsilon}{2}$. Consider the sequence

$$K_j = \{x \in F : |x| \leq j\}.$$

Then each K_j is closed and bounded in $\mathbb{R}^n$ and thus compact. Moreover, $K_n \nearrow F$ which implies $(V \setminus K_n) \searrow (V \setminus F)$. Since $\mu(A) < \infty$ and $\mu(V \setminus A) \leq \mu(V \setminus F) < \frac{\varepsilon}{2}$ by monotonicity, we see that $\mu(V) < \infty$. Therefore by continuity from above,

$$\mu(V \setminus K_n) \searrow \mu(V \setminus F).$$

Thus there is a sufficiently large $J \in \mathbb{N}$ such that $K = K_J$ satisfies $K \subset A \subset V$ and

$$\mu(V \setminus K) = \mu(V \setminus K_J) = \mu(V \setminus K_J) - \mu(V \setminus F) + \mu(V \setminus F) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves (ii).

(iii) Finally, if $\mu(A) < \infty$, the two equalities follow from (i) and (ii), respectively. In the case that $\mu(A) = \infty$, the closed set F from (i) may be chosen so that $\mu(A \setminus F) < 1$, which implies $\mu(F) = \infty$. Then the same sequence K_j of compact sets in (ii) will work. $\square$

We next prove an analog of the extension theorem for elementary classes.

Lemma 17.2.16. *Let $\mathcal{E}$ be an elementary class on X. Then $\mathcal{A}(\mathcal{E})$ consists of finite disjoint unions of sets in $\mathcal{E}$.*

Proof. Let $\mathcal{A}$ denote the collection of all finite disjoint unions of sets in $\mathcal{E}$. Clearly $\varnothing, X \in \mathcal{A}$ and $\mathcal{A}$ is closed under finite unions. Take $A \in \mathcal{A}$ and write $A = \bigcup_{i=1}^n E_i$, where the $E_i \in \mathcal{E}$ are disjoint. Then

$$A^C = \bigcap_{i=1}^n E_i^C = \bigcap_{i=1}^n \bigcup_{k=1}^{N_i} F_{ik} \quad \text{for some } F_{ik} \in \mathcal{E}$$

which is possible since $\mathcal{E}$ is an elementary class. This can alternatively be written

$$A^C = \bigcup_{k_1,\ldots,k_n=1}^{N_1,\ldots,N_n} (F_{1k_1} \cap \cdots \cap F_{nk_n}).$$

It is clear that $A^C \in \mathcal{A}$ so $\mathcal{A}$ is an algebra. Since $\mathcal{E} \subseteq \mathcal{A}$, we have $\mathcal{A}(\mathcal{E}) \subseteq \mathcal{A}$. By definition, $\mathcal{A}(\mathcal{E})$ consists of all intersections of finite unions from $\mathcal{E}$, so $\mathcal{A}(\mathcal{E}) \supseteq \mathcal{A}$ and therefore they are equal. $\square$

Theorem 17.2.17. *Let $\mathcal{E}$ be an elementary class on X and suppose $\mu_0 : \mathcal{E} \to [0, \infty]$ is an additive set function. Then*

(1) There exists a finitely additive measure $\mu_1 : \mathcal{A}(\mathcal{E}) \to [0, \infty]$ such that $\mu_1|_{\mathcal{E}} = \mu_0$.

(2) If μ_0 is subadditive on $\mathcal{E}$ then μ_1 is σ-additive on $\mathcal{A}(\mathcal{E})$ and $\mu_1(\varnothing) = 0$.

(3) If μ_0 is subadditive on $\mathcal{E}$ then there exists a measure $\mu : \sigma(\mathcal{E}) \to [0,\infty]$ such that $\mu|_{\mathcal{E}} = \mu_0$.

Proof. (1) If $A \in \mathcal{A}(\mathcal{E})$ then $A = \coprod_{i=1}^n E_i$ for disjoint sets $E_i \in \mathcal{E}$ by Lemma 17.2.16. Set $\mu_1(A) = \sum_{i=1}^n \mu_0(E_i)$. Then clearly μ_1 is finitely additive and $\mu_1|_{\mathcal{E}} = \mu_0$ so we need only check that μ_1 does not depend on the E_i chosen to represent A. Suppose that $A = \coprod_{j=1}^m F_j$ for $F_j \in \mathcal{E}$ as well. For $1 \le i \le n$, we can write

$$E_i = E_i \cap A = E_i \cap \left(\bigcup_{j=1}^m F_j\right) = \bigcup_{j=1}^m (E_i \cap F_j).$$

Then additivity gives us $\mu_0(E_i) = \sum_{j=1}^m \mu_0(E_i \cap F_j)$. Finally, summing over the E_i, we obtain

$$\sum_{i=1}^n \mu_0(E_i) = \sum_{i=1}^n \sum_{j=1}^m \mu_0(E_i \cap F_j) = \sum_{j=1}^m \sum_{i=1}^n \mu_0(E_i \cap F_j) = \sum_{j=1}^m \mu_0(F_j).$$

Hence $\mu_1(A)$ is well-defined for all $A \in \mathcal{A}(\mathcal{E})$.

(2) The property $\mu_1(\varnothing) = 0$ follows directly from subadditivity. Suppose $A_k \in \mathcal{A} := \mathcal{A}(\mathcal{E})$ such that $A = \coprod_{k=1}^\infty A_k \in \mathcal{A}$. Then for any N, we can write A as a disjoint union:

$$A = \left(\coprod_{k=1}^N A_k\right) \cup \left(A \setminus \coprod_{k=1}^N A_k\right)$$

and both pieces are in $\mathcal{A}$, so we have

$$\mu_1(A) = \sum_{k=1}^N \mu_1(A_k) + \sum_1 \left(A \setminus \coprod_{k=1}^N A_k\right) \ge \sum_{k=1}^N \mu_1(A_k).$$

Since this holds for all N, we can take a limit to obtain

$$\mu_1(A) \ge \sum_{k=1}^\infty \mu_1(A_k).$$

On the other hand, $A = \coprod_{j=1}^N E_j$ for some $E_j \in \mathcal{E}$, and each A_k can be written $A_k = \coprod_{i=1}^{N_k} F_{ik}$ for $F_{ik} \in \mathcal{E}$. Then

$$A = \coprod_{j=1}^N E_j = \coprod_{k=1}^\infty A_k = \coprod_{k=1}^\infty \coprod_{i=1}^{N_k} F_{ik}$$

$$\implies E_j = E_j \cap A = \coprod_{k=1}^\infty \coprod_{i=1}^{N_k} (F_{ik} \cap E_j) \in \mathcal{E}.$$

Since μ_1 is subadditive on $\mathcal{E}$, $\mu_1(E_j) \le \sum_{k=1}^\infty \sum_{i=1}^{N_k} \mu_1(F_{ik} \cap E_j)$ for each j, so we have

$$\mu_1(A) = \sum_{j=1}^N \mu_1(E_j) \le \sum_{k=1}^\infty \sum_{i=1}^{N_k} \sum_{j=1}^N \mu_1(F_{ik} \cap E_j) = \sum_{k=1}^\infty \mu_1(A_k)$$

(using Tonelli's Theorem to switch the sums in the last step). Hence μ_1 is σ-additive on $\mathcal{A}$.

(3) follows directly from the extension theorem when we observe that $\sigma(\mathcal{E}) = \sigma(\mathcal{A})$. $\square$

17.3 Borel Measures

For this section, let $\mathcal{E} = \{(a,b] \cap \mathbb{R} : -\infty \leq a < b \leq \infty\}$, $\mathcal{A} = \mathcal{A}(\mathcal{E})$ and $\mathcal{B} = \sigma(\mathcal{A}) = \sigma(\mathcal{E})$, the Borel σ-algebra on $\mathbb{R}$. Our goal is to classify all measures $\mu : \mathcal{B} \to [0,\infty]$ such that $\mu((a,b]) < \infty$ for all $-\infty < a < b < \infty$.

If μ is such a measure, define the function $F_\mu : \mathbb{R} \to \mathbb{R}$ by

$$F_\mu(x) = \begin{cases} \mu((0,x] \cap \mathbb{R}) & 0 \leq x \leq \infty \\ -\mu((x,0]) & -\infty \leq x < 0. \end{cases}$$

This is called a *(cumulative) distribution function* for μ on $\mathbb{R}$, since it is of the form $F_\mu(x) = \mu((-\infty,x])$ for all $x \in \mathbb{R}$.

Lemma 17.3.1. *Let $F = F_\mu$ be the distribution function for a measure μ on $\mathcal{B}$. Then*

(1) F is nondecreasing.

(2) F is right continuous and is continuous at $\pm\infty$.

(3) $F(0) = 0$.

(4) For all $-\infty \leq a < b \leq \infty$, $\mu((a,b]) = F(b) - F(a)$.

Proof. (1) follows from monotonicity of μ.

(4) First suppose $a < 0 < b$. Then

$$\mu((a,b]) = \mu((a,0] \cup (0,b]) = \mu((a,0]) + \mu((0,b]) = -F(a) + F(b).$$

Now suppose $0 < a < b$. Then

$$\mu((a,b]) = \mu((0,b] \setminus (0,a]) = \mu((0,b]) - \mu((0,a]) = F(b) - F(a).$$

The other case when $a < b < 0$ is similar.

(2) Suppose $b \in \mathbb{R}$ and b_n is a sequence in $\mathbb{R}$ which converges to b from above. Then

$$\begin{aligned} (a,b_n] \searrow (a,b] &\implies \mu((a,b_n]) \searrow \mu((a,b]) \quad \text{by continuity from above} \\ &\implies \lim_{n\to\infty} (F(b_n) - F(a)) = F(b) - F(a) \quad \text{by (4)} \\ &\implies \lim_{n\to\infty} F(b_n) = F(b). \end{aligned}$$

Thus F is continuous from the right. Continuity at $\pm\infty$ is easily checked.

(3) follows from right continuity and the definition of F for $x \geq 0$. □

Theorem 17.3.2. *If $F : \mathbb{R} \to \mathbb{R}$ is right continuous and nondecreasing, there exists a unique measure $\mu : \mathcal{B} \to [0,\infty]$ such that $\mu((a,b] \cap \mathbb{R}) = F(b) - F(a)$ for all $-\infty \leq a < b \leq \infty$.*

Proof. We will use Theorem 17.2.17 to construct $\mu : \mathcal{B} \to [0,\infty]$. Let $\mathcal{E}$ be the elementary class defined above and define $\mu_0 : \mathcal{E} \to [0,\infty]$ by $\mu_0((a,b] \cap \mathbb{R}) = F(b) - F(a)$. First we prove μ_0 is finitely additive. Take $(a,b] \in \mathcal{E}$ and write it as a disjoint union:

$$(a,b] = \bigcup_{i=1}^{N} (a_i, b_i] \quad \text{where } a = a_1 < b_1 = a_2 < b_2 = \ldots < b_N = b.$$

Then we have

$$\sum_{i=1}^{N} \mu_0((a_i,b_i]) = \sum_{i=1}^{N} (F(b_i) - F(a_i)) = F(b) - F(a) = \mu_0((a,b]).$$

This implies finite additivity for all disjoint unions in $\mathcal{E}$.

To prove μ_0 is subadditive, it suffices to prove subadditivity for disjoint unions, since any union in $\mathcal{E}$ can be written $\bigcup_{i=1}^{\infty} B_i = \coprod_{i=1}^{\infty} A_i$ where $A_1 = B_1$ and $A_i = B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j\right)$. Again choose $(a,b] \in \mathcal{E}$ and write $(a,b] = \coprod_{i=1}^{\infty}(a_i,b_i]$. For the moment, assume $a, b \in \mathbb{R}$. Fix $\tilde{a} \in (a,b]$ and for each $i \in \mathbb{N}$, choose $\tilde{b}_i > b_i$. Then

$$[\tilde{a}, b] \subseteq \bigcup_{i=1}^{\infty} (a_i, \tilde{b}_i)$$

is an open cover of a compact set in $\mathbb{R}$. By the Heine-Borel theorem, there is some $N \in \mathbb{N}$ such that

$$(\tilde{a}, b] \subset [\tilde{a}, b] \subseteq \bigcup_{i=1}^{N} (a_i, \tilde{b}_i).$$

By finite subadditivity and monotonicity, we have

$$\begin{aligned} \mu_0((\tilde{a},b]) &\le \sum_{i=1}^{N} \mu_0((a_i,\tilde{b}_i]) \le \sum_{i=1}^{\infty} \mu_0((a_i,\tilde{b}_i]) \\ \implies F(b) - F(a) &\le \sum_{i=1}^{\infty} (F(\tilde{b}_i) - F(a_i)). \end{aligned}$$

Let $\tilde{a}$ decrease to a. Then by right continuity of F, we get

$$F(b) - F(a) \le \sum_{i=1}^{\infty} (F(\tilde{b}_i) - F(b_i)) + \sum_{i=1}^{\infty} (F(b_i) - F(a_i)).$$

Finally, letting the sequence b_i decrease to b, right continuity of F once more gives us

$$\mu_0((a,b]) = F(b) - F(a) \le \sum_{i=1}^{\infty} (F(b_i) - F(a_i)) = \sum_{i=1}^{\infty} \mu_0((a_i,b_i]).$$

This establishes subadditivity in the finite case; the infinite cases are proven similarly. Therefore by Theorem 17.2.17, there is a unique measure extending μ_0 to $\mathcal{B} = \sigma(\mathcal{E})$. $\square$

Definition. *Given a nondecreasing, right continuous function $F : \mathbb{R} \to \mathbb{R}$, the unique measure $\mu : \mathcal{B} \to [0, \infty]$ satisfying $\mu((a, b]) = F(b) - F(a)$ for all $-\infty \le a < b \le \infty$ is called the* **Lebesgue-Stieltjes measure** *associated to F.*

Take a Lebesgue-Stieltjes measure μ with distribution function F. The *completion* of $\mathcal{B}$ with respect to μ is $\overline{\mathcal{B}}_\mu := \sigma(\mathcal{B} \cup \mathcal{N})$, where $\mathcal{N} = \{F \subset X \mid F \subset N \in \mathcal{B}, \mu(N) = 0\}$. We extend μ to $\overline{\mathcal{B}}_\mu$ by

$$\bar{\mu}(A \cup F) = \mu(A) \text{ for all } A \in \mathcal{B}, F \in \mathcal{N}.$$

As the name suggests, $(\mathbb{R}, \overline{\mathcal{B}}_\mu, \bar{\mu})$ is a complete measure space.

Lemma 17.3.3. *For every set $E \in \overline{\mathcal{B}}_\mu$,*

$$\bar{\mu}(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j)) : E \subset \bigcup_{j=1}^{\infty} (a_j, b_j) \right\}.$$

Theorem 17.3.4. *For every set $E \in \overline{\mathcal{B}}_\mu$,*

$$\bar{\mu}(E) = \inf\{\mu(U) : E \subset U, U \text{ is open in } X\} = \sup\{\mu(K) : K \subset E, K \text{ is compact in } X\}.$$

In other words, Theorem 17.3.4 says that μ extends to $\overline{\mathcal{B}}_\mu$ as both an inner and an outer measure.

Let G_δ denote the collection of countable intersections of closed sets in $\mathcal{B}$, and let F_σ denote the collection of countable unions of open sets in $\mathcal{B}$. The next result gives a characterization of $\bar{\mu}$-measurable sets where μ is some Lebesgue-Stieltjes measure on $\mathbb{R}$.

Theorem 17.3.5. *Let μ be a Lebesgue-Stieltjes measure on $\mathbb{R}$. For any $E \subseteq \mathbb{R}$, the following are equivalent:*

(1) $E \in \overline{\mathcal{B}}_\mu$.

(2) $E = V \setminus N_1$ for some G_δ set V and $N_1 \in \mathcal{B}$ such that $\mu(N_1) = 0$.

(3) $E = H \cup N_2$ for some F_σ set H and $N_2 \in \mathcal{B}$ such that $\mu(N_2) = 0$.

Definition. *When $F(x) = x$, the Lebesgue-Stieltjes measure $\lambda := \mu_F$ is called the* **Lebesgue measure** *on $\mathbb{R}$.*

Notice that $\lambda((a, b]) = b - a$ for all $a, b \in \mathbb{R}$, so Lebesgue measure is precisely the 'length' measure we have been after all along. The following proposition states that all of the properties of such a length function that we expect hold for Lebesgue measure.

Proposition 17.3.6. *Let λ be the Lebesgue measure on $\mathbb{R}$. Then*

(1) For all $b \in \mathbb{R}$, $\lambda(\{b\}) = 0$.

(2) For all $a, b \in \mathbb{R}$,

$$\lambda((a, b)) = \lambda([a, b)) = \lambda([a, b]) = \lambda((a, b]) = b - a.$$

(3) λ is invariant under translations: if $a \in \mathbb{R}$ then for all $B \in \mathcal{B}$, $\lambda(a+B) = \lambda(B)$.

(4) λ is invariant under dilations: if $c \in \mathbb{R}$ then for all $B \in \mathcal{B}$, $\lambda(cB) = |c|\lambda(B)$

Proof. (1) Write $\{b\} = \bigcap_{n=1}^{\infty} \left(b - \frac{1}{n}, b\right]$. Then for each $n \in \mathbb{N}$, $\{b\} \subset \left(b - \frac{1}{n}, b\right]$ and $\lambda\left(b - \frac{1}{n}, b\right] = \frac{1}{n}$. So $\lambda(\{b\}) < \frac{1}{n}$ for all $n \in \mathbb{N}$, and thus $\lambda(\{b\}) = 0$.

(2) follows from (1) and additivity, since each of the different intervals can be written as a disjoint union of (a, b) with one or both of its endpoints.

(3) It's easy to check that the function $\lambda_a(B) = \lambda(a+B)$ is a measure on B. Then

$$\lambda_a((c,d]) = \lambda(a + (c,d]) = \lambda((a+c, a+d]) = (a+d) - (a+c) = d - c = \lambda((c,d]).$$

By uniqueness, we have $\lambda_a = \lambda$ on all of $\mathcal{B}$.

(4) Similarly, define $\lambda_c(B) = \frac{1}{|c|}\lambda(cB)$ for all $B \in \mathcal{B}$. One can show λ_c is a measure on $\mathcal{B}$ that satisfies $\lambda_c((a,b]) = b - a$ so uniqueness again gives the result. □

Example 17.3.7. We can generalize Lebesgue measure to $\mathbb{R}^2$, and even $\mathbb{R}^n$, as we will see in later sections. If $(\mathbf{a}, \mathbf{b}] = (a_1, b_1] \times (a_2, b_2]$ is a half-open rectangle in $\mathbb{R}^2$, define the distribution function $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(\mathbf{x}) = F(x, y) = xy$. Then the Lebesgue measure on $\mathbb{R}^2$ is defined as

$$\lambda((\mathbf{a}, \mathbf{b}]) = \lambda((a_1, b_1] \times (a_2, b_2]) = (a_2 - a_1)(b_2 - b_1).$$

This captures the notion of area on $\mathbb{R}^2$, as we would expect. Generalizing this to $\mathbb{R}^n$ for $n \geq 3$ is done in an analogous fashion.

Example 17.3.8. Let $D \subseteq \mathbb{R}$ be any set and define a measure $\mu_D : \mathcal{B} \to [0, \infty]$ by $\mu_D(A) = \#(A \cap D)$, called the *counting measure* with respect to D. We can alternatively express this as $\mu_D = \sum_{x \in D} \delta_x$. Consider the counting measures μ_{D_1} and μ_{D_2} defined from $D_1 = \mathbb{Q}$ and $D_2 = \left\{\frac{m}{2^n} : m \text{ is odd and } n \in \mathbb{N}\right\}$. Note that both D_1 and D_2 are dense in $\mathbb{R}$ and $\mu_{D_1}((a,b]) = \infty = \mu_{D_2}((a,b])$ for all distinct $a, b \in \mathbb{R}$, but $\mu_{D_1} \neq \mu_{D_2}$. Although μ_{D_1} and μ_{D_2} are both σ-finite on $\mathcal{B}$, they are not σ-finite on the algebra generated by $\mathcal{E} = \{(a,b] : a < b\}$. This is a counterexample to the extension theorem when the condition of σ-finiteness is removed.

17.4 Measurable Functions

First we recall the following facts from the theory of metric spaces.

Definition. *If $\rho : X \times X \to [0, \infty)$ is a metric on X, a set $V \subseteq X$ is* **open** *with respect to ρ if for every $x \in V$, there is some $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq V$.*

Lemma 17.4.1. *The collection $\mathcal{T}_\rho = \{V \subseteq X \mid V \text{ is open with respect to } \rho\}$ is a topology on X, and is equal to the topology generated by $\{B(x, \varepsilon) \mid x \in X, \varepsilon > 0\}$.*

Proposition 17.4.2. *Let X be a space.*

(1) If $\mathcal{E} \subseteq \mathbb{P}(X)$ is countable, then $\sigma(\mathcal{E}) = \sigma(\mathcal{T}(\mathcal{E}))$.

(2) If (X, ρ) is a metric space containing a countable dense subset $D \subset X$, then

$$\mathcal{B}_X = \sigma(\mathcal{T}_\rho) = \sigma(\{B(x, \varepsilon) \mid x \in X, \varepsilon > 0\}).$$

Proof. (1) Clearly $\mathcal{E} \subset \mathcal{T}(\mathcal{E})$ so $\sigma(\mathcal{E}) \subseteq \sigma(\mathcal{T}(\mathcal{E}))$. For the other containment, consider a set $A \in \mathcal{T}(\mathcal{E})$. Then $A = \bigcup_{\alpha \in I} \bigcap_{E \in \mathbb{F}_\alpha} E$ where I is an arbitrary index set and $\mathbb{F}_\alpha \subset \mathcal{E}$ is finite. Since $\mathcal{E}$ is countable, any such index set I may be chosen to be countable. Therefore $A \in \sigma(\mathcal{E})$ so $\sigma(\mathcal{E}) = \sigma(\mathcal{T}(\mathcal{E}))$.

(2) Note that $\mathcal{T}_\rho = \mathcal{T}\left(\left\{B\left(x, \frac{1}{n}\right) : x \in D, n \in \mathbb{N}\right\}\right)$. Applying (1) gives the result. $\square$

Lemma 17.4.3. *For metric spaces (X, ρ) and (Y, d), and any function $f : X \to Y$, the following are equivalent:*

(1) f is continuous: for all $\varepsilon > 0$, there is a $\delta > 0$ such that if $\rho(x_1, x_2) < \delta$, then $d(f(x_1), f(x_2)) < \varepsilon$.

(2) $f^{-1}(V)$ is open in X for all open subsets $V \subset Y$.

(3) $f^{-1}(C)$ is closed in X for all closed subsets $C \subset Y$.

(4) $\lim_{n \to \infty} f(x_n) = f(x)$ for all convergent sequences $(x_n) \to x \in X$.

For any $\mathcal{E} \subseteq \mathbb{P}(X)$ and $\mathbb{F} \subseteq \mathbb{P}(Y)$ and a function $f : X \to Y$, we write $f^{-1}(\mathbb{F}) = \{f^{-1}(V) : V \in \mathbb{F}\}$ to denote the *pullback* of $\mathbb{F}$ under f and $f_*(\mathcal{E}) = \{A \subset Y : f^{-1}(A) \in \mathcal{E}\}$ to denote the *pushforward* of $\mathcal{E}$ under f. The next four results are usually proven in a general topology course. We restate them here in order to compare to similar statements about measurable spaces to follow.

Lemma 17.4.4. *Let $f : X \to Y$ be a function between spaces.*

(1) If $\mathbb{F}$ is a topology on Y then $f^{-1}(\mathbb{F})$ is a topology on X.

(2) If $\mathcal{T}$ is a topology on X then $f_*(\mathcal{T})$ is a topology on Y.

If $\mathbb{F}$ is a topology on Y, then $\mathcal{T} = f^{-1}(\mathbb{F})$ is the smallest topology on X such that f is continuous with respect to $\mathcal{T}$ and $\mathbb{F}$. Similarly, if $\mathcal{T}$ is a topology on X, then $\mathbb{F} = f_*(\mathcal{T})$ is the largest topology on Y such that f is continuous with respect to $\mathcal{T}$ and $\mathbb{F}$.

Proposition 17.4.5. *Let $\mathcal{E} \subset \mathbb{P}(Y)$ and let $f : X \to Y$ be any function. Then $\mathcal{T}(f^{-1}(\mathcal{E})) = f^{-1}(\mathcal{T}(\mathcal{E}))$.*

Corollary 17.4.6. *If $f : (X, \mathcal{T}) \to (Y, \mathbb{F})$ is a function between topological spaces and $\mathcal{E} \subseteq \mathbb{P}(X)$ is a basis for $\mathbb{F}$, then f is continuous if and only if $\mathcal{T}(f^{-1}(\mathcal{E})) \subset \mathcal{T}$.*

Lemma 17.4.7. *If $(X, \mathcal{T}) \xrightarrow{f} (Y, \mathbb{F}) \xrightarrow{g} (Z, \mathcal{G})$ are continuous functions between topological spaces, then $g \circ f : (X, \mathcal{T}) \to (Z, \mathcal{G})$ is also continuous.*

It is useful to think of a topology not just as an abstract collection of sets, but as a system on which we can study continuous functions between spaces. Analogously, σ-algebras are the system on which we can study *measurable functions*, which we define next.

Definition. *Let $(X, \mathcal{M})$ and $(Y, \mathbb{F})$ be measurable spaces and let $f : X \to Y$ be a function. We say f is* **$(\mathcal{M}, \mathbb{F})$-measurable** *if $f^{-1}(V) \in \mathcal{M}$ for every $V \in \mathbb{F}$.*

We now prove analogs to Lemma 17.4.4, Proposition 17.4.5, Corollary 17.4.6 and Lemma 17.4.7 for measurable functions.

Lemma 17.4.8. *Let $f : X \to Y$ be a function between spaces.*

(1) If $\mathbb{F}$ is a σ-algebra on Y then $f^{-1}(\mathbb{F})$ is a σ-algebra on X.

(2) If $\mathcal{M}$ is a σ-algebra on X then $f_*(\mathcal{M})$ is a σ-algebra on Y.

Proof. Exercise. □

As in the topological case, if $\mathbb{F}$ is a σ-algebra on Y, then $\mathcal{M} = f^{-1}(\mathbb{F})$ is the smallest σ-algebra on X such that f is measurable with respect to $\mathcal{M}$ and $\mathbb{F}$. Similarly, if $\mathcal{M}$ is a σ-algebra on X, then $\mathbb{F} = f_*(\mathcal{M})$ is the largest σ-algebra on Y such that f is measurable with respect to $\mathcal{M}$ and $\mathbb{F}$.

Proposition 17.4.9. *Let $\mathcal{E} \subset \mathbb{P}(Y)$ and let $f : X \to Y$ be any function. Then $\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E}))$.*

Proof. Clearly $f^{-1}(\mathcal{E}) \subseteq f^{-1}(\sigma(\mathcal{E}))$ so $\sigma(f^{-1}(\mathcal{E})) \subseteq f^{-1}(\sigma(\mathcal{E}))$. Conversely, consider the pushforward $\mathcal{M} := f_*(\sigma(f^{-1}(\mathcal{E}))) = \{A \subseteq Y : f^{-1}(A) \in \sigma(f^{-1}(\mathcal{E}))\}$. Clearly $\mathcal{E} \subseteq \mathcal{M}$ so $\sigma(\mathcal{E}) \subseteq \mathcal{M}$. That is, if $A \in \sigma(\mathcal{E})$ then $f^{-1}(A) \in \sigma(f^{-1}(\mathcal{E}))$. This proves $f^{-1}(\sigma(\mathcal{E})) \subseteq \sigma(f^{-1}(\mathcal{E}))$, so we're done. □

Corollary 17.4.10. *If $f : (X, \mathcal{M}) \to (Y, \mathbb{F})$ is a function between measurable spaces and $\mathcal{E} \subseteq \mathbb{P}(X)$ such that $\mathcal{M} = \sigma(\mathcal{E})$, then f is measurable if and only if $\sigma(f^{-1}(\mathcal{E})) \subset \mathcal{M}$.*

Proof. Apply Proposition 17.4.9. □

Lemma 17.4.11. *If $(X, \mathcal{M}) \xrightarrow{f} (Y, \mathbb{F}) \xrightarrow{g} (Z, \mathcal{G})$ are measurable functions between measurable spaces, then $g \circ f : (X, \mathcal{M}) \to (Z, \mathcal{G})$ is also measurable.*

Proof. This follows from the fact that $(g \circ f)^{-1}(\mathcal{G}) = f^{-1}(g^{-1}(\mathcal{G})) \subseteq f^{-1}(\mathbb{F}) \subseteq \mathcal{M}$. □

Definition. *If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, a function $f : X \to Y$ is* **Borel measurable** *if $f^{-1}(\mathcal{B}_Y) = \mathcal{B}_X$.*

Example 17.4.12. A function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable (we usually shorten this to measurable for functions $\mathbb{R} \to \mathbb{R}$) if $f^{-1}(\mathcal{B}) \subseteq \mathcal{B}$. In addition, we call $f : \mathbb{R} \to \mathbb{R}$ *Lebesgue measurable* if $f^{-1}(\mathcal{B}) \subseteq \mathcal{L} := \overline{\mathcal{B}}_\lambda$, where λ is the Lebesgue measure on $\mathbb{R}$ and $\mathcal{L}$ is the completion of $\mathcal{B}$ with respect to λ. Notice that Borel measurability implies Lebesgue measurability, but the converse is false. In particular, if $f, g : \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable functions, it is not guaranteed that their composition $g \circ f$ is Lebesgue measurable.

Lemma 17.4.13. *Every continuous function $f : X \to Y$ between topological spaces is Borel measurable with respect to $\mathcal{B}_X$ and $\mathcal{B}_Y$.*

Proof. By Proposition 17.4.9, $f^{-1}(\mathcal{B}_Y) = f^{-1}(\sigma(\mathcal{T}_Y)) = \sigma(f^{-1}(\mathcal{T}_Y)) \subseteq \sigma(\mathcal{T}_X)$. □

Definition. *The* **Borel σ-algebra on the extended reals** *$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is $\overline{\mathcal{B}} := \sigma(\{[a, \infty] : a \in \overline{\mathbb{R}}\})$.*

It is useful to write $\overline{\mathcal{B}}$ in a couple different ways:

$$\mathcal{B} = \{A \subseteq \overline{\mathbb{R}} : A \cap \mathbb{R} \in \mathcal{B}\} = \{B, B \cup \{\infty\}, B \cup \{-\infty\}, B \cup \{\pm\infty\} : B \in \mathcal{B}\}.$$

Proposition 17.4.14. *Let $f_n : (X, \mathcal{M}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be a sequence of measurable functions. Then*

(1) $\sup_n\{f_n(x)\}$ *is measurable.*

(2) $\inf_n\{f_n(x)\}$ *is measurable.*

(3) $\limsup f_n(x)$ *and* $\liminf f_n(x)$ *are measurable.*

Proof. (1) Set $f_+(x) = \sup\{f_n(x)\}$. Then

$$\begin{aligned} f_+^{-1}([-\infty, b]) &= \{x \mid f_+(x) \le b\} = \{x \mid f_n(x) \le b \text{ for all } n\} \\ &= \bigcap_{n=1}^{\infty} \{x \mid f_n(x) \le b\} \\ &= \bigcap_{n=1}^{\infty} f_n^{-1}([-\infty, b]) \end{aligned}$$

which lies in $\mathcal{M}$. Since the sets $[-\infty, b]$ for $b \in \overline{\mathbb{R}}$ generate $\overline{\mathcal{B}}$, we're done.

(2) is similar to (1), using the sets $[a, \infty]$ to generate $\overline{\mathcal{B}}$.

(3) Write the limsup and liminf as $\limsup f_n(x) = \inf_n\{\sup f_k(x) \mid k \ge n\}$ and $\liminf f_n(x) = \sup_n\{\inf f_k(x) \mid k \ge n\}$. Then measurability of these functions follows from (1) and (2). □

Lemma 17.4.15. *If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.*

Proof. $\mathcal{B}$ is generated by the sets (a,∞), and $f^{-1}((a,\infty)) = [b,\infty)$ or (b,∞) for some $b \in \mathbb{R}$. $\square$

Definition. *Let X be a space and A an index set such that for all $\alpha \in A$, $(Y_\alpha, \mathbb{F}_\alpha)$ are measurable (resp. topological) spaces and there are given functions $f_\alpha : X \to Y_\alpha$. Then*

$$\sigma(f_\alpha \mid \alpha \in A) := \sigma\left(\bigcup_{\alpha\in A} f_\alpha^{-1}(\mathbb{F}_\alpha)\right) \quad \textit{and} \quad \mathcal{T}(f_\alpha \mid \alpha \in A) := \mathcal{T}\left(\bigcup_{\alpha\in A} f_\alpha^{-1}(\mathbb{F}_\alpha)\right)$$

*are the σ-**algebra** (resp. **topology**) **generated by the** f_α. Each is equal to the smallest σ-algebra (resp. topology) on X with respect to which each f_α is measurable (resp. continuous).*

Proposition 17.4.16. *Let $f_\alpha : X \to (Y, \mathbb{F}_\alpha)$ be functions. A function $g : (Z, \mathcal{M}) \to X$ between measurable spaces is measurable with respect to $\mathcal{M}$ and $\sigma(f_\alpha \mid \alpha \in A)$ if and only if $f_\alpha \circ g$ is $(\mathcal{M}, \mathbb{F}_\alpha)$-measurable for each $\alpha \in A$. Similarly, g is continuous on $\mathcal{M}$ and $\mathcal{T}(f_\alpha \mid \alpha \in A)$ if and only if $f_\alpha \circ g$ is continuous on $\mathcal{M}$ and $\mathbb{F}_\alpha$ for each $\alpha \in A$.*

The most important case of the above situation is when $X = \prod_{\alpha\in A} Y_\alpha$ and $f_\alpha = \pi_\alpha : X \to Y_\alpha$ is the canonical projection for each $\alpha \in A$.

Definition. *For $X = Y_1 \times Y_2$, with $(Y_1, \mathcal{M}_1)$ and $(Y_2, \mathcal{M}_2)$ measurable spaces and $\pi_1 : X \to Y_1$ and $\pi_2 : X \to Y_2$ the canonical projections, the* **product σ-algebra** *on X is $\mathcal{M}_1 \otimes \mathcal{M}_2 := \sigma(\pi_1, \pi_2)$. Similarly, if $\mathcal{T}_i$ is a topology on $Y_i, i = 1, 2$, the* **product topology** *on X is $\mathcal{T}_1 \otimes \mathcal{T}_2 := \mathcal{T}(\pi_1, \pi_2)$.*

Example 17.4.17. If $X = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, the Borel σ-algebra on $\mathbb{R}^2$ is equal to $\mathcal{B} \otimes \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. In general, $\mathcal{B}^n := \mathcal{B}_{\mathbb{R}^n} = \underbrace{\mathcal{B} \otimes \cdots \otimes \mathcal{B}}_{n} = \mathcal{B}^{\otimes n}$.

Proposition 17.4.18. *Suppose $f, g : (X, \mathcal{M}) \to (\mathbb{R}, \mathcal{B})$ are measurable functions. Then*

(1) $fg : X \to \mathbb{R}$, defined by $(fg)(x) = f(x)g(x)$ for all $x \in X$, is measurable.

(2) $f \pm g : X \to \mathbb{R}$, defined by $(f \pm g)(x) = f(x) \pm g(x)$ for all $x \in X$, are measurable.

(3) If $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g} : X \to \mathbb{R}$, defined by $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$ for all $x \in X$, is measurable.

Proof. Define $F : X \to \mathbb{R}^2$ by $F(x) = (f(x), g(x))$. Then F is measurable since it's measurable componentwise, by Proposition 17.4.16. The functions $\mathbb{R}^2 \to \mathbb{R}$ given by $(a, b) \mapsto ab, a \pm b$ or $\frac{a}{b}$ are continuous and therefore measurable by Lemma 17.4.13, so their compositions with F are measurable. These compositions are precisely $fg, f \pm g$ and, when $g \neq 0$, $\frac{f}{g}$. $\square$

For $i = 1, 2$, let X_i be a space, $\mathcal{E}_i \subseteq \mathbb{P}(X_i)$ a collection of subsets such that $X_i \in \mathcal{E}_i$, $\mathcal{T}_i = \mathcal{T}(\mathcal{E}_i)$ and $\mathcal{M}_i = \sigma(\mathcal{E}_i)$. Also let $\pi_i : X_1 \times X_2 \to X_i$ be the canonical coordinate projections. Then we have a few ways of writing the product σ-algebra and topology on $X_1 \times X_2$:

$$\begin{aligned}
\mathcal{M}_1 \otimes \mathcal{M}_2 &= \sigma(\pi_1, \pi_2) \\
&= \sigma(\{A \times X_2 \mid A \in \mathcal{M}_1\} \cup \{X_1 \times B \mid B \in \mathcal{M}_2\}) \\
&= \sigma(\{A \times B \mid A \in \mathcal{M}_1, B \in \mathcal{M}_2\}); \\
\text{and} \quad \mathcal{T}_1 \otimes \mathcal{T}_2 &= \mathcal{T}(\pi_1, \pi_2) \\
&= \mathcal{T}(\{A \times X_2 \mid A \in \mathcal{T}_1\} \cup \{X_1 \times B \mid B \in \mathcal{T}_2\}) \\
&= \mathcal{T}(\{A \times B \mid A \in \mathcal{T}_1, B \in \mathcal{T}_2\}).
\end{aligned}$$

Each description is useful in certain scenarios, so it is good to state them now.

Theorem 17.4.19. *For X_1 and X_2 with the setup above,*

(1) $\mathcal{M}_1 \otimes \mathcal{M}_2 = \sigma(\mathcal{E}_1 \times \mathcal{E}_2)$.

(2) $\mathcal{T}_1 \otimes \mathcal{T}_2 = \mathcal{T}(\mathcal{E}_1 \times \mathcal{E}_2)$.

(3) If $\mathcal{E}_1$ and $\mathcal{E}_2$ are countable, then $\sigma(\mathcal{E}_i) = \sigma(\mathcal{T}_i)$ for each $i = 1, 2$. In particular, $\mathcal{B}_{X_1 \times X_2} = \sigma(\mathcal{T}_1 \otimes \mathcal{T}_2) = \mathcal{B}_1 \otimes \mathcal{B}_2$.

Proof. (1) On one hand, $\mathcal{E}_1 \times \mathcal{E}_2 \subseteq \{A \times B \mid A \in \mathcal{M}_1, B \in \mathcal{M}_2\}$ so by the preceding remarks, $\sigma(\mathcal{E}_1 \times \mathcal{E}_2) \subseteq \sigma(\mathcal{M}_1 \times \mathcal{M}_2) = \mathcal{M}_1 \otimes \mathcal{M}_2$. To show the other containment, it suffices to show that all sets of the form $A \times X_2$ and $X_1 \times B$ are in $\sigma(\mathcal{E}_1 \times \mathcal{E}_2)$. For any $A \in \mathcal{M}_1$, $A \times X_2 \in \sigma(\mathcal{E}_1 \times X_2) \subseteq \sigma(\mathcal{E}_1 \times \mathcal{E}_2)$ since $X_2 \in \mathcal{E}_2$. Likewise, for any $B \in \mathcal{M}_2$, $X_1 \times B \in \sigma(X_1 \times \mathcal{E}_2) \subseteq \sigma(\mathcal{E}_1 \times \mathcal{E}_2)$. Hence $\sigma(\mathcal{E}_1 \times \mathcal{E}_2) = \mathcal{M}_1 \otimes \mathcal{M}_2$.

(2) is proven similarly.

(3) Clearly $\sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{T}_1)$ and $\sigma(\mathcal{E}_2) \subseteq \sigma(\mathcal{T}_2)$ since $\mathcal{T}_i = \mathcal{T}(\mathcal{E}_i)$ for each $i = 1, 2$. □

Example 17.4.20. The Borel σ-algebra on $\mathbb{R}$ can be generated by the collection of intervals with *rational* endpoints, which is a countable collection. By (3) of Theorem 17.4.19, the Borel σ-algebra $\mathcal{B}^n$ on $\mathbb{R}^n$ can be generated by the (countable) collection of n-hyperrectangles with rational coordinates, or alternatively by the open n-balls with rational center and radius.

Corollary 17.4.21. *For any $m, n \geq 1$, $\mathcal{B}^{m+n} = \mathcal{B}^m \otimes \mathcal{B}^n$.*

Chapter 18

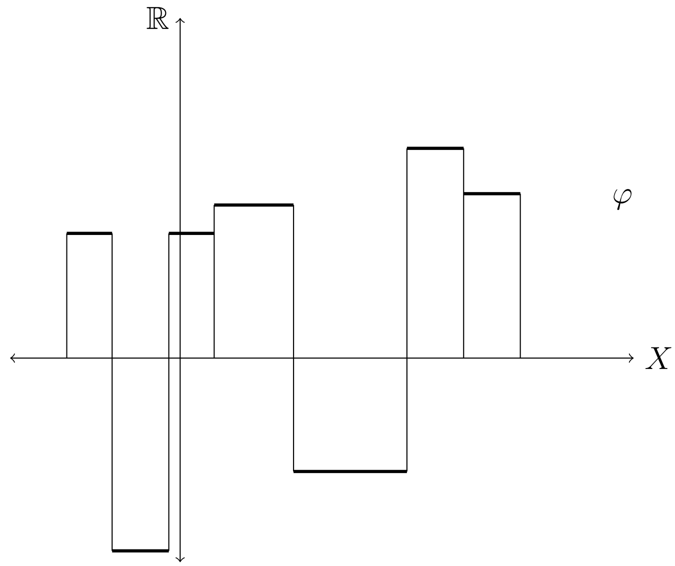
Integration Theory

The main topic in measure theory is integration, which both generalizes and vastly improves upon classical Riemann integration on $\mathbb{R}^n$. We define the integral with respect to a measure and prove the most important theorems: Monotone Convergence, Fatou's Lemma, Dominated Convergence, Tonelli's and Fubini's theorems and the change-of-variables formula. We will prove that Lebesgue integration properly captures the Riemann integral.

18.1 Lebesgue Integration

Let $(X, \mathcal{M}, \mu)$ be a measure space and consider a function $f : X \to \mathbb{C}$. We want to be able to integrate f by mimicking the approximation method used to construct the Riemann integral, but this may not always be possible if f is not reasonably 'nice'. We start by defining a class of functions that *are* nice.

Definition. *A function $\varphi : X \to \mathbb{C}$ is* **simple** *if φ is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$-measurable and the set $\{\varphi(x) \mid x \in X\}$ is finite, that is, φ has finite range.*



We will assume all functions in this section are $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$-measurable. Consider a simple function $\varphi : X \to \mathbb{C}$. Then for each $a \in \mathbb{C}$, $\varphi^{-1}(\{a\}) \in \mathcal{M}$.

Definition. *For a set $A \subset X$, the* **characteristic function** *of A is*

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

Clearly χ_A is measurable if and only if $A \in \mathcal{M}$. Notice that if $A \in \mathcal{M}$, then χ_A is a simple function. If $\varphi : X \to \mathbb{C}$ is any simple function, then φ can be represented as a sum of characteristic functions:

$$\varphi(x) = \sum_{a \in \mathbb{C}} a \chi_{\varphi^{-1}(\{a\})}.$$

This sum is finite since $\chi_{\varphi^{-1}(\{a\})} \neq 0$ for only finitely many $a \in \mathbb{R}$.

We now proceed to define integration on simple functions, then use this to construct Lebesgue integrals for nonnegative functions and finally for all measurable functions.

Definition. *The* **Lebesgue integral of a simple function** *$\varphi : X \to \mathbb{R}$ is*

$$\int_X \varphi \, d\mu = \sum_{a \in \mathbb{R}} a \mu(\varphi^{-1}(\{a\})).$$

Definition. *If $f : X \to [0, \infty]$ is a nonnegative measurable function on a measure space $(X, \mathcal{M}, \mu)$, the* **Lebesgue integral** *of f over X is*

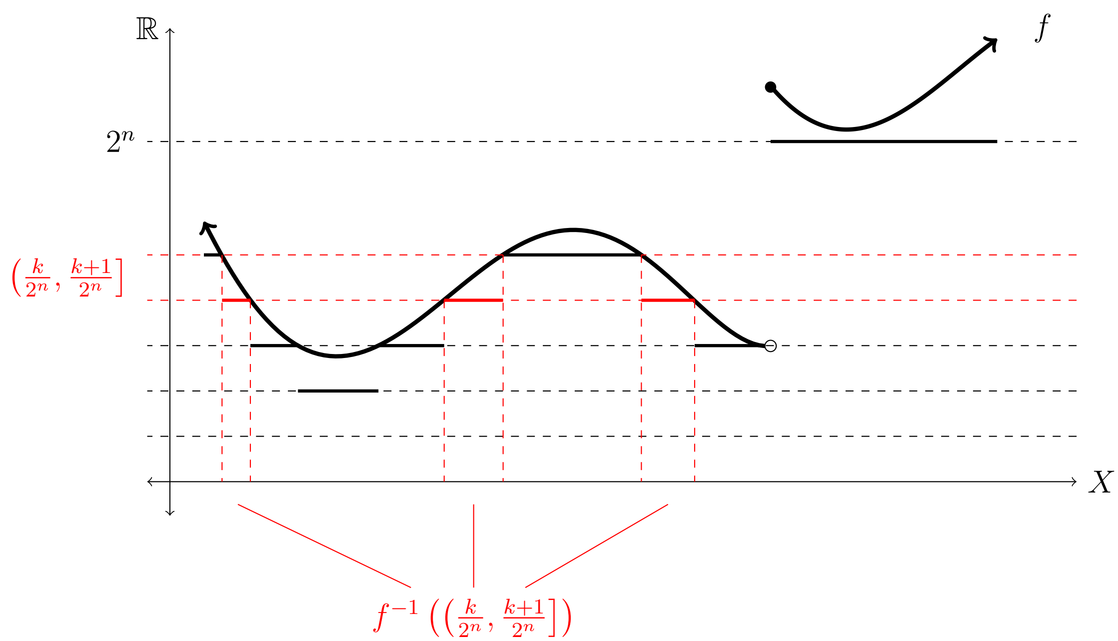
$$\int_X f\, d\mu = \sup\left\{\int_X \varphi\, d\mu : \varphi \text{ is simple, } 0 \le \varphi(x) \le f(x) \text{ for all } x \in X\right\}.$$

We immediately observe that $\int_X f\, d\mu$ exists (although it may be infinite) for all measurable functions $f : X \to [0, \infty]$, since for every simple function $\varphi : X \to [0, \infty]$, $\int_X \varphi\, d\mu$ is defined and the set of all integrals for $0 \le \varphi \le f$ is bounded.

Lemma 18.1.1. *Let $f : X \to [0, \infty]$ be a measurable function on X. Then there exists a sequence of simple functions $\varphi_n : X \to [0, \infty)$ such that φ_n converge pointwise from below to f as $n \to \infty$.*

Proof. (Sketch) For each $n \ge 1$, define the function φ_n by

$$\varphi_n = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \chi_{f^{-1}\left(\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right)} + 2^n \chi_{f^{-1}((2^n, \infty])}.$$



Clearly each φ_n is simple – they are finite sums of characteristic functions. By construction $\varphi_n \le f$ pointwise and φ_n is a monotonically increasing sequence, so it converges. Further, one can show that the φ_n converge pointwise to f. □

Definition. *If $A \subseteq X$ is a subset, we define* **integration over** *A for simple functions φ by*

$$\int_A \varphi\, d\mu = \int_X \chi_A \varphi\, d\mu$$

and extend to nonnegative functions by taking suprema.

It is clear that for any simple function φ, $\chi_A\varphi$ is still a simple function, so the integral over A is well-defined.

In the next proposition, we collect some basic properties of the Lebesgue integral for simple functions.

Proposition 18.1.2. *Let $\varphi, \psi : X \to [0,\infty]$ be simple functions. Then*

(1) For any real number $\lambda \geq 0$, $\displaystyle\int_X \lambda\varphi\,d\mu = \lambda\int_X \varphi\,d\mu$.

(2) $\displaystyle\int_X (\varphi+\psi)\,d\mu = \int_X \varphi\,d\mu + \int_X \psi\,d\mu$.

(3) (Monotonicity) If $\varphi \leq \psi$ pointwise then $\displaystyle\int_X \varphi\,d\mu \leq \int_X \psi\,d\mu$.

(4) Define $\nu(A) = \displaystyle\int_A \varphi\,d\mu$ for each set $A \in \mathcal{M}$. Then ν is a measure on X.

Proof. (1) First if $\lambda = 0$ then $0\varphi = 0$ which is a simple function. By definition of the Lebesgue integral for simple functions, we have $\int_X 0\,d\mu = 0$, so the statement holds. Now assume $\lambda > 0$. Then we have

$$\begin{aligned}\int_X \lambda\varphi\,d\mu &= \sum_{a\in\mathbb{R}} a\mu((\lambda\varphi)^{-1}(\{a\})) = \sum_{a\in\mathbb{R}} a\mu\left(\varphi^{-1}\left(\left\{\tfrac{a}{\lambda}\right\}\right)\right) = \sum_{b\in\mathbb{R}} \lambda b\mu(\varphi^{-1}(\{b\})) \\ &= \lambda\sum_{b\in\mathbb{R}} b\mu(\varphi^{-1}(\{b\})) = \lambda\int_X \varphi\,d\mu.\end{aligned}$$

(2) The sum of simple functions is clearly still a simple function, so we have

$$\int_X (\varphi+\psi)\,d\mu = \sum_{a\in\mathbb{R}} a\mu((\varphi+\psi)^{-1}(\{a\})).$$

The preimage can be written

$$(\varphi+\psi)^{-1}(\{a\}) = \{x \in X \mid \varphi(x)+\psi(x) = a\} = \bigcup_{b\geq 0}[\varphi^{-1}(\{b\}) \cap \psi^{-1}(\{a-b\})].$$

This union is disjoint and finite since φ is simple. Therefore we can write

$$\begin{aligned}\int_X (\varphi+\psi)\,d\mu &= \sum_{a\in\mathbb{R}} a\sum_{b\in\mathbb{R}} \mu(\varphi^{-1}(\{b\}) \cap \psi^{-1}(\{a-b\})) \quad \text{by additivity} \\ &= \sum_{b,c\in\mathbb{R}} (b+c)\mu(\varphi^{-1}(\{b\}) \cap \psi^{-1}(\{c\})) \quad \text{letting } c = a-b \\ &= \sum_{b,c\in\mathbb{R}} b\mu(\varphi^{-1}(\{b\}) \cap \psi^{-1}(\{c\})) + \sum_{b,c\in\mathbb{R}} c\mu(\varphi^{-1}(\{b\}) \cap \psi^{-1}(\{c\})).\end{aligned}$$

Consider the first term:

$$\begin{aligned}\sum_{b,c\in\mathbb{R}} b\mu(\varphi^{-1}(\{b\})\cap\psi^{-1}(\{c\})) &= \sum_{b\in\mathbb{R}} b\sum_{c\in\mathbb{R}}\mu(\varphi^{-1}(\{b\})\cap\psi^{-1}(\{c\}))\\ &= \sum_{b\in\mathbb{R}} b\mu(\varphi^{-1}(\{b\})) = \int_X \varphi\,d\mu.\end{aligned}$$

Likewise,

$$\begin{aligned}\sum_{b,c\in\mathbb{R}} c\mu(\varphi^{-1}(\{b\})\cap\psi^{-1}(\{c\})) &= \sum_{c\in\mathbb{R}} c\sum_{b\in\mathbb{R}}\mu(\varphi^{-1}(\{b\})\cap\psi^{-1}(\{c\}))\\ &= \sum_{c\in\mathbb{R}} c\psi^{-1}(\{c\}) = \int_X \psi\,d\mu.\end{aligned}$$

So in the full sum we have $\displaystyle\int_X(\varphi+\psi)\,d\mu = \int_X\varphi\,d\mu + \int_X\psi\,d\mu$.

(3) Suppose $0\le\varphi\le\psi$ pointwise. Then by definition of the Lebesgue integral for simple functions, we have

$$\begin{aligned}\int_X\varphi\,d\mu = \sum_{a\in\mathbb{R}} a\mu(\varphi^{-1}(\{a\})) &= \sum_{a\in\mathbb{R}} a\sum_{b\in\mathbb{R}}\mu(\varphi^{-1}(\{a\})\cap\psi^{-1}(\{b\})) \quad \text{by additivity}\\ &\le \sum_{a\in\mathbb{R}}\sum_{b\in\mathbb{R}} b\mu(\varphi^{-1}(\{a\})\cap\psi^{-1}(\{b\})) \quad \text{since } \varphi\le\psi\\ &= \sum_{b\in\mathbb{R}}\sum_{a\in\mathbb{R}} b\mu(\varphi^{-1}(\{a\})\cap\psi^{-1}(\{b\}))\\ &= \sum_{b\in\mathbb{R}} b\sum_{a\in\mathbb{R}}\mu(\varphi^{-1}(\{a\})\cap\psi^{-1}(\{b\}))\\ &= \sum_{b\in\mathbb{R}} b\mu(\psi^{-1}(\{b\})) = \int_X\psi\,d\mu.\end{aligned}$$

(4) It's enough to check additivity. Let $\{a_1,\ldots,a_N\}$ be the range of f and, for each $1\le i\le N$, write $B_i=\varphi^{-1}(\{a_i\})$. Then $\varphi=\sum_{i=1}^N a_i\chi_{B_i}$ so for any $A\in\mathcal{M}$,

$$\nu(A) = \int_X \chi_A\sum_{i=1}^N a_i\chi_{B_i}\,d\mu = \int_X\sum_{i=1}^N a_i\chi_{A\cap B_i}\,d\mu = \sum_{i=1}^N a_i\mu(A\cap B_i),$$

by (2). Now if $A=\bigcup_{k=1}^\infty A_k$ for disjoint sets $A_k\in\mathcal{M}$, we have

$$\begin{aligned}\nu(A) = \sum_{i=1}^N a_i\mu(A\cap B_i) &= \sum_{i=1}^N a_i\sum_{k=1}^\infty\mu(A_k\cap B_i) \quad \text{by } \sigma\text{-additivity of } \mu\\ &= \sum_{k=1}^\infty\sum_{i=1}^N a_i\mu(A_k\cap B_i) \quad \text{by Tonelli's theorem (16.1.4)}\\ &= \sum_{k=1}^\infty\int_{A_k}\varphi\,d\mu = \sum_{k=1}^\infty\nu(A_k).\end{aligned}$$

Thus ν is a measure. $\square$

Example 18.1.3. Let X be a countable space and set $\mathcal{M} = \mathbb{P}(X)$. For any nonnegative function $\rho : X \to [0, \infty]$, we can define a measure $\mu : \mathcal{M} \to [0, \infty]$ by

$$\mu(A) = \sum_{x \in A} \rho(x) = \sum_X \rho(x)\chi_A(x).$$

Proposition 18.1.4. *Let X be countable and $\mathcal{M} = \mathbb{P}(X)$, and suppose ρ and μ are defined as above. For any measurable function $f : X \to [0, \infty]$,*

$$\int_X f\, d\mu = \sum_X \rho f.$$

Proof. First, if $f(x_0) = \infty$ for some $x_0 \in X$ and $\rho(x_0) > 0$, then $\int_X f\, d\mu = \infty = \sum_X \rho f$. Now assume f is finite anywhere that ρ is positive. For any simple function $\varphi : X \to [0, \infty]$,

$$\begin{aligned}
\int_X \varphi\, d\mu &= \sum_{a \in \mathbb{R}} a\mu(\varphi^{-1}(\{a\})) = \sum_{a \in \mathbb{R}} a \sum_{\substack{x \in X \\ \varphi(x) = a}} \rho(x) \\
&= \sum_{a \in \mathbb{R}} \sum_{x \in X} a\rho(x)\chi_{\varphi^{-1}(\{a\})}(x) \\
&= \sum_{x \in X} \sum_{a \in \mathbb{R}} a\rho(x)\chi_{\varphi^{-1}(\{a\})}(x) \quad \text{by Tonelli's theorem (16.1.4)} \\
&= \sum_X \rho\varphi.
\end{aligned}$$

So the formula holds for simple functions. Now if $0 \le \varphi \le f$, $\int_X \varphi\, d\mu = \sum_X \rho\varphi leq \sum_X \rho f$ so taking the supremum over all such $\varphi \le f$ gives

$$\int_X f\, d\mu = \sup\left\{\int_X \varphi\, d\mu : 0 \le \varphi \le f, \varphi \text{ simple}\right\} \le \sum_X \rho f.$$

On the other hand, let $\Lambda \subset X$ be a finite subset and set $\varphi_\Lambda = f\chi_\Lambda$, which is a simple function. Then

$$\sum_\Lambda \rho f = \sum_X \rho\varphi_\Lambda = \int_X \varphi_\Lambda\, d\mu \le \int_X f\, d\mu.$$

Taking the supremum over all finite Λ yields

$$\sum_X \rho f = \sup_{\substack{\text{finite} \\ \Lambda \subset X}} \rho f \le \int_X f\, d\mu.$$

Hence we have $\int_X f\, d\mu = \sum_X \rho f$. $\square$

We now prove the Monotone Convergence Theorem (MCT) for Lebesgue integrals. This generalizes the analagous theorem for discrete sums (16.1.2).

Theorem 18.1.5 (Monotone Convergence Theorem)**.** *Suppose $f_n : X \to [0, \infty]$ is a sequence of measurable functions such that for all $x \in X$, $\lim_{n\to\infty} f_n(x) = f(x)$ exists and $f_n(x) \leq f_{n+1}(x)$ for all $n \in \mathbb{N}$, that is, $f_n \nearrow f$ pointwise. Then*

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X \lim_{n\to\infty} f_n \, d\mu = \int_X f \, d\mu.$$

Proof. First note that convergence from below implies

$$\int_X f_n \, d\mu \leq \int_X f_{n+1} \, d\mu \leq \int_X f \, d\mu$$

for all $n \in \mathbb{N}$ by Proposition 18.1.2(b), so $L = \lim_{n\to\infty} \int_X f_n \, d\mu$ exists by the classic monotone convergence theorem. Moreover, $L \leq \int_X f \, d\mu$, where both may be infinite. Let φ be a simple function such that $0 \leq \varphi \leq f$ and let $\alpha \in (0, 1)$. Define $E_n = \{x \in X \mid f_n(x) \geq \alpha\varphi(x)\}$. Note that $E_n \nearrow X$ by the hypotheses on f_n and φ. Also, for each $n \in \mathbb{N}$,

$$\int_X f_n \, d\mu \geq \int_{E_n} f_n \, d\mu = \int_X f_n \chi_{E_n} \, d\mu \geq \int_X \alpha\varphi\chi_{E_n} \, d\mu = \alpha \int_{E_n} \varphi \, d\mu$$

by Proposition 18.1.2(b) and (a). Taking $n \to \infty$, the inequality is preserved:

$$L = \lim_{n\to\infty} \int_X f_n \, d\mu \geq \alpha \sup \left\{ \int_X \varphi \, d\mu : 0 \leq \varphi \leq f, \varphi \text{ simple} \right\} = \alpha \int_X f \, d\mu.$$

Finally, taking the supremum over all $\alpha \in (0, 1)$, we have $L \geq \int_X f \, d\mu$. Therefore

$$L = \lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

□

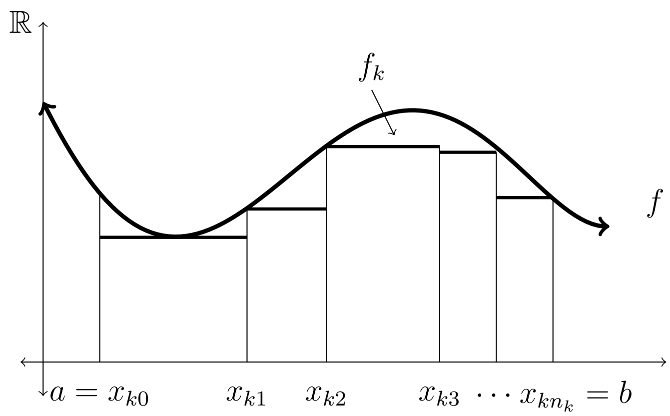
The next proposition says that the Lebesgue integral coincides with the Riemann integral, at least for continuous, nonnegative functions.

Proposition 18.1.6. *Let $f : [a, b] \to [0, \infty]$ be continuous and let λ be the Lebesgue measure on $\mathbb{R}$. Then*

$$\int_{[a,b]} f \, d\lambda = \int_a^b f(x) \, dx.$$

Proof. Choose a sequence of partitions $P_k = \{a = x_{k0} < x_{k1} < \ldots < x_{kn_k} = b\}$ such that $P_k \subseteq P_{k+1}$ for each $k \geq 1$ and the mesh, $\max\{|x_{ki} - x_{k,i+1}| : 0 \leq i \leq n_k - 1\}$ approaches 0 as $k \to \infty$. For each partition P_k, let $f_k : [a, b] \to [0, \infty]$ be defined by

$$f_k(x) = \sum_{i=0}^{n_k - 1} \min\{f(y) \mid y \in [x_{ki}, x_{k,i+1}]\}\chi_{[x_{ki}, x_{k,i+1}]}(x).$$



By construction, $f_k \nearrow f$ pointwise, so by the Monotone Convergence Theorem,

$$\lim_{k\to\infty} \int_{[a,b]} f_k \, d\lambda = \int_{[a,b]} f \, d\lambda.$$

For each k, the integral over $[a, b]$ of f_k is

$$\int_{[a,b]} f_k \, d\lambda = \sum_{j=0}^{n_k-1} \left(\min_{x_j \le x \le x_{j+1}} f(x) \right) (x_{j+1} - x_j)$$

which is clearly a Riemann sum. Therefore

$$\lim_{k\to\infty} \int_{[a,b]} f_k \, d\lambda = \int_a^b f(x) \, dx.$$

Improper integrals are handled in the same way as for Riemann integrals. □

18.2 Properties of Integration

In this section we prove the important general theorems for Lebesgue integration.

Theorem 18.2.1 (Fatou's Lemma)**.** *Suppose $f_n : X \to [0, \infty]$ is a sequence of nonnegative measurable functions. Then*

$$\int_X \liminf f_n \, d\mu \leq \liminf \int_X f_n \, d\mu.$$

Proof. Define $g_k = \inf\{f_n \mid n \geq k\}$ so that the sequence (g_k) increases from below to $\liminf f_n$. In particular, $g_k \leq f_n$ for all $n \geq k$, so

$$\int_X g_k \leq \int_X f_n \text{ for all } n \geq k.$$

By the Monotone Convergence Theorem,

$$\int_X \liminf f_n = \int_X \lim_{k\to\infty} g_k = \int_{k\to\infty} \sum_X g_k \leq \liminf \int_X f_n.$$

□

This generalizes Theorem 16.1.5 to Lebesgue integrals. Next we extend integration to functions $f : X \to \mathbb{R}$ and $X \to \mathbb{C}$.

Definition. *Let $f : X \to \mathbb{R}$ be a real-valued function. If $\int_X |f| \, d\mu < \infty$, we say f is* **integrable** *with respect to μ and define the* **Lebesgue integral** *of f to be*

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu,$$

where $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$. The definition extends to complex-valued functions $g : X \to \mathbb{C}$ by

$$\int_X g \, d\mu = \int_X \operatorname{Re} g \, d\mu + i \int_X \operatorname{im} g \, d\mu.$$

Theorem 18.2.2. *If $f_n : X \to [0, \infty]$ is a sequence of measurable, nonnegative functions, then*

$$\int_X \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

Proof. First consider two functions, $f_1, f_2 : X \to [0, \infty]$. By Lemma 18.1.1, there exist sequences $0 \leq \varphi_k \leq f_1$ and $0 \leq \psi_k \leq f_2$ of simple functions such that $\varphi_k \nearrow f_1$ and $\psi_k \nearrow f_2$. Then $(\varphi_k + \psi_k) \nearrow (f_1 + f_2)$, and by Proposition 18.1.2(b), $\int_X (\varphi_k + \psi_k) \, d\mu = \int_X \varphi_k \, d\mu + \int_X \psi_k \, d\mu$ for all $k \in \mathbb{N}$. Thus by the Monotone Convergence Theorem,

$$\int_X (f_1 + f_2) \, d\mu = \int_X f_1 \, d\mu + \int_X f_2 \, d\mu.$$

By induction, this holds for all finite sums. For each $N \in \mathbb{N}$, set $F_N = \sum_{n=1}^{\infty} f_n$, so that $F_N \nearrow \sum_{n=1}^{\infty} f_n$ as $N \to \infty$. Then by the Monotone Convergence Theorem and the above work, we have

$$\int_X \sum_{n=1}^{\infty} f_n \, d\mu = \int_X \lim_{N\to\infty} F_N = \lim_{N\to\infty} \int_X F_N \, d\mu = \lim_{N\to\infty} \sum_{n=1}^{N} \int_X f_n = \sum_{n=1}^{\infty} \int_X f_n.$$

□

The next proposition extends linearity of the Lebesgue integral from Proposition 18.1.2(a) and (b) to real- and complex-valued functions.

Proposition 18.2.3. *Suppose $f, g : X \to \mathbb{C}$ are integrable functions and $\lambda \in \mathbb{R}$. Then*

$$\int_X (f + \lambda g) \, d\mu = \int_X f \, d\mu + \lambda \int_X g \, d\mu.$$

Proof. Considering the real and imaginary parts of each function separately, we may assume f and g are real-valued. The case when $\lambda = 0$ is the same as in Proposition 18.1.2(a). To obtain the result for any $\lambda \neq 0$, it suffices to consider when $\lambda = 1$. Let $h = f + g$ and write

$$f = f^+ - f^-, \quad g = g^+ - g^-, \quad h = h^+ - h^- = (f^+ + g^+) - (f^- + g^-).$$

Then $h^+ + f^- + g^- = h^- + f^+ + g^+$ so by Proposition 18.1.2(b),

$$\begin{aligned} \int_X h^+ \, d\mu + \int_X f^- \, d\mu + \int_X g^- \, d\mu &= \int_X (h^+ + f^- + g^-) \, d\mu \\ &= \int_X (h^- + f^+ + g^+) \, d\mu = \int_X h^- \, d\mu + \int_X f^+ \, d\mu + \int_X g^+ \, d\mu. \end{aligned}$$

Since each function is integrable, we can subtract to obtain

$$\begin{aligned} \int_X h \, d\mu &= \int_X h^+ \, d\mu - \int_X h^- \, d\mu \\ &= \left(\int_X f^+ \, d\mu - \int_X f^- \, d\mu \right) + \left(\int_X g^+ \, d\mu - \int_X g^- \, d\mu \right) = \int_X f \, d\mu + \int_X g \, d\mu. \end{aligned}$$

□

One of the most useful results in forming integral estimations is Chebyshev's lemma.

Theorem 18.2.4 (Chebyshev's Lemma)**.** *Suppose $f : X \to [0, \infty]$ is measurable and $M \in (0, \infty)$. Then*

$$\mu(f^{-1}([M, \infty))) \leq \frac{1}{M} \int_X f \, d\mu.$$

Proof. First note that $\chi_{f^{-1}([M,\infty))} \leq \chi_{f^{-1}([M,\infty))} \frac{f}{M} \leq \frac{f}{M}$ pointwise. Then monotonicity,

$$\mu(f^{-1}([M, \infty))) = \int_X \chi_{f^{-1}([M,\infty))} \, d\mu \leq \int_X \frac{f}{M} \, d\mu = \frac{1}{M} \int_X f \, d\mu.$$

□

We say a condition holds *almost everywhere* (abbreviated a.e.) if that condition holds on a subset of X whose complement has measure zero. One of the key properties of Lebesgue integration is that it 'ignores' function values on sets of measure zero. In this way, the Lebesgue integral generalizes the corresponding property for the classic Riemann integral, which assigns zero weight to point-sets.

Lemma 18.2.5. *Suppose f and g are nonnegative functions on a measure space $(X, \mathcal{M}, \mu)$. Then*

(1) $\int_X f\,d\mu = 0$ if and only if $f = 0$ a.e.

(2) If $f \leq g$ a.e. then $\int_X f\,d\mu \leq \int_X g\,d\mu$.

(3) If $f = g$ a.e. then $\int_X f\,d\mu = \int_X g\,d\mu$.

Proof. (1) If $f = 0$ a.e. then $\mu(f^{-1}(0,\infty]) = 0$. If φ is a simple function such that $0 \leq \varphi \leq f$ then $\varphi^{-1}(0,\infty] \subseteq f^{-1}(0,\infty]$ so by monotonicity (Proposition 17.2.1), $\mu(\varphi^{-1}(0,\infty]) = 0$. Thus $\varphi = 0$ a.e. as well. Now

$$\int_X \varphi\,d\mu = \sum_{a\geq 0} a\mu(\varphi^{-1}(\{a\})) = 0 + \sum_{a>0} a\mu(\varphi^{-1}(\{a\})) = 0.$$

Taking the supremum over all such simple functions gives us $\int_X f\,d\mu$. Conversely, if $\int_X f\,d\mu = 0$ then by Chebyshev's lemma, for all $n \in \mathbb{N}$,

$$\mu\left(f^{-1}\left[\frac{1}{n},\infty\right)\right) \leq \frac{1}{n}\int_X f\,d\mu = 0.$$

Using subadditivity (Proposition 17.2.1), we can write

$$\mu(f^{-1}((0,\infty))) = \mu\left(\bigcup_{n=1}^{\infty} f^{-1}\left(\left[\frac{1}{n},\infty\right)\right)\right) \leq \sum_{n=1}^{\infty}\mu\left(f^{-1}\left(\left[\frac{1}{n},\infty\right)\right)\right) = \sum_{n=1}^{\infty} 0 = 0.$$

Lastly, note that $\mu(f^{-1}(\{\infty\})) = 0$ so we get $\mu(f^{-1}((0,\infty])) = 0$, i.e. $f = 0$ a.e.

(2) Let $E = \{x \in X : f(x) \leq g(x)\}$ so that $\mu(E^C) = 0$. Write $f = f\chi_E + f\chi_{E^C}$ and $g = g\chi_E + g\chi_{E^C}$. By linearity (Proposition 18.2.3), we have

$$\begin{aligned}
\int_X f\,d\mu &= \int_X f\chi_E\,d\mu + \int_X f\chi_{E^C}\,d\mu \\
&= \int_X f\chi_E\,d\mu + 0 \quad \text{by (1)} \\
&\leq \int_X g\chi_E\,d\mu \quad \text{by hypothesis} \\
&= \int_X g\chi_E\,d\mu + \int_X g\chi_{E^C}\,d\mu = \int_X g\,d\mu.
\end{aligned}$$

(3) Apply (1) and Proposition 18.2.3 to the function $f - g$. □

Corollary 18.2.6. *For any measurable functions $f, g : X \to \mathbb{R}$, if $f \le g$ a.e. then $\int_X f\,d\mu \le \int_X g\,d\mu$.*

Proof. Apply Lemma 18.2.5 to each part of the expressions $f = f^+ - f^-$ and $g = g^+ - g^-$. $\square$

Theorem 18.2.7 (Lebesgue Dominated Convergence Theorem)**.** *Suppose $f_n : X \to \mathbb{C}$ are measurable functions such that $f_n \to f$ pointwise on X. If $|f_n| \le g$ a.e. for some integrable function $g : X \to [0, \infty]$ and for all $n \in \mathbb{N}$, then f_n and f are all integrable, and*

$$\lim_{n\to\infty} \int_X f_n\,d\mu = \int_X f\,d\mu.$$

Proof. First, Lemma 18.2.5 gives us $\int_X |f_n|\,d\mu \le \int_X g\,d\mu < \infty$ so all the f_n are integrable. Moreover, since f_n converges to f, by Proposition 17.4.14 f is measurable. Therefore $|f| \le g$ a.e. implies $\int_X |f|\,d\mu < \infty$ by monotonicity, so f is integrable. Writing $f = \operatorname{Re} f + i \operatorname{im} f$ and considering the real and imaginary parts separately, we may assume f is real-valued. We have $g + f_n \ge 0$ and $g - f_n \ge 0$ for all $n \in \mathbb{N}$. By Fatou's Lemma,

$$\int_X g\,d\mu + \int_X f\,d\mu = \int_X (g+f)\,d\mu \le \liminf \int_X (g+f_n)\,d\mu = \int_X g\,d\mu + \liminf \int_X f_n\,d\mu.$$

Since g is integrable, we can subtract the finite integral $\int_X g\,d\mu$ from each side to obtain

$$\int_X f\,d\mu \le \liminf \int_X f_n\,d\mu.$$

Similarly,

$$\int_X g\,d\mu - \int_X f\,d\mu = \int_X (g-f)\,d\mu \le \liminf \int_X (g-f_n)\,d\mu = \int_X g\,d\mu - \limsup \int_X f_n\,d\mu.$$

Here subtracting $\int_X g\,d\mu$ from each side gives

$$\int_X f\,d\mu \ge \limsup \int_X f_n\,d\mu.$$

Therefore $\limsup \int_X f_n\,d\mu \le \int_X f\,d\mu \le \liminf \int_X f_n\,d\mu$, which shows that $\lim_{n\to\infty} \int_X f_n\,d\mu$ exists and equals

$$\lim_{n\to\infty} \int_X f_n\,d\mu = \int_X f\,d\mu.$$

$\square$

Proposition 18.2.8. *Suppose $f : X \to \mathbb{C}$ is integrable. Then*

$$\left|\int_X f\,d\mu\right| \le \int_X |f|\,d\mu.$$

Proof. Set $\int_X f\,d\mu = re^{i\theta}$ for some $r \in \mathbb{R}, \theta \in [0, 2\pi)$. Then by Proposition 18.2.3,

$$\left|\int_X f\,d\mu\right| = r = e^{-i\theta}\int_X f\,d\mu = \int_X e^{-i\theta} f\,d\mu = \int_X \mathrm{Re}(e^{-i\theta}f)\,d\mu + i\int_X \mathrm{im}(e^{-i\theta}f)\,d\mu.$$

Since $r \in \mathbb{R}$, we must have $i\int_X \mathrm{im}(e^{-i\theta}f)\,d\mu = 0$. Moreover, $\mathrm{Re}(e^{-i\theta}f) \le |e^{-i\theta}f| = |f|$ so by monotonicity,

$$\left|\int_X f\,d\mu\right| = \int_X \mathrm{Re}(e^{-i\theta}f)\,d\mu \le \int_X |f|\,d\mu.$$

□

The following theorem says that Riemann integrable functions may always be approximated by Borel measurable functions.

Theorem 18.2.9. *Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable. Then there exist Borel measurable functions $g, G : [a,b] \to \mathbb{R}$ such that*

(1) $g \le f \le G$ *pointwise.*

(2) $g = G$ *a.e.*

(3) $\displaystyle\int_a^b f(x)\,dx = \int_{[a,b]} g(x)\,d\lambda.$

18.3 Types of Convergence

Let $(X, \mathcal{M}, \mu)$ be a measure space.

Definition. *For any $p \geq 1$, the p*th **Lebesgue space** *of functions on X is*

$$L^p(\mu) := \left\{ f : X \to \mathbb{C} \text{ measurable} : \int_X |f|^p \, d\mu < \infty \right\}.$$

Definition. *For each $p \geq 1$, define the* L^p**-norm** *(sometimes just p-norm) of a function $f \in L^p(\mu)$ by*

$$||f||_{L^p} := \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

Examples.

1. For $p = 1$, the space $L^1(\mu)$ consists of all integrable functions $X \to \mathbb{C}$ with respect to μ. The L^1-norm on this space is just $\int_X |f| \, d\mu$, so $L^1(\mu)$ alternatively consists of all complex-valued functions on X having finite L^1-norm.

2. By convention, we let L^0 denote the class of measurable functions $X \to \mathbb{C}$, with L^+ further denoting the class of *nonnegative* measurable functions $X \to [0, \infty]$.

For the next three definitions, let $f_n : X \to \mathbb{C}$ be a sequence of measurable functions and let $f : X \to \mathbb{C}$ be any function.

Definition. *We say f_n* **converges almost everywhere** *to f, written $f_n \xrightarrow{a.e.} f$, if there exists a set $E \in \mathcal{M}$ such that $\mu(E) = 0$ and for all $x \in E^C$,*

$$\lim_{n \to \infty} f_n(x) = f(x).$$

Definition. *The sequence f_n* **converges in measure** *to f, denoted $f_n \to_\mu f$, if for every $\varepsilon > 0$, $\mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) \longrightarrow 0$ as $n \to \infty$. This is sometimes also called* L^0**-convergence**.

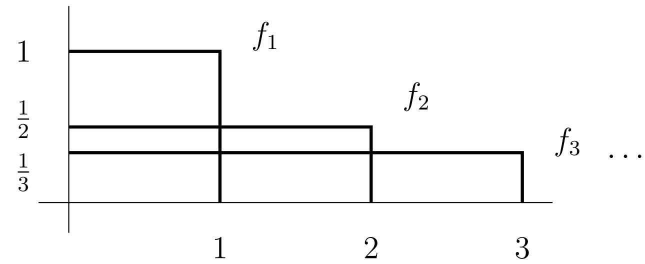
Definition. *For any $p \geq 1$, we say f_n has* L^p**-convergence**, *written $f_n \to_p f$, if*

$$||f_n - f||_{L^p} \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Remark. Uniform convergence implies pointwise convergence, which implies convergence almost everywhere. Additionally, the Dominated Convergence Theorem shows that dominated almost everywhere convergence, i.e. $f_n \to f$ a.e. and $|f_n| \leq g$ for some measurable function g, implies L^1-convergence: $f_n \to_1 f$. Finally, Chebyshev's lemma (18.2.4) shows that L^p-convergence implies convergence in measure.

The following examples show that some of the implications in the Remark are strict.

Example 18.3.1. For $X = [0, \infty)$ with Lebesgue measure λ, define the sequence of functions $f_n = \frac{1}{n}\chi_{[0,n]}$. The first three functions of this sequence are illustrated below.

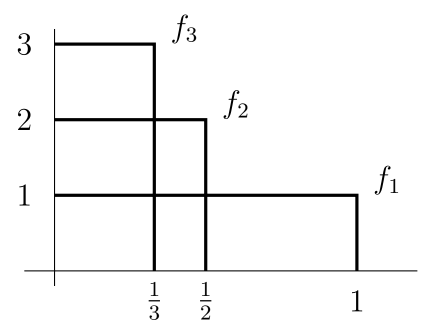


(In the literature, this sequence is sometimes referred to as the "steamroller".) It's easy to see that $f_n \to 0$ uniformly, but $\int_X f_n \, d\lambda = 1$ for all $n \in \mathbb{N}$, so the integrals $\int_X f_n \, d\lambda$ do not converge to 0. This is a counterexample to the Dominated Convergence Theorem when no dominating function $g \geq |f_n|$ exists. So f_n does not converge in L^1, but notice that for any $p > 1$,

$$\int_X |f_n|^p \, d\lambda = \int_X f_n^p \, d\lambda = \frac{1}{n^p} \longrightarrow 0$$

as $n \to \infty$, so f_n converges to the zero function in L^p. Also, $\lambda(\{x \in X : |f_n(x)| > \varepsilon\}) = 0$ for any $\varepsilon > 0$ when n is large enough, so we see that $f_n \to_\lambda 0$, that is, f_n converges to 0 in measure.

Example 18.3.2. Let $X = [0,1]$ be a measure space with Lebesgue measure λ. Define the sequence $f_n = n\chi_{[0,\frac{1}{n}]}$, called the "teepee".



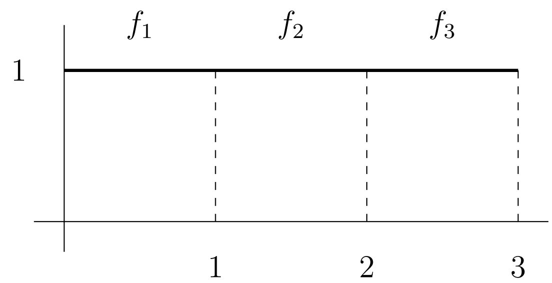
Let $E = \{0\}$. Then $f_n(x) \to 0$ on E^C, so $f_n \to 0$ pointwise a.e. However, $\int f_n \, d\mu = 1$ for all n, so $f_n \to 0 \neq 1$ in L^1 (or in L^p for any $p \geq 1$). One can show that f_n converges to 0 in measure as well.

Example 18.3.3. Again consider $X = [0,1]$ with Lebesgue measure λ. Using dyadic partitions of $[0,1]$, we can define a sequence of functions

$$\begin{aligned}
&f_1 = \chi_{[0,\frac{1}{2}]} \quad f_2 = \chi_{[\frac{1}{2},1]} \\
&f_3 = \chi_{[0,\frac{1}{4}]} \quad f_4 = \chi_{[\frac{1}{4},\frac{1}{2}]} \quad f_5 = \chi_{[\frac{1}{2},\frac{3}{4}]} \quad f_6 = \chi_{[\frac{3}{4},1]} \\
&\text{etc.}
\end{aligned}$$

Then f_n does not converge pointwise for any $x \in [0,1]$, but for any $p \geq 1$, $\int f_n^p \, d\mu \to 0$ as $n \to \infty$, meaning $f_n \to 0$ in L^p.

Example 18.3.4. Let $X = [0, \infty)$ with Lebesgue measure λ and define the sequence of functions $f_n = \chi_{[n-1,n]}$, called the "train".



Here $f_n \to 0$ pointwise (and hence pointwise a.e.) but for all n, $|f_n(x) - f(x)| = 1$ on a set of positive measure, so f_n does not converge to 0 in measure. Likewise, f_n does not converge to 0 in L^p.

Definition. *Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\{f_n : X \to \mathbb{C}\}_{n=1}^\infty$ is a sequence of complex-valued functions. We say the sequence $\{f_n\}$ is* **Cauchy in measure** *if for all $\varepsilon > 0$, $\mathcal{M}(\{x \in X : |f_n(x) - f_m(x)| > \varepsilon\}) \to 0$ as $n, m \to \infty$.*

Lemma 18.3.5. *Suppose $\{a_n\}_{n=1}^\infty \subset \mathbb{C}$ is a sequence such that*

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty.$$

Then $\lim_{n\to\infty} a_n$ exists in $\mathbb{C}$.

Proof. Without loss of generality suppose $m > n \geq 1$. Then

$$\begin{aligned} |a_m - a_n| &= |a_m - a_{m-1} + a_{m-1} - \ldots + a_{n+1} - a_n| \\ &= \sum_{k=n+1}^{m} |a_k - a_{k-1}| \\ &\leq \sum_{k=n+1}^{\infty} |a_k - a_{k-1}|. \end{aligned}$$

The latter is the tail of a convergent series, so it must approach 0 as $n \to \infty$. This shows that $|a_m - a_n| \to 0$ as $n \to \infty$, and in particular the sequence $\{a_n\}$ is Cauchy and hence it converges. □

Proposition 18.3.6. *Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $\{f_n : X \to \mathbb{C}\}_{n=1}^\infty$ is Cauchy in measure. Then there exists a subsequence $g_k = f_{n_k}$ such that $\lim_{k\to\infty} g_k$ exists a.e. Moreover, if E is the negligible set outside of which g_k converges, and $f(x) = \lim_{k\to\infty} \chi_{E^C} g_k$, then $g_k \to f$ pointwise a.e. and $f_n \to f$ in measure μ.*

Proof. Since f_n is Cauchy in measure, we may choose $\varepsilon_k > 0$ for each $k \in \mathbb{N}$ such that $\sum_{k=1}^\infty \varepsilon_k < \infty$ and define the subsequence $g_k = f_{n_k}$ such that

$$\mu(\{x \in X : |g_{k+1}(x) - g_k(x)| > \varepsilon_k\}) < \varepsilon_k$$

for each k. Set $E_k = \{x \in X : |g_{k+1}(x) - g_k(x)| > \varepsilon_k\}$. Then by construction,

$$\sum_{k=1}^{\infty} \mu(E_k) \le \sum_{k=1}^{\infty} \varepsilon_k < \infty.$$

Note that by Tonelli's theorem, this sum can be written

$$\sum_{k=1}^{\infty} \mu(E_k) = \sum_{k=1}^{\infty} \int_X \chi_{E_k}\, d\mu = \int_X \sum_{k=1}^{\infty} \chi_{E_k}\, d\mu.$$

Let $E = \limsup E_k = \{x \in X : x \in E_k \text{ i.o.}\}$. Then for any $N \in \mathbb{N}$, $\mu(E) \le \sum_{j=N}^{\infty} \mu(E_j) \le \sum_{j=N}^{\infty} \varepsilon_j$ which must approach 0 as $N \to \infty$ since $\sum_{j=1}^{\infty} \varepsilon_j$ converges. Thus $\mu(E) = 0$. Now for each $x \in E^C$, there exists an $N \in \mathbb{N}$ such that for all $k \ge N$, $|g_{k+1}(x) - g_k(x)| \le \varepsilon_k$. Thus

$$\sum_{k=N}^{\infty} |g_{k+1}(x) - g_k(x)| \le \sum_{k=N}^{\infty} \varepsilon_k < \infty.$$

By Lemma 18.3.5, this implies $\lim_{k\to\infty} g_k(x)$ exists for all $x \in E^C$.

Set $f(x) = \lim_{k\to\infty} \chi_{E^C} g_k(x)$ and consider the set $\{x \in X : |f_n(x) - f(x)| > \varepsilon\}$ for a given $\varepsilon > 0$. By the triangle inequality, we have

$$\begin{aligned}\{x \in X : |f_n(x) - f(x)| > \varepsilon\} &\subseteq \{x \in X : |f_n(x) - g_k(x)| + |g_k(x) - f(x)| > \varepsilon\} \\ &\subseteq \left\{x \in X : |f_n(x) - g_k(x)| > \tfrac{\varepsilon}{2}\right\} \cup \left\{x \in X : |g_k(x) - f(x)| > \tfrac{\varepsilon}{2}\right\}.\end{aligned}$$

Now by monotonicity and subadditivity of μ, we can write

$$\begin{aligned}\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) &\le \mu\left(\left\{x : |f_n(x) - g_k(x)| > \tfrac{\varepsilon}{2}\right\} \cup \left\{x : |g_k(x) - f(x)| > \tfrac{\varepsilon}{2}\right\}\right) \\ &\le \mu\left(\left\{x : |f_n(x) - g_k(x)| > \tfrac{\varepsilon}{2}\right\}\right) + \mu\left(\left\{x : |g_k(x) - f(x)| > \tfrac{\varepsilon}{2}\right\}\right) \\ &\longrightarrow 0 + \mu\left(\left\{x : |g_k(x) - f(x)| > \tfrac{\varepsilon}{2}\right\}\right) \quad \text{by the Cauchy hypothesis} \\ &\longrightarrow 0 \quad \text{since } \lim_{k\to\infty} g_k = f.\end{aligned}$$

Hence $f_n \to f$ in measure. □

Theorem 18.3.7 (Egoroff's Theorem)**.** *Suppose $(X, \mathcal{M}, \mu)$ is a finite measure space and f_n is a sequence of functions converging pointwise a.e. to f. Then for all $\varepsilon > 0$, there exists a subset $E \subseteq X$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on E^C.*

Proof. For each $n, k \in \mathbb{N}$, let $E_n^{(k)}$ be the set of $x \in X$ such that $|f_m(x) - f(x)| > \frac{1}{k}$ for some $m \ge n$. For a fixed k, if there were an element $x \in E_n^{(k)}$ for all n then for every $N \ge 1$, we would have $|f_n(x) - f(x)| > \frac{1}{k}$ for some $n \ge N$. Then f_n would not converge pointwise to f at x. Therefore by the convergence a.e. hypothesis, the set $E^{(k)} = \bigcup_{n=1}^{\infty} E_n^{(k)}$ has measure zero. Let $\varepsilon > 0$. Then for each k, there is an N_k such that for all $n \ge N_k$, $\mu(E_n^{(k)}) < \frac{\varepsilon}{2^k}$. Finally, let $E = \bigcup_{k=1}^{\infty} E_{N_k}^{(k)}$. By subadditivity,

$$\mu(E) \le \sum_{k=1}^{\infty} \mu(E_{N_k}^{(k)}) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$$

so E has Lebesgue measure $< \varepsilon$ as desired. Moreover, let $K \in \mathbb{N}$ such that $\frac{1}{K} < \varepsilon$ by the Archimedean principle. Then for all $k \geq K$ and $x \in E^C$,

$$|f_n(x) - f(x)| \leq \frac{1}{k} \leq \frac{1}{K} < \varepsilon.$$

Therefore f_n converges uniformly to f on E^C. □

Definition. *A sequence of functions f_n is said to have* **almost uniform convergence***, written a.u. convergence, if for any $\varepsilon > 0$, there is a set $E \subset X$ such that $\mu(E) < \varepsilon$ and f_n converges uniformly on E^C.*

Remark. Almost uniform convergence implies both convergence a.e. and convergence in measure. Egoroff's theorem says that on a finite measure space, pointwise convergence a.e. implies a.u. convergence, so the two are equivalent.

18.4 Product Measures

In this section we prove the theorems of Tonelli and Fubini, which generalize Theorems 16.1.4 and 16.1.10 for any arbitrary spaces. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be σ-finite measure space.

Lemma 18.4.1. *There is a unique measure π on $(X \times Y, \mathcal{M} \otimes \mathbb{F})$ such that $\pi(A \times B) = \mu(A)\nu(B)$ for all $A \in \mathcal{M}$ and $B \in \mathbb{F}$.*

Proof. By Proposition 17.2.3, $\mathcal{E} := \mathcal{M} \times \mathbb{F}$ is an elementary class. Define $\pi_0 : \mathcal{E} \to [0, \infty]$ by $\pi_0(A \times B) = \mu(A)\nu(B)$ for all $A \in \mathcal{M}$ and $B \in \mathbb{F}$. Then π_0 inherits σ-finiteness and σ-additivity from μ and ν. Therefore by Theorem 17.2.17, π_0 extends uniquely to a measure on $\mathcal{M} \otimes \mathbb{F}$ satisfying the desired property. $\square$

Informally, Lemma 18.4.1 says that there is a unique measure on the product σ-algebra $\mathcal{M} \otimes \mathbb{F}$ that integrates 'rectangles' $A \times B \in \mathcal{M} \times \mathbb{F}$ correctly, i.e. by multiplying their individual measures. We will write this unique measure as $\pi = \mu \times \nu$.

Theorem 18.4.2 (Tonelli's Theorem)**.** *Suppose $f : X \times Y \to [0, \infty]$ is $(\mathcal{M} \otimes \mathbb{F})$-measurable. Then*

(i) For all $x \in X$, the assignment $y \mapsto f(x, y)$ is an $\mathbb{F}$-measurable function.

(ii) For all $y \in Y$, the assignment $x \mapsto f(x, y)$ is an $\mathcal{M}$-measurable function.

(iii) The function $y \mapsto \int_X f(x, y)\, d\mu(x)$ is $\mathbb{F}$-measurable.

(iv) The function $x \mapsto \int_Y f(x, y)\, d\nu(y)$ is $\mathcal{M}$-measurable.

(v) $\displaystyle\int_{X\times Y} f\, d\pi = \int_X \int_Y f\, d\nu\, d\mu = \int_Y \int_X f\, d\mu\, d\nu.$

Proof. Recall that by Theorem 17.4.19, $\mathcal{M} \otimes \mathbb{F} = \sigma(\mathcal{M} \times \mathbb{F})$. For any 'rectangle' $A \times B$, with $A \in \mathcal{M}$ and $B \in \mathbb{F}$, it is clear that $\chi_{A\times B}(x, y) = \chi_A(x)\chi_B(y)$. It follows immediately that $\chi_{A\times B}(-, y)$ is $\mathcal{M}$-measurable and $\chi_{A\times B}(x, -)$ is $\mathbb{F}$-measurable, so (i) and (ii) hold for indicator functions – and for all simple functions after extending by linearity. Further, $y \mapsto \int_X \chi_{A\times B}(x, y)\, d\mu$ is just equal to the function $\chi_B(y)\mu(A)$ which is a simple function and hence $\mathbb{F}$-measurable. Similarly, $x \mapsto \int_Y \chi_{A\times B}(x, y)\, d\nu$ is $\mathcal{M}$-measurable, so (iii) and (iv) are seen to hold for indicator (and, by linearity, simple) functions. We have

$$\int_X \int_Y \chi_{A\times B}(x, y)\, d\nu\, d\mu = \nu(B)\mu(A) = \mu(A)\nu(B) = \int_Y \int_X \chi_{A\times B}(x, y)\, d\mu\, d\nu$$

but by Lemma 18.4.1,

$$\int_{X\times Y} \chi_{A\times B}(x, y)\, d\pi = \pi(A \times B) = \mu(A)\nu(B).$$

Hence Tonelli's theorem holds for all indicator, and therefore simple, functions.

Set $\mathcal{A} = \mathcal{A}(\mathcal{E})$ where $\mathcal{E} = \mathcal{M} \times \mathbb{F}$ and consider a set $E = \prod_{k=1}^{\infty}(A_k \times B_k) \in \mathcal{A}_\sigma$, where $A_k \in \mathcal{M}$ and $B_k \in \mathbb{F}$ for each k. Then by the same argument as above,

$$\chi_E(x,y) = \sum_{k=1}^{\infty} \chi_{A_k}(x)\chi_{B_k}(y)$$

so we have

$$\begin{aligned}\int_Y \chi_E(x,y)\,d\nu &= \sum_{k=1}^{\infty} \int_Y \chi_{A_k}(x)\chi_{B_k}(y)\,d\nu \quad \text{by Theorem 16.1.4}\\ &= \sum_{k=1}^{\infty} \nu(B_K)\chi_{A_k}(x)\end{aligned}$$

which is $\mathcal{M}$-measurable since sums and limits of measurable functions are measurable. Likewise, $\int_X \chi_E(x,y)\,d\mu$ is $\mathbb{F}$-measurable. If $\mathcal{C}$ is the collection of measurable sets $E \in \mathcal{M} \otimes \mathbb{F}$ such that χ_E satisfies conditions (i) – (v), then we have already shown $\mathcal{A} \subseteq \mathcal{C}$. If $\mu(X)$ and $\nu(Y)$ are finite, one can show that $\mathcal{C}$ is a monotone class, so the Monotone Class Theorem says that $\mathcal{M} \otimes \mathbb{F} \subseteq \mathcal{C}$, meaning (i) – (v) hold for all indicator functions. Extending by linearity, we have that (i) – (v) hold for all simple functions.

To generalize to the case when μ and ν are σ-finite, take sequences $X_n \nearrow X$ and $Y_n \nearrow Y$ such that $\mu(X_n), \nu(Y_n) < \infty$ and define

$$\mu_n(A) := \mu(A \cap X_n), \qquad \nu_n(B) := \nu(B \cap Y_n) \quad \text{and} \quad \mu_n \times \nu_n(E) := \pi(E \cap (X_n \times Y_n))$$

for all $A \in \mathcal{M}, B \in \mathbb{F}, E \in \mathcal{M} \otimes \mathbb{F}$ and $n \in \mathbb{N}$. For any function $f : X \to [0,\infty]$, we have

$$\int_X f\,d\mu_n = \int_X f\chi_{X_n}\,d\mu \quad \text{and} \quad \int_Y f\,d\nu_n = \int_Y f\chi_{Y_n}\,d\nu,$$

and $\chi_{X_n} \nearrow \chi_X = 1$ and $\chi_{Y_n} \nearrow \chi_Y = 1$ be construction. So we get Tonelli's theorem for all simple functions when μ and ν are σ-finite. To generalize to all nonnegative functions, use approximating sequences of simple functions $\varphi_k \nearrow f$ and the Monotone Convergence Theorem. $\square$

Theorem 18.4.3 (Fubini's Theorem). *Suppose $f \in L^1(\mu \times \nu)$. Then*

(i) For all $x \in X$, $y \mapsto f(x,y)$ is an $L^1(\nu)$ function.

(ii) For all $y \in Y$, $x \mapsto f(x,y)$ is an $L^1(\mu)$ function.

(iii) The function $y \mapsto \int_X f(x,y)\,d\mu(x)$ is in $L^1(\nu)$.

(iv) The function $x \mapsto \int_Y f(x,y)\,d\nu(y)$ is in $L^1(\mu)$.

(v) $\displaystyle\int_{X\times Y} f\,d\pi = \int_X \int_Y f\,d\nu\,d\mu = \int_Y \int_X f\,d\mu\,d\nu.$

Proof. Write $f = \operatorname{Re} f + i \operatorname{im} f$ and separate each part into its positive and negative parts. Then the result follows from Tonelli's theorem on each piece. $\square$

18.5 Lebesgue Integration on $\mathbb{R}^n$

Let $n \geq 1$ and consider the measurable space $(\mathbb{R}^n, \mathcal{B}^n)$ where $\mathcal{B}^n = \mathcal{B}_{\mathbb{R}^n}$ is the Borel σ-algebra on $\mathbb{R}^n$. Define $\lambda_n := \underbrace{\lambda \times \cdots \lambda}_{n}$ where λ is Lebesgue measure on $\mathbb{R}$.

Definition. *The function $\lambda_n : \mathcal{B}^n \to [0, \infty]$ is called the* **(n-dimensional) Lebesgue measure** *on $\mathbb{R}^n$.*

Note that λ_n is the *unique* measure that gives the correct hypervolume to hyperrectangles:

$$\lambda_n\left(\prod_{k=1}^{n}(a_k, b_k]\right) = \prod_{k=1}^{n}(b_k - a_k).$$

Lemma 18.5.1. *λ_n is translation invariant.*

Proof. For all $A \in \mathcal{B}^n$ and $x \in \mathbb{R}^n$, define $\mu_x(A) = \lambda_n(x + A)$. Then $\mu_x(B) = \lambda_n(B)$ on hyperrectangles $B = \prod_{k=1}^{n}(a_k, b_k]$. Then $\mu_x = \lambda_n$ by uniqueness. □

Definition. *Let $\Omega \subset \mathbb{R}^n$ be an open subset and $T : \Omega \to \mathbb{R}^n$. Then T is a* **C^1-diffeomorphism** *if $T = (T_1, \ldots, T_n)$ where each T_k is a C^1-function, T is one-to-one, $T(\Omega) \subseteq \mathbb{R}^n$ is open and $T^{-1} : T(\Omega) \to \Omega$ is also C^1.*

One of the most important theorems in calculus is the change-of-variables formula for integration. Here we prove the general statement for Lebesgue integrals.

Theorem 18.5.2. *If $T : \Omega \to \mathbb{R}^n$ is a C^1-diffeomorphism, then for all Lebesgue measurable functions $f : T(\Omega) \to [0, \infty]$,*

$$\int_{T(\Omega)} f(y)\, d\lambda_n(y) = \int_{\Omega} f \circ T(x) |\det D_x T|\, d\lambda_n(x),$$

where $D_x T : \mathbb{R}^n \to \mathbb{R}^n$ is the differential operator $D_x T(v) = \frac{d}{dt}\big|_{t=0} T(x + tv)$.

Proof. (Sketch) Set $\lambda = \lambda_n$. First suppose the result holds for C^1-diffeomorphisms $\Omega \xrightarrow{T} T(\Omega) \xrightarrow{S} \mathbb{R}^n$. Then by the chain rule,

$$\det(D_x(S \circ T)) = \det(D_{T(x)} S \circ D_x T) = \det(D_{T(x)} S) \det(D_x T).$$

Assuming the theorem holds, we then have

$$\begin{aligned}\int_{\Omega} (f \circ S \circ T)(x) |\det(D_x(S \circ T))|\, d\lambda(x) &= \int_{T(\Omega)} (f \circ S)(y) |\det(D_y S)|\, d\lambda(y) \\ &= \int_{S(T(\Omega))} f(z)\, d\lambda(z).\end{aligned}$$

Thus it suffices to prove the change-of-variables formula for invertible operators. If T is an invertible linear operator, $D_x T = T$ and the result follows for T by Tonelli's theorem (18.4.2), viewing T as the composition of elementary transformations (i.e. operators represented by elementary matrices). Now for an arbitrary diffeomorphism T, take a Borel measurable set $B \subseteq T(\Omega)$ and set $A = T^{-1}(B) \in \mathcal{B}^n$. We want to show

$$\lambda(B) = \lambda(T(A)) = \int_A |\det D_x T| \, d\lambda = |\det D_x T| \lambda(A),$$

since then we have

$$\lambda(B) = \int_B d\lambda = \int_{T(\Omega)} \chi_B \, d\lambda = \int_\Omega \chi_B \circ T |\det D_x T| \, d\lambda.$$

Approximate $\lambda(T(A))$ by using the linear operator $D_x T$:

$$\lambda(T(A)) \approx \lambda(D_x T(A \smallsetminus \{x\})) = |\det D_x T| \lambda(A \smallsetminus \{x\}) = |\det D_x T| \lambda(A).$$

Now for a set $Q \subseteq \Omega$, partition Q by $\mathcal{M}$-sets: $Q = \bigcup_{\substack{A \in \mathcal{M} \\ A \subseteq Q}} A$. Then

$$\lambda(T(Q)) = \sum_{\substack{A \in \mathcal{M} \\ A \subseteq Q}} \lambda(T(A)) \approx \sum_{\substack{A \in \mathcal{M} \\ A \subseteq Q}} |\det D_x T| \lambda(A) \longrightarrow \int_Q |\det D_x T| \, d\lambda$$

as the mesh of the partition goes to 0. This proves the theorem. $\square$

Chapter 19

Signed Measures and Differentiation

In this chapter we develop the notion of abstract differentiation as an analog to differentiation and integration in the classical case. To do so, we must first introduce signed measures, complex measures and signed and complex integration. The principal result in this chapter is the Lebesgue-Radon-Nikodym theorem, which then allows us to define the derivative of a measure.

19.1 Signed Measures

Definition. *A* **signed measure** *on a measurable space* $(X, \mathcal{M})$ *is a* σ*-additive function* $\nu : \mathcal{M} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ *such that*

(1) $\nu(\varnothing) = 0$.

(2) Either $\nu(\mathcal{M}) \subseteq [-\infty, \infty)$ *or* $\nu(\mathcal{M}) \subseteq (-\infty, \infty]$.

Note that finite measures $\mu : \mathcal{M} \to [0, \infty)$ are signed measures. We will call these *positive measures* to distinguish them from the general case.

Examples.

1. If $\mu_1, \mu_2 : \mathcal{M} \to [0, \infty)$ are (positive) measures on X, then $\nu(A) = \mu_1(A) - \mu_2(A)$ is a signed measure on X.

2. If μ is a positive measure on X and $f : X \to [-\infty, \infty]$ is a function such that either $\int_X f^+ \, d\mu < \infty$ or $\int_X f^- \, d\mu < \infty$, then

$$\nu(A) = \int_A f \, d\mu = \int_X \chi_A f \, d\mu$$

is a signed measure on X. We denote this $d\nu = f \, d\mu$.

We will show that these are in some sense the only types of signed measures.

Proposition 19.1.1. *Suppose* $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ *is a signed measure. Then*

(1) (Continuity from below) If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$ *converges to* A *from below, then* $\nu(A_n)$ *converges to* $\nu(A)$.

(2) (Continuity from above) If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$ *converges to* A *from above, then* $\nu(A_n)$ *converges to* $\nu(A)$.

Proof. Similar to Proposition 17.2.1. □

Definition. *Let* ν *be a signed measure on* $(X, \mathcal{M})$ *and let* $A \in \mathcal{M}$ *be a measurable set. We say* A *is* **null** *(with respect to* ν*) if* $\nu(B) = 0$ *for any measurable subset* $B \subseteq A$*. Similarly,* A *is* **positive** *if* $\nu(B) \geq 0$ *for any measurable* $B \subseteq A$*; and* A *is* **negative** *if* $\nu(B) \leq 0$ *for any measurable* $B \subseteq A$.

Proposition 19.1.2. *Let* $(X, \mathcal{M})$ *be a measurable space with signed measure* $\nu : \mathcal{M} \to \overline{\mathbb{R}}$. *Then*

(1) If $\{P_n\}_{n=1}^{\infty} \subset \mathcal{M}$ *is a sequence of positive sets, then* $\bigcup_{n=1}^{\infty} P_n$ *is positive.*

(2) If $\{E_n\}_{n=1}^{\infty} \subset \mathcal{M}$ *is a sequence of negative sets, then* $\bigcup_{n=1}^{\infty} E_n$ *is negative.*

(3) If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$ *is a sequence of null sets, then* $\bigcup_{n=1}^{\infty} A_n$ *is null.*

Proof. We prove (1); the proofs of (2) and (3) are similar. Suppose $\{P_n\}$ is a sequence of positive sets. Set $\widetilde{P}_1 = P_1$ and for all $n \geq 2$, set $\widetilde{P}_n = P_n \smallsetminus \bigcup_{k=1}^{n-1} P_k$. Then for any measurable set $A \subseteq \bigcup_{n=1}^\infty P_n$, we have $A = \coprod_{n=1}^\infty A \cap \widetilde{P}_n$, so by σ-additivity,

$$\nu(A) = \bigcup_{n=1}^\infty \nu(A \cap \widetilde{P}_n) \geq 0$$

since each $\widetilde{P}_n \subseteq P_n$ is positive. Hence $\bigcup_{n=1}^\infty P_n$ is positive. $\square$

Lemma 19.1.3. *If $A \subseteq X$ is a subset and ν is a signed measure on X such that $\nu(A) \in (0, \infty)$, then there exists a positive set $P \subseteq A$ such that $\nu(P) \geq \nu(A)$.*

Proof. Set $n(A) = \min(1, \sup\{-\nu(B) : B \in \mathcal{M}, B \subseteq A\})$. Then $n(A) \geq 0$ with equality if and only if A itself is positive, so assume $n(A) > 0$. Then there exists some $B_1 \subseteq A$ with $-\nu(B_1) \geq \frac{1}{2}\nu(A)$. Take $E_1 = A \smallsetminus B_1$. Now for $n \geq 2$, suppose $B_1, \ldots, B_n$ and $E_1, \ldots, E_{n-1}$ have been given. Set $E_n = A \smallsetminus \bigcup_{k=1}^n B_k$ and choose $B_{n+1} \subseteq E_n$ such that $-\nu(B_{n+1}) \geq \frac{1}{2}\nu(E_n)$. Then

$$\begin{aligned} A &= \left(\coprod_{n=1}^\infty B_n\right) \cup \left(A \smallsetminus \coprod_{n=1}^\infty B_n\right) \\ \implies \nu(A) &= \sum_{n=1}^\infty \nu(B_n) + \nu\left(A \smallsetminus \coprod_{n=1}^\infty B_n\right) \quad \text{by additivity} \\ \implies \sum_{n=1}^\infty -\nu(B_n) &= \nu\left(A \smallsetminus \coprod_{n=1}^\infty B_n\right) - \nu(A) \quad \text{since } \nu(A) < \infty. \end{aligned}$$

But now $-\nu(B_n) \to 0$ as $n \to \infty$, so $\frac{1}{2}\nu(E_n) \to 0$. Thus $P = A \smallsetminus \coprod_{n=1}^\infty B_n$ is the desired positive set. $\square$

Lemma 19.1.4. *If $A \subseteq X$ is a positive set with respect to a signed measure $\nu : \mathcal{M} \to \overline{\mathbb{R}}$, then $\nu|_A$ is a positive measure.*

Theorem 19.1.5 (Hahn Decomposition)**.** *For every signed measure $\nu : \mathcal{M} \to \overline{\mathbb{R}}$, there exists a measurable set $P \in \mathcal{M}$ such that P is positive and $N = P^C$ is negative. As a result, $X = P \cup N$ is a decomposition of X into a positive and a negative set.*

Proof. Without loss of generality, suppose $\nu(A) \in [-\infty, \infty)$ for all $A \in \mathcal{M}$. Take $s = \sup\{\nu(E) : E \in \mathcal{M}\} \geq 0$. If $s = 0$, take $P = \varnothing$ and we're done. Otherwise, $s > 0$ so there exists a sequence $\{A_j\}_{j=1}^\infty \subset \mathcal{M}$ such that $\nu(A_j) > 0$ and $\nu(A_j) \to s$. By Lemma 19.1.3, for each j there is a positive set $P_j \subseteq A_j$ with $\nu(P_j) \geq \nu(A_j)$. We have $\nu(P_j) \leq s$ by definition, so $\nu(P_j) \to s$ by the squeeze theorem. Take $P = \bigcup_{j=1}^\infty P_j$. Then P is positive by Proposition 19.1.2 and we have $s \geq \nu(P)$. Moreover, $\nu|_P$ is a positive measure by Lemma 19.1.4, so by monotonicity, $s \geq \nu(P) \geq \nu(P_n) \to s$. Hence $\nu(P) = s > 0$.

It remains to show $N = P^C$ is a negative set. If not, there is some $E \subseteq N$ such that $\nu(E) > 0$. Then by σ-additivity,

$$\nu(P \cup E) = \nu(P) + \nu(E) = s + \nu(E) > s.$$

Of course this is a contradiction, so N is negative. This proves that $X = P \cup N$ is a decomposition of X into a positive and a negative set. $\square$

Definition. *If ν is a signed measure on X, a pair $\{P, N\}$ such that P is positive, N is negative and $X = P \cup N$ is called a* **Hahn decomposition** *of X.*

Remark. Hahn decompositions are unique up to null sets. If $\{P, N\}$ is a Hahn decomposition with respect to a signed measure ν, then for any other decomposition $\{P', N'\}$ where P' is positive and N' is negative, we have $\nu(P' \triangle P) = 0$ and $\nu(N' \triangle N) = 0$.

Definition. *Two signed measures $\mu, \nu : \mathcal{M} \to \overline{\mathbb{R}}$ are said to be* **mutually singular***, denoted $\mu \perp \nu$, if there is a set $E \in \mathcal{M}$ such that $\mu(E) = 0$ and $\nu(E^C) = 0$.*

Corollary 19.1.6 (Jordan Decomposition)**.** *Let ν be a signed measure on X. Then $\nu = \nu^+ - \nu^-$ for unique positive measures ν^+ and ν^- satisfying $\nu^+ \perp \nu^-$.*

Proof. Suppose $X = P \cup N$ is a Hahn decomposition of X. Define $\nu^+(A) = \nu(A \cap P)$ and $\nu^-(A) = -\nu(A \cap N)$ for all measurable sets A. By Lemma 19.1.4, ν^+ and ν^- are positive measures, and clearly $\nu^+ \perp \nu^-$ (using P and $N = P^C$), so we need only check that this decomposition of ν is independent of choice of Hahn decomposition. Suppose $\tilde{\nu}^+$ and $\tilde{\nu}^-$ is another pair of positive measures such that $\nu = \tilde{\nu}^+ - \tilde{\nu}^-$. Let $\widetilde{P} \in \mathcal{M}$ be a set such that $\tilde{\nu}^-(\widetilde{P}) = 0$ and $\tilde{\nu}^+(\widetilde{N}) = 0$, where $\widetilde{N} = \widetilde{P}^C$. Note that $\{\widetilde{P}, \widetilde{N}\}$ is a Hahn decomposition of X. Then for any $A \in \mathcal{M}$,

$$\tilde{\nu}^+(A) = \nu(A \cap \widetilde{P}) = \nu(A \cap P) = \nu^+(A)$$

and similarly

$$\tilde{\nu}^-(A) = -\nu(A \cap \widetilde{N}) = -\nu(A \cap N) = \nu^-(A).$$

Therefore $\nu = \nu^+ - \nu^-$ is well-defined. Uniqueness is guaranteed by uniqueness of Hahn decomposition up to null sets. $\square$

Definition. *For a signed measure ν on X, the decomposition $\nu = \nu^+ - \nu^-$ is called the* **Jordan decomposition** *of ν; ν^+ is called the* **positive variation** *of ν; ν^- is the* **negative variation** *of ν; and $|\nu| := \nu^+ + \nu^-$ is the* **total variation** *of ν.*

Definition. *A signed measure $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ is said to be* **finite** *if $|\nu|$ is a finite measure.*

Example 19.1.7. If $f \in L^1(X, \mathcal{M}, \mu)$ where μ is a positive measure and f is real-valued, then $\nu(A) = \int_A f\,d\mu$ is a signed measure, written $d\nu = f\,d\mu$. In particular, if f is nonnegative then ν is a positive measure. Notice that $f \in L^1(\mu)$ implies $|\nu| = \nu^+ + \nu^- < \infty$. A Hahn decomposition for ν can be written down explicitly in terms of f:

$$P = \{x \in X : f(x) > 0\} \quad \text{and} \quad N = \{x \in X : f(x) \leq 0\}.$$

Then the Jordan decomposition of ν is

$$\nu^+(A) = \int_A f^+\,d\mu \quad \text{and} \quad \nu^-(A) = \int_A f^-\,d\mu$$

with total variation $|\nu|(A) = \int_A |f|\,d\mu$.

Remark. In some sense, the types of signed measures in Example 19.1.7 are the *only* signed measures, for if ν is a signed measure with Hahn decomposition $\{P, N\}$, we can write $\nu(A) = \nu^+(A) - \nu^-(A) = \int_A g\, d|\nu|$, where $g = \chi_P - \chi_N$.

Definition. *For a signed measure $\nu = \nu^+ - \nu^-$, the space of* **signed integrable functions** *is $L^1(\nu) := L^1(\nu^+) \cap L^1(\nu^-)$. For any $f \in L^1(\nu)$, we define the* **signed integral** *of f to be*

$$\int_X f\, d\nu := \int_X f\, d\nu^+ - \int_X f\, d\nu^-.$$

Remark. If $d\nu = g\, d\mu$ for some positive measure μ and integrable function g, then for any μ-integrable function f,

$$\int_X f\, d\nu = \int_X fg^+\, d\mu - \int_X fg^-\, d\mu = \int_X f(g^+ - g^-)\, d\mu = \int_X fg\, d\mu.$$

This shows that $f \in L^1(\nu^+)$ if and only if $\int_X |f| g^+\, d\mu < \infty$, and similarly $f \in L^1(\nu^-)$ if and only if $\int_X |f| g^-\, d\mu < \infty$. Hence $L^1(\nu) = L^1(|g|\, d\mu)$.

Lemma 19.1.8. *Let ν be a signed measure on $(X, \mathcal{M})$. Then*

(a) $L^1(\nu) = L^1(|\nu|)$.

(b) If $f \in L^1(\nu)$, $\left|\int f\, d\nu\right| \le \int |f|\, d|\nu|$.

(c) If $E \in \mathcal{M}$, $|\nu|(E) = \sup\left\{\left|\int_E f\, d\nu\right| : |f| \le 1\right\}$.

Proof. (a) By definition $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$. On the other hand, write $|\nu| = \nu^+ + \nu^-$. Then $\int |f|\, d|\nu| < \infty$ if and only if $\int |f|\, d\nu^+ + \int |f|\, d\nu^- < \infty$ which is valid if and only if $\int |f|\, d\nu^+ < \infty$ and $\int |f|\, d\nu^- < \infty$. Hence $L^1(|\nu|) = L^1(\nu^+) \cap L^1(\nu^-)$ as well.

(b) Let $f \in L^1(\nu)$. Then

$$\begin{aligned}
\left|\int f\, d\nu\right| &= \left|\int f\, d\nu^+ - \int f\, d\nu^-\right| \le \left|\int f\, d\nu^+\right| + \left|\int f\, d\nu^-\right| \\
&\le \int |f|\, d\nu^+ + \int |f|\, d\nu^- \quad \text{since } \nu^+, \nu^- \text{ are positive measures} \\
&= \int |f|\, d|\nu|.
\end{aligned}$$

(c) For any measurable function f such that $|f| \le 1$, we have

$$\int f\chi_E\, d|\nu| = \int_E |f|\, d|\nu| \le \int_E d|\nu| = |\nu|(E) < \infty.$$

In particular, this shows $f\chi_E \in L^1(|\nu|) = L^1(\nu)$ by (a). Now (b) gives us

$$\left|\int_E f\, d\nu\right| = \left|\int f\chi_E\, d\nu\right| \le \int |f|\, |\chi_E|\, d|\nu| \le \int \chi_E\, d|\nu| = |\nu|(E).$$

Taking the sup over all such f yields $|\nu|(E) \geq \sup\left\{\left|\int_E f\, d\nu\right| : |f| \leq 1\right\}$. On the other hand, use Hahn decomposition to decompose $X = P \cup N$ into a positive set P and a negative set $N = P^C$. Then the function $g = \chi_{P\cap E} - \chi_{N\cap E}$ satisfies $|g| \leq 1$. Moreover, indicator functions are measurable so $g \in L^1(\nu)$. Now

$$\begin{aligned}\left|\int_E g\, d\nu\right| &= \left|\int_E \chi_{P\cap E}\, d\nu - \int_E \chi_{N\cap E}\, d\nu\right| = |\nu(P\cap E) + \nu(N\cap E)| \\ &= |\nu^+(E) + \nu^-(E)| \quad \text{by Jordan decomposition} \\ &= |\nu|(E).\end{aligned}$$

Then $\sup\left\{\left|\int_E f\, d\nu\right| : |f| \leq 1\right\} \geq \left|\int_E g\, d\nu\right| = |\nu|(E)$ so we have both inequalities. $\square$

19.2 Lebesgue-Radon-Nikodym Theorem

Definition. *If ν is a signed measure and μ is a positive measure, both defined on $(X, \mathcal{M})$, then we say ν is* **absolutely continuous** *with respect to μ, denoted $\nu << \mu$, provided for all $A \in \mathcal{M}$ such that $\mu(A) = 0$, $\nu(A) = 0$ as well.*

Example 19.2.1. If $f \in L^1(\mu)$ then the signed measure $\nu(A) = \int_A f \, d\mu$ is absolutely continuous with respect to μ.

Lemma 19.2.2. *For a signed measure $\nu = \nu^+ - \nu^-$ and a positive measure μ on $(X, \mathcal{M})$, the following are equivalent:*

(a) $\nu << \mu$.

(b) $|\nu| << \mu$.

(c) $\nu^+ << \mu$ and $\nu^- << \mu$.

Proof. Throughout the proof, let E denote a measurable set. Use Hahn decomposition to write $X = P \cup N$, so that $\nu^+(E) = \nu(P \cap E)$ and $\nu^-(E) = -\nu(N \cap E)$.

(a) $\implies$ (c) First, suppose $\nu << \mu$. If $\mu(E) = 0$ then $P \cap E \subseteq E$ so by monotonicity of positive measures, $\mu(P \cap E) = 0$, and $\nu << \mu$ implies $\nu^+(E) = \nu(P \cap E) = 0$. Similarly, $N \cap E \subseteq E$ implies $\mu(N \cap E) = 0$ which implies $\nu^-(E) = -\nu(N \cap E) = 0$. Hence $\nu^+ << \mu$ and $\nu^- << \mu$.

(c) $\implies$ (b) Now assume $\nu^+ << \mu$ and $\nu^- << \mu$. Then if $\mu(E) = 0$, we have $\nu^+(E) = 0$ and $\nu^-(E) = 0$, so $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0 + 0 = 0$. Hence $|\nu| << \mu$.

(b) $\implies$ (a) Finally, assume $|\nu| << \mu$. Then if $\mu(E) = 0$, we have $0 = |\nu|(E) = \nu^+(E) + \nu^-(E)$. Thus $\nu^+(E) = -\nu^-(E)$, but since ν^+, ν^- are positive measures, this is only possible if $\nu^+(E) = \nu^-(E) = 0$. Hence $\nu(E) = \nu^+(E) - \nu^-(E) = 0 - 0 = 0$ and we have $\nu << \mu$. In total, we have proven $(\nu << \mu) \implies (\nu^+, \nu^- << \mu) \implies (|\nu| << \mu) \implies (\nu << \mu)$, so all three statements are equivalent. □

Lemma 19.2.3. *Suppose $\{\nu_j\}_{j=1}^\infty$ is a sequence of positive measures. Then*

(a) If $\nu_j \perp \mu$ for all j, then $\sum_{j=1}^\infty \nu_j \perp \mu$.

(b) If $\nu_j << \mu$ for all j, then $\sum_{j=1}^\infty \nu_j << \mu$.

Proof. (a) Suppose $\nu_j \perp \mu$ for all $j \in \mathbb{N}$. This means there exists a sequence of sets $\{E_j\}_{j=1}^\infty$ in our σ-algebra such that for each $j \in \mathbb{N}$, $\nu_j(E_j) = 0$ and $\mu(E_j^C) = 0$. Set $E = \bigcap_{j=1}^\infty E_j$, so that $E^C = \bigcup_{j=1}^\infty E_j^C$ by deMorgan's Laws. Then by monotonicity of positive measures, $\nu_j(E) \leq \nu_j(E_j) = 0$ and hence $\nu_j(E) = 0$ for each $j \in \mathbb{N}$. So

$$\sum_{j=1}^\infty \nu_j(E) = \sum_{j=1}^\infty 0 = 0.$$

On the other hand, subadditivity of μ gives us

$$\mu(E^C) = \mu\left(\bigcup_{j=1}^\infty E_j^C\right) \leq \sum_{j=1}^\infty \mu(E_j^C) = \sum_{j=1}^\infty 0 = 0.$$

Hence $\sum_{j=1}^{\infty} \nu_j \perp \mu$.

(b) Now assume $\nu_j << \mu$ for all $j \in \mathbb{N}$. Then if $\mu(E) = 0$ for some measurable set E, this means $\nu_j(E) = 0$ for each $j \in \mathbb{N}$, and hence $\sum_{j=1}^{\infty} \nu_j(E) = \sum_{j=1}^{\infty} 0 = 0$. Therefore $\sum_{j=1}^{\infty} \nu_j << \mu$. $\square$

Lemma 19.2.4. *If ν is a finite signed measure and μ is a positive measure on $(X, \mathcal{M})$, then $\nu << \mu$ if and only if for all $\varepsilon > 0$, there is a $\delta > 0$ such that for all $A \in \mathcal{M}$, $\mu(A) < \delta$ implies $|\nu(A)| < \varepsilon$.*

Proof. Since $|\nu(A)| \leq |\nu|(A)$ for all measurable sets A, we may assume ν is a positive measure.

($\Longleftarrow$) is trivial: if $\mu(A) = 0 < \delta$ for all $\delta > 0$, then $\nu(A) = 0 < \varepsilon$ for all $\varepsilon > 0$.

($\Longrightarrow$) We prove the contrapositive. Suppose the conclusion does not hold. Then we can find an $\varepsilon > 0$ such that for each $n \in \mathbb{N}$, there is a set $A_n \in \mathcal{M}$ with $\nu(A_n) \geq \varepsilon$ but $\mu(A_n) < \frac{1}{2^n}$. Let $A = \limsup A_n = \{x \in X : x \in A_n \text{ i.o.}\}$. Then by continuity from above, $\nu(A) = \lim_{N\to\infty} \nu\left(\bigcup_{k=N}^{\infty} A_k\right) \geq \varepsilon > 0$ but for all $N \in \mathbb{N}$,

$$\mu(A) \leq \mu\left(\bigcup_{k=N}^{\infty} A_n\right) \leq \sum_{n=N}^{\infty} \mu(A_n) \leq \frac{1}{2^{N-1}}$$

which approaches 0 as $N \to \infty$. Therefore $\mu(A) = 0$. Hence ν is not absolutely continuous with respect to μ. $\square$

Lemma 19.2.5. *If μ and ν are finite, positive measures on $(X, \mathcal{M})$ then either $\mu \perp \nu$ or there exist $\varepsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$ and $\nu \geq \varepsilon\mu$ on E.*

Proof. For each $n \in \mathbb{N}$, $\nu - \frac{1}{n}\mu$ is a signed measure, so there is a Hahn decomposition $X = P_n \cup N_n$ for each of these. Set $P = \bigcup_{n=1}^{\infty} P_n$ and $N = \bigcap_{n=1}^{\infty} N_n = P^C$. Then $0 \leq \nu(N) \leq \frac{1}{n}\mu(N)$ for all n, so $\nu(N) = 0$ and the same logic shows that N is a negative set for each $\nu - \frac{1}{n}\mu$. If $\mu(P) = 0$, then $\nu \perp \mu$. On the other hand, if $\mu(P) > 0$ then $\mu(P_n) > 0$ for some n, so P_n is a positive set for $\nu - \frac{1}{n}\mu$. Taking $\varepsilon = \frac{1}{n}$ and $E = P_n$, the second statement is seen to hold. $\square$

Theorem 19.2.6 (Lebesgue-Radon-Nikodym)**.** *Given two σ-finite measures μ and ν on $(X, \mathcal{M})$, with μ positive and ν signed, there are unique signed measures ρ and λ such that $\rho << \mu$, $\lambda \perp \mu$ and $\nu = \rho + \lambda$. Moreover, there exists an (extended) μ integrable function $f : X \to \overline{\mathbb{R}}$ such that $d\rho = f\,d\mu$.*

Proof. First suppose that ν is positive and ν and μ are both finite measures. For any μ-integrable function $f : X \to [0, \infty]$, define $\mu_f = \int_X f\,d\mu$ which is a positive measure by Example 19.1.7. Set $\mathbb{F} = \{f \in L^1(\mu) : \mu_f \leq \nu\}$. If $f, g \in \mathbb{F}$ then $h = \max\{f, g\} \in \mathbb{F}$ as well, since for any $A \in \mathcal{M}$, we have

$$\begin{aligned}\mu_h(A) = \int_A h\,d\mu &= \int_{A\cap\{x:f(x)>g(x)\}} f\,d\mu + \int_{A\cap\{x:f(x)\leq g\}} g\,d\mu \\ &\leq \nu(A \cap \{x : f(x) > g(x)\}) + \nu(A \cap \{x : f(x) \leq g(x)\}) = \nu(A).\end{aligned}$$

Let $\mathcal{M} = \sup\{\mu_f(X) : f \in \mathbb{F}\}$ and for each $n \in \mathbb{N}$, choose $f_n \in \mathbb{F}$ such that $\mu_{f_n}(X) \to M$. Since $\mathbb{F}$ is closed under taking maxima, each $\tilde{f}_n = \max\{f_1, \ldots, f_n\}$ lies in $\mathbb{F}$. Then $M \geq \mu_{\tilde{f}_n}(X) \geq \mu_{f_n}(X)$ for each n, so it follows that $\mu_{f_n}(X) \to M$ as well. We also have

$$M = \lim_{n\to\infty} \mu_{f_n}(X) = \lim_{n\to\infty} \int_X f_n \, d\mu = \int_X \lim_{n\to\infty} f_n \, d\mu$$

by the Monotone Convergence Theorem, so $f = \lim_{n\to\infty} \tilde{f}_n \in \mathbb{F}$. Set $\rho = \int_X f \, d\mu = \mu_f$. Then by Example 19.2.1, $\rho << \mu$. It remains to check that $\lambda = \nu - \rho$ satisfies $\lambda \perp \mu$. Since $f \in \mathbb{F}$, it is clear that $\lambda \geq 0$. Suppose there is an $\varepsilon > 0$ and a set $E \in \mathcal{M}$ such that $\mu(E) > 0$ and $\lambda - \varepsilon\mu$ is positive on E. Then

$$\int_E \varepsilon \, d\mu \leq \int_E d\lambda \leq \int_X d\lambda = \lambda(X) = \nu(X) - \int_X f \, d\mu$$

since $\lambda \geq 0$. This implies $\int_X (f + \varepsilon\chi_E) \, d\mu \leq \nu(X)$, or in other words $f + \varepsilon\chi_E \in \mathbb{F}$. So $\mu_{f+\varepsilon\chi_E}(X) \leq M$, but on the other hand,

$$\begin{aligned}\mu_{f+\varepsilon\chi_E}(X) &= \int (f + \varepsilon\chi_E) \, d\mu = \int f \, d\mu + \varepsilon \int_E d\mu \quad \text{by linearity} \\ &= \int f \, d\mu + \varepsilon\mu(E) = M + \varepsilon\mu(E) > M.\end{aligned}$$

This is a contradiction, so no such ε and E exist. Hence by Lemma 19.2.5, $\lambda \perp \mu$.

To prove ρ and λ are unique, suppose $\nu = \rho' + \lambda'$ with $\rho' << \mu$, $d\rho' = f' \, d\mu$ and $\lambda' << \mu$. Then $\rho + \lambda = \rho' + \lambda'$ implies

$$\lambda' - \lambda = \rho - \rho' = f \, d\mu - f' \, d\mu = (f - f') \, d\mu.$$

Since $\lambda' - \lambda = (f - f') \, d\mu$ is simultaneously mutually singular and absolutely continuous with respect to μ, it must be the zero measure. Hence $\lambda' = \lambda$, $\rho' = \rho$ and the theorem holds in the positive finite case.

In the σ-finite case, there are countable sequences of μ-finite and ν-finite sets increasing to X, so we may intersect these and get a sequence $X_n \nearrow X$ such that $\mu(X_n) < \infty$ and $\nu(X_n) < \infty$ for all $n \in \mathbb{N}$. Define $\mu_n(E) = \mu(E \cap X_n)$ and $\nu_n(E) = \nu(E \cap X_n)$, so that each μ_n and ν_n is a finite measure. Then by the first part of the proof, for each $n \in \mathbb{N}$, $\nu_n = \int f_n \, d\mu_n + \lambda_n$ for some $f_n \in L^1(\mu_n)$ and $\lambda_n \perp \mu_n$. By Tonelli's theorem, $\rho = \sum_{n=1}^\infty \int f_n \, d\mu_n = \int \sum_{n=1}^\infty f_n \, d\mu_n$ so if $f = \sum_{n=1}^\infty$, then $f \in L^1(\mu)$ and $d\rho = f \, d\mu$ is σ-finite. We also have $\lambda_n(X_n) = \nu_n(X_n) - \int_{X_n} f_n \, d\mu_n < \infty$. So $\lambda = \sum_{n=1}^\infty \lambda_n$ is σ-finite. Finally, $\nu = \rho + \lambda$ and by Lemma 19.2.3, $\rho << \mu$ and $\lambda \perp \mu$, so this is the desired decomposition.

If ν is a signed measure, apply the results above to the positive measures ν^+ and ν^- and subtract. $\square$

Definition. *For a signed measure ν which is absolutely continuous with respect to a positive measure μ, the* **Lebesgue decomposition** *of ν with respect to μ is $\nu = \rho + \lambda$, where $\rho << \mu$ and $\lambda \perp \mu$ as in the Lebesgue-Radon-Nikodym theorem.*

Corollary 19.2.7. *If $\nu << \mu$ then there is an extended $f \in L^1(\mu)$ which is unique up to a.e. equivalence such that $d\nu = f\,d\mu$.*

Definition. *For $\nu << \mu$, the function f such that $d\nu = f\,d\mu$ is called the* **Radon-Nikodym derivative** *of ν with respect to μ, denoted $f = \frac{d\nu}{d\mu}$.*

Proposition 19.2.8. *Suppose ν, ν_1, ν_2 are σ-finite signed measures and μ, λ are σ-finite positive measures on $(X, \mathcal{M})$. Then*

(a) If $\nu_1, \nu_2 << \mu$ then $\frac{d(\nu_1+\nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}$.

(b) If $\nu << \mu$ and $\mu << \lambda$ then $\nu << \lambda$ and $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$ a.e.

Proposition 19.2.9. *For $j = 1, 2$, let μ_j, ν_j be σ-finite measures on $(X_j, \mathcal{M}_j)$ such that $\nu_j << \mu_j$. Then $\nu_1 \times \nu_2 << \mu_1 \times \mu_2$ and*

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2).$$

Proof. Let $\mu = \mu_1 \times \mu_2, \nu = \nu_1 \times \nu_2, \mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2$ and suppose $\nu_j << \mu_j$ for $j = 1, 2$. Then for any $E \in \mathcal{M}$, Fubini's theorem implies that the function $x_1 \mapsto \int_{X_2} \chi_E(x_1, x_2)\,d\mu_2$ is μ_1-measurable and $x_2 \mapsto \int_{X_1} \chi_E(x_1, x_2)\,d\mu_1$ is μ_2-measurable. For any $x_1 \in X_1, x_2 \in X_2$, set $E_{x_1} = \{y_2 \in X_2 \mid (x_1, y_2) \in E\}$ and $E_{x_2} = \{y_1 \in X_1 \mid (y_1, x_2) \in E\}$. Then Fubini's theorem allows us to write

$$\mu(E) = \int_{X_1 \times X_2} \chi_E\,d\mu = \int_{X_2}\int_{X_1} \chi_E(x_1, x_2)\,d\mu_1\,d\mu_2 = \int_{X_2}\int_{X_1} \chi_{E_{x_2}}(x_1)\,d\mu_2 = \int_{X_2} \mu_1(E_{x_2})\,d\mu_2$$

and

$$\nu(E) = \int_{X_1 \times X_2} \chi_E\,d\nu = \int_{X_2}\int_{X_1} \chi_E(x_1, x_2)\,d\nu_1\,d\nu_2 = \int_{X_2}\int_{X_1} \chi_{E_{x_2}}(x_1)\,d\nu_2 = \int_{X_2} \nu_1(E_{x_2})\,d\nu_2$$

If $\mu(E) = 0$ then by the top line,

$$\int_{X_2} \mu_1(E_{x_2})\,d\mu_2 = 0$$

which means that $\mu_1(E_{x_2})$ as a function on X_2 is 0 a.e. Thus there is some set $A \in \mathcal{M}_2$ such that $\mu_2(A) = 0$ and $\mu_1(E_{x_2}) = 0$ for all $x_2 \in A^C$. By hypothesis, $\nu_2(A) = 0$ and $\nu_1(E_{x_2}) = 0$ for all $x_2 \in A^C$. Then the second line above becomes

$$\nu(E) = \int_{X_2} \nu_1(E_{x_2})\,d\nu_2 = \int_A \nu_1(E_{x_2})\,d\nu_2 + \int_{A^C} \nu_1(E_{x_2})\,d\nu_2 = 0 + 0 = 0.$$

Therefore $\nu << \mu$ as claimed.

Now set $F = \frac{d\nu}{d\mu}, f_1 = \frac{d\nu_1}{d\mu_1}$ and $f_2 = \frac{d\nu_2}{d\mu_2}$. (Since $\nu << \mu$ from above, F is defined.) Then by definition of Radon-Nikodym derivatives, the functions F, f_1 and f_2 satisfy

$$\nu(E) = \int_E F\,d\mu, \quad \nu_1(E_1) = \int_{E_1} f_1\,d\mu_1, \quad \nu_2(E_2) = \int_{E_2} f_2\,d\mu_2$$

for any $E \in \mathcal{M}$ and $E_j \in \mathcal{M}_j, j = 1, 2$. By the Lebesgue-Radon-Nikodym theorem, $F \in L^1(\mu)$ and $f_j \in L^1(\mu_j)$ for $j = 1, 2$. Then by Fubini's theorem, for any $E = E_1 \times E_2 \in \mathcal{M}_1 \times \mathcal{M}_2$,

$$\nu(E) = \int_E F \, d\mu = \int_{E_1} \int_{E_2} f_1 f_2 \, d\mu_1 \, d\mu_2 = \int_{E_1} f_1 \, d\mu_1 \int_{E_2} f_2 \, d\mu_2 = \nu_1(E_1)\nu_2(E_2).$$

So $\nu(E) = \nu_1(E_1)\nu_2(E_2)$ whenever $E_1 \in \mathcal{M}_1, E_2 \in \mathcal{M}_2$. But by Lemma 18.4.1, ν is the unique measure on $\mathcal{M}$ such that $\nu(E_1 \times E_2) = \nu_1(E_1)\nu_2(E_2)$ for all $E_j \in \mathcal{M}_j, j = 1, 2$. Thus for any $E \in \mathcal{M}$,

$$\int_E F \, d\mu = \int_E f_1 f_2 \, d\mu$$

by the Fubini's theorem argument above, and since Radon-Nikodym derivatives are unique (up to a.e.), we must have $F = f_1 f_2$ as required. $\square$

19.3 Complex Measures

Definition. *A* **complex measure** *on a measurable space* $(X, \mathcal{M})$ *is a complex-valued set function* $\nu : \mathcal{M} \to \mathbb{C}$ *which is* σ*-additive and satisfies* $\nu(\varnothing) = 0$*. The* **real** *and* **imaginary parts** *of* ν *are defined as* $\nu_r = \operatorname{Re}\nu$ *and* $\nu_i = \operatorname{im}\nu$*, so that* $\nu(A) = \nu_r(A) + i\nu_i(A)$ *for all* $A \in \mathcal{M}$*.*

Definition. *For a complex measure* ν*, we say a function* $f : X \to \mathbb{C}$ *is* **integrable** *if* $f \in L^1(\nu) := L^1(\nu_r) \cap L^1(\nu_i)$*. For* $f \in L^1(\nu)$*, we define its* **complex integral** *by*

$$\int_X f\,d\nu = \int_X f\,d\nu_r + i\int_X f\,d\nu_i.$$

Remark. By the definition of signed integrable functions, for any complex measure ν

$$L^1(\nu) = L^1(\nu_r) \cap L^1(\nu_i) = L^1(\nu_r^+) \cap L^1(\nu_r^-) \cap L^1(\nu_i^+) \cap L^1(\nu_i^-).$$

Example 19.3.1. For any positive measure $\mu : \mathcal{M} \to [0, \infty]$ and complex-valued, μ-integrable function $g \in L^1(\mu)$, the measure $\nu(A) = \int_A g\,d\mu$ is a complex measure.

Lemma 19.3.2. *Let* ν *be a complex measure and* μ *be a positive measure. Then*

(1) $\nu << \mu$ *if and only if* $\nu_r << \mu$ *and* $\nu_i << \mu$*.*

(2) $\nu \perp \mu$ *if and only if* $\nu_r \perp \mu$ *and* $\nu_i \perp \mu$*.*

Theorem 19.3.3 (Lebesgue-Radon-Nikodym for Complex Measures)**.** *Given two* σ*-finite measures* μ *and* ν *on* $(X, \mathcal{M})$*, with* μ *positive and* ν *complex, there are unique complex measures* ρ *and* λ *such that* $\nu = \rho + \lambda$*,* $\rho << \mu$*,* $\lambda \perp \mu$ *and* $d\rho = f\,d\mu$ *for some* $f \in L^1(\mu)$*.*

Proof. Apply the Lebesgue-Radon-Nikodym theorem from Section 19.2 to the signed measures ν_r and ν_i and add together the results using linearity and Lemma 19.3.2. □

Definition. *For a complex measure* ν *and* $\mu \geq 0$ *such that* $\nu << \mu$*,* $\nu = \rho + \lambda$ *is called the* **Lebesgue decomposition** *of* ν *with respect to* μ*.*

Definition. *If* ν *is a complex measure and* $\nu << \mu$*, the a.e.-unique function* f *such that* $d\nu = f\,d\mu$ *is called the* **Radon-Nikodym derivative** *of* ν *with respect to* μ*, denoted* $f = \frac{d\nu}{d\mu}$*.*

Corollary 19.3.4. *For every complex measure* ν*, there is a positive measure* μ *and complex-valued function* $g \in L^1(\mu)$ *such that* $d\nu = g\,d\mu$*.*

Proof. Take $\mu = |\nu_r| + |\nu_i|$. Then μ is positive and $\nu << \mu$, so the Lebesgue-Radon-Nikodym theorem for complex measures implies the result. □

Definition. *Let* ν *be a complex measure with* $d\nu = g\,d\mu$ *for finite, positive* μ *and* $g \in L^1(\mu)$*. Then the* **total variation** $|\nu|$ *of* ν *is defined by*

$$|\nu|(A) = \int_A |g|\,d\mu$$

for all $A \in \mathcal{M}$*.*

Proposition 19.3.5. *The total variation of a complex measure ν is independent of the choices of μ and g such that $d\nu = g\,d\mu$.*

Proof. Suppose $d\nu = g_1\,d\mu_1 = g_2\,d\mu_2$ for finite, positive μ_1, μ_2 and $f_j \in L^1(\mu_j)$, for $j = 1, 2$. Then by Proposition 19.2.8,

$$g_1 \frac{d\mu_1}{d(\mu_1+\mu_2)}\, d(\mu_1+\mu_2) = g_1\,d\mu_1 = \nu = g_2\,d\mu_2 = g_2 \frac{d\mu_2}{d(\mu_1+\mu_2)}\, d(\mu_1+\mu_2).$$

So by uniqueness of the Radon-Nikodym derivative, $g_1 \frac{d\mu_1}{d(\mu_1+\mu_2)} = g_2 \frac{d\mu_2}{d(\mu_1+\mu_2)}$ a.e. with respect to $\mu_1 + \mu_2$. Since μ_1 and μ_2 are positive, it follows that

$$|g_1|\,d\mu_1 = |g_1| \frac{d\mu_1}{d(\mu_1+\mu_2)}\, d(\mu_1+\mu_2) = |g_2| \frac{d\mu_2}{d(\mu_1+\mu_2)}\, d(\mu_1+\mu_2) = |g_2|\,d\mu_2.$$

Hence the definition of $|\nu|$ is well-defined. □

Proposition 19.3.6. *Let ν be a complex measure on $(X, \mathcal{M})$. Then*

(a) For all $A \in \mathcal{M}$, $|\nu(A)| \le |\nu|(A)$.

(b) $\nu << |\nu|$ and if $f = \frac{d\nu}{d|\nu|}$ then $|f| = 1$ a.e. with respect to $|\nu|$.

(c) $L^1(\nu) = L^1(|\nu|)$.

(d) For every $f \in L^1(\nu)$, $\left|\int_X f\,d\nu\right| \le \int_X |f|\,d|\nu|$.

Proof. (a) By Corollary 19.3.4, we may write $d\nu = g\,d\mu$ for some positive μ and $g \in L^1(\mu)$. Then for any $A \in \mathcal{M}$,

$$|\nu(A)| = \left|\int_A f\,d\mu\right| \le \int_A |f|\,d\mu = |\nu|(A)$$

by Proposition 18.2.8 and the definition of total variation.

(b) $\nu << |\nu|$ follows from (a). Then $f = \frac{d\nu}{d|\nu|}$ is defined and $g\,d\mu = d\nu = f\,d|\nu| = f|g|\,d\mu$ which shows that $g = f|g|$ μ-a.e. Hence $f = \frac{g}{|g|}$ $|\nu|$-a.e. Since μ is positive, $|g| > 0$ μ-a.e. which implies $|f| = 1$ $|\nu|$-a.e.

(c) – (d) Similar to the proof of Lemma 19.1.8(a) and (b). □

Theorem 19.3.7. *Let $\mathcal{A} \subseteq \mathbb{P}(X)$ be an algebra, $\mathcal{M} = \sigma(\mathcal{A})$ and suppose ν is a complex measure on $(X, \mathcal{M})$. Define*

$$\mu_0(E) = \sup\left\{\sum_{j=1}^n |\nu(E_j)| : E_j \in \mathcal{A}_E \textit{ is a partition of } E\right\}$$

$$\mu_1(E) = \sup\left\{\sum_{j=1}^n |\nu(E_j)| : E_j \in \mathcal{M}_E \textit{ is a partiion of } E\right\}$$

$$\mu_2(E) = \sup\left\{\sum_{j=1}^\infty |\nu(E_j)| : E_j \in \mathcal{M}_E \textit{ is a partition of } E\right\}$$

$$\mu_3(E) = \sup\left\{\left|\int_E f\,d\mu\right| : |f| \le 1\right\}.$$

Then $\mu_0 = \mu_1 = \mu_2 = \mu_3 = |\nu|$.

Proof. First consider

$$\begin{aligned}\left|\int_E f\,d\nu\right| &= \left|\int_E fg\,d|\nu|\right| \\ &\le \int_E |f|\,|g|\,d|\nu| \\ &\le |\nu|(E) \quad \text{since } g(E)\subseteq S^1.\end{aligned}$$

Thus $\mu_3(E) \le |\nu|(E)$. On the other hand, take $f = \bar{g} = g^{-1}$ to see that $\mu_3(E) = |\nu|(E)$ for all $E \in \mathcal{M}$.

Next, it is clear that $\mu_0 \le \mu_1 \le \mu_2$. Note that if $\{E_j\}$ is a partition of E, then

$$\begin{aligned}\sum_{j=1}^{\infty} |\nu(E_j)| &= \sum_{j=1}^{\infty}\left|\int_E g\,d|\nu|\right| \\ &\le \sum_{j=1}^{\infty}\int_{E_j} |g|\,d|\nu| \\ &= \sum_{j=1}^{\infty} |\nu|(E_j) \\ &= |\nu|(E) = \mu_3(E).\end{aligned}$$

So $\mu_2 \le \mu_3$.

Finally, we complete the proof by showing $|\nu| \le \mu_0$. Recall (as in Lemma 18.1.1) that there are simple functions

$$g_n = \sum_{k=1}^{N_n} z_k^n \chi_{A_k^n}$$

for $z_k^n \in D^2$ and $A_k^n \in \mathcal{A}_E$ disjoint such that $g_n \nearrow g$ pointwise and $\int |g - g_n|\,d|\nu| \to 0$. Then

$$|\nu|(E) = \int_E 1\,d|\nu| = \int_E \bar{g}\,d\nu = \lim_{n\to\infty}\int_E \bar{g}_n\,d\nu$$

and

$$\begin{aligned}\left|\int_E \bar{g}_n\,d\nu\right| &= \left|\sum_{k=1}^{N_n} \bar{z}_k^n \nu(E\cap A_k^n)\right| \\ &\le \sum_{k=1}^{N_n} |z_k^n|\,|\nu(E\cap A_k^n)| \\ &\le \sum_{k=1}^{N_n} |\nu(E\cap A_k^n)| \le \mu_0(E).\end{aligned}$$

Taking the limit as $n \to \infty$ gives

$$|\nu|(E) = \lim_{n\to\infty}\int_E \bar{g}_n\,d\nu \le \mu_0(E).$$

Hence all five measures are equal. ∎

19.4 Complex Lebesgue Integration

Let ν be a complex measure on $\mathbb{R}^n$ and λ be Lebesgue measure on $\mathbb{R}^n$. Then by the Lebesgue-Radon-Nikodym theorem (19.3.3), $\nu = \rho + \lambda'$ for measures $\rho << \lambda$ and $\lambda' \perp \lambda$, and there is some $g \in L^1(\lambda)$ such that $d\rho = g\,d\lambda$. Write $d\nu = g d\lambda + d\lambda'$. Our goal is to prove:

Theorem 19.4.1. *If $f(x) = x$ a.e. with respect to Lebesgue measure, then*

$$\lim_{r \searrow 0} \frac{\nu(B(f(x), r))}{\lambda(B(f(x), r))} = g(x).$$

Lemma 19.4.2 (Covering Lemma)**.** *Let $\mathcal{E}$ be a collection of open balls in $\mathbb{R}^n$ and let $U = \bigcup_{B \in \mathcal{E}} B$. If $c < \lambda(U)$ then there exist disjoint $B_1, \ldots, B_k \in \mathcal{E}$ such that*

$$c < 3^n \sum_{j=1}^{k} \lambda(B_j).$$

Proof. By Corollary 17.2.15(ii), choose is a compact set $K \subseteq U$ such that $\lambda(K) > c$ and let $\mathcal{E}_1 \subseteq \mathcal{E}$ be a finite subcover of K. There is some $B_1 \in \mathcal{E}_1$ of largest diameter (though it need not be unique). Let $\mathcal{E}_2 = \{A \in \mathcal{E}_1 \mid A \cap B_1 = \varnothing\}$. If $\mathcal{E}_2 \neq \varnothing$, choose $B_2 \in \mathcal{E}_2$ with largest diameter and repeat until $\mathcal{E}_{k+1} = \varnothing$. This gives us our $B_1, \ldots, B_k \in \mathcal{E}$. For any ball $B = B(x, r)$, let $B^* = B(x, 3r)$. We will show that $K \subseteq \bigcup_{j=1}^{k} B_j^*$ to prove the desired formula. For each $A \in \mathcal{E}_1$, there is a smallest $1 \leq j \leq k$ such that $B_j \cap A \neq \varnothing$. In this case, $\operatorname{diam}(A) \leq \operatorname{diam}(B_j)$ so $A \subseteq B_j^*$. Therefore $A \subseteq \bigcup_{j=1}^{k} B_j^*$ so $K \subseteq \bigcup_{A \in \mathcal{E}_1} A \subseteq \bigcup_{j=1}^{k} B_j^*$. Finally, subadditivity yields

$$\lambda(K) \leq \sum_{j=1}^{k} \lambda(B_j^*) \leq 3^n \sum_{j=1}^{k} \lambda(B_j).$$

$\square$

Definition. *A function $f : \mathbb{R}^n \to \mathbb{C}$ is* **locally integrable** *if $f\chi_K \in L^1(\lambda)$ for all bounded sets $K \in \mathcal{B}^n$. The set of all such f is denoted $L^1_{loc}(\lambda)$.*

Definition. *For $f \in L^1_{loc}(\lambda), x \in \mathbb{R}^n$ and $r > 0$, we define the* **averaging integral**

$$(A_r f)(x) = \frac{1}{\lambda(B(x, r))} \int_{B(x,r)} f\,d\lambda.$$

Lemma 19.4.3. *$A_\bullet f : (0, \infty) \times \mathbb{R}^n \to \mathbb{C}$ is jointly continuous in $r > 0$ and $x \in \mathbb{R}^n$.*

Let $E_r = B(x, r)$. We begin proving Theorem 19.4.1 through a series of refinements.

Lemma 19.4.4. *Let ρ, λ' be complex measures.*

(1) If $\rho << \lambda$, then $\frac{\rho(E_r)}{\lambda(E_r)} \to \frac{d\rho}{d\lambda}(x)$ as $r \searrow 0$.

(2) If $\lambda' \perp \lambda$, then $\frac{\lambda'(E_r)}{\lambda(E_r)} \to 0$ as $r \searrow 0$.

Proof. (2) Suppose $\lambda' \geq 0$. Then $\lambda' \perp \lambda$ means there exists some $A \in \mathcal{B}$ such that $\lambda'(A) = 0$ and $\lambda(A^C) = 0$. Given $\varepsilon > 0$, there exists $V_\varepsilon \subset \mathbb{R}^n$ such that $A \subseteq V_\varepsilon$ and $\lambda'(V_\varepsilon) < \varepsilon$ by Corollary 17.2.15(iii). Let

$$F_k = \left\{ x \in A : \limsup_{r\to 0} \frac{\lambda'(E_r)}{\lambda(E_r)} > \frac{1}{k} \right\}$$

and set $F = \bigcap_{k=1}^\infty F_k$. For $x \in F_k$, there exists $r_x > 0$ such that $B(x, r_x) \subseteq V_\varepsilon$ and

$$\lambda'(B(x, r_x)) > \frac{1}{k}\lambda(B(x, r_x)).$$

Consider the collection $\mathcal{E} = \{B(x, r_x) \mid x \in F\}$. Let $U = \bigcup_{B\in\mathcal{E}} B \subseteq V_\varepsilon$ and note that $F_k \subseteq U$. So for all $c < \lambda(U)$, the covering lemma (19.4.2) produces $B_1, \ldots, B_N \in \mathcal{E}$ such that $c < 3^n \sum_{j=1}^N \lambda(B_j)$. Thus

$$c < 3^n \sum_{j=1}^N \lambda(B_j) < 3^n k \sum_{j=1}^N \lambda'(B_j) \leq 3^n k \lambda'(U) < 3^n k\varepsilon.$$

Taking the limit as $c \nearrow \lambda(U)$, we get $\lambda(U) < 3^n k\varepsilon$ so since $\varepsilon > 0$ was arbitrary, we must have $\lambda(U) = 0$. Hence $\lambda(F_k) = 0$ for all $k \geq 1$ since $F_k \subseteq U$ and $\lambda \geq 0$. This implies $\lambda(F) = 0$ so $\frac{\lambda'(E_r)}{\lambda(E_r)} \to 0$ as required. For general λ', the statement follows from the fact that λ' is regular (in the sense of Corollary 17.2.15) if and only if $|\lambda'|$ is.

(1) is proven similarly. ☐

Lemma 19.4.5. *If f is a continuous function, then $\lim_{r\to 0} A_r f(x) = f(x)$ pointwise a.e.*

Proof. For continuous f and any $x \in \mathbb{R}^n$, we have

$$\begin{aligned} |A_r f(x) - f(x)| &= \frac{\left|\int_{B(x,r)} (f(y) - f(x))\, dy\right|}{|\lambda(B(x,r))|} \\ &\leq \frac{\int_{B(x,r)} |f(y) - f(x)|\, dy}{\lambda(B(x,r))} \\ &\leq \sup_{y\in B(x,r)} |f(y) - f(x)| \end{aligned}$$

and the latter converges to 0 as $r \to 0$ by continuity of f. ☐

Definition. *The* **Hardy-Littlewood maximal function** *of any $f \in L^1_{loc}(\lambda)$ is*

$$Hf(x) = \sup_{r>0} A_r|f|(x).$$

Lemma 19.4.6. *For any $f \in L^1_{loc}(\lambda)$, $Hf(x)$ is measurable.*

Proof. By Lemma 19.4.3, $A_r|f|(x)$ is jointly continuous in r and x, hence measurable by Lemma 17.4.13. Apply Proposition 17.4.14. ☐

Theorem 19.4.7 (Maximal Theorem). *If $f \in L^1_{loc}(\lambda)$ then for all $\alpha > 0$,*

$$\lambda\left(\{x : Hf(x) > \alpha\}\right) \leq \frac{3^n}{\alpha}\int_{\mathbb{R}^n} |f|\, d\lambda.$$

Proof. Set $E_\alpha = \{x : Hf(x) > \alpha\}$. For all $x \in E_\alpha$, there exists some r_x such that $A_{r_x}|f|(x) > \alpha$ by Corollary 17.2.15(iii). Thus

$$\lambda(B(x, r_x)) < \frac{1}{\alpha}\int_{B(x,r_x)} |f|\, d\lambda.$$

Then $E_\alpha \subseteq \bigcup_{x\in E_\alpha} B(x, r_x)$ and if $c < \lambda(E_\alpha) < \lambda\left(\bigcup_{x\in E_\alpha} B(x, r_x)\right)$. The covering lemma (19.4.2) says that there exist $x_1, \ldots, x_N \in E_\alpha$ such that $B_i = B(x_i, r_{x_i})$ satisfy

$$\begin{aligned}
c < \sum_{i=1}^N 3^n \lambda(B_i) &< \frac{3^n}{\alpha}\sum_{i=1}^N \int_{B_i} |f|\, d\lambda \\
&= \frac{3^n}{\alpha}\int_{\mathbb{R}^n}\sum_{i=1}^N \chi_{B_i}|f|\, d\lambda \quad \text{by Tonelli's theorem (18.4.2)} \\
&< \frac{3^n}{\alpha}\int_{\mathbb{R}^n} |f|\, d\lambda.
\end{aligned}$$

Now let $c \nearrow \lambda(E_\alpha)$ to finish. □

Theorem 19.4.8. *For any $f \in L^1_{loc}(\lambda)$, $\lim_{r\to 0} A_r f(x) = f(x)$ pointwise a.e.*

Proof. If $f \in L^1_{loc}(\lambda)$, then $\tilde{f} := f\chi_{[-N,N]}$ is integrable for every $N \in \mathbb{N}$, so it suffices to prove the theorem for $f \in L^1(\lambda)$. A generalization of Theorem 18.2.9 shows that we can find a continuous $g \in L^1(\lambda)$ approximating f with respect to Lebesgue measure. Then by Lemma 19.4.5, $A_r g(x) \to g(x)$ pointwise a.e. so

$$\begin{aligned}
|A_r f(x) - f(x)| &\leq |A_r f(x) - A_r g(x)| + |A_r g(x) - g(x)| + |g(x) - f(x)| \\
&\leq \sup_{r>0} A_r|f - g|(x) + 0 + |f - g|(x).
\end{aligned}$$

For $\alpha > 0$, let

$$E_\alpha = \left\{x : \limsup_{r>0} |A_r f(x) - f(x)| > \alpha\right\} \subseteq \left\{x : H|f-g|(x) > \frac{\alpha}{2}\right\} \cup \left\{x : |f - g| > \frac{\alpha}{2}\right\}.$$

Then by the maximal theorem,

$$\begin{aligned}
\lambda(E_\alpha) &\leq \lambda\left(\left\{x : H|f-g| > \frac{\alpha}{2}\right\}\right) + \lambda\left(\left\{x : |f-g| > \frac{\alpha}{2}\right\}\right) \\
&\leq \frac{3^n}{\alpha/2}\int_{\mathbb{R}^n} |f - g|\, d\lambda + \frac{1}{\alpha/2}\int_{\mathbb{R}^n} |f-g|\, d\lambda
\end{aligned}$$

which can be made small since g approximates f. Since $\alpha > 0$ was arbitrary, $\lambda(E_\alpha) = 0$. Hence $\lim_{r\to 0} A_r f(x) = f(x)$ for all $x \notin \bigcup_{n=1}^\infty E_{1/n}$ so the converges is pointwise a.e. □

Definition. *For $f \in L^1_{loc}(\lambda)$, the* **Lebesgue set** *of f is*

$$L_f = \left\{ x : \lim_{r \to 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, dy = 0 \right\}.$$

The proof of Theorem 19.4.8 implies:

Corollary 19.4.9. *For all $f \in L^1_{loc}(\lambda)$, $\lambda(L_f^C) = 0$.*

Definition. *We say a family of sets $\{E_r\}$* **shrinks nicely to** *$x \in \mathbb{R}^n$ if for every r, $E_r \subseteq B(x,r)$ and there exists $\alpha > 0$ such that $\lambda(E_r) > \alpha\lambda(B(x,r))$.*

Corollary 19.4.10 (Lebesgue Differentiation)**.** *For every $f \in L^1_{loc}(\lambda)$ and $x \in L_f$,*

$$\lim_{r \to 0} \frac{1}{\lambda(E_r)} \int_{E_r} f \, d\lambda = f(x)$$

for every family $\{E_r\}$ shrinking nicely to x.

We now prove Theorem 19.4.1 by proving the following generalization.

Theorem 19.4.11. *Let ν be a regular complex Borel measure on $\mathbb{R}^n$ and let $d\nu = f \, d\lambda + d\lambda'$. Then for almost every $x \in \mathbb{R}^n$,*

$$\lim_{r \to 0} \frac{\nu(E_r)}{\lambda(E_r)} = f(x)$$

for every family $\{E_r\}$ shrinking nicely to x.

Proof. Note that if ν is regular, then λ' and $f \, d\lambda$ are as well and thus $f \in L^1_{loc}(\lambda)$. By Corollary 19.4.10, it then suffices to show that for almost x,

$$\lim_{r \to 0} \frac{\lambda'(E_r)}{\lambda(E_r)} = 0$$

for every family shrinking nicely to x. We have this already for $E_r = B(x,r)$ (Lemma 19.4.4), but for any $\{E_r\}$ shrinking nicely to x, there is some $\alpha > 0$ such that

$$\left| \frac{\lambda'(E_r)}{\lambda(E_r)} \right| \leq \frac{|\lambda'|(E_r)}{\lambda(E_r)} \leq \frac{|\lambda|(B(x,r))}{\lambda(E_r)} \leq \frac{|\lambda|(B(x,r))}{\alpha\lambda(B(x,r))}.$$

Therefore the limit for E_r follows from the limit for $B(x,r)$ and we are done. □

19.5 Functions of Bounded Variation

Recall (Section 17.3) that for finite, positive Borel measures μ on $\mathbb{R}$, there is a distribution function $F(x) = \mu((-\infty, x])$. If λ is Lebesgue measure on $\mathbb{R}$ and $\mu << \lambda$, this distribution can be written

$$F(x) = \int_{(-\infty,x]} f \, d\lambda$$

where $f = \frac{d\mu}{d\lambda}$ is the Radon-Nikodym derivative of μ. Recall that F also satisfies $\mu((a, b]) = F(b) - F(a)$.

Theorem 19.5.1. *Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function and define*

$$G(x) = F(x^+) = \lim_{y \to x^+} F(y).$$

Then

(1) G is increasing and right continuous.

(2) For $x \in \mathbb{R}$, $G(x) = \lim_{y \to x^+} F(y^-)$.

(3) The set of discontinuities of F is countable.

(4) There exists a unique Borel measure $\mu = \mu_G$ on $\mathbb{R}$ such that $\mu((a, b]) = G(b) - G(a)$ for all $a < b$.

(5) For $x \in \mathbb{R}$, $F'(x)$ and $G'(x)$ exist a.e. and $F'(x) = G'(x)$ a.e.

(6) $F'(x) \in L^1_{loc}(\lambda)$ and there exists a unique positive Borel measure ν_s on $\mathbb{R}$ such that $\nu_s \perp \lambda$ and

$$G(b) - G(a) = F(b^+) - F(a^+) = \int_{(a,b]} F' \, d\lambda + v_s((a, b]).$$

Moreover, if F is bounded then $F' \in L^1(\lambda)$.

We want to generalize this for complex measures and ultimately obtain a generalized version of the fundamental theorem of calculus. Let μ be a complex measure on $(\mathbb{R}, \mathcal{B})$ and define a distribution F by $F(x) = \mu((-\infty, x])$ for all $x \in \mathbb{R}$. Then as in Lemma 17.3.1, we have:

- $\mu((a, b]) = F(b) - F(a)$ for all $a < b$.
- F is right continuous.
- $|\mu|((a, b]) = \sup_P \sum_{x_i \in P} |F(x_{i+1}) - F(x_i)|$ where the supremum is over all partitions $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$.

Definition. *Given a function $F : \mathbb{R} \to \mathbb{C}$, the* **total variation** *of F is*

$$T_F((a,b]) = \sup_P \sum_{x_i \in P} |F(x_{i+1}) - F(x_i)|$$

for partitions P and $a < b$,

$$\begin{aligned} T_F((-\infty, b]) &= \lim_{a\to\infty} T_F((a,b]) \quad \textit{for all } b \in \mathbb{R} \\ \textit{and} \quad T_F(\mathbb{R}) &= \lim_{b\to\infty} T_F((-\infty, b]). \end{aligned}$$

In particular, when F is a distribution for a complex measure $\mu : \mathcal{B} \to \mathbb{C}$, we have $|\mu| = T_F$.

Definition. *A function $F : \mathbb{R} \to \mathbb{C}$ has* **bounded variation** *if $T_F(\mathbb{R}) < \infty$. Set*

$$\begin{aligned} BV &= \{F : \mathbb{R} \to \mathbb{C} \mid F \textit{ has bounded variation}\} \\ NBV &= \{F \in BV \mid F(-\infty) = 0\}. \end{aligned}$$

For any $F \in BV$, the function $G(x) = F(X^+) - F(-\infty)$ is right continuous and $F = G$ a.e.

Theorem 19.5.2. *There is a bijective correspondence*

$$\left\{ \begin{pmatrix} NBV \textit{ right continuous functions} \\ F : \mathbb{R} \to \mathbb{C} \end{pmatrix} \right\} \longleftrightarrow \left\{ \begin{pmatrix} \textit{finite complex Borel} \\ \textit{measures } \mu \end{pmatrix} \right\}$$
$$F(x) \longmapsto \mu_F \textit{ such that } \mu_F((-\infty, x]) = F(x).$$

To prove this, first consider $F : \mathbb{R} \to \mathbb{R}$ with total variation $T_F(x) = \mu((-\infty, b])$.

Lemma 19.5.3. *$T_F + F$ is nondecreasing on $\mathbb{R}$.*

Proof. Let $x < y$ and take $\varepsilon > 0$. Choose a partition P of $(-\infty, x]$ such that

$$\sum_P |F(x_i) - F(x_{i-1})| \geq T_F(x) - \varepsilon.$$

Then $P \cup \{y\}$ is a partition of $(-\infty, y]$, so

$$T_F(y) \geq \sum_P |F(x_i) - F(x_{i-1})| + |F(y) - F(x)|.$$

Now we have

$$\begin{aligned} T_F(y) + F(y) &\geq \sum_P |F(x_i) - F(x_{i-1})| + |F(y) - F(x)| + F(y) \\ &= \sum_P |F(x_i) - F(x_{i-1})| + |F(y) - F(x)| + (F(y) - F(x)) + F(x) \\ &\geq \sum_P |F(x_i) - F(x_{i-1})| + F(x) \\ &\geq T_F(x) + F(x) - \varepsilon. \end{aligned}$$

Taking $\varepsilon \to 0$, we get that $T_F(y) - F(y) \geq T_F(x) - F(x)$. ☐

Similarly:

Lemma 19.5.4. *$T_F(x) - F(x)$ is nondecreasing on $\mathbb{R}$.*

Definition. *For a function $F \in BV$, we define its* **positive variation** *by $F_+ = \frac{1}{2}(T_F + F)$ and its* **negative variation** *by $F_- = \frac{1}{2}(T_F - F)$.*

Clearly $F = F_+ - F_-$, so any $F \in BV$ can be expressed as the difference of two nondecreasing functions of bounded variation. This implies Theorem 19.5.2 for $F : \mathbb{R} \to \mathbb{R}$ as follows. Write $F = F_+ - F_-$ for nondecreasing, right continuous functions $F_+, F_- \in BV$. Then by Theorem 17.3.2, there exist unique, finite, positive measures μ_{F_+} and μ_{F_-} on $\mathcal{B}$ corresponding to F_+ and F_-, respectively. Then the signed measure $\mu_F := \mu_{F_+} - \mu_{F_-}$ satisfies $\mu_F((-\infty, x]) = F(x)$ by definition. We call $F = F_+ - F_-$ and $\mu_F = \mu_{F_+} - \mu_{F_-}$ the *Jordan decomposition* of F (and/or μ_F).

Now suppose $F : \mathbb{R} \to \mathbb{C}$ is a function of bounded variation. Then $\operatorname{Re} F, \operatorname{im} F \in BV$ as well. Applying the result for each of these gives us unique positive measures $\mu_{\operatorname{Re} F}$ and $\mu_{\operatorname{im} F}$ on $\mathcal{B}$, and $\mu_F := \mu_{\operatorname{Re} F} + i\mu_{\operatorname{im} F}$ is the desired complex measure satisfying $\mu_F((-\infty, x]) = F(x)$ whenever $x \in \mathbb{R}$. This completes the proof of Theorem 17.3.2.

To define differentiation for functions of bounded variation, start again with $F : \mathbb{R} \to \mathbb{R}$ and write $F = F_+ - F_-$. Then by Theorem 19.5.1, F has a countable set of discontinuities and F_+' and F_-' are defined almost everywhere.

Definition. *The* **derivative** *of $F : \mathbb{R} \to \mathbb{R}$ is $F' = F_+' - F_-'$.*

Note that μ_F has a Lebesgue decomposition by Theorem 19.2.6: $\mu_F = \int g \, d\lambda + \lambda'$ for $g \in L^1(\lambda)$ and $\lambda' \perp \lambda$. Moreover, for almost every $x \in \mathbb{R}$,

$$g(x) = \lim_{r \searrow 0} \frac{\mu_F(E_r)}{\lambda(E_r)}$$

for any family E_r shrinking nicely to $\{x\}$, by Corollary 19.4.10. In particular, these limits exist for $E_r^+ = (x, x+r]$ and $E_r^- = (x-r, x]$ and we are able to define left and right derivatives for F.

Definition. *For $F \in BV$, the* **right derivative** *of F is*

$$\frac{d^+}{dx} F = \lim_{r \searrow 0} \frac{\mu_F(E_r^+)}{\lambda(E_r^+)} = \lim_{r \searrow 0} \frac{F(x+r) - F(x)}{r}$$

and the **left derivative** *of F is*

$$\frac{d^-}{dx} F = \lim_{r \searrow 0} \frac{\mu_F(E_r^-)}{\lambda(E_r^-)} = \lim_{r \searrow 0} \frac{F(x) - F(x-r)}{r}.$$

In particular, if $g = \frac{d\mu_F}{d\lambda}$ then $g(x) = \frac{d^+}{dx} F(x) = \frac{d^-}{dx} F(x)$ and $F' = g$ almost everywhere. For $a < b$, we *almost* have the fundamental theorem of calculus:

$$F(b) - F(a) = \int_a^b F'(x) \, d\lambda(x) + \lambda((a, b]).$$

Theorem 19.5.5 (Fundamental Theorem of Calculus for Lebesgue Integration). *Let $F : \mathbb{R} \to \mathbb{R}$ be a function of bounded variation. Then*

$$\int_a^b F'(x)\, d\lambda(x) = F(b) - F(a)$$

if and only if $\mu_F << \lambda$.

Recall from Lemma 19.2.4 that $\mu_F << \lambda$ if and only if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\mu_F(E)| < \varepsilon$ for all measurable sets E having $\lambda(E) < \delta$. Taking $E = \coprod_{i=1}^n (a_i, b_i]$ motivates the following definition.

Definition. *A function $F : \mathbb{R} \to \mathbb{R}$ is* **absolutely continuous** *if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $a_i, b_i \in \mathbb{R}$ for which $(a_i, b_i]$ are disjoint,*

$$\sum_{i=1}^n (b_i - a_i) < \delta \qquad \textit{implies} \qquad \left|\sum i = 1^n (F(b_i) - F(a_i))\right| < \varepsilon.$$

Proposition 19.5.6. *F is absolutely continuous if and only if $\mu_F << \lambda$.*

Lemma 19.5.7. *Suppose F is absolutely continuous on $\mathbb{R}$. Then*

(a) F is continuous on $\mathbb{R}$.

(b) $T_F((a, b]) < \infty$ for all $a < b$ in $\mathbb{R}$.

Theorem 19.5.8. *If $F \in NBV$, then $F' \in L^1(\lambda)$ and*

(1) $\mu_F \perp \lambda$ if and only if $F' = 0$ a.e.

(2) $\mu_F << \lambda$ if and only if $F(x) = \int_{-\infty}^x F'(t)\, d\lambda(t)$ for all $x \in \mathbb{R}$.

Corollary 19.5.9 (Fundamental Theorem of Calculus). *For any function $F : [a, b] \to \mathbb{C}$, the following are equivalent:*

(1) F is absolutely continuous on $[a, b]$.

(2) There exists some $f \in L^1([a, b])$ such that

$$F(x) - F(a) = \int_a^x f(t)\, dt.$$

(3) F is differentiable a.e. on $[a, b]$, $F' \in L^1([a, b])$ and

$$F(x) - F(a) = \int_a^x F'(t)\, dt.$$

Chapter 20

Function Spaces

20.1 More on Banach Spaces

Let $\mathbb{F}$ be a field (usually taken to be $\mathbb{R}$ or $\mathbb{C}$) and let X be a normed linear space over $\mathbb{F}$. Recall that X is a Banach space if it is complete with respect to the metric $\rho(x,y) = ||x-y||$.

Example 20.1.1. Suppose $(X, \mathcal{M}, \mu)$ is a measure space. For any Borel measurable function $f : X \to \mathbb{F}$, define

$$||f||_\infty := \inf\{M > 0 : |f| \le M \ \mu\text{-a.e.}\}.$$

Then $L^\infty(\mu) = \{$a.e. equivalence classes of Borel measurable $f : X \to \mathbb{F} : ||f||_\infty < \infty\}$ is a Banach space with the norm $||\cdot||_\infty$.

Definition. *For a normed linear space* $(X, ||\cdot||)$ *over* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$*, the* **dual space** *of* X *is* $X^* = \mathcal{L}(X, \mathbb{F})$*. Elements of* X^* *are called* **linear functionals** *on* X*.*

Example 20.1.2. Consider the normed linear space $X = C[0,1]$ over $\mathbb{C}$. An example of a functional on this space is $L \in \mathcal{L}(X, \mathbb{C})$ defined for each $f \in C[0,1]$ by

$$L(f) = \int_0^1 f(x)\,dx.$$

More generally, for a finite or complex Borel measure μ on $[0,1]$, define an operator

$$L_\mu(f) := \int_0^1 f\,d\mu.$$

Then L_μ is bounded, since

$$\begin{aligned} ||L_\mu(f)|| &\le \int_0^1 |f|\,d|\mu| \le ||f||_u |\mu|([0,1]) \\ \implies ||L_\mu||_{op} &\le |\mu|([0,1]) < \infty. \end{aligned}$$

Hence $L_\mu \in C[0,1]^*$.

Remark. To distinguish between types of dual spaces, we write X^* for the topological dual (defined above) and $X' = \{$linear operators $f : X \to \mathbb{F}\}$ for the algebraic dual.

The following result allows us to reduce from studying complex functionals to real functionals. Let $X^*_{\mathbb{R}}$ denote the real dual space to $X_{\mathbb{R}}$.

Proposition 20.1.3. *If X is a complex vector space, then*

(1) If $f \in X^*$ and $u = \operatorname{Re} f$, then for all $x \in X$, $f(x) = u(x) - iu(ix)$.

(2) Conversely, then for any $u \in X^*_{\mathbb{R}}$, $f(x) = u(x) - iu(ix)$ is an element of the dual X^*.

(3) If X is a normed linear space, then $||f||_{X^*} = ||u||_{X^*_{\mathbb{R}}}$ for $u = \operatorname{Re} f$.

Proof. (1) Clearly u is (real) linear and $\operatorname{im} f(x) = -\operatorname{Re}(if(x)) = -u(ix)$, so $f(x) = u(x) - iu(ix)$.

(2) Since u is real linear, $f(x) = u(x) - iu(ix)$ is also real linear. Moreover, $f(ix) = u(ix) - iu(-x) = u(ix) + iu(x) = i(u(x) - iu(ix)) = if(x)$ so f is complex linear as well.

(3) For any $x \in X$, $|u(x)| = |\operatorname{Re} f(x)| \le |f(x)|$ implies that $||u|| \le ||f||$. On the other hand, choose $x \in X$ such that $f(x) \neq 0$ and choose $\lambda \in S^1 \subseteq \mathbb{C}$ such that $\lambda f(x) = |f(x)|$. Then $|f(x)| = f(\lambda x) = \operatorname{Re} f(\lambda x) = u(\lambda x)$ so

$$\frac{|f(x)|}{||x||} = \frac{u(\lambda x)}{|\lambda|\,||x||} = \frac{u(\lambda x)}{||\lambda x||} \le \sup_{y\neq 0} \frac{u(y)}{||y||} = ||u||.$$

Taking the sup over all $x \neq 0$, we get $||f|| \le ||u||$ so they are equal. □

Definition. *A* **sublinear** *(or* **Minkowski***)* **functional** *on an $\mathbb{F}$-vector space X is a function $P : X \to [0, \infty)$ such that*

(1) $P(x + y) \le P(x) + P(y)$ for any $x, y \in X$.

(2) $P(\lambda y) = \lambda P(y)$ for all $\lambda \ge 0$ in $\mathbb{F}$.

Example 20.1.4. Every seminorm is a sublinear functional.

Theorem 20.1.5 (Hahn-Banach)**.** *Suppose X is a real vector space, $P : X \to [0, \infty)$ is a sublinear functional and $M \subseteq X$ is a subspace such that $f \in M'$ and $f \le P$ on M. Then there exists a functional $F \in X'$ such that $F|_M = f$ and $F \le P$ on X.*

Proof. Given $x \notin M$, we will find an extension F of f to $M \oplus \mathbb{R}x$ such that $F \le P$ on $M \oplus \mathbb{R}x$. If such an F exists, set $\alpha = F(x)$. Then $F(m + \lambda x) = f(m) + \lambda\alpha$ for all $m \in M, \lambda \in \mathbb{R}$. So if we can find α such that $f(m) + \lambda\alpha \le P(m + \lambda x)$ for all $m \in M, \lambda \in \mathbb{R}$, then we'll be done. This inequality is equivalent to each of the following inequalities:

$$\alpha \le \frac{P(m + \lambda x) - f(m)}{\lambda} \quad \text{for } \lambda > 0 \tag{20.1}$$

$$\alpha \ge \frac{f(\tilde{m}) - P(\tilde{m} - \tilde{\lambda}x)}{\tilde{\lambda}} \quad \text{for } \tilde{\lambda} > 0. \tag{20.2}$$

In turn, (1) and (2) are equivalent to

$$\begin{aligned} \frac{f(\tilde{m}) - P(\tilde{m} - \tilde{\lambda}x)}{\tilde{\lambda}} &\le \alpha \le \frac{P(m + \lambda x) - f(m)}{\lambda} \\ \iff f(\lambda\tilde{m} + \tilde{\lambda}m) &\le \tilde{\lambda}P(m + \lambda x) + \lambda P(\tilde{m} - \tilde{\lambda}x) \\ &= P(\tilde{\lambda}m + \lambda\tilde{\lambda}x) + P(\lambda\tilde{m} - \lambda\tilde{\lambda}x). \end{aligned}$$

But by assumption,

$$\begin{aligned} &= P(\lambda\tilde{m} - \lambda\tilde{\lambda}x + \lambda\tilde{\lambda}x + \tilde{\lambda}m) \\ &= P(\lambda\tilde{m} - \lambda\tilde{\lambda}x) + P(\lambda\tilde{\lambda}x + \tilde{\lambda}m). \end{aligned}$$

So such an α exists and therefore so does the extension F to $M \oplus \mathbb{R}x$.

Now we use Zorn's Lemma to extend this to all of X. For $g : X \to \mathbb{R}$, let $D(g) \subseteq X$ denote any subspace on which g is linear. For $f, g : X \to \mathbb{R}$ which are linear on some subspace of X, we will write $f < g$ if g extends f, i.e. $D(f) \subseteq D(g)$ and $g|_{D(f)} = f$. Set

$$\mathcal{F} = \{F : X \to \mathbb{R} \mid F \leq P \text{ on } D(f) \text{ and } f < F\}.$$

Then $(\mathcal{F}, <)$ is a poset. If $\Phi \subseteq \mathcal{F}$ is a totally ordered subset, then

$$D = \bigcup_{g \in \Phi} D(g)$$

must be a subspace of X. Define F on $D = D(F)$ by $F(x) = g(x)$ if $x \in D(g)$. So $F \in \mathcal{F}$ and $g < F$ for all $g \in \Phi$. Hence every totally ordered subset of $\mathcal{F}$ has an upper bound so we can apply Zorn's Lemma to see that $\mathcal{F}$ contains a maximal element F. That is, F has no extension in $\mathcal{F}$. However, the first construction implies $D(F) = X$, so we are finished. $\square$

Corollary 20.1.6 (Complex Hahn-Banach)**.** *If X is a complex normed linear space, $P : X \to \mathbb{R}_{\geq 0}$ is a seminorm and $f : X \to \mathbb{C}$ is a linear functional such that $|f| \leq P$ on $D(f)$, then there exists a functional $F \in X'$ such that $F|_{D(f)} = f$ and $|F| \leq P$ on X.*

Proof. First let $u = \operatorname{Re} f$ and note that $u \leq |\operatorname{Re} f| \leq |f| \leq P$ on $D(f) = D(u)$, so by Theorem 20.1.5, there exists a real functional $U \in X'_{\mathbb{R}}$ such that $U|_{D(u)} = u$ and $U \leq P$. Now $-U(x) = U(-x) \leq P(-x) = P(x)$ so we have $|U| \leq P$ on X. By the proof of Proposition 20.1.3, $F(x) = U(x) - iU(ix)$ is the extension we are looking for. $\square$

Corollary 20.1.7. *Suppose $(X, ||\cdot||)$ is a normed linear space and $M \subseteq X$ is a closed subspace. For $x \in X \setminus M$, let $\delta = \operatorname{dist}(x, M) = \inf\{||x - y|| : y \in M\}$. Then there exists a linear functional $f \in X^*$ satisfying:*

(1) $||f||_{op} = 1$.

(2) $f|_M = 0$.

(3) $f(x) = \delta$.

Proof. Define $F(m + \lambda x) = \lambda\delta$ for all $\lambda \in \mathbb{C}, m \in M$. Then $F \in (M \oplus \mathbb{C}x)'$ and obviously $F|_M = 0$. Notice that for all $\lambda \neq 0$ and $m \in M$, $\frac{m}{\lambda} \in M$ implies

$$||m + \lambda x|| = |\lambda| \left|\left|\frac{m}{\lambda} + x\right|\right| \geq |\lambda|\delta = |F(m + \lambda x)|.$$

Thus the operator norm of F on $M \oplus \mathbb{C}x$ may be computed:

$$||F||_{op} = \sup_{\substack{m \in M \\ \lambda \in \mathbb{C}}} \frac{|F(m + \lambda x)|}{||m + \lambda x||} = \sup_{\substack{m \in M \\ \lambda \in \mathbb{C}}} \frac{|\lambda|\delta}{||m + \lambda x||} = \frac{\delta}{\delta} = 1.$$

In particular, $|F| \leq ||\cdot||$ so the complex Hahn-Banach theorem (Corollary 20.1.6) gives us $f \in X'$ such that $f|_{M\oplus\mathbb{C}x} = F$ and $|f(z)| \leq ||z||$ for all $z \in X$. This in fact means $||f||_{op} \leq 1$, so $f \in X^*$. On the other hand,

$$\begin{aligned} ||f||_{op} &= \sup_{z\in X} \frac{|f(z)|}{||z||} \\ &\geq \sup_{\substack{m\in M\\ \lambda\in\mathbb{C}}} \frac{|f(m+\lambda x)|}{||m+\lambda x||} \\ &= \sup_{\substack{m\in M\\ \lambda\in\mathbb{C}}} \frac{|F(m+\lambda x)|}{||m+\lambda x||} = ||F||_{op} = 1. \end{aligned}$$

So $||f||_{op} = 1$. □

Corollary 20.1.8. *For every $x \in X$, there exists a functional $f \in X^*$ such that $f(x) = ||x||$ and $||f||_{op} = 1$.*

Proof. Apply Corollary 20.1.7 to the subspace $M = 0$. □

Corollary 20.1.9. *The map*

$$\begin{aligned} X &\longrightarrow X^{**} \\ x &\longmapsto (\hat{x} : f \mapsto f(x)) \end{aligned}$$

is an isometry and in particular an injection.

Proof. For any $f \in X^*$, we have

$$|\hat{x}(f)| = |f(x)| \leq ||f||_{X^*}||x||_X = ||x||_X||f||_{op}.$$

So $||\hat{x}||_{X^{**}} \leq ||x||_X$. On the other hand, if $f \in X^*$ satisfies the conditions of Corollary 20.1.8, then

$$|\hat{x}(f)| = |f(x)| = ||x||_X = ||x||_X||f||_{X^*}$$

so that $||\hat{x}||_{X^{**}} \geq ||x||_X$. Hence $||\hat{x}||_{X^{**}} = ||x||_X$ for all $x \in X$, so $x \mapsto \hat{x}$ is indeed an isometry. □

Definition. *A Banach space X is said to be* **reflexive** *if $X \to X^{**}, x \mapsto \hat{x}$ is also surjective.*

Remark. Denote the image of X in X^{**} under the isometry in Corollary 20.1.9 by $\widehat{X}$. This is sometimes called the *completion* of X. If X is a Banach space, then $\widetilde{X}$ is closed in X^{**}. In particular, a Banach space X is reflexive if and only if it embeds as a dense subset of X^{**}.

More generally, we say an arbitrary normed linear space X is reflexive if $\overline{\widehat{X}} = X^{**}$.

20.2 Hilbert Spaces

Definition. *An* **inner product space** $(H, \langle \cdot, \cdot \rangle)$ *consists of a vector space H over $\mathbb{C}$ and a map $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ satisfying*

(1) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ *for all* $x, y \in H$.

(2) For all $y \in H$, $x \mapsto \langle x, y \rangle$ is linear and $\langle x, ay + bz \rangle = \bar{a}\langle x, y \rangle + \bar{b}\langle x, z \rangle$.

(3) For all $x \in H$, $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

Example 20.2.1. Let $H = L^2(\mu)$ be the Lebesgue space associated to a measure space $(X, \mathcal{M}, \mu)$, as defined in Section 18.3. Then the pairing

$$\langle f, g \rangle = \int_X f(x)\overline{g(x)}\, d\mu(x)$$

makes $L^2(\mu)$ into an inner product space.

Example 20.2.2. As an example of the above, let $X = \{1, 2, \ldots, n\}$ and let μ be the counting measure (Example 17.3.8). Then $H = \mathbb{C}^n$ is an inner product space with inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \bar{y}_i.$$

This extends naturally to ℓ^2 by taking infinite sums.

Proposition 20.2.3. *Define $||x|| = \sqrt{\langle x, x \rangle}$. Then $(H, ||x||)$ is a normed linear space.*

Lemma 20.2.4 (Cauchy-Schwarz Inequality)**.** *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space. Then for any $x, y \in H$, $|\langle x, y \rangle| \leq ||x||\, ||y||$.*

Proof. Suppose $x, y \neq 0$. Then by orthogonal projection,

$$\begin{aligned}
0 &\leq \left|\left| y - \frac{\langle y, x \rangle}{||x||^2} x \right|\right|^2 \\
&= ||y||^2 + 2\,\mathrm{Re}\left\langle y, \frac{\langle y, x \rangle}{||x||^2} x \right\rangle + \left|\left| \frac{\langle y, x \rangle}{||x||^2} x \right|\right|^2 \\
&= ||y||^2 - 2\,\mathrm{Re}\left(\frac{\overline{\langle y, x \rangle}}{||x||^2} \langle y, x \rangle \right) + \frac{|\langle y, x \rangle|}{||x||^4} ||x||^2 \\
&= ||y||^2 - \frac{|\langle y, x \rangle|^2}{||x||^2}.
\end{aligned}$$

Rearranging, we get $|\langle y, x \rangle|^2 \leq ||y||^2 ||x||^2$. ∎

Lemma 20.2.5 (Parallelogram Law)**.** *For any $x, y \in H$, $||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2$. Conversely, if $||\cdot||$ is a norm on H satisfying the parallelogram law, then there exists an inner product $\langle \cdot, \cdot \rangle$ on H such that $||x||^2 = \langle x, x \rangle$.*

Definition. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space. Two vectors $x, y \in H$ are* **orthogonal** *if $\langle x, y \rangle = 0$. More generally, a set $S \subseteq H$ is orthogonal if x, y are orthogonal for all $x \neq y$ in S.*

Proposition 20.2.6 (Pythagorean Theorem)**.** *If S is a finite orthogonal set, then*

$$\left|\left|\sum_{x \in S} x\right|\right|^2 = \sum_{x \in S} ||x||^2.$$

Proof. Using the properties of an inner product, we have

$$\left|\left|\sum_{x \in S} x\right|\right|^2 = \left\langle \sum_{x \in S} x, \sum_{y \in S} y \right\rangle = \sum_{x \in S} \sum_{y \in S} \langle x, y \rangle = \sum_{x \in S} \langle x, x \rangle$$

since all cross-terms are zero. □

Definition. *An inner product space $(H \langle \cdot, \cdot \rangle)$ is called a* **Hilbert space** *if H is complete with respect to the topology induced by the norm $|| \cdot ||$.*

If $A \subseteq H$ is a subset of an inner product space, denote by $A^\perp$ the set of all vectors $x \in H$ orthogonal to every element of A.

Lemma 20.2.7. *If $A \subseteq H$ is any set, then $A^\perp$ is a closed subspace of H.*

Proof. For each $x \in H$, let p_x be the linear map $p_x(y) = \langle y, x \rangle$. Then

$$A^\perp = \bigcap_{x \in A} \ker p_x = \bigcap_{x \in A} p_x^{-1}(\{0\}).$$

□

Definition. *A set $K \subseteq H$ is* **convex** *if for all $x, y \in K$, $tx + (1-t)y$ lies in K for all $t \in [0, 1]$.*

Example 20.2.8. Any vector subspace of H is convex.

Theorem 20.2.9. *If H is a Hilbert space and K is a closed, convex subset, then for all $x \in H$, there exists a unique $y \in K$ so that $||x - y|| = \operatorname{dist}(x, K)$.*

Proof. Let $\delta = \operatorname{dist}(x, K)$ and assume $x \notin K$. Then there exists a sequence $(y_n) \subseteq K$ such that $||x - y_n|| \to \delta$ as $n \to \infty$. Let $n, m \geq 1$ and consider

$$\begin{aligned} 2||x - y_n||^2 + 2||x - y_m||^2 &= ||2x - y_n - y_m||^2 + ||y_n - y_m||^2 \quad \text{by Lemma 20.2.5} \\ &= 4\left|\left|x - \frac{y_n + y_m}{2}\right|\right|^2 + ||y_n - y_m||^2. \end{aligned}$$

Now $\frac{y_n + y_m}{2} \in K$ by convexity, so

$$4\left|\left|x - \frac{y_n + y_m}{2}\right|\right|^2 + ||y_n - y_m||^2 \geq 4\delta^2 + ||y_n - y_m||^2.$$

Taking the limsup as $n, m \to \infty$, we get

$$2\delta^2 + 2\delta^2 \geq 4\delta^2 + \limsup_{n,m\to\infty} ||y_n - y_m||^2$$

which implies $||y_n - y_m|| \to 0$. Hence (y_n) is Cauchy in H, meaning (y_n) converges to some $y \in H$ by completeness. Moreover, since K is closed, $y \in K$. Now by continuity of $||\cdot||$,

$$||x - y|| = \lim_{n\to\infty} ||x - y_n|| = \delta.$$

To see y is unique, suppose $z \in K$ also satisfies $||x - z|| = \delta$. Then $\frac{y+z}{2} \in K$ by convexity, so

$$\begin{aligned} 4\delta^2 &= 2||x-y||^2 + 2||x-z||^2 \\ &= ||2x - y - z||^2 + ||y - z||^2 \\ &= 4\left|\left|x - \frac{y+z}{2}\right|\right|^2 + ||y-z||^2 \\ &= 4\delta^2 + ||y-z||^2. \end{aligned}$$

Therefore $||y - z|| = 0$, so $y = z$. □

Corollary 20.2.10. *If M is a closed subspace of H, then for all $x \in H$ there exists a unique $y \in M$ such that $||x - y|| = \operatorname{dist}(x, M)$. Moreover, y is the unique element of M having $x - y \in M^\perp$.*

Proof. The first statement follows immediately from Theorem 20.2.9. For the second statement, suppose $y \in M$ is such that $x - y \in M^\perp$. Then for all $z \in M$,

$$\begin{aligned} ||x-z||^2 &= ||(x-y) + (y-z)||^2 \\ &= ||x-y||^2 + ||y-z||^2 \quad \text{by Prop. 20.2.6} \\ &\geq ||x-y||^2. \end{aligned}$$

Moreover, this is an equality only if $||y - z|| = 0$. Thus y, if it exists, is the unique element of M minimizing $||x - y||$. Conversely, suppose $y \in M$ is such that $||x - y|| = \delta = \operatorname{dist}(x, M)$. Then for any $z \in M$, define $g_z(t) = ||x - y + tz||^2$. Since M is convex, $-y + tz \in M$ and

$$g_z(t) = ||x-y||^2 + 2t\operatorname{Re}\langle x - y, z\rangle + t^2||z||^2.$$

Thus $g_z(t)$ has a minimum at $t = 0$, so $\operatorname{Re}\langle x - y, z\rangle = 0$ for all $z \in M$. Hence $x - y \in M^\perp$. □

Definition. *If M is a closed subspace of H and $x \in H$, the unique $y \in M$ such that $x - y \in M^\perp$ is called the* **orthogonal projection** *of x onto M, denoted $\operatorname{proj}_M x$.*

Proposition 20.2.11. *Let M be a closed subspace and $P = \operatorname{proj}_M$ the orthogonal projection map $H \to H$. Then*

(1) P is linear.

(2) $\operatorname{im} P = M$.

(3) $\ker P = M^\perp$.

(4) $P^2 = P$.

(5) $P = P^*$.

Proof. (1) For $x, z \in H$ and $\lambda \in \mathbb{C}$,

$$x + \lambda z - Px - \lambda Pz = x - Px + \lambda(z - Pz) \in M^\perp.$$

In particular, $P(x + \lambda z) = Px + \lambda Pz$ by uniqueness.

(2) It's clear that $\operatorname{im} P \subseteq M$. On the other hand, if $m \in M$, then $Pm = m$ so $M \subseteq \operatorname{im} P$.

(3) $x \in \ker P \iff Px = 0 \iff x = x - 0 \in M^\perp$.

(4) is obvious from the definition of P.

(5) For any $x, y \in H$,

$$\begin{aligned}\langle Px, y\rangle &= \langle Px, y - Py + Py\rangle \\ &= \langle Px, y - Py\rangle + \langle Px, Py\rangle \\ &= 0 + \langle Px, Py\rangle = \langle Px, Py\rangle.\end{aligned}$$

Similarly $\langle x, Py\rangle = \langle Px, Py\rangle$ so we get that $\langle Px, y\rangle = \langle x, Py\rangle$. In other words, $P = P^*$. □

Corollary 20.2.12 (Orthogonal Decomposition)**.** *If M is a proper closed subspace of a Hilbert space H, then $H = M \oplus M^\perp$.*

Proof. If $x \in H$, then $x = y + (x - y)$ for $y = \operatorname{proj}_M x$. Then $y \in M$ and $x - y \in M^\perp$, so we see that $H = M + M^\perp$. Further, it is clear that 0 is the unique vector orthogonal to all vectors in H, so we get $M \cap M^\perp = \{0\}$. □

Theorem 20.2.13 (Riesz Representation)**.** *Let H be a Hilbert space. Then the map $\varphi : H \to H^*$ given by $\varphi(z) = \langle \cdot, z\rangle$ is a conjugate-linear isometric isomorphism.*

Proof. The fact that φ is conjugate-linear follows from the definition of the inner product. Since isometries are always injective, it suffices to show that φ is a surjective isometry. Fix $z \in H$. Then

$$||\varphi(z)||_{H^*} = ||\langle \cdot, z\rangle||_{H^*} = \sup_{||y||=1} |\langle y, z\rangle| \le \sup_{||y||=1} ||y|| \cdot ||z|| = ||z||,$$

by the Cauchy-Schwarz inequality (Lemma 20.2.4). Hence $||\varphi(z)||_{H^*} \le ||z||$. Also, equality is achieved when $y = z$: $(\varphi(z))(z) = \langle z, z\rangle = ||z||^2$, so $||\varphi(z)||_{H^*} = ||z||$. Therefore φ is an isometry.

Now to show surjectivity, choose a nonzero linear functional $f \in H^*$. Let $M = \ker f$ and take $x_0 \in M^\perp = (\ker f)^\perp$. Since $M^\perp$ is a subspace, we may assume $||x_0|| = 1$. For any $x \in H$, we have that $u = f(x)x_0 = f(x_0)x \in \ker f$, so

$$0 = \langle u, x_0\rangle = f(x)||x_0||^2 - f(x_0)\langle x, x_0\rangle = f(x) - \langle x, \overline{f(x_0)}x_0\rangle.$$

Therefore $f(x) = \langle x, \overline{f(x_0)}x_0\rangle$ for all $x \in H$. Taking $x_f = \overline{f(x_0)}x_0$, we have that $f = \langle \cdot, x_f\rangle = \varphi(x_f)$ so φ is surjective. □

Suppose that X and Y are normed vector spaces and $T \in \mathcal{L}(X,Y)$.

Definition. *Define $T^\dagger : Y^* \to X^*$ by $T^\dagger f = f \circ T$. Then $T^\dagger$ is called the* **adjoint** *or* **transpose** *of T.*

Lemma 20.2.14. *$T^\dagger \in \mathcal{L}(Y^*, X^*)$ and $||T^\dagger|| = ||T||$.*

Proof. Set $\mathcal{L} = \mathcal{L}(X,Y)$. For any $f \in Y^*$ and $x \in X$,

$$|T^\dagger f(x)| = |f \circ T(x)| \leq ||f||_{Y^*}||T(x)||_Y \leq ||f||_{Y^*}||T||_{\mathcal{L}}||x||_X.$$

Then since $f : Y \to \mathbb{F}$ is a bounded linear operator to the ground field and $T \in \mathcal{L}$, $||f||_{Y^*}$ and $||T||_{\mathcal{L}}$ are both defined (and finite). The above then shows that $||T^\dagger f||_{X^*} \leq ||T||_{\mathcal{L}}||f||_{Y^*}$ for all $f \in Y^*$, so it follows that $T^\dagger \in \mathcal{L}(Y^*, X^*)$, with $||T^\dagger||_{\mathcal{L}(Y^*,X^*)} \leq ||T||_{\mathcal{L}}$. On the other hand, for any $\varepsilon > 0$ there is some $x \in X$, $||x||_X = 1$, such that $||Tx||_Y > ||T||_{\mathcal{L}} - \varepsilon$ by definition of the $\mathcal{L}$-norm. For this x, Corollary 20.1.8 says there is some $f \in Y^*$ with $||f||_{Y^*} = 1$ and $f(Tx) = ||Tx||_Y$. Then we have

$$|T^\dagger f(x)| = |f \circ T(x)| = ||Tx||_Y > ||T||_{\mathcal{L}} - \varepsilon.$$

So $||T^\dagger f||_{X^*} > ||T||_{\mathcal{L}} - \varepsilon$, and since $||f||_{Y^*} = 1$, this shows $||T||_{\mathcal{L}} \leq \sup_{||f||=1} ||T^\dagger f||_{X^*} = ||T^\dagger||_{\mathcal{L}(Y^*,X^*)}$. Hence $||T^\dagger||_{\mathcal{L}(Y^*,X^*)} = ||T||_{\mathcal{L}}$ as claimed. $\square$

Applying the adjoint operation twice, one obtains $T^{\dagger\dagger} \in \mathcal{L}(X^{**}, Y^{**})$. The following states that bounded linear operators satisfy a type of duality.

Lemma 20.2.15. *If X and Y are identified with their natural images $\widehat{X}$ and $\widehat{Y}$ in X^{**} and Y^{**}, then $T^{\dagger\dagger}|_X = T$.*

Proof. For any $\hat{x} \in \widehat{X}$, let $x \in X$ such that $\hat{x}(f) = f(x)$ for all $f \in X^*$. Then for any such $f \in X^*$,

$$(T^{\dagger\dagger}\hat{x})(f) = (\hat{x} \circ T^\dagger)(f) = \hat{x}(f \circ T) = f \circ T(x) = \widehat{Tx}(f).$$

This shows that $T^{\dagger\dagger}\hat{x} = \widehat{Tx}$ in $\widehat{Y}$, and since $\hat{x} \in \widehat{X}$ was arbitrary, $T^{\dagger\dagger} = T$. $\square$

Lemma 20.2.16. *$T^\dagger$ is injective iff the range of T is dense in Y.*

Proof. ($\implies$) If $T(X)$ is not dense in Y, there exists a $y \in Y$ such that $y \notin \overline{T(X)}$. By Corollary 20.1.7, there exists $f \in Y^*$ satisfying $||f||_{Y^*} = 1, f|_{\overline{T(X)}} = 0$ and $f(y) = \operatorname{dist}(y, \overline{T(X)}) \neq 0$. Applying $T^\dagger$, we see that for any $x \in X$, $T^\dagger f(x) = f(Tx) = 0$ since $Tx \in T(X)$. This implies $T^\dagger f = 0$, so $T^\dagger$ is not injective.

($\impliedby$) If $T^\dagger$ is not injective, $\ker T^\dagger \neq 0$, meaning we can find a nonzero functional $f \in \ker T^\dagger$. Since f is nonzero, $f(y) \neq 0$ for some $y \in Y$, but by continuity of linear operators, f is nonzero on some (nonempty) neighborhood $V \subset Y$ containing y. Of course $V \neq Y$ since f is linear, meaning $f(0) = 0$. By assumption $T^\dagger f = 0$, i.e. $f(Tx) = 0$ for any $x \in X$, so none of the elements of V can be of the form Tx for $x \in X$. In other words, $T(X) \subseteq V^C$, but since V is an open neighborhood, V^C is closed. It follows that $\overline{T(X)} \subseteq V^C$ but $V^C \neq Y$ since V is nonempty. In particular, $\overline{T(X)} \neq Y$. $\square$

Lemma 20.2.17. *If the range of $T^\dagger$ is dense in X^*, then T is injective; the converse is true if X is reflexive.*

Proof. Suppose T is not injective. Then there is some nonzero $x \in X$ with $Tx = 0$ in Y. We may scale x so that $||x||_X = 1$, since $T\left(\frac{x}{||x||}\right) = 0$ by linearity. By Corollary 20.1.8, there is a functional $f \in X^*$ with $||f||_{X^*} = 1$ and $f(x) = ||x||_X = 1$. In particular, for any $g \in B(f, 1)$ in X^*, i.e. $||f - g||_{X^*} < 1$, we get

$$|1 - g(x)| = |f(x) - g(x)| \leq ||f - g||_{X^*}||x||_X = ||f - g||_{X^*} < 1.$$

If $g(x) = 0$, we get $1 < 1$, a contradiction, so $g(x) \neq 0$. However, notice that for all $h \in Y^*$, $T^\dagger h(x) = h(Tx) = h(0) = 0$ so $g \neq T^\dagger h$ for any $h \in Y^*$. Since $g \in B(f, 1)$ was arbitrary, we get $T^\dagger(Y^*) \subseteq B(f, 1)^C$. As in Lemma 20.2.16, $B(f, 1)^C$ is a closed proper subset of X^*, so $\overline{T^\dagger(Y^*)} \subseteq B(f, 1)^C \subsetneq X^*$. Hence $T^\dagger(Y^*)$ is not dense in X^*.

For the second statement, assume X is reflexive but $T^\dagger(Y^*)$ is not dense in X^*. Then we can find some $f \in X^*$ such that $f \notin \overline{T^\dagger(Y^*)}$, which means $\delta = \text{dist}(f, \overline{T^\dagger(Y^*)}) > 0$. By Corollary 20.1.7, choose $\hat{x} \in X^{**}$ such that $||\hat{x}||_{X^{**}} = 1$, $\hat{x}|_{T^\dagger(Y^*)} = 0$ and $\hat{x}(f) = \delta \neq 0$. Since X is reflexive, $\hat{x}$ is the image of some $x \in X$ under the isometry $X \to X^{**}$. Then $||x||_X = ||\hat{x}||_{X^{**}} = 1 \neq 0$ so $x \neq 0$. Suppose $Tx \neq 0$. Then by Corollary 20.1.8, there is some $g \in Y^*$ so that $g(Tx) \neq 0$. However, $g(Tx) = \hat{x}(g \circ T) = \hat{x}(T^\dagger g) = 0$, contradicting the fact that $\hat{x}$ is 0 on $T^\dagger(Y^*)$. Thus we must have $Tx = 0$, but x was nonzero, so T is not injective. □

Now let H be a Hilbert space and $T \in \mathcal{L}(H, H)$. Let $V : H \to H^*$ be the conjugate-linear isomorphism $(Vy)(x) = \langle x, y \rangle$ in Theorem 20.2.13.

Definition. *The operator $T^* = V^{-1}T^\dagger V \in \mathcal{L}(H, H)$ is called the* **adjoint** *of T.*

Proposition 20.2.18. *For every $T \in \mathcal{L}(H, H)$, T^* is the unique element of $\mathcal{L}(H, H)$ satisfying $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in H$.*

Proof. Letting $x, y \in H$, we have

$$\langle x, T^*y \rangle = (V(T^*y))(x) = (V(V^{-1}T^\dagger Vy))(x) = (T^\dagger Vy)(x) = (Vy)(Tx) = \langle Tx, y \rangle.$$

For uniqueness, suppose $S \in \mathcal{L}(H, H)$ is another operator satisfying $\langle Tx, y \rangle = \langle x, Sy \rangle$ for all $x, y \in H$. For a fixed $y \in H$, $\langle x, Sy \rangle = \langle x, T^*y \rangle$ for all x, so $V(Sy) = V(T^*y)$, but since V is an isomorphism, $Sy = T^*y$. Since y was arbitrary, we must have $S = T^*$, so the adjoint is unique. □

Lemma 20.2.19. *The adjoint $T^* : H \to H$ satisfies the following properties:*

(i) $||T^*|| = ||T||$.

(ii) $||T^*T|| = ||T||^2$.

(iii) $(aS + bT)^* = \bar{a}S^* + \bar{b}T^*$.

(iv) $(ST)^* = T^*S^*$.

(v) $T^{**} = T$.

Proof. (v) Let $x, y \in H$. Then by sesquilinearity, $\langle T^*x, y\rangle = \overline{\langle y, T^*x\rangle} = \overline{\langle Ty, x\rangle} = \langle x, Ty\rangle$. Since T^{**} is the unique operator satisfying $\langle T^*x, y\rangle = \langle x, T^{**}y\rangle$ by Proposition 20.2.18, we must have $T = T^{**}$.

(i) For all $x \in H$,

$$\begin{aligned} ||Tx||^2 &= \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle \\ &\leq ||x|| \cdot ||T^*Tx|| \quad \text{by Cauchy-Schwarz (20.2.4} \\ &\leq ||x|| \cdot ||T^*|| \cdot ||Tx||. \end{aligned}$$

From this we see that $||Tx|| \leq ||x|| \cdot ||T^*||$, and since this holds for all x, we get $||T|| \leq ||T^*||$. Applying this once more to T and $T^{**} = T$, we get $||T^*|| \leq ||T^{**}|| = ||T||$ so we have equality: $||T^*|| = ||T||$.

(ii) From the proof of (i), we have

$$||x|| \cdot ||T^*Tx|| \leq ||x|| \cdot ||T^*|| \cdot ||Tx|| = ||x|| \cdot ||T|| \cdot ||Tx|| \leq ||T||^2||x||^2.$$

So it follows that $||T^*T|| \leq ||T||^2$. On the other hand, let $y \in H$ with $||y|| = 1$. The calculation in (i) shows that $||Ty||^2 \leq ||T^*T||\,||y||^2 = ||T^*T||$ so we get $||T||^2 \leq ||T^*T||$. Hence $||T^*T|| = ||T||^2$ as claimed.

(iii) Let $a, b \in \mathbb{C}$ and $S, T \in \mathcal{L}(H, H)$. For any $f \in H^*$, we have $(aS + bT)^\dagger f = f \circ (aS + bT) = f \circ (aS) + f \circ (bT) = a(f \circ S) + b(f \circ T) = aS^\dagger f + bT^\dagger f = (aS^\dagger + bT^\dagger)f$ by linearity. Thus we see that $(aS + bT)^\dagger = aS^\dagger + bT^\dagger$. Now by definition of the adjoint,

$$\begin{aligned} (aS + bT)^* &= V(aS + bT)^\dagger V^{-1} = V(aS^\dagger + bT^\dagger)V^{-1} \\ &= V(aS^\dagger)V^{-1} + V(bT^\dagger)V^{-1} \\ &= \bar{a}VS^\dagger V^{-1} + \bar{b}VT^\dagger V^{-1} \quad \text{since } V \text{ is conjugate linear} \\ &= \bar{a}S^* + \bar{b}T^*. \end{aligned}$$

(iv) Let $S, T \in \mathcal{L}(H, H)$. Notice that for any $f \in H^*$, $(ST)^\dagger f = f \circ (ST) = (f \circ S) \circ T = T^\dagger(f \circ S) = T^\dagger(S^\dagger f)$. So $(ST)^\dagger = T^\dagger S^\dagger$. Now by definition of the adjoint,

$$(ST)^* = V(ST)^\dagger V^{-1} = VT^\dagger S^\dagger V^{-1} = VT^\dagger V^{-1}VS^\dagger V^{-1} = T^*S^*.$$

□

Corollary 20.2.20. *For any $T \in \mathcal{L}(H, H)$, T is unitary iff T is invertible and $T^{-1} = T^*$.*

Proof. ($\Longrightarrow$) Every unitary transformation is an isometry, so in particular invertible. For any $x, y \in H$, we see that $\langle x, T^{-1}y\rangle = \langle Tx, TT^{-1}y\rangle = \langle Tx, y\rangle$. By the uniqueness of T^* from Proposition 20.2.18, this implies $T^{-1} = T^*$.

($\Longleftarrow$) Suppose T is invertible with $T^{-1} = T^*$. Then for all $x, y \in H$, we must show $\langle x, y\rangle = \langle Tx, Ty\rangle$. But we have $\langle Tx, Ty\rangle = \langle x, T^*Ty\rangle = \langle x, T^{-1}Ty\rangle = \langle x, y\rangle$ so T is indeed unitary. □

Definition. *If $\mathcal{B} = \{u_\alpha\}_{\alpha\in A}$ is a subset of a Hilbert space H, we say $\mathcal{B}$ is* **orthogonal** *if $\langle u_\alpha, u_\beta\rangle = 0$ for every $\alpha \neq \beta$ in A. Further, $\mathcal{B}$ is* **orthonormal** *if $\langle u_\alpha, u_\beta\rangle = \delta_{\alpha\beta}$ for all $\alpha, \beta \in A$, where $\delta_{\alpha\beta}$ denotes the Kronecker delta.*

Example 20.2.21. Let $\{e_1, e_2, e_3\}$ be the standard basis in $\mathbb{R}^3$. Considering $\mathbb{R}^3$ with the standard inner product (the dot product), $\{e_1, e_2, e_3\}$ is an orthonormal set. for any vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we have $\langle x, e_j\rangle = x_j$ for each of $j = 1, 2, 3$. Notice that

$$||x||^2 = x_1^2 + x_2^2 + x_3^2 \geq x_1^2 + x_2^2 = ||P_M(x)||^2$$

where $M = \mathrm{Span}(e_1, e_2)$. This is generalized by Bessel's inequality, proven next.

Proposition 20.2.22 (Bessel's Inequality)**.** *Let $\{u_\alpha\}_{\alpha\in A}$ be an orthonormal set in a Hilbert space H. Then for all $x \in H$,*

$$\sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2 \leq ||x||^2.$$

Proof. Let $\Gamma \subset A$ be a finite subset. Then

$$\begin{aligned}
0 &\leq \left|\left| x - \sum_{\alpha\in\Gamma} \langle x, u_\alpha\rangle u_\alpha \right|\right|^2 \\
&= ||x||^2 - 2\mathrm{Re} \sum_{\alpha\in\Gamma} \overline{\langle x, u_\alpha\rangle}\langle x, u_\alpha\rangle + \left|\left| \sum_{\alpha\in\Gamma} \langle x, u_\alpha\rangle u_\alpha \right|\right|^2 \\
&= ||x||^2 - 2\sum_{\alpha\in\Gamma} |\langle x, u_\alpha\rangle|^2 + \sum_{\alpha\in\Gamma} ||\langle x, u_\alpha\rangle u_\alpha||^2 \quad \text{by the Pythagorean theorem} \\
&= ||x||^2 - \sum_{\alpha\in\Gamma} |\langle x, u_\alpha\rangle|^2.
\end{aligned}$$

Rearranging, we obtain $\sum_{\alpha\in\Gamma} |\langle x, u_\alpha\rangle|^2 \leq ||x||^2$. Finally, taking the supremum over all finite subsets $\Gamma \subset A$ gives the result. $\square$

Definition. *Let X be a normed linear space and suppose $\{v_\alpha\}_{\alpha\in A}$ is a sequence in X. We say the limit $s = \sum_{\alpha\in A} v_\alpha$ exists in X if for all $\varepsilon > 0$, there is some finite subset $\Gamma_\varepsilon \subset A$ such that*

$$\left|\left| s - \sum_{\alpha\in\Lambda} v_\alpha \right|\right| < \varepsilon$$

for all finite subsets $\Lambda \subset A$ with $\Gamma_\varepsilon \subseteq \Lambda$.

Lemma 20.2.23. *Suppose X is a Banach space and $\{v_\alpha\}_{\alpha\in A}$ is a sequence in X. Consider the formal expression $s = \sum_{\alpha\in A} v_\alpha$.*

(1) If s exists in X, then $\{\alpha \in A \mid v_\alpha \neq 0\}$ is at most countable.

(2) If s exists, then every linear operator $T \in \mathcal{L}(X, Y)$ on X satisfies

$$T\left(\sum_{\alpha\in A} v_\alpha\right) = \sum_{\alpha\in A} T v_\alpha.$$

(3) If $\sum_{\alpha\in A}||v_\alpha||^2<\infty$ then $s=\sum_{\alpha\in A}v_\alpha$ exists in X.

Proposition 20.2.24. *If H is a Hilbert space and $A\subseteq H$ is an orthogonal set, then $s=\sum_{v\in A}v$ exists in H if and only if $\sum_{v\in A}||v||^2<\infty$. Moreover, if $\sum_{v\in A}||v||^2<\infty$,*

(1) $||s||=\sum_{v\in A}||v||^2$.

(2) For all $x\in H$, $\langle s,x\rangle=\sum_{v\in A}\langle v,x\rangle$.

Example 20.2.25. If $\{v_n\}_{n=1}^\infty\subseteq H$ is an orthogonal sequence in a Hilbert space, then $s=\sum_{n=1}^\infty v_n=\lim_{N\to\infty}\sum_{n=1}^N v_n$ exists in H if and only if the series $\sum_{n=1}^\infty||v_n||^2$ converges.

Corollary 20.2.26. *For a Hilbert space H, suppose $\beta\subseteq H$ is an orthonormal set and let $M=Span(\beta)$. Then for all $x,y\in H$,*

(1) $P_Mx=\sum_{u\in\beta}\langle x,u\rangle u$.

(2) $\sum_{u\in\beta}\langle x,u\rangle\langle u,y\rangle=\langle P_mx,y\rangle$.

(3) $\sum_{u\in\beta}|\langle x,u\rangle|^2=||P_Mx||^2$.

Proof. (1) By Bessel's inequality (Proposition 20.2.22), $\sum_{u\in\beta}||\langle x,u\rangle u||^2=\sum_{u\in\beta}|\langle x,u\rangle|^2\le||x||^2$ so $\sum_{u\in\beta}\langle x,u\rangle u$ is convergent in H by Proposition 20.2.24. Define an operator P by $Px=\sum_{u\in\beta}\langle x,u\rangle u$ for all $x\in H$. We want to show $P=P_M$. Clearly $Px\in M$ for all $x\in H$, so it's enough to show $x-Px\in M^\perp$. Take $u_0\in\beta$ and consider

$$\begin{aligned}\langle x-Px,u_0\rangle=\langle x,u_0\rangle-\langle Px,u_0\rangle&=\langle x,u_0\rangle-\sum_{u\in\beta}\langle x,u\rangle\langle u,u_0\rangle\\&=\langle x,u_0\rangle-\langle x,u_0\rangle\langle u_0,u_0\rangle\quad\text{by orthogonality}\\&=\langle x,u_0\rangle-\langle x,u_0\rangle=0\quad\text{since }||u_0||=1.\end{aligned}$$

Thus $x-Px$ is orthogonal to u_0, but since $u_0\in\beta$ was arbitrary and $M=\text{Span}(\beta)$, it follows that $x-Px\in M^\perp$. By uniqueness of orthogonal decompositions (Corollary 20.2.10), we must have $Px=P_Mx$ for all x, and so $P=P_M$.

(2) and (3) are easy to prove using (1). □

Definition. *An orthonormal set $\beta\subseteq H$ is* **complete** *(or* **maximal***, or an* **orthonormal basis***) if $x\in\beta^\perp$ implies $x=0$.*

Theorem 20.2.27. *Let H be a Hilbert space and $\beta\subseteq H$ an orthonormal set. Then the following are equivalent:*

(1) β is complete.

(2) For all $x\in H$, $x=\sum_{u\in\beta}\langle x,u\rangle u$.

(3) For all $x,y\in H$, $\langle x,y\rangle=\sum_{u\in\beta}\langle x,u\rangle\langle u,y\rangle$.

(4) For all $x\in H$, $||x||^2=\sum_{u\in\beta}|\langle x,u\rangle|^2$.

Proof. (1) $\implies$ (2) Let $M = \overline{\text{Span}(\beta)}$. By completeness, $M = H$ so applying (1) of Corollary 20.2.26 gives us $x = P_M x = \sum_{u\in\beta}\langle x, u\rangle u$.

(2) $\implies$ (3) $\implies$ (4) are straightforward.

(4) $\implies$ (1) If $x \in \beta^\perp$ then $\langle x, u\rangle = 0$ for all $u \in \beta$, so using the assumption in (4), we have $||x||^2 = \sum_{u\in\beta} |\langle x, u\rangle|^2 = \sum_{u\in\beta} 0 = 0$ but this is only possible if $x = 0$. □

We briefly recall the Gram-Schmidt orthonormalization process from linear algebra. Suppose $\{v_n\}_{n=1}^\infty$ is a linearly independent set of vectors in an inner product space $(X, \langle\cdot,\cdot\rangle)$. Then one can "build" an orthonormal basis of the subspace $M = \text{Span}\{v_n\}_{n\in\mathbb{N}}$ as follows:

(1) Set $u_1 = \frac{v_1}{||v_1||}$.

(2) Given $u_1, \ldots, u_{k-1}$, set $u_k' = v_k - \sum_{j=1}^{k-1}\langle v_k, u_j\rangle u_j$. Then set $u_k = \frac{u_k'}{||u_k'||}$.

(3) The resulting vectors $\{u_k\}_{k\in\mathbb{N}}$ are an orthonormal basis of M.

Example 20.2.28. Consider the Hilbert space $H = L^2([-1, 1], m)$, where m is Lebesgue measure. For the basis consisting of monomials, $\{1, x, x^2, \ldots\}$, the Gram-Schmidt algorithm produces an orthonormal basis

$$\left\{1, x, \frac{1}{2}(3x^2 - 1), \ldots\right\},$$

called the *Legendre polynomials* on $[-1, 1]$.

20.3 L^p Spaces

Let $(X, \mathcal{M}, \mu)$ be a measure space. Recall the following definitions from Section 18.3.

Definition. *For $p \geq 0$ and a measurable function $f : X \to \mathbb{C}$, the L^p-**norm** of f is*

$$||f||_p = \left(\int_X |f(x)|^p \, d\mu\right)^{1/p}.$$

Extend this to $p = \infty$ by

$$||f||_\infty = \inf\{M \geq 0 : |f| \leq M \ \mu\text{-a.e.}\}.$$

Then the pth **Lebesgue space** *for $(X, \mathcal{M}, \mu)$ is*

$$L^p = L^p(X, \mathcal{M}, \mu) = \{f : X \to \mathbb{C} \mid f \text{ is } \mu\text{-measurable and } ||f||_p < \infty\}.$$

Theorem 20.3.1. *Let $0 \leq p \leq \infty$. Then*

(1) $||\cdot||_p$ is a norm on L^p.

(2) $(L^p, ||\cdot||_p)$ is a Banach space for all $1 \leq p \leq \infty$.

Proof. We will prove $p = \infty$; the cases where p is finite require some further results first.

(1) It suffices to show the triangle inequality holds for $||\cdot||_\infty$. For any $f \in L^\infty$, $|f| \leq ||f||_\infty$ μ-a.e. so given measurable functions f and g, we see that

$$\begin{aligned} &|f+g| \leq |f| + |g| \leq ||f||_\infty + ||g||_\infty \ \mu\text{-a.e.} \\ \implies &||f+g||_\infty \leq ||f||_\infty + ||g||_\infty. \end{aligned}$$

(2) Take a sequence $(f_n) \subset L^\infty$ witih $\sum ||f_n||_\infty < \infty$. We will show $\sum f_n$ exists in L^∞. Set $M_n = ||f_n||_\infty$. Then $\sum M_n < \infty$ and the sets $E_n = \{x : |f_n(x)| > M_n\}$ hae $\mu(E_n) = 0$ for all n. Further, by subadditivity, the set $E = \bigcup_{n=1}^\infty E_n$ has measure 0. Now for $x \in E^C$, $|f_n(x)| \leq M_n$ for all n and therefore

$$\sum_{n=1}^\infty |f_n(x)| \leq \sum_{n=1}^\infty M_n < \infty.$$

This shows that the function

$$S(x) = \begin{cases} \sum_{n=1}^\infty f_n(x), & x \in E^C \\ 0, & x \in E \end{cases}$$

is defined on $X \to \mathbb{C}$. Also, $||S||_\infty < \infty$ because for all $x \in E^C$,

$$|S(x)| \leq \sum_{n=1}^\infty |f_n(x)| \leq \sum_{n=1}^\infty M_n < \infty.$$

Finally, note that for all $x \in E^C$,

$$\left|S(x) - \sum_{n=1}^{N} f_n(x)\right| = \left|\sum_{n=N+1}^{\infty} f_n(x)\right| \leq \sum_{n=N+1}^{\infty} |f_n(x)| \leq \sum_{n=N+1}^{\infty} M_n$$

which tends to 0 as $N \to \infty$. So $\left|S(x) - \sum_{n=1}^{N} f_n(x)\right| \to 0$ as $N \to \infty$ on E^C, meaning μ-a.e. since $\mu(E) = 0$. Hence

$$\left\|S - \sum_{n=1}^{N} f_n\right\|_\infty \leq \sum_{n=N+1}^{\infty} M_n \longrightarrow 0 \text{ as } N \to \infty,$$

so in particular, $\sum_{n=1}^{N} f_n \to S$ in L^∞. This finishes the proof for $p = \infty$. $\square$

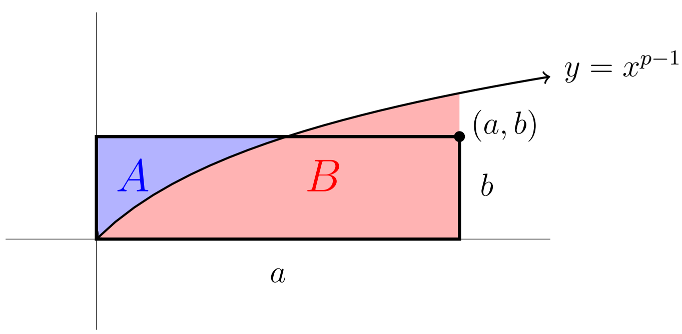
For the finite cases, we prove the following sequence of inequalities which generalize Lemmas 13.2.2 and 20.3.4.

Lemma 20.3.2 (Young's Inequality). *For any $p, q \in \mathbb{N}$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, the inequality*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

holds for all $a, b \geq 0$.

Proof. Consider the function $y = x^{p-1}$ and the point (a, b) on the graph below:



The areas of the regions A and B shown above are given by

$$B = \int_0^a x^{p-1}\,dx = \frac{1}{p}a^p$$
$$A = \int_0^b y^{1/(p-1)}\,dy = \int_0^b y^{q-1}\,dy = \frac{1}{q}b^q.$$

Then by examination $ab \leq A + B = \frac{1}{p}a^p + \frac{1}{q}b^q$. $\square$

The following easy equivalences will be useful.

Lemma 20.3.3. *For $p, q \in \mathbb{N}$, the following are equivalent:*

(a) $\frac{1}{p} + \frac{1}{q} = 1$.

(b) $q = \frac{p}{p-1}$.

(c) $\frac{q}{p} = \frac{1}{p-1}$.

(d) $\frac{q}{p} = q - 1$.

(e) $q - 1 = \frac{1}{p-1}$.

Formally, if $p = 1$ we say $q = \infty$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$. The next step is to prove Hölder's inequality which generalizes the Cauchy-Schwarz inequality (Lemma 20.2.4), the $p = q = 2$ case.

Lemma 20.3.4 (Hölder's Inequality)**.** *For any $p, q \in \mathbb{N} \cup \{\infty\}$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$,*

$$||fg||_1 \leq ||f||_p ||g||_q$$

for all $f, g : X \to \mathbb{C}$.

Proof. If $p = 1$, we have

$$||fg||_1 = \int |fg|\, d\mu \leq ||g||_\infty \int |f|\, d\mu = ||f||_1 ||g||_\infty.$$

If $p > 1$, note that

$$\begin{aligned}
\int \left(\frac{|f(x)|}{||f||_p}\right)\left(\frac{|g(x)|}{||g||_q}\right) d\mu &\leq \int \left(\frac{1}{p} \cdot \frac{|f(x)|^p}{||f||_p^p} + \frac{1}{q} \cdot \frac{|g(x)|^q}{||g||_q^q}\right) d\mu \quad \text{by Young's inequality} \\
&= \int \frac{1}{p} \cdot \frac{|f(x)|^p}{||f||_p^p}\, d\mu + \int \frac{1}{q} \cdot \frac{|g(x)|^q}{||g||_q^q} \\
&= \frac{1}{p} + \frac{1}{q} = 1.
\end{aligned}$$

Multiplying both sides by $||f||_p ||g||_q$, we get $||fg||_1 \leq ||f||_p ||g||_q$ as desired. □

We are now ready to give the proof of Theorem 20.3.1 for $p < \infty$.

Proof. (1) For all $f, g \in L^p$,

$$\begin{aligned}
||f+g||_p^p &= \int |f+g|^p\, d\mu = \int |f+g|\, |f+g|^{p-1}\, d\mu \\
&\leq \int (|f| + |g|)|f+g|^{p-1}\, d\mu \quad \text{by the triangle inequality} \\
&= \int |f|\, |f+g|^{p-1}\, d\mu + \int |g|\, |f+g|^{p-1}\, d\mu \\
&\leq \left(\int |f|^p\, d\mu\right)^{1/p} \left(\int |f+g|^{(p-1)q}\, d\mu\right)^{1/q} + \left(\int |g|^p\, d\mu\right)^{1/p} \left(\int |f+g|^{(p-1)q}\, d\mu\right)^{1/q} \\
&\qquad \text{by Hölder's inequality, with } q = \frac{p}{p-1} \\
&= ||f||_p^p ||f+g||_q^{p/q} + ||g||_p^p ||f+g||_q^{p/q} \\
&= \left(||f||_p^p + ||g||_p^p\right) ||f+g||_q^{p/q}.
\end{aligned}$$

Raising each side to $\frac{1}{p} = 1 - \frac{1}{q}$ gives the triangle inequality – this is also sometimes called *Minkowski's inequality*.

(2) It remains to show that for $p \in [1, \infty)$, L^p is a Banach space. Let $(f_n) \subseteq L^p$ be a Cauchy sequence. By Chebyshev's inequality (Theorem 18.2.4), for any $\varepsilon > 0$ we have

$$\begin{aligned}\mu\left(\{x \in X : |f_n - f_m| > \varepsilon\}\right) &= \mu\left(\{x \in X; |f_n - f_m|^p > \varepsilon^p\}\right)\\ &\leq \frac{1}{\varepsilon^p}\int |f_n - f_m|^p \, d\mu\\ &= \frac{1}{\varepsilon^p}||f_n - f_m||_p^p\end{aligned}$$

which tends to 0 as $n, m \to \infty$ since the sequence is Cauchy. This shows that $\{f_n\}$ is Cauchy in measure, so by Proposition 18.3.6, there exists a subsequence $g_k = f_{n_k}$ such that $f(x) := \lim_{k\to\infty} g_k(x)$ exists μ-a.e. Consider

$$\begin{aligned}||f - g_k||_p^p &= \int \liminf_{n\to\infty} |g_n - g_k|^p \, d\mu\\ &\leq \liminf_{n\to\infty} \int |g_n - g_k|^p \, d\mu \quad \text{by Fatou's lemma (Theorem 18.2.1)}\\ &= \liminf_{n\to\infty} ||g_n - g_k||_p^p\end{aligned}$$

which goes to 0 as $k \to \infty$ since (g_k) is Cauchy. Thus we see that g_k converges to f in L^p. As in the proof for $p = \infty$, this implies $f \in L^p$ (use the triangle inequality on a similarly defined $S(x)$) and that $f_n \to f$ in L^p since (f_n) is Cauchy. Hence every Cauchy sequence in L^p converges, so L^p is complete. □

Lemma 20.3.5 (Hölder's Inequality II). *For all $0 < p, q, r \leq \infty$ and measurable functions $f, g : X \to \mathbb{C}$, we have*

$$||fg||_r \leq ||f||_p ||g||_q.$$

Proof. The proof when $p = q = r = \infty$ is easy. Assume $0 < p, q, r < \infty$. Then $1 = \frac{r}{p} + \frac{r}{q} = \frac{1}{p/r} + \frac{1}{q/r}$ so by the first version of Hölder's inequality (Lemma 20.3.4),

$$\begin{aligned}||fg||_r^r = \int |f|^r |g|^r \, d\mu = \int ||f|^r |g|^r| &= |||f|^r |g|^r||_1\\ \leq |||f|^r||_{p/r} |||g|^r||_{q/r} &= ||f||_p^r ||g||_q^r.\end{aligned}$$

Taking the rth root on either side finishes the proof. □

Corollary 20.3.6. *For any $0 < p < q < r \leq \infty$, $L^q \subseteq L^p + L^r$.*

Example 20.3.7. Let $X = \mathbb{N}$ with the counting measure μ (Example 17.3.8). We typically denote the Lebesgue space $L^p(\mathbb{N}, \mu)$ by ℓ^p. Here, the norm of a function $f : \mathbb{N} \to \mathbb{C}$ is given by

$$||f||_p = \left(\sum_{n=1}^{\infty} |f(n)|^p\right)^{1/p}.$$

Then for any $0 < p \leq q \leq \infty$ and any $0 < r \leq \infty$, we can write

$$||f||_p = ||f \cdot 1||_p \leq ||f||_q ||1||_q$$

by the second version of Hölder's inequality. Since $||1||_r < \infty$, we get that $f \in \ell^q$ if $f \in \ell^p$. Hence $\ell^p \subseteq \ell^q$.

This result generalizes:

Proposition 20.3.8. *Suppose (X, μ) is a finite measure space. Then for all $0 < p \leq q \leq \infty$, $L^p \subseteq L^q$.*

To close, we describe the dual of an L^p space. Fix $\frac{1}{p} + \frac{1}{q} = 1$, $g \in L^q(\mu)$ and define $\varphi_g \in L^p(\mu)^*$ by $\varphi_g(f) = \int fg\,d\mu$ for $f \in L^p(\mu)$. This map is valid because Hölder's inequality (the first version) gives us

$$|\varphi_g(f)| \leq \int |fg|\,d\mu = ||fg||_1 \leq ||f||_p ||g||_q,$$

so $||\varphi_g||_{L^q(\mu)^*} \leq ||g||_q < \infty$.

Proposition 20.3.9. *For a finite measure space X, $||\varphi_g|| = ||g||_q$ for all $g \in L^q(\mu)$ and $1 \leq p \leq \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. In particular, the map $\varphi : L^q \to (L^p)^*$ is injective.*

Proof. First suppose $p = \infty$ so that $q = 1$. Fix $g \in L^1$ and set $f = \overline{\operatorname{sgn} g}$ where $\operatorname{sgn} g = \frac{g}{|g|}\chi_{D(g)}$ and $D(g) = \{x \in X : g(x) \neq 0\}$. Then $||f||_\infty = 1$ by construction and we have

$$\varphi_g(f) = \int fg\,d\mu = \int |g|\,d\mu = ||g||_1 = ||f||_\infty ||g||_1.$$

Thus $||\varphi_g|| \geq ||g||_1$ which implies $||\varphi_g|| = ||g||_1$.

Now suppose $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. For $g \in L^q$, set $f = \overline{\operatorname{sgn} g}\,|g|^{q/p} \in L^p$. Then

$$\varphi_g(f) = \int |g|^{q/p+1}\,d\mu = \int |g|^q\,d\mu = ||g||_q^q$$

$$\text{and} \quad ||f||_p^p = \int |g|^q\,d\mu = ||g||_q^q$$

$$\text{which implies} \quad \frac{\varphi_g(f)}{||f||_p} = \frac{||g||_q^q}{||g||_q^{q/p}} = ||g||_q.$$

Thus $||\varphi_g|| \geq ||g||_q$, so $||\varphi_g|| = ||g||_q$.

Finally, suppose $p = 1$ and $q = \infty$. Let X_n be a sequence converging to X from below so that $\mu(X_n) < \infty$ for all n. Let $M = ||g||_\infty < \infty$ and suppose $\varepsilon > 0$. Set

$$f = \overline{\operatorname{sgn} g}\chi_{X_n \cap \{x : |g(x)| \geq M - \varepsilon\}}.$$

Note that we make take n large enough so that $\mu\left(X_n \cap \{x : |g(x)| \geq M - \varepsilon\}\right) > 0$. Then

$$\begin{aligned} \varphi_g(f) &= \int_{X_n \cap \{x : |g(x)| \geq M - \varepsilon\}} |g|\,d\mu \\ &\geq (M - \varepsilon)\mu\left(X_n \cap \{x : |g(x)| \geq M - \varepsilon\}\right) \\ &= (M - \varepsilon)\,||f||_1. \end{aligned}$$

Thus for all $\varepsilon > 0$, $||\varphi_g|| \geq ||g||_\infty - \varepsilon$ which implies $||\varphi_g|| \geq ||g||_\infty$. Hence they are equal. $\square$

Theorem 20.3.10. *For all σ-finite measure spaces (X, μ), the map*

$$\begin{aligned} \varphi : L^q(\mu) &\longrightarrow L^p(\mu)^* \\ g &\longmapsto \varphi_g \end{aligned}$$

is an isometry for all $1 \le p, q < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We proved in Proposition 20.3.9 that φ is injective and preserves norms, so it suffices to show the map is surjective. First assume $\mu(X) < \infty$. Let $\psi \in (L^p)^*$ and define $\nu(A) = \psi(\chi_A)$ for all $A \in \mathcal{M}$. Because μ is finite, we get $L^\infty \subseteq L^p$ for all $1 \le p < \infty$. Write $A = \coprod_{k=1}^\infty A_k$ for disjoint, μ-measurable sets A_k. Then

$$\chi_A = \sum_{k=1}^\infty \chi_{A_k} \in L^p(\mu).$$

By linearity and continuity of ψ, we have

$$\nu(A) = \psi(\chi_A) = \psi\left(\sum_{k=1}^\infty \chi_{A_k}\right) = \sum_{k=1}^\infty \psi(\chi_{A_k}) = \sum_{k=1}^\infty \nu(A_k).$$

So ν is σ-additive. Clearly also $\nu(\varnothing) = 0$, so ν is a measure on $(X, \mathcal{M})$. Further, if $\mu(A) = 0$ then $\chi_A = 0$ in $L^p(\mu)$, so $\nu(A) = \psi(\chi_A) = \psi(0) = 0$ by linearity. Hence $\nu << \mu$. By the Lebesgue-Radon-Nikodym theorem (19.2.6), there is a function $g \in L^1(\mu)$ such that $d\nu = g\,d\mu$. That is, $\psi(\chi_A) = \int g\chi_A\,d\mu$ for all $A \in \mathcal{M}$. By linearity, $\psi(s) = \int gs\,d\mu$ for all simple functions $s \in L^p(\mu)$. Suppose $p = 1$ and choose $f \in L^\infty(\mu)$. Then by Lemma 18.1.1, there exists a sequence s_n of simple functions in $L^1(\mu)$ converging pointwise to f from below and satisfying $|s_n| \le |f|$ for all n. By the dominated convergence theorem (18.2.7), $s_n \to f$ in $L^1(\mu)$ so $\psi(f) = \int fg\,d\mu$.

Now, if $p > 1$, we need to show that the Radon-Nikodym derivative $g = \frac{d\nu}{d\mu}$ lies in $L^q(\mu)$. For $M < \infty$, let $g_M = g\chi_{\{x:|g(x)|\le M\}} \in L^\infty(\mu) \subseteq L^q(\mu)$. Then

$$\psi_M(f) = \psi(f\chi_{\{x:|g(x)|\le M\}}) = \int fg\chi_{\{x:|g(x)|\le M\}}\,d\mu = \psi_{g_M}(f).$$

So

$$\begin{aligned} |\psi_M(f)| &= |\psi(f\chi_{\{x:|g|\ge M\}})| \\ &\le ||\psi||_{(L^p)^*}||f\chi_{\{x:|g|\ge M\}}||_p \\ &\le ||\psi||_{(L^p)^*}||f||_p. \end{aligned}$$

In particular, $||\psi_M||_{(L^p)^*} \le ||\psi||_{(L^p)^*}$. By Proposition 20.3.9, $||g_M||_q = ||\psi_M||_{(L^p)^*} \le ||\psi||_{(L^p)^*}$ for all $M < \infty$. Taking the limit as $M \to \infty$ and using the monotone convergence theorem (18.1.5), we get $||g||_q \le ||\psi||_{(L^p)^*} < \infty$. Hence $g \in L^q(\mu)$. Finally, repeat the proof that $\psi(f) = \int fg\,d\mu$ for all $f \in L^p(\mu)$ using the sequence $s_n \to f$ and the dominated convergence theorem.

Now assume μ is a σ-finite measure on X. Choose sets $X_n \in \mathcal{M}$ such that $\mu(X_n) < \infty$ for each n and $X_n \nearrow X$. For $f \in L^p(X_n, \mu)$, we identify f with $f\chi_{X_n} \in L^p(X, \mu)$, so one may consider $L^p(X_n, \mu)$ as a subspace of $L^p(X, \mu)$ for each $1 \leq p < \infty$ and $n \geq 1$. By the finite case, there exists $g_n \in L^q(X_n, \mu)$ such that $\psi(f) = \int_{X_n} fg\, d\mu$ for all $f \in L^p(X_n, \mu)$. Then

$$||g_n||_q = \sup\{|\psi(f)| : f \in L^p(X_n, \mu), ||f||_p = 1\}.$$

In particular, $||g_n||_q \leq ||\psi||_{(L^p)^*}$. It's clear that $g_n = g_m$ a.e. on $X_n \cap X_m$ for all $n, m \geq 1$. This means that $g := \lim g_n$ exists μ-a.e. By Fatou's lemma (Theorem 18.2.1),

$$||g||_q \leq \liminf_{n \to \infty} ||g_n||_q \leq ||\psi||_{(L^p)^*}.$$

Now since $\psi(f) = \int_{X_n} fg\, d\mu = \int fg_n\, d\mu$ for all $f \in L^p(X_n, \mu)$ and $\bigcup_{n=1}^\infty L^p(X_n, \mu)$ is dense in $L^p(X, \mu)$, it follows from continuity that any $f \in L^p(X, \mu)$ satisfies $\psi(f) = \int fg\, d\mu$. Hence $\varphi_g = \psi$ in all cases, so φ is surjective. $\square$

Part V

Probability Theory

Chapter 21

Introduction

The contents of Part V were compiled from a course on probability theory taught by Dr. Sarah Raynor in Spring 2015 at Wake Forest University. The companion text for the course is *Probability and Measure*, 4^{th} ed., by P. Billingsley.

One of the best examples to illustrate the nuance of measure theory and probability is the Cantor set. The Cantor set C is defined as follows. Let $A_0 = [0, 1]$, the unit interval. Let A_1 be the set $A_0 - \left(\frac{1}{3}, \frac{2}{3}\right)$ formed by deleting the middle third of A_0. Next, A_2 is similarly formed by deleting the middle thirds $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$ from each component of A_1. The process is continued to define a sequence

$$A_n = A_{n-1} - \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n}\right).$$

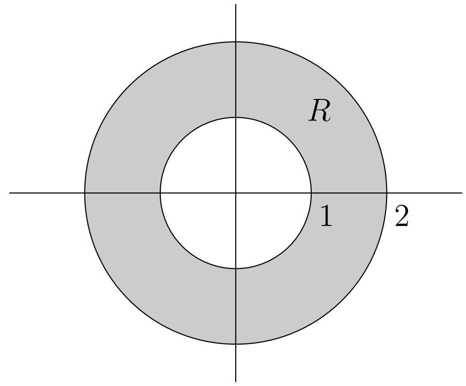
Finally, the Cantor set is the subset of $[0, 1]$ given by $C = \bigcap_{n=0}^{\infty} A_n$, that is, the points remaining in the unit interval after iterating this process over the natural numbers.

Length is our first idea of measure, from which many others will stem. If we take the usual length of an interval on the real line to be end point minus starting point, then the unit interval $[0, 1]$ has length 1. One may then ask: How long is the Cantor set? To measure the length of C, we instead calculate the length of its complement and subtract it from 1. This is the following infinite sum:

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \dots$$

which is a geometric series converging to $\dfrac{1/3}{1 - 2/3} = 1$. Thus the complement of the Cantor set has length 1, but the total unit interval has length 1 meaning the Cantor set has length 0. This is our first example of a set of measure zero.

Area, volume, hypervolume, etc. are all extensions of length to higher dimensions — these are also examples of measures. For example, the area of an annulus is easy to compute. Consider the following region R.



We compute the area A by $A = 4\pi - \pi = 3\pi$. However, one may also want to compute the mass of the annulus, say if it were made of aluminum or steel. Given a density function, e.g. $\rho = e^{-r^2} kg/cm^2$, find the mass of the annular region. This is computed by a double integral,

$$\iint_R \rho \, dA = \int_0^{2\pi} \int_1^2 e^{-r^2} r \, dr \, d\theta = -\frac{1}{2} \int_0^{2\pi} (e^{-2} - e^{-1}) \, d\theta = \pi(e^{-1} - e^{-2}).$$

If we think of a double integral as the limit of the process of breaking the region into smaller regions and adding together all their masses, we see the same concept at work as in the Cantor set example.

How does this relate to probability?

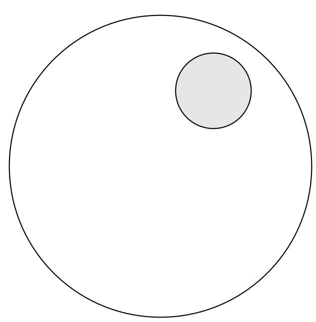
Example 21.0.1. What is the probability of rolling a prime number on a standard six-sided die? This can be computed by the same divide-and-conquer approach:

$$P(\text{prime}) = P(2) + P(3) + P(5) = \frac{3}{6} = \frac{1}{2}.$$

Example 21.0.2. When playing craps (rolling two dice), what is the probability of rolling either a 7 or an 11?

$$P(7 \text{ or } 11) = P(7) + P(11) = \frac{6}{36} + \frac{2}{36} = \frac{2}{9}.$$

Example 21.0.3. Given a dartboard of unit area, the probability of hitting a small region on the board with your dart is precisely the area of that region:



There is a common theme among the above examples, which is that the calculation of probability relies on our ability to measure things and compare the relative measures. We will use the following version of the definition of a measure from Chapter 17.

Definition. *A* **measure** μ *on a set* S *is a function* $\mu : \mathbb{P}(S) \to [0, \infty]$ *such that* μ *is countably additive, that is, if* A *is a subset of* S *of the form* $\bigcup_{n=1}^{\infty} A_n$ *and* $A_n \cap A_m = \varnothing$ *for all* $n \neq m$*, then* $\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$*.*

Some interesting questions arise from defining a measure this way:

1. Is every subset of S measurable? When the set is finite, the answer is yes. However, for the unit interval with length as a measure, the answer is *no.* A counterexample is difficult to produce at this time.

2. The Banach-Tarski Paradox (sometimes called the Banach-Tarski Theorem) says that it is possible to take a solid ball of any size, say the size of a basketball, decompose it into finitely many pieces and put them back together only using rigid motions to get a ball the size of the sun. How is this possible?

The domain of a measure must have a special structure, which is called a σ-field (sometimes σ-algebra in the literature).

Definition. *Let* S *be a set and* $\mathcal{F}$ *a collection of subsets of* S*.* $\mathcal{F}$ *is a* **σ-algebra** *provided*

(1) $S \in \mathcal{F}$*.*

(2) If $A \in \mathcal{F}$ *then* $A^C \in \mathcal{F}$ *as well.*

(3) If $A_1, A_2, \ldots \in \mathcal{F}$ *(this may be a countable list) then* $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$*.*

It turns out that this is just enough structure to allow us to define a measure on $\mathcal{F}$. This will be the main ‘universe’ in which we work, defining probability measures and developing their applications.

Chapter 22

Probability and Normal Numbers

In these notes we will denote a sample space by Ω and a particular event taken from this sample space by ω. Our prototypical example will have $\Omega = (0, 1]$. For technical reasons we will always assume an interval of the real line is of the form $(a, b]$ so that collections of intervals may be chosen disjointly (so they don't overlap at the endpoints). If $I = (a, b]$ we will denote the usual notion of length by $|I| = |b - a|$.

Suppose $A = \bigcup_{i=1}^{n} I_i$ where $I_i = (a_i, b_i]$ are pairwise disjoint intervals in the sample space $\Omega = (0, 1]$. We define

Definition. *The* **probability** *of event A occuring within the sample space Ω is*

$$P(A) := \sum_{i=1}^{n} |I_i| = \sum_{i=1}^{n} |b_i - a_i|.$$

At this point we are carefully avoiding complicated subsets of Ω, such as the Cantor set in the introduction. These will be the focus in later chapters.

If A and B are disjoint subsets of Ω and each of A, B is a finite disjoint union of intervals, then

$$P(A \cup B) = P(A) + P(B).$$

This is called the finite additivity of probability. So far we have brushed over something important: is our definition of $P(A)$ well-defined? That is, if A has two different representations as finite disjoint unions of intervals in Ω, do they both give the same probability? Well suppose $A = \bigcup_{i=1}^{n} I_i = \bigcup_{j=1}^{m} J_j$. We create a collection of intervals $K_{ij} = I_i \cap J_j$, called a **refinement** of the I_i and J_j. Notice that

$$A = \bigcup_{j=1}^{m} \bigcup_{i=1}^{n} K_{ij} = \bigcup_{j=1}^{m} \bigcup_{i=1}^{n} (I_i \cap J_j).$$

This implies well-definedness of our definition of $P(A)$.

Example 22.0.1. This relates to the Riemann integral in an important way. For a subset $A \subset \Omega$ which is a disjoint union of finitely many intervals in $\Omega = (0, 1]$, define the **characteristic function**

$$f_A = \sum_{i=1}^{n} \chi_{I_i} \qquad \text{where} \qquad \chi_{I_i}(x) = \begin{cases} 1 & x \in I_i \\ 0 & x \notin I_i. \end{cases}$$

Similarly define $g_B = \sum_{j=1}^{m} \chi_{J_j}$. Then finite additivity of probability implies the additive property of Riemann integrals:

$$\int_0^1 (f_A + g_B)\, dx = \int_0^1 f_A\, dx + \int_0^1 g_B\, dx.$$

This is because

$$\int_0^1 \chi_I(x)\, dx = |I| = b - a.$$

Keep in mind that for the moment we are only dealing with event spaces that are finite disjoint unions of half-open intervals; when we encounter more complicated subsets of Ω, Riemann integration breaks down. In that case we will need to use Lebesgue integration, one of the main tools in modern integration theory.

Our next goal is to equate the probabilistic notion of selecting points from the unit interval with the physical act of flipping an infinite number of coins and counting heads and tails. Define $d_i(\omega)$ to be the result of the ith flip of the infinite sequence of coin flips; we will denote this numerically by

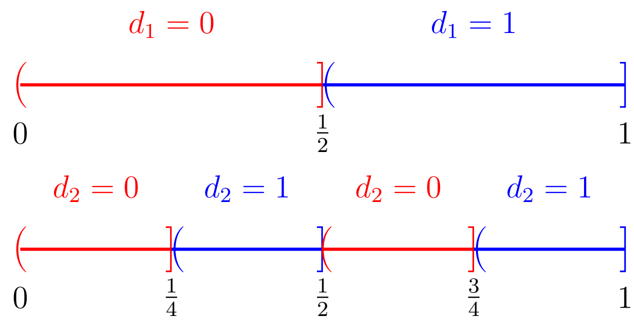
$$d_i(\omega) = \begin{cases} 1 & \text{if heads} \\ 0 & \text{if tails.} \end{cases}$$

The event ω can be represented as a sequence of 1's and 0's: $(d_1(\omega), d_2(\omega), d_3(\omega), \ldots)$. We will also make use of the dyadic (binary) representation

$$\omega = \sum_{i=1}^{\infty} d_i(\omega) 2^{-i}.$$

Each sequence of 0's and 1's corresponds to a unique real number in the interval $[0, 1]$. However, not every real number in $[0, 1]$ has a unique dyadic representation. For example, $\frac{5}{8}$ can be represented by $0.101000\ldots$ but also by the non-terminating $0.100111\ldots$ It is convention to prefer the *non-terminating* representation, since this will coincide with our other preference for half-open intervals $(a, b]$. Notice that picking only non-terminating decimal representations excludes $0 = 0.000\ldots$ from our probability space, so we are indeed constructing $(0, 1]$.

Now, drawing at random with uniform probability from $\Omega = (0, 1]$ is equivalent to the dyadic representation of an infinite sequence of coin flips. The reason is that $P[d_i(\omega) = 1]$ is equal to the sum of the lengths of $\frac{2^i}{2}$ intervals, each of which has length $\frac{1}{2^i}$. This is illustrated below.



These are sometimes called **dyadic intervals**. From this we can see that the probability of any single flip coming up heads is $\frac{1}{2}$, since at any level, half of the 2^i intervals are included in this event. The 2^n intervals of length 2^{-n} for any n are called the set of **rank n dyadic intervals**; they have the nice property of being nested. Formally, if $n > m$ and I_i is an interval of rank n, there is a *unique* J_j of rank m such that $I_i \subset J_j$.

Example 22.0.2. The binomial formula expresses the probability that k heads will be flipped in n trials. Using the interval construction above, we see that

$$\begin{aligned} P(k \text{ heads in the first } n \text{ flips}) &= P(k \text{ of the first } n \text{ bigits are } 1) \\ &= \#\Big\{\text{subsets of } \{1, \ldots, n\} \text{ with } k \text{ elements}\Big\} \cdot 2^{-n} \\ &= \binom{n}{k} 2^{-n} \end{aligned}$$

which is exactly the same as provided by the binomial formula.

Notice that if $\{I_i\}_{i \in \mathbb{N}}$ is a collection of rank n dyadic intervals and $n \geq m$, then $d_m(x)$ is constant on I_i for all I_i of rank n.

22.1 The Weak Law of Large Numbers

This brings us to the Law of Large Numbers. In probability theory, the LLN states that a sequence of random trials converges to a particular value or outcome: the expected value (EV). In this course, we will distinguish between two different versions of the LLN.

Theorem 22.1.1 (Weak Law of Large Numbers)**.** *Let ω be an event in the sample space $\Omega = (0,1]$ which may be expressed as a finite disjoint union of intervals. Then for any $\varepsilon > 0$,*

$$\lim_{n\to\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^{n} d_i(\omega) - \frac{1}{2}\right| > \varepsilon\right) = 0.$$

Proof. To prove the Weak LLN, we first define the **Rademacher functions**,

$$r_i(\omega) = 2d_i(\omega) - 1 = \begin{cases} 1 & \text{if heads} \\ -1 & \text{if tails} \end{cases}$$

for each i, and the **cumulative Rademacher function** of rank n,

$$s_n(\omega) = \sum_{i=1}^{n} r_i(\omega).$$

In this language, the above probability may be expressed as

$$\lim_{n\to\infty} P\left(\left|\tfrac{1}{n}s_n(\omega)\right| > \varepsilon\right) = 0.$$

In addition, the Rademacher functions are defined so $P[r_i = 1] = \frac{1}{2}$ and $P[r_i = -1] = \frac{1}{2}$, meaning they have an average value of 0:

$$\int_0^1 r_i(\omega)\, d\omega = 0.$$

This implies that the cumulative function also has average value 0. Furthermore, it's easy to see that whenever $i \neq j$,

$$\int_0^1 r_i(\omega) r_j(\omega)\, d\omega = 0$$

by looking at their graphs. However, $r_i^2(\omega) = 1$ for all i, so

$$\int_0^1 r_i^2(\omega)\, d\omega = \int_0^1 1\, d\omega = 1.$$

Putting this all together, we get

$$\begin{aligned}
\int_0^1 s_n^2(\omega)\,d\omega &= \int_0^1 \left(\sum_{i=1}^n r_i(\omega)\right)^2 d\omega \\
&= \int_0^1 \sum_{i=1}^n r_i(\omega) \sum_{j=1}^n r_j(\omega)\,d\omega \\
&= \int_0^1 \left[\sum_{i=1}^n r_i^2(\omega) + \sum_{\substack{i,j=1 \\ i\neq j}}^n r_i(\omega) r_j(\omega)\right] d\omega \\
&= \sum_{i=1}^n 1 + \sum_{i=1}^n 0 = n + 0 = n.
\end{aligned}$$

Finally, we need the following facts: if f is a nonnegative step function and $\alpha > 0$, then

(1) The set $\{\omega \mid f(\omega) > \alpha\}$ is a finite union of intervals, and

(2) $P[\omega \mid f(\omega) > \alpha] \leq \dfrac{1}{\alpha} \displaystyle\int_0^1 f(\omega)\,d\omega$. (This is Chebyshev's Inequality (Theorem 18.2.4).)

This is used in the following calculation of the original probability limit:

$$\begin{aligned}
P\left(\left|\frac{1}{n} s_n(\omega)\right| > \varepsilon\right) &= P[|s_n(\omega)| > n\varepsilon] \\
&= P[s_n^2(\omega) > n^2\varepsilon^2] \\
&\leq \frac{1}{n^2\varepsilon^2} \int_0^1 s_n^2(\omega)\,d\omega \quad \text{by Chebyshev's inequality} \\
&= \frac{1}{n^2\varepsilon^2} n = \frac{1}{n\varepsilon^2}
\end{aligned}$$

which converges to 0 as $n \to \infty$. This completes the proof of the Weak LLN. □

22.2 The Strong Law of Large Numbers

Definition. *A* **normal number** *is a real number* $\omega \in (0,1]$ *such that*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} d_i(\omega) = \frac{1}{2}.$$

The Strong Law of Large Numbers says the following about normal numbers:

Theorem 22.2.1 (Strong Law of Large Numbers)**.** *The set N of normal numbers has probability $P(N) = 1$.*

Instead of proving the SLLN, we will instead be proving an equivalent statement known as *Borel's Theorem on Normal Numbers* (22.2.5). The only measure theory we need so far is the definition of a measure zero set:

Definition. *A set $W \subset \Omega$ is said to be* **negligible** *(alternatively, has* **measure zero***) if for every $\varepsilon > 0$, there is a countable collection of intervals $\{I_k\}_{k=1}^{\infty}$ such that*

(a) $W \subset \displaystyle\bigcup_{k=1}^{\infty} I_k$.

(b) $\displaystyle\sum_{k=1}^{\infty} |I_k| < \varepsilon$.

Lemma 22.2.2. *Suppose $\{W_i\}_{i=1}^{\infty}$ are all negligible sets in Ω. Then $W = \displaystyle\bigcup_{i=1}^{\infty} W_i$ is also negligible. That is, the countable union of negligible sets is negligible.*

Proof. Let $\varepsilon > 0$ and let $i \in \mathbb{N}$. Since W_i is negligible, there is a countable collection $\{I_k^i\}_{k=1}^{\infty}$ so that $W_i \subset \displaystyle\bigcup_{k=1}^{\infty} I_k^i$, and $\displaystyle\sum_{k=1}^{\infty} |I_k^i| < \frac{\varepsilon}{2^i}$. Then

$$W \subset \bigcup_{i=1}^{\infty}\bigcup_{k=1}^{\infty} I_k^i \qquad \text{and} \qquad \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} |I_k^i| < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon.$$

Since the doubly-indexed collection $\{I_k^i\}_{(i,k)\in\mathbb{N}^2}$ is countable, we conclude that W is negligible. □

This has an immediate and important consequence.

Corollary 22.2.3. *All countable sets are negligible.*

Proof. If we can prove that any singleton set is negligible, then Lemma 22.2.2 immediately applies since a countable set is just the countable union of singletones. This is easy to establish, since for any singleton $\{x\} \subset \Omega$, $\left(x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}\right]$ covers $\{x\}$ and has length ε. □

Example 22.2.4. By Corollary 22.2.3, $\mathbb{Q}$ is a negligible set. This is odd, however, since we know that $\mathbb{Q}$ is an example of a set that is dense in the real numbers. It turns out that there are even larger sets than $\mathbb{Q}$ that are still negligible.

Corollary 22.2.3 suggests an interesting question: Is every negligible set countable? The answer turns out to be no; for example, the Cantor set is uncountable, but as we saw in the introduction, C has measure 0.

Recall that we want to prove the complement of the normal numbers has measure 0. It would be nice if N^C were countable, but unfortunately this is not the case. To see this, what are some numbers in $N^C = \left\{\omega : \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} d_i(\omega) \neq \frac{1}{2}\right\}$? Some obvious examples are $(0,1,1,1,1,1,\ldots)$ and $(1,0,1,1,1,1,\ldots)$, but a more interesting one is $(1,0,0,1,0,0,1,0,0,1,0,0,\ldots)$. In fact, anything of the form $(a,0,0,b,0,0,c,0,0,d,0,0,\ldots)$ is in N^C because

$$\sum_{i=1}^{n} d_i \leq \frac{1}{3}.$$

Now $(a,b,c,d,\ldots)$ is an infinite sequence of 0's and 1's, of which there are *uncountably many.* Therefore there are uncountably many sequences of the form $(a,0,0,b,0,0,c,0,0,\ldots)$ in N^C. Hence N^C must be uncountable.

Nevertheless, there is a way to prove $P(N^C) = 0$, which is given below.

Theorem 22.2.5 (Borel's Normal Number Theorem)**.** *N^C is negligible.*

Proof. Clearly $N = \left\{\omega \mid \frac{1}{n}s_n(\omega) \to 0\right\}$ in the language of Rademacher functions. Then the theorem may be restated as

$$P[\omega : |s_n(\omega)| > \varepsilon n] = 0$$

for any $\varepsilon > 0$. By Chebyshev's Inequality (see Theorem 24.3.4),

$$\begin{aligned} P[\omega : |s_n(\omega)| > \varepsilon n] &= P[\omega : |s_n^4(\omega)| > \varepsilon^4 n^4] \\ &\leq \frac{1}{n^4\varepsilon^4}\int_0^1 s_n^4(\omega)\, d\omega. \end{aligned}$$

To evaluate this integral, note that

$$\begin{aligned} s_n^4(\omega) &= \left(\sum_{\alpha=1}^{n} r_\alpha(\omega)\right)\left(\sum_{\beta=1}^{n} r_\beta(\omega)\right)\left(\sum_{\gamma=1}^{n} r_\gamma(\omega)\right)\left(\sum_{\delta=1}^{n} r_\delta(\omega)\right) \\ &= \sum_{\alpha,\beta,\gamma,\delta=1}^{n} r_\alpha(\omega) r_\beta(\omega) r_\gamma(\omega) r_\delta(\omega). \end{aligned}$$

Recall that even powers of the $r(\omega)$ functions are equal to 1, while odd powers are equal to -1. The possible values of the Rademacher functions, as well as what they contribute for the integral, are shown in the table below.

different values	functions	integral	number of instances
1 ($\alpha = \beta = \gamma = \delta$)	$r_i^4(\omega)$	$\int_0^1 r_i^4(\omega)\, d\omega = 1$	n
2	$r_i^2(\omega)r_j^2(\omega)$	$\int_0^1 r_i^2(\omega)r_j^2(\omega)\, d\omega = 1$	$3n(n-1)$
2	$r_i^3(\omega)r_j(\omega)$	$\int_0^1 r_i^3(\omega)r_j(\omega)\, d\omega = 0$	$\vdots$
3	$r_i^2(\omega)r_j(\omega)r_k(\omega)$	$\int_0^1 r_i^2(\omega)r_j(\omega)r_k(\omega)\, d\omega = 0$	
4	$r_i(\omega)r_j(\omega)r_k(\omega)r_l(\omega)$	$\int_0^1 r_i(\omega)r_j(\omega)r_k(\omega)r_l(\omega)\, d\omega = 0$	

Thus the only parts that don't integrate to 0 are the first two:

$$s_n^4(\omega) = 1 \cdot n + 1 \cdot 3n(n-1) + 0 + 0 = 3n^2 - 2n < 3n^2.$$

This gives us $P[\omega : |s_n(\omega)| < \varepsilon n] < \dfrac{1}{\varepsilon^4 n^4} \cdot 3n^2 = \dfrac{3}{\varepsilon^4 n^2}$. For a given n, choose $\varepsilon_n = n^{1/8}$. Then we have

$$P[\omega : |s_n(\omega)|] < \frac{3}{\varepsilon_n^4 n^2} = \frac{3}{\left(n^{1/8}\right)^4 n^2} < \frac{3}{n^{3/2}}$$

and this tends to 0 as $n \to \infty$. We will use this calculation in a moment.

Let $A_n = \left\{\omega : \frac{1}{n}|s_n(\omega)| > \varepsilon_n\right\}$. We need to verify three things:

(i) $N^C \subset \displaystyle\bigcup_{n=m}^{\infty} A_n$ for any $m \leq n$;

(ii) The A_n are all finite unions of intervals; and

(iii) $\displaystyle\sum_{n=m}^{\infty} |A_n|$ is sufficiently small.

First, note that (i) is the same as $N \supset \displaystyle\bigcap_{n=m}^{\infty} A_n^C$ by DeMorgan's Laws. If $\omega_0 \in A_n^C$ and r is some number such that $n > r \geq m$, then

$$\omega_0 \in \left\{\omega : \frac{1}{n}|s_n(\omega)| \leq \varepsilon_n\right\} \subseteq \left\{\omega : \frac{1}{n}|s_n(\omega)| \leq \varepsilon_r\right\}.$$

So $\displaystyle\lim_{n\to\infty} \frac{1}{n}|s_n(\omega)| \leq \varepsilon_r$ and as $r \to \infty$, $\varepsilon_r \to 0$ which makes this limit go to 0. Hence ω is normal.

(ii) For a fixed n, we claim that A_n is a finite union of disjoint intervals. From this it will follow that $\displaystyle\bigcup_{n=m}^{\infty} A_n$ is a countable union of intervals, since countable unions of finite unions are countable. Consider $A_n = \left\{\omega : \frac{1}{n}|s_n(\omega)| > \varepsilon_n\right\}$. For any n, $s_n(\omega)$ is a step function and therefore so is $\frac{1}{n}|s_n(\omega)|$. Hence by (1) of Chebyshev's inequality (Theorem 18.2.4), A_n is a finite union of intervals.

(iii) By (ii), A_n is a finite union of some intervals $\{I_k\}$. Note that $\sum_{k=1}^{\infty}|I_k| \leq \sum_{n=m}^{\infty} P(A_n)$ since all the intervals I_k on the left appear in the sum on the right. By our work above,

$$\sum_{n=m}^{\infty} P(A_n) < \sum_{n=m}^{\infty} \frac{3}{n^{3/2}} \leq \frac{3c}{\sqrt{m}}$$

for some constant c, by the integral test. Given $\varepsilon > 0$, we can choose m sufficiently large so that $\varepsilon > \frac{3c}{\sqrt{m}}$. Hence $\sum_{k=1}^{\infty}|I_k| < \varepsilon$ which completes the proof that N^C is negligible. □

We now have at our disposal a 'weak' and 'strong' law of large numbers; naturally from the way they are named, the Strong LLN implies the Weak LLN. However, at the moment we don't have the tools to prove this.

So far we have several good examples of negligible sets, so one might be tempted to think that all sets are negligible. Of course that isn't true, or else the very concept of negligibility would be meaningless, so let's find some non-negligible sets.

Proposition 22.2.6. $\Omega = (0, 1]$ *is not negligible.*

Proof. Given $\{I_k\}_{k=1}^{\infty}$ a countable collection of intervals such that $\Omega \subset \bigcup_{k=1}^{\infty} I_k$, we want to show that $\sum_{k=1}^{\infty}|I_k| \geq |\Omega| = 1$. This results from the more general theorem below. □

Theorem 22.2.7. *Let I and $\{I_k\}_{k=1}^{\infty}$ be intervals.*

(1) If $\bigcup_{k=1}^{\infty} I_k \subset I$ and the I_k are disjoint, then $\sum_{k=1}^{\infty}|I_k| \leq |I|$.

(2) If $\bigcup_{k=1}^{\infty} I_k \supset I$ then $\sum_{k=1}^{\infty}|I_k| \geq |I|$.

(3) If $\bigcup_{k=1}^{\infty} I_k = I$ and the I_k are disjoint, then $\sum_{k=1}^{\infty}|I_k| = |I|$.

Proof. First note that (1) and (2) imply (3). We use induction to prove (1) and (2). First suppose there's only one interval $I_1 \subset I$. Then clearly $|I_1| \leq |I|$. Now assume inductively that the conclusion holds for $I_1, \ldots, I_n$ and consider the case for $n+1$ intervals. Write $\{I_k\}_{k=1}^{n+1}$ in order, using the fact that they are disjoint:

$$a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots < b_n \leq a_{n+1} < b_{n+1} \leq b$$

where $I = (a, b]$ and $I_k = (a_k, b_k]$. Since $\bigcup_{k=1}^{n} I_k \subset (a, b_n]$, the inductive hypothesis gives us

$$\sum_{k=1}^{\infty} |I_k| \leq |b_n - a| \leq |a_{n+1} - a|.$$

Then

$$\begin{aligned}\sum_{k=1}^{n+1} |I_k| &= \sum_{k=1}^{n} |I_k| + |b_{n+1} - a_{n+1}| \\ &\leq |a_{n+1} - a| + |b_{n+1} - a_{n+1}| \\ &\leq |a_{n+1} - a| + |b - a_{n+1}| \\ &= |b - a|\end{aligned}$$

since the differences are all positive by our chosen ordering. So property (1) holds for finite collections of disjoint intervals. Now consider a countable collection of disjoint intevals $\{I_k\}_{k=1}^{\infty}$. For any finite n, $\sum_{k=1}^{\infty} |I_k| \leq |I|$ and taking the limit as $n \to \infty$ preserves the inequality (e.g. by the monotone convergence theorem); hence $\sum_{k=1}^{\infty} |I_k| \leq |I|$ as well.

(2) The case with one interval is the same as above. Assume the inequality holds for n intervals and let $\{I_k\}_{k=1}^{n+1}$ be a collection of intervals, not necessarily disjoint, so that $\bigcup_{k=1}^{n+1} I_k \supset I$. This is the same as $(a, b] \subset \bigcup_{k=1}^{n+1} (a_k, b_k]$. Because the intervals are not disjoint, we can't order the a_k and b_k like we did last time; however we can write

$$b_{n+1} \geq b_n \geq \cdots b_1 \qquad \text{with} \quad b_{n+1} \geq b.$$

If $a_{n+1} \leq a$ we're done since then $(a, b] \subset (a_{n+1}, b_{n+1}]$. Otherwise $(a, a_{n+1}] \subset \bigcup_{k=1}^{n} (a_k, b_k]$. By the inductive hypothesis, $|a_{n+1} - a| \leq \sum_{k=1}^{n} |b_k - a_k|$ so

$$\begin{aligned}|b - a_{n+1}| + |a_{n+1} - a| &\leq \left(\sum_{k=1}^{n} |b_k - a_k|\right) + |b - a_{n+1}| \\ \implies |b - a| &\leq \left(\sum_{k=1}^{n} |b_k - a_k|\right) + |b - a_{n+1}| \\ &\leq \left(\sum_{k=1}^{n} |b_k - a_k|\right) + |b_{n+1} - a_{n+1}| \\ &= \sum_{k=1}^{n+1} |b_k - a_k|.\end{aligned}$$

Hence (2) holds for finite collections of intervals.

To prove the property for countable collections, we will exploit the completeness of $\mathbb{R}$ via the Heine-Borel Theorem (13.2.13), which states that any closed, bounded interval of real numbers is compact. Thus any open cover of such an interval will have a finite subcover and we can reduce to the finite case. At the moment though, we have neither a closed interval nor an open cover of the interval. To remedy this, let $\varepsilon > 0$ be arbitrarily small. Enlarge I to a closed interval $[a+\varepsilon, b]$ and consider the open cover

$$\bigcup_{k=1}^{\infty} \left(a_k, b_k + \frac{\varepsilon}{2^k}\right) \supset [a+\varepsilon, b].$$

The Heine-Borel Theorem implies there is a finite subcover

$$\bigcup_{j=1}^{J} \left(a_{k_j}, b_{k_j} + \frac{\varepsilon}{2^{k_j}}\right) \supset [a+\varepsilon, b]$$

so the finite case above tells us that

$$\begin{aligned}
|b-a-\varepsilon| &\leq \sum_{j=1}^{J} \left| b_{k_j} + \frac{\varepsilon}{2^{k_j}} - a_{k_j} \right| \\
|b-a| - \varepsilon &\leq \sum_{j=1}^{J} \left(|b_{k_j} - a_{k_j}| + \frac{\varepsilon}{2^{k_j}} \right) \\
&\leq \sum_{k=1}^{\infty} \left(|b_k - a_k| + \frac{\varepsilon}{2^k} \right) = \sum_{k=1}^{\infty} |I_k| + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ proves (2) in the countable case. □

We obtain two immediate and useful results of this theorem.

Corollary 22.2.8. *Any finite interval on the real line is not negligible. In particular,* $\Omega = (0,1]$ *is not negligible.*

Corollary 22.2.9. *If* A *is negligible then* A^C *is not negligible.*

Corollary 22.2.8 is subtle but quite vital to the foundations of probability theory: now that we know our universe ($\Omega = (0,1]$ in this section) is not negligible, we can show sets have full measure by demonstrating that their complements are negligible (Corollary 22.2.9).

22.3 Properties of Normal Numbers

In Section 22.1 we proved that N is not negligible by showing that its complement N^C is negligible. In this section we explore some consequences of this fact and highlight the difference between negligibility and various other measures of 'smallness'.

Proposition 22.3.1. *N and N^C are both dense in $(0,1]$.*

Proof. Suppose $\omega \in (0,1]$. Given $\varepsilon > 0$, let j be the natural number such that $2^{-j} < \varepsilon/2$. Write $\omega = \sum_{i \in \mathbb{N}} 2^{-i} d_i(\omega)$. Then the number $n = \sum_{i=1}^{j-1} 2^{-i} d_i(\omega) + \sum_{i=j}^{\infty} 2^{-i} e_i$, where $e_i = 0$ when i is odd and 1 when i is even, is a normal number since the tail looks like $\ldots 0101010101 \ldots$ In addition,

$$\begin{aligned}
|\omega - n| &= \left| \sum_{i=j}^{\infty} 2^{-i}(d_i(\omega) - e_i) \right| \\
&\leq \sum_{i=j}^{\infty} 2^{-i} |d_i(\omega) - e_i| \quad \text{by the triangle inequality} \\
&\leq \sum_{i=j}^{\infty} 2^{-i} \cdot 1 = \frac{2^{-j}}{1 - 1/2} = \quad \text{by geometric series} \\
&= 2^{-j+1} < \varepsilon \qquad \text{by our choice of } j \text{ above.}
\end{aligned}$$

Hence N is dense in $(0,1]$.

On the other hand, given the same $\omega \in (0,1]$, we can append the sequence $100100100100\ldots$ since we know this makes the average term go to $\frac{1}{3} \neq \frac{1}{2}$. By the same logic as above, this new number is within ε of ω so N^C is dense in $(0,1]$. □

Definition. *A set A is* **trifling** *if for each ε there exists a* finite *sequence of intervals I_k satisfying*

(i) $A \subset \bigcup_k I_k$ *and*

(ii) $\sum_k |I_k| < \varepsilon$.

Clearly a trifling set is negligible, and finite unions of trifling sets are trifling. Trifling sets are especially 'small' and have some nice properties.

Proposition 22.3.2. *If A is trifling then Cl(A) is trifling.*

Proof. Suppose A if trifling, so that there is a finite collection of intervals $\{I_k\}_{k=1}^n$ covering A and their total length is less than $\frac{\varepsilon}{2}$. Write $I_k = (a_k, b_k]$. We enlarge each I_k to form a new collection $\{J_k\}_{k=1}^n$ given by

$$J_k = \left(a_k - \frac{\varepsilon}{2^{k-1}}, b_k\right].$$

Notice that the maximum length added to the length of the I_k is

$$\sum_{k=1}^{n} \frac{\varepsilon}{2^{k-1}} \le \varepsilon \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = \frac{\varepsilon}{2}$$

so that their total length is still less than $\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Also, since each $I_k \subset J_k$ and the I_k cover A, $\{J_k\}_{k=1}^{n}$ is also a finite cover of A. Moreover, each closed interval $[a_k, b_k] \subset J_k$ and therefore their union $\bigcup_{k=1}^{n} [a_k, b_k]$ contains the closure of A. This proves that $\mathrm{Cl}(A)$ is trifling if A is trifling. $\square$

Examples.

1. By Corollary 22.2.3, $\mathbb{Q} \cap (0,1]$ is negligible. Since $\mathbb{Q}$ is dense in the reals, the closure of $\mathbb{Q} \cap (0,1]$ is $(0,1]$ so the contrapositive to Proposition 22.3.2 implies that $\mathbb{Q} \cap (0,1]$ is not trifling.

2. The Cantor set C, as defined in the Introduction, is an uncountable set. However, C is trifling. To see this, let $\varepsilon > 0$ be given and take n to be the natural number such that $\left(\frac{2}{3}\right)^n < \varepsilon$. Then part (b) of this problem shows that at the nth level, every number in C is contained in one of 2^n intervals, each of which has length 3^{-n}. Clearly the sum of the lengths of these intervals is $\left(\frac{2}{3}\right)^n < \varepsilon$ by our choice of n, and 2^n is finite so the collection of these intervals satisfies conditions (i) and (ii), proving C is trifling.

Definition. *A set $A \subset \Omega$ is* **nowhere dense** *if for every interval $J \subset \Omega$, there is a subinterval $I \subset J$ such that $I \cap A = \varnothing$.*

Proposition 22.3.3. *$A \in \Omega$ is nowhere dense $\iff$ the interior of the closure of A is empty.*

Proof omitted. $\square$

Proposition 22.3.4. *A trifling set is nowhere dense.*

Proof. Suppose $x \in \mathrm{Int}(\mathrm{Cl}(A))$. Then there exists an $\varepsilon > 0$ such that $J = (x - \varepsilon, x + \varepsilon) \subset \mathrm{Cl}(A)$. Since the closure of a trifling set is trifling, we can cover $\mathrm{Cl}(A)$ with a collection of intervals $\{I_k\}_{k=1}^{n}$ such that $\sum_{k=1}^{n} |I_k| < \varepsilon$. However, notice that $|J| = |(x+\varepsilon) - (x-\varepsilon)| = 2\varepsilon > \varepsilon$ so by (2) of Theorem 22.2.7,

$$J \not\subseteq \bigcup_{k=1}^{n} I_k,$$

a contradiction. Hence A is nowhere dense. $\square$

Proposition 22.3.5. *A compact negligible set is trifling.*

Proof. Suppose $A \subset \Omega$ is compact and negligible. For a given $\varepsilon > 0$, negligible implies there exists a collection $\{I_k\}_{k=1}^{\infty}$ with the following properties:

- For each k, $I_k = (a_k, b_k]$.
- $\displaystyle\bigcup_{k=1}^{\infty} I_k \supseteq A$.
- $\displaystyle\sum_{k=1}^{\infty} |I_k| < \frac{\varepsilon}{2}$.

From this collection we form an open cover $\{J_k\}_{k=1}^{\infty}$ where $J_k = \left(a_k, b_k + \frac{\varepsilon}{2^{k+1}}\right)$. Observe that the most length we could have added to the original collection of I_k's is

$$\sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} = \frac{\varepsilon}{2} \qquad \text{by geometric series.}$$

So $\displaystyle\sum_{k=1}^{\infty} |J_k| < \varepsilon$ and $\{J_k\}$ is an open cover of A. By compactness, there exists a finite (open) subcover $\{J_{k_i}\}_{i=1}^{n}$ of A. Then the collection $\{J_{k_i}^*\}_{i=1}^{n}$, where $J_{k_i}^* = \left(a_{k_i}, b_{k_i} + \frac{\varepsilon}{2^{k_i+1}}\right]$, is a finite collection of intervals covering A and satisfying

$$\sum_{i=1}^{n} |J_{k_i}^*| = \sum_{i=1}^{n} |J_{k_i}| \leq \sum_{k=1}^{\infty} |J_k| < \varepsilon.$$

Hence A is trifling. □

Example.

3. Let $B = \bigcup_n (r_n - 2^{-n-2}, r_n + 2^{-n-2}]$, where $r_1, r_2, \ldots$ is an enumeration of the rationals in $(0,1]$. Then $B^C = (0,1] - B$ is nowhere dense but not trifling (or even negligible).

   Proof. To prove B^C is nowhere dense, we will prove that every interval contains a subinterval which is contained in $(B^C)^C = B$. An additional fact we will exploit is that, given an enumeration $r_1, r_2, \ldots$ of the rationals in $(0,1]$, the set $\{r_m\}_{m=n}^{\infty}$ for any $n \geq 1$ is dense in $(0,1]$.

   Suppose J is an interval in Ω; denote the midpoint of J by x and let $|J| = \varepsilon$. To buy ourselves some space away from the endpoints of J, we will consider the subinterval $J^* = \left(x - \frac{\varepsilon}{4}, x + \frac{\varepsilon}{4}\right]$. Let n be a natural number such that $2^{-n-2} < \frac{\varepsilon}{2}$, which can be written $2^{-n-1} < \frac{\varepsilon}{4}$. By the comments above, $\{r_m\}_{m=n}^{\infty}$ is dense in $(0,1]$ for this choice of n, so a rational $r_m, m \geq n$ can be found in J^*. Then the interval $I = (r_m - 2^{-m-2}, r_m + 2^{-m-2}]$ is contained in B, that is, $I \cap B^C = \varnothing$. What's more, $|I| = 2^{-m-1} < 2^{-n-1} < \frac{\varepsilon}{4}$ and since $r_m \in J^*$, I extends at most $\frac{\varepsilon}{4}$ to the right or left of the endpoints of J^*. This shows that $I \subset J$ and therefore we conclude that B^C is nowhere dense in Ω. □

Definition. *We say a set is* **of the first category** *if the set can be represented as a countable union of nowhere dense sets.*

This is a topological notion of smallness, just as negligibility is a metric notion of smallness. The following examples illustrate that neither of these conditions implies the other.

Examples.

4 The non-negligible set N of normal numbers is of the first category.

Proof. To prove the statement, we will show that $A_m = \bigcap_{n=m}^{\infty} \left[\omega : |n^{-1}s_n(\omega)| < \frac{1}{2}\right]$ is nowhere dense and $N \subset \bigcup_m A_m$. Consider A_m for a fixed $m \in \mathbb{N}$. To prove A_m is nowhere dense in $\Omega = (0,1]$, we will show that for any $J \subset \Omega$ there is a subinterval $I \subset J$ such that $I \cap A_m = \varnothing$. Given such an interval $J = (a,b]$, choose a dyadic interval $J \subset I$ of order $n_0 > m$ such that

$$\tfrac{1}{n_0}|s_{n_0}(\omega)| > \frac{1}{2} \quad \text{for all } \omega \in I.$$

Such a choice of I is possible because specifying I is equivalent to a choice of the first n_0 of every $\omega \in I$ (so that $I \subset J$), and taking the rest of the digits to be 1's, with n_0 large enough so that $\frac{1}{n_0}|s_{n_0}(\omega)|$ is sufficiently close to 1 for all $\omega \in I$. We then claim that $I \cap A_m = \varnothing$. To see this, recall the definition of A_m:

$$A_m = \cap_{n=m}^{\infty} \left[\omega : \tfrac{1}{n}|s_n(\omega)| < \tfrac{1}{2}\right].$$

If $\omega \in I$ then our choice of n_0 means that $\omega \notin \left[\omega : \frac{1}{n_0}|s_{n_0}(\omega)| < \frac{1}{2}\right]$ and therefore ω does not lie in the intersection defining A_m. Hence A_m and I are disjoint.

Now to prove $N \subset \bigcup_m A_m$, recall that the set of normal numbers may be defined by

$$\begin{aligned} N &= \left\{\omega : \lim_{n\to\infty} \tfrac{1}{n}|s_n(\omega)| = 0\right\} \\ &= \left\{\omega : \text{ for all } \varepsilon > 0, \tfrac{1}{n}|s_n(\omega)| < \varepsilon \text{ for all } n \geq \text{some } n_0\right\}. \end{aligned}$$

Let $\omega \in N$ and choose $\varepsilon = \frac{1}{2}$. Then for all $n \geq n_0$ for the appropriate choice of n_0, $\frac{1}{n}|s_n(\omega)| < \frac{1}{2}$. This shows that $\omega \in A_{n_0}$ and so $\omega \in \bigcup_m A_m$. Hence $N \subset \bigcup_m A_m$. □

5 On the other hand, the negligible set N^C is not of the first category.

Proof. If N^C were a countable union of nowhere dense sets, then $(0,1] = N \cup N^C$ would be as well since by example 4, N is of the first category. However, a famous theorem of Baire (see Royden's *Real Analysis*) says that a nonempty interval is not of the first category, so it follows that N^C is not of the first category. □

Chapter 23

Probability Measures

23.1 Probability Measures

All the discussion about negligible sets and normal numbers highlights the utility of computing 'lengths' of more complicated sets than finite unions of intervals. If we try to define a function giving length for *any* subset of $(0,1]$, there are immediate logical contradictions (e.g. the Banach-Tarski Paradox in $\mathbb{R}^3$ and similar issues in other dimensions). So we need to restrict our attention to certain types of subsets. For instance, we want to calculate the length of the normal numbers, which may be written

$$N = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ \omega : \frac{1}{n}|s_n(\omega)| < \frac{1}{k} \right\}.$$

Notice that for any particular n and k, the set inside all the unions and intersections is a finite union of intervals. Measure theory comes to the rescue with the following definition. Let $\Omega = (0,1]$ as before.

Lemma 23.1.1. *$\mathcal{B}_0 = \{A \subset \Omega \mid A$ is a finite union of intervals$\}$ is an algebra.*

Proof. Note that $\Omega \in \mathcal{B}_0$ and that the complement of a finite union of intervals is a finite union of intervals: if

$$A = \bigcup_{i=1}^{n} (a_i, b_i]$$

then $A^C = \bigcap_{i=1}^{n} (a_i, b_i]^C = \bigcap_{i=1}^{n} \Big((0, a_i] \cup (b_i, 1] \Big)$ which is a finite union of intervals. The proof that $\mathcal{B}_0$ is closed under finite unions is similar. $\square$

Note that $\mathcal{B}_0$ is *not* a σ-algebra, since for example $\left(0, \frac{1}{2}\right)$ is not in $\mathcal{B}_0$ but this can be expressed as a countable union of sets in $\mathcal{B}_0$:

$$\left(0, \frac{1}{2}\right) = \bigcup_{n=1}^{\infty} \left(0, \frac{1}{2} - \frac{1}{n}\right].$$

Definition. *The* **Borel σ-algebra** *on $\Omega = (0,1]$ is defined as $\mathcal{B} = \sigma(\mathcal{B}_0)$. An element of $\mathcal{B}$ is called a* **Borel set**. *Any countable sequence of set operations applied to an interval $(a,b] \subset \Omega$ will produce a Borel set.*

It is important to note that not every Borel set is obtained in this way; that is, not every Borel set is the result of applying countably many set operations on an interval.

The Borel σ-algebra is our favourite σ-algebra in probability theory, as it allows us to define measures in a meaningful way, i.e. so that they are compatible with all Borel sets. However, the one important type of set that we have seen so far – a negligible or measure zero set – is *not always* a Borel set.

Example 23.1.2. Open sets (and therefore closed sets) are Borel sets. This is because an open set $U \subset (0,1]$ contains a countable, dense set $\mathbb{Q} \cap U$. Therefore for any $x \in U$, there exist rationals $p_x, q_x \in \mathbb{Q}$ such that

$$x - \varepsilon \leq p_x < x \leq q_x < x + \varepsilon.$$

Then $x \in (p_x, q_x] \subset (x-\varepsilon, x+\varepsilon] \subset U$. We can thus express U as a countable union of intervals:

$$U = \bigcup_{x\in U}(p_x, q_x].$$

Hence U is Borel.

Example 23.1.3. The set of normal numbers N is a Borel set.

Definition. *Let Ω be a space and let $\mathcal{F}$ be an algebra on Ω. A measure $P : \mathcal{F} \to \mathbb{R}$ is called a* **probability measure** *provided*

(1) $0 \le P(A) \le 1$ *for all* $A \in \mathcal{F}$.

(2) $P(\Omega) = 1$ *and* $P(\varnothing) = 0$.

Remark. If $A \in \mathcal{F}$ then by (2),

$$P(A) + P(A^C) = P(A \cup A^C) = P(\Omega) = 1.$$

Hence $P(A^C) = 1 - P(A)$ for all A. This actually implies $P(\varnothing) = 0$, so stating it in the definition was redundant. Further, since $P(A^C) = 1 - P(A) \ge 0$, $P(A) \le 1$ for all $A \in \mathcal{F}$ so this was redundant in the definition too.

Definition. *Let $\mathcal{F}$ be an algebra on Ω and $P : \mathcal{F} \to \mathbb{R}$ be a probability measure on $\mathcal{F}$. The triple $(\Omega, \mathcal{F}, P)$ is called a* **probability space***.*

Definition. *If $F \in \mathcal{F}$ such that $P(F) = 1$, F is called a* **support** *of P on $\mathcal{F}$.*

Example 23.1.4. Let $\Omega = \mathbb{N} = \{1, 2, 3, \ldots\}$ and define the function $p : \Omega \to \mathbb{R}$ by $p(n) = \frac{1}{2^n}$. Notice that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. For any $A \subset \Omega$, we define a probability measure P by

$$P(A) = \sum_{a\in A} p(a).$$

The right σ-algebra to use here is $\mathcal{F} = \mathbb{P}(\Omega)$, which makes $(\Omega, \mathcal{F}, P)$ into what is called a *discrete probability space* (see below).

Example 23.1.5. Let $\Omega = (0, \infty)$ and consider the σ-algebra $\mathcal{F} = \mathbb{P}(\Omega)$. We can use exactly the same formula for p to define

$$P(A) = \sum_{a\in A\cap\mathbb{Z}} p(a).$$

For example, $P\left(\left[\frac{1}{2}, 10\frac{1}{2}\right]\right) = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{10}}$. Oddly, many intervals have zero probability in this space: $P\left(\left[\frac{1}{3}, \frac{1}{2}\right]\right) = 0$. In this example, $\mathbb{Z}$ is a support for P.

Definition. *Let Ω be a countable space. For a nonnegative function $p : \Omega \to [0, \infty)$ such that $\sum_{\omega\in\Omega} p(\omega) = 1$, define $P(A) = \sum_{\omega\in A} p(\omega)$. Then (Ω, P) is called a* **discrete probability space***.*

Lemma 23.1.6. *A discrete probability space cannot contain an infinite sequence $A_1, A_2, \dots$ of independent events each of probability $\frac{1}{2}$.*

Proof. Suppose Ω is such a space. Consider a number $\omega \in \Omega$. Then ω must lie in one of the following sets:

$$A_1 \cap A_2 \qquad A_1 \cap A_2^C \qquad A_1^C \cap A_2 \qquad \text{or} \quad A_1^C \cap A_2^C,$$

each of which has probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ by independence. Thus $P(\omega) \leq \frac{1}{4}$. Likewise at the "$n$th" level we have a collection of intersections

$$A_1 \cap \dots \cap A_n \qquad A_1 \cap \dots \cap A_n^C \qquad \cdots \qquad A_1^C \cap \dots \cap A_n^C$$

which partition Ω, and by independence each of these sets has probability 2^{-n}. Then ω must lie in one of these intersections, so again $P(\omega) \leq 2^{-n}$. Taking $n \to \infty$, we conclude $P(\omega) = 0$ but this is not possible in a discrete probability space, since we would have

$$P(\Omega) = \sum_{\omega \in \Omega} P(\omega) = \sum_{\omega \in \Omega} 0 = 0 \neq 1.$$

□

This can be generalized as follows.

Lemma 23.1.7. *Suppose that $0 \leq p_n \leq 1$, and put $\alpha_n = \min\{p_n, 1 - p_n\}$. If $\sum_n \alpha_n$ diverges, then no discrete probability space can contain independent events $A_1, A_2, \dots$ such that A_n has probability p_n.*

Proof. As above, define at the nth level a collection of intersections

$$\{B_1 \cap \dots \cap B_n\}$$

where B_i is a choice of either A_i or A_i^C. For a particular choice of the B_i, the intersection $B = \cap_{i=1}^n B_i$ has probability

$$P(B) \leq \prod_{i=1}^n (1 - \alpha_i)$$

by independence, since $1 - \alpha_n$ corresponds to the *maximum* of $\{p_n, 1 - p_n\}$. However notice that for each i, $1 - \alpha_i \leq e^{-\alpha_i}$ ($y = e^{-x}$ is increasing on $(0, 1]$). Thus the above product is bounded by

$$\prod_{i=1}^n (1 - \alpha_i) \leq \prod_{i=1}^n e^{-\alpha_i} = e^{-\sum_{i=1}^n \alpha_i}.$$

Since the series in the exponent above is assumed to diverge, the term on the right approaches zero as $n \to \infty$. This shows that $P(B) \to 0$ as n gets large, so in particular for any $\omega \in \Omega$, ω must lie in such an intersection as B at every level n, and so $P(\omega) = 0$ for all $\omega \in \Omega$. As in the previous proof, this cannot happen if Ω is a discrete probability space. Hence no such sequence $A_1, A_2, \dots$ exists. □

Let $\Omega = (0,1]$ with the Lebesgue measure λ defined on $\mathcal{B}_0$. Then Theorem 17.2.7 says that there exists an extension of λ to $\mathcal{B} = \sigma(\mathcal{B}_0)$, which we also call the Lebesgue measure λ. Notice that $\mathcal{I} = \{(a,b] : 0 \leq a < b \leq 1\}$ is a π-system, and further, $\sigma(\mathcal{I}) = \mathcal{B}$ because $\sigma(\mathcal{I}) \supset \mathcal{B}_0$. Then by Theorem 17.2.12, λ is the *only* measure on Ω that restricts to the length function on the collection $\mathcal{I}$ of intervals. This means that whenever we make a choice of measure on $(0,1]$ and want it to coincide with the natural notion of length of an interval, we have no choice but to choose the Lebesgue measure.

Example 23.1.8. Suppose $\{r_i\}_{i=1}^{\infty}$ is an enumeration of $\mathbb{Q} \cap (0,1]$. Let $\varepsilon > 0$ and for each $i \in \mathbb{N}$, let $I_i = \left(r_i - \frac{\varepsilon}{2^{i+1}}, r_i + \frac{\varepsilon}{2^{i+1}}\right) \cap (0,1]$. Then I_i is open so $A = \bigcup_{i=1}^{\infty} I_i$ is open and

$$\lambda(A) \leq \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon$$

by subadditivity. On the other hand, $\lambda(A)$ is clearly positive since A contains nonempty intervals of nonzero length. It turns out that A^C is nowhere dense, but $\lambda(B) \geq 1 - \varepsilon$, i.e. $\lambda(B)$ is *almost always 1.*

Example 23.1.9. Let $A_n = \{\omega \in \Omega \mid d_i(\omega) = d_{n+i}(\omega) = d_{2n+i}(\omega) \text{ for all } i = 1, \dots, n\}$. For example, an element of A_5 might look like

$$0.010110101101011\underbrace{01000110100\dots}_{\text{anything}}$$

Then $P(A_n) = \frac{2^n}{(2^3)^n}$ for each n, and if $A = \bigcup_{n=1}^{\infty} A_n$, $\lambda(A) \leq \frac{1}{3}$. As in the previous example, A^C turns out to be nowhere dense, but with (relatively) large measure: $\lambda(A^C) = \frac{2}{3}$.

23.2 Convergence in Probability

Assume $(\Omega, \mathcal{F}, P)$ is a generic probability space for a σ-field $\mathcal{F}$. We will assume in this section that all sets are $\mathcal{F}$-sets. Our goal is to explore some more complicated concepts in probability with infinite sequences of subsets (events) in a probability space Ω.

Proposition 23.2.1. *Let $\{A_n\}_{n=1}^{\infty}$ be a countable collection of sets in Ω.*

(1) If $P(A_n) = 0$ for all n then $P\left(\bigcup_{n=1}^{\infty} A_n\right) = 0$.

(2) If $P(B_n) = 1$ for all n then $P\left(\bigcap_{n=1}^{\infty} B_n\right) = 1$.

Proof. (1) is proven by subadditivity, and (1) $\implies$ (2) by DeMorgan's Law. $\square$

Definition. *For a sequence $\{A_n\}_{n=1}^{\infty}$ of measurable sets in Ω, the* **limit superior** *and* **limit inferior** *of the A_n are*

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m \quad \text{and} \quad \liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m.$$

If there exists $A \in \mathcal{F}$ such that $\limsup A_n = \liminf A_n = A$, we say the A_n **converge** *to A.*

Remark. Notice that $x \in \limsup A_n$ if for every $n \in \mathbb{N}$ there is some $m \geq n$ such that $x \in A_m$, or equivalently, if x lies in some subsequence of $\{A_n\}$. It is sometimes said that such an x lies in the collection $\{A_n\}$ *infinitely often* (i.o.).

On the other hand, $x \in \liminf A_n$ if there exists an $n \in \mathbb{N}$ such that for every $m \geq n$, $x \in A_m$, or alternatively if x lies in every A_m beyond some cutoff A_n. It is also said that x lies in *all but finitely many of the A_n*.

For fixed n,

$$\begin{aligned} \bigcup_{m=n}^{\infty} A_m \supset \bigcap_{m=n}^{\infty} A_m &\implies \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m \supset \bigcap_{m=n}^{\infty} A_m \\ &\implies \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m \supset \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m. \end{aligned}$$

Hence $\limsup A_n \supset \liminf A_n$, just as is the case with the numerical lim sup and lim inf.

Example 23.2.2. For $\omega \in \Omega$, define the value $\ell_n(\omega)$ to be the length of the run of heads in a sequence of coin tosses starting at the nth flip. Explicitly,

$$\ell_n(\omega) = \{k \mid d_{n+k}(\omega) = 1 \text{ and } d_{n+i}(\omega) = 0 \text{ for } i = 0, \ldots, k-1\}.$$

Notice that for any nonzero values of k and r,

$$P[\ell_n(\omega) = k] = \prod_{i=1}^{k+1} \frac{1}{2} = \frac{1}{2^{k+1}}$$

$$\text{and} \quad P[\ell_n(\omega) \geq r] = \sum_{k=r}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2^r}.$$

Set $A_n = \{\omega \mid \ell_n(\omega) \geq r\}$. Then for any $\omega \in \Omega$,

$$\begin{aligned}
\omega \in A_n \text{ infinitely often } &\iff \omega \in \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m \\
&\iff \text{for every } n \text{ there is some } m \geq n \text{ such that } \omega \in A_m \\
&\iff \text{there is an infinite subsequence } \{A_m\} \text{ containing } \omega.
\end{aligned}$$

Theorem 23.2.3. *For any collection of sets* $\{A_n\}_{n=1}^{\infty}$*,*

(1) $P(\liminf A_n) \leq \liminf P(A_n) \leq \limsup P(A_n) \leq P(\limsup A_n)$.

(2) If $\lim A_n$ *exists, the above inequalities are equalities.*

Proof. Note that (2) follows from (1) since if a limit exists, it equals $\liminf A_n$ and $\limsup A_n$. The middle inequality in (1) is obvious. Further, the third inequality is obtained by taking complements in the first inequality, so it suffices to prove the latter. Note that if $B_n = \bigcap_{m=n}^{\infty} A_m$ so that

$$\liminf A_n = \bigcup_{n=1}^{\infty} \left(\bigcap_{m=n}^{\infty} A_m \right) = \bigcup_{n=1}^{\infty} B_n,$$

then $B_n \subset A_n$ for all n. So by monotonicity, $P(B_n) \leq P(A_n)$. Moreover, the B_n are ascending and their limit is $\liminf A_n$. By continuity from below (Proposition 17.2.1), $\lim P(B_n) = P(\liminf A_n)$ so together this gives us $P(\liminf A_n) = \lim P(B_n) \leq \liminf P(A_n)$. □

Remark. Theorem 23.2.3 only holds in the finite measure case, that is, when $P(\omega) < \infty$. When Ω is a space of infinite measure, we cannot take complements and subtract the probability from 1, so the arguments above – including continuity from below – do not apply.

In Example 23.2.2, we can apply Theorem 23.2.3 to see that

$$P[\omega \mid \ell_n(\omega) \geq r \text{ i.o.}] = P(\limsup A_n) \geq \limsup P(A_n) = \frac{1}{2^r}.$$

23.3 Independence

Definition. *Suppose $P(A) > 0$. Then the* **conditional probability** *of B given A is*

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}.$$

This definition can be rearranged as $P(A \cap B) = P(A)P(B \mid A)$ and inducted on to produce the formula

$$P(A \cap B \cap C) = P(A)P(B \mid A)P(C \mid A \cap B), \text{ and so on.}$$

Proposition 23.3.1. *If $\{A_n\}_{n=1}^{\infty}$ partition Ω then for any $B \in \Omega$,*

$$P(B) = \sum_{n=1}^{\infty} P(B \mid A_n).$$

Proof sketch. Since B may be expressed as a disjoint, countable union $B = \bigcup_{n=1}^{\infty}(B \cap A_n)$, use countable additivity to produce the result. □

Definition. *Two events $A, B \in \Omega$ are said to be* **independent** *if $P(A \cap B) = P(A)P(B)$.*

If $P(A), P(B) > 0$, the definition of independence is equivalent to

$$P(B) = P(B \mid A) \quad \text{and} \quad P(A) = P(A \mid B).$$

Definition. *A (finite) collection $\{A_i\}_{i=1}^{n}$ is an* **independent collection** *if for any subcollection $\{A_{k_j}\}_{j=1}^{m}$,*

$$\prod_{j=1}^{m} P(A_{k_j}) = P\left(\bigcap_{j=1}^{m} A_{k_j}\right).$$

Example 23.3.2. Note that simple pairwise independence *is not the same* as independence on the whole collection. For instance, define $B_{uv} = \{\omega \mid d_u(\omega) = d_v(\omega)\}$. Then B_{12}, B_{13} and B_{23} are pairwise independence events, but the collection of all three is not independent.

Remarks.

1. For a collection of n sets, there are $2^n - n - 1$ possible conditions on those sets.

2. Any subcollection of a collection of independent events is also independent.

3. Independence is invariant under reordering of the sets in a collection.

4. We say an arbitrary collection of sets $\{A_\theta\}_{\theta \in \Theta}$ is independent if and only if every finite subcollection is independent.

Definition. *Given two collections $\mathcal{A}_1$ and $\mathcal{A}_2$, we say these are* **independent collections**, *or $\mathcal{A}_1$ is independent of $\mathcal{A}_2$, if for every $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, $P(A_1 \cap A_2 = P(A_1)P(A_2)$. Similarly, a collection of collections $\{\mathcal{A}_k\}_{k=1}^{n}$ is independent if $A_1, \ldots, A_n$ are independent whenever $A_k \in \mathcal{A}_k$. This can be extended to arbitrary collections in the same way as before.*

Example 23.3.3. Let $H_n = \{\omega \mid d_n(\omega) = 0\}$, which represents the event of getting heads on the nth coin flip (if we identify heads with 0 and tails with 1). Then $\{H_n\}_{n=1}^{\infty}$ is an independent collection of events. Furthermore, if we define $\mathcal{A}_1 = \{H_{2k+1} \mid k \in \mathbb{N}\}$ the collection of heads on odd flips and $\mathcal{A}_2 = \{H_{2k} \mid k \in \mathbb{N}\}$ the collection of heads on even flips, then $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent.

Theorem 23.3.4. *Suppose $\{\mathcal{A}_k\}_{k=1}^{n}$ is an independent collection of collections, where $\mathcal{A}_k$ is a π-system for $k = 1, \ldots, n$. Then $\{\sigma(\mathcal{A}_k)\}_{k=1}^{\infty}$ is independent.*

Proof. For each k define $\mathcal{B}_k = \mathcal{A}_k \cup \{\Omega\}$ which are all still π-systems. The fact that the $\mathcal{A}_k$ are independent is equivalent to

$$\prod_{k=1}^{n} P(B_k) = P\left(\bigcap_{k=1}^{n} B_k\right) \quad \text{for all } B_k \in \mathcal{B}_k.$$

Fix $B_2 \in \mathcal{B}_2, \ldots, B_n \in \mathcal{B}_n$ and define

$$\mathcal{L} = \left\{ B \in \mathcal{F} : P(B) \prod_{k=2}^{n} P(B_k) = P\left(B \cap \left(\bigcap_{k=2}^{n} B_k\right)\right)\right\}.$$

Clearly $\mathcal{L}$ contains $\mathcal{B}_1$. We claim that $\mathcal{L}$ is a λ-system.

(1) $\Omega \in \mathcal{B}_1 \implies \Omega \in \mathcal{L}$.

(2) For any $B \in \mathcal{L}$,

$$\begin{aligned}
P(B^C) \prod_{k=2}^{n} P(B_k) &= (1 - P(B)) \prod_{k=2}^{n} P(B_k) \\
&= P(\Omega) \prod_{k=2}^{n} P(B_k) - P(B) \prod_{k=2}^{n} P(B_k) \\
&= P\left(\Omega \cap \left(\bigcap_{k=2}^{n} B_k\right)\right) - P\left(B \cap \left(\bigcap_{k=2}^{n} B_k\right)\right) \\
&\quad \text{since } \Omega, B \in \mathcal{L} \\
&= P\left((\Omega \setminus B) \cap \left(\bigcap_{k=2}^{n} B_k\right)\right) \quad \text{by additivity} \\
&= P\left(B^C \cap \left(\bigcap_{k=2}^{n} B_k\right)\right).
\end{aligned}$$

Hence $B^C \in \mathcal{L}$.

(3) follows similarly, again breaking $A \cup B$ into sets in $\mathcal{L}$ and using additivity.

Hence $\mathcal{L}$ is a λ-system, so by the π-λ theorem (17.2.11), $\mathcal{L} \supset \sigma(\mathcal{B}_1)$. This shows that $\sigma(\mathcal{B}_1)$ is independent of $\mathcal{B}_2, \ldots, \mathcal{B}_n$.

Now, since $\sigma(\mathcal{B}_1)$ is a π-system, a similar argument can be made by fixing $B_1 \in \sigma(\mathcal{B}_1), B_3 \in \mathcal{B}_3, \ldots, B_n \in \mathcal{B}_n$ to show that $\{\sigma(\mathcal{B}_1), \sigma(\mathcal{B}_2), \mathcal{B}_3, \ldots, \mathcal{B}_n\}$ are independent. Repeating shows that $\{\sigma(\mathcal{B}_k)\}$ are independent, but notice that $\sigma(\mathcal{A}_k) = \sigma(\mathcal{B}_k)$ for all k, so we're done. □

Corollary 23.3.5. *If $\{\mathcal{A}_\theta\}_{\theta\in\Theta}$ is an independent, arbitrary collection of π-systems then $\{\sigma(\mathcal{A}_\theta)\}_{\theta\in\Theta}$ is also independent.*

Corollary 23.3.6. *Given a matrix of events*

$$\begin{matrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

such that the collection $\{A_{ij}\}_{(i,j)\in\mathbb{N}^2}$ is independent, define $\mathcal{F}_i$ to be the σ-algebra of the events in the ith row. Then the collection of $\mathcal{F}_i$ is an independent collection.

Proof. Let $\mathcal{A}_i$ be the collection of finite intersections of events in the ith row. This is a π-system by construction. Clearly $\sigma(\mathcal{A}_i) \subset \mathcal{F}_i$, and since a single set is a (trivial) intersection, $\mathcal{F}_i \subset \sigma(\mathcal{A}_i)$ as well, which shows $\sigma(\mathcal{A}_i) = \mathcal{F}_i$ for each i. Now we verify that the $\mathcal{A}_i$ are independent. Given $B_1 \in \mathcal{A}_{k1}, \ldots, B_j \in \mathcal{A}_{kj}$, each one is a finite intersection of the A_{ij} so we have

$$\prod_{i=1}^{j} P(B_i) = \prod_{i=1}^{j}\prod_{r=1}^{m_i} P(A_{ir}) = P\left(\bigcap_{i=1}^{j}\bigcap_{r=1}^{m_i} A_{ir}\right) = P\left(\bigcap_{i=1}^{j} B_i\right).$$

Therefore Theorem 23.3.4 applies to give us independence of the $\mathcal{F}_i$. □

Examples.

1. For the even/odd coin flips, we have a matrix

$$\begin{matrix} H_1 & H_3 & H_5 & \cdots \\ H_2 & H_4 & H_6 & \cdots \end{matrix}$$

which is independent by Example 23.3.3. By Corollary 23.3.6, the σ-algebras of the two rows are independent, but these just correspond to the even and odd flips. What this says is that any event constructed with the odd-numbered coin flips is independent of any other event constructed with the even flips.

2. Recall $B_{uv} = \{\omega \mid d_u(\omega) = d_v(\omega)\}$. The collections $\mathcal{A}_1 = \{B_{12}, B_{13}\}$ and $\mathcal{A}_2 = \{B_{23}\}$ are independent, but we know their union is not, so $\sigma(\mathcal{A}_1)$ and $\sigma(\mathcal{A}_2)$ must not be independent. This suggests that some condition of Theorem 23.3.4 fails to be met – in fact, $\mathcal{A}_1$ is not a π-system.

3. Suppose $\mathcal{A} = \{A_i\}$ is a partition of Ω and for some set $B \in \mathcal{F}$, $P(B \mid A_i) = p$ for each i such that $P(A_i) > 0$. Then $P(B) = p$ by Proposition 23.3.1 and $\mathcal{B} = \{B\}$ is independent of $\sigma(\mathcal{A})$.

It is common in probability theory to only have partial information about an event. This can mean several things, such as knowing some class of sets the event lies in or knowing conditions on the probability of the event based on other events. For a probability space $(\Omega, \mathcal{F}, P)$, such partial information information can be represented in terms of a *subalgebra* $\mathcal{A} \subset \mathcal{F}$ where $\mathcal{A}$ is a σ-algebra (or sometimes just a algebra). If $B \in \mathcal{F}$ is independent of $\mathcal{A}$, then $P(B) = P(B \cap A)P(A)$ for every $A \in \mathcal{A}$. This shows that knowing information about $P(A)$ *does not* necessarily tell you anything about $P(B)$.

Define an equivalence relation $\sim_{\mathcal{A}}$ on Ω by $\omega \sim_{\mathcal{A}} \omega'$ if and only if for every $A \in \mathcal{A}$, $\chi_A(\omega) = \chi_A(\omega')$, where χ denotes the characteristic function on a set:

$$\chi_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

As an example of so-called 'partial information' in probability theory, even if we know everything about the characteristic functions on a collection $\mathcal{A}$, we still cannot distinguish between ω and ω' that satisfy $\omega \sim \omega'$.

Proposition 23.3.7. *For a subalgebra $\mathcal{A} \subset \mathcal{F}$, $\sim_{\mathcal{A}}$ and $\sim_{\sigma(\mathcal{A})}$ are the same equivalence relation.*

Proof. It is clear that $\sim_{\sigma(\mathcal{A})}$ is a finer partition of Ω than $\sim_{\mathcal{A}}$, since $\mathcal{A} \subset \sigma(\mathcal{A})$. Fix $\omega, \omega' \in \Omega$. Then $\mathcal{A}_{\omega,\omega'} = \{A \in \mathcal{F} : \chi_A(\omega) = \chi_A(\omega')\}$ is a σ-algebra. Notice that if $\omega \sim_{\mathcal{A}} \omega'$ then $\mathcal{A}_{\omega,\omega'} \supset \mathcal{A}$ so by the π-λ theorem, $\mathcal{A}_{\omega,\omega'} \supset \sigma(\mathcal{A})$. Hence $\omega \sim_{\sigma(\mathcal{A})} \omega'$. □

Examples.

1. Consider $\mathcal{A} = \{H_{2n}\}_{n \in \mathbb{N}}$ where H_{2n} are as defined in Example 23.3.3. Then $\omega \sim_{\mathcal{A}} \omega'$ if and only if ω and ω' have the same results on even coin flips, but this tells us nothing about the odd flips of either event.

2. Let $\mathcal{A}$ be the σ-algebra generated by countable and cocountable subsets of Ω, which is a subalgebra of the Borel σ-algebra $\mathcal{B}$ on $\Omega = (0, 1]$. Then $\mathcal{B}$ is independent of $\mathcal{A}$ so we get practically no information about events in $\mathcal{A}$; in fact, every countable set has measure 0 and every cocountable set has measure 1, but this does nothing to distinguish events.

   On the other hand, $\mathcal{A}$ is generated by the singletons, so $\omega \sim_{\mathcal{A}} \omega' \iff \omega = \omega'$. In this sense, we have *all* the information on events in $\mathcal{A}$, paradoxically. The difference between these two situations, where we either have no information or complete information is that there is a measure present on $\mathcal{B}$ but not on $\mathcal{A}$. The mathematics is actually contained in the statement of independence, and while it may be useful to think of σ-algebras and subalgebras in terms of 'information', it is clear that this analogy breaks down in situations such as this.

23.4 The Borel-Cantelli Lemmas

Lemma 23.4.1. *If $\sum_{n=1}^{\infty} P(A_n)$ converges then $P(\limsup A_n) = 0$.*

Proof. Fix n. Then $\limsup A_n \subset \bigcup_{k=n}^{\infty} A_k$ so by subadditivity we have

$$P(\limsup A_n) \leq \sum_{k=n}^{\infty} P(A_k).$$

But the tail of a convergent series tends to 0 as n gets big, so the right side of this inequality goes to 0. Hence $P(\limsup A_n) = 0$. □

The first Borel-Cantelli Lemma would have been useful at the end of the proof of Borel's Normal Number Theorem (22.2.5); however, at the time we did not have the constructions required to describe this lemma. The second Borel-Cantelli Lemma is

Lemma 23.4.2. *If $\sum_{n=1}^{\infty} P(A_n)$ diverges and the events $\{A_n\}_{n=1}^{\infty}$ are independent, then $P(\limsup A_n) = 1$.*

Proof. We want to show that $P(\limsup A_n) = P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right) = 1$. Note that this is the same as showing

$$P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right)^C = P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^C\right) = 0,$$

which will be true if $P\left(\bigcap_{k=n}^{\infty} A_k^C\right) = 0$ for all $n \geq 1$. By independence, we have

$$\begin{aligned} P\left(\bigcap_{k=n}^{\infty} A_k^C\right) &= \prod_{k=n}^{\infty} P(A_k^C) = \prod_{k=n}^{\infty}(1 - P(A_k)) \\ &\leq \prod_{k=n}^{\infty} e^{-P(A_k)} \quad \text{since } 1 - x \leq e^{-x} \text{ on } (0,1] \\ &= e^{-\sum_{k=n}^{\infty} P(A_k)} = 0 \end{aligned}$$

since $\sum_{k=n}^{\infty} P(A_k)$ diverges. □

Examples.

1. Recall the funtion $\ell_n(\omega)$ which denotes the length of the run of 0's (heads) starting at the nth term in the sequence expression of ω. Let (r_n) be a sequence of real numbers such that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{r_n}}$ converges. Then $P[\omega : \ell_n(\omega) \geq r_n \text{ i.o.}] = 0$. To see this, note that

$$P[\omega : \ell_n(\omega) \geq r_n \text{ i.o.}] = P(\limsup\{\omega \mid \ell_n(\omega) \geq r_n\})$$

and we have computed $P[\omega : \ell_n(\omega) \geq r_n] = \frac{1}{2^{r_n}}$. Applying the first Borel-Cantelli Lemma gives the result.

For $\varepsilon > 0$, set $r_n = (1+\varepsilon)\log_2 n$ so that $2^{r_n} = n^{1+\varepsilon}$. Then $\sum_{n=1}^{\infty} 2^{-r_n}$ *barely* converges, but the first Borell-Cantelli Lemma says that

$$\begin{aligned} &P[\omega : \ell_n(\omega) \geq (1+\varepsilon)\log_2 n \text{ i.o.}] = 0 \\ \implies &P\left[\omega : \limsup \left(\frac{\ell_n(\omega)}{\log_2 n}\right) > 1\right] = 0. \end{aligned}$$

2 Notice that the collection of $\{\omega : \ell_n(\omega) = 0\} = \{\omega : d_n(\omega) = 1\}$ for all n are independent events, each with probability $\frac{1}{2}$. By the second Borel-Cantelli Lemma, $P[\omega : \ell_n(\omega) = 0 \text{ i.o.}] = 1$. On the other hand, define

$$A_n = \{\omega : \ell_n(\omega) = 1\} = \{\omega : d_n(\omega) = 0, d_{n+1}(\omega) = 1\}.$$

Then $P(A_n) = \frac{1}{4}$ so $\sum_{n=1}^{\infty} P(A_n)$ diverges, but we cannot directly apply BC2 since the A_n are not independent. However, the collection $\{A_{2n}\}_{n=1}^{\infty}$ *is independent* with $P(A_{2n}) = \frac{1}{4}$ for all n, so BC2 tells us $P[\omega \mid \omega \in A_{2n} \text{ i.o.}] = 1$. Further, we see that $\{\omega \mid \omega \in A_{2n} \text{ i.o.}\} \subseteq \{\omega : \ell_n(\omega) = 1 \text{ i.o.}\}$ so by subadditivity, $P[\omega : \ell_n(\omega) = 1 \text{ i.o.}] = 1$ as well. In the same way, one can prove that $P[\omega : \ell_n(\omega) = k \text{ i.o.}] = 1$ for any $k \in \mathbb{N}$.

3 Suppose (r_n) is a nondecreasing sequence such that $\sum_{n=1}^{\infty} \frac{1}{r_n 2^{r_n}}$ diveges. We claim that $P[\omega : \ell_n(\omega) \geq r_n \text{ i.o.}] = 1$. First, it is known that $\sum_{n=1}^{\infty} \frac{1}{r_n 2^{r_n}}$ diverges if and only if $\sum_{n=1}^{\infty} \frac{1}{s_n 2^{s_n}}$ diverges, where $s_n = \lceil r_n \rceil$. Thus without loss of generality we may assume the r_n are integers.

Define $n_1 = 1$ and $n_{k+1} = n_k + r_{n_k}$ for each $k \geq 2$. The events $A_k = \{\omega : \ell_{n_k}(\omega) \geq r_{n_k}\}$ for all $k \in \mathbb{N}$ are independent events. By BC2, $P[\omega \mid \omega \in A_k \text{ i.o.}] = 1$ if $\sum_{k=1}^{\infty} P(A_k)$ diverges, but this series can be written

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{2^{r_{n_k}}} &= \sum_{k=1}^{\infty} \frac{1}{2^{n_{k+1}-n_k}} \\ &= \sum_{k=1}^{\infty} \frac{n_{k+1}-n_k}{r_{n_k} 2^{r_{n_k}}} \\ &= \sum_{k=1}^{\infty} \sum_{n=n_k+1}^{n_{k+1}} \frac{1}{r_{n_k} 2^{r_{n_k}}} \\ &\geq \sum_{k=1}^{\infty} \sum_{n=n_k+1}^{n_{k+1}} \frac{1}{r_n 2^{r_n}} \quad \text{since the } r_n \text{ are nondecreasing} \\ &= \sum_{n=1}^{\infty} \frac{1}{r_n 2^n} \end{aligned}$$

which diverges. Hence $P[\omega \mid \omega \in A_k \text{ i.o.}] = 1$, and since $\{\omega : \ell_n(\omega) \geq r_n \text{ i.o.}\} \supseteq \limsup A_k$, we have shown that $P[\omega : \ell_n(\omega) \geq r_n \text{ i.o.}] = 1$.

4 Let $r_n = \log_2 n$ as before. Then $\sum_{n=1}^{\infty} \frac{1}{r_n 2^{r_n}} = \sum_{n=1}^{\infty} \frac{1}{n \log_2 n}$ diverges, so BC2 tells us that $P[\omega : \ell_n(\omega) \geq r_n \text{ i.o.}] = 1$, which implies

$$P\left[\omega : \limsup \left(\frac{\ell_n(\omega)}{\log_2 n}\right) \geq 1\right] = 1$$

but before we showed that

$$P\left[\omega : \limsup \left(\frac{\ell_n(\omega)}{\log_2 n}\right) > 1\right] = 0$$

(notice the strict inequality). By additivity, this implies that

$$P\left[\omega : \limsup \left(\frac{\ell_n(\omega)}{\log_2 n}\right) = 1\right] = 1.$$

Definition. *Suppose* $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$ *and set*

$$\mathcal{T} = \bigcap_{k=1}^{\infty} \sigma(A_k, A_{k+1}, A_{k+2}, \ldots).$$

$\mathcal{T}$ *is called the* **tail σ-algebra** *of the* A_n.

The most important theorem related to the tail σ-algebra is stated below. This is sometimes known as Kolmogorov's 0-1 Law.

Theorem 23.4.3 (Kolmogorov)**.** *Suppose* $\{A_n\}_{n=1}^{\infty}$ *are independent and* $A \in \mathcal{T}$. *Then either* $P(A) = 0$ *or* $P(A) = 1$.

Proof. Clearly $\sigma(A_1), \sigma(A_2), \ldots, \sigma(A_n), \sigma(A_{n+1}, A_{n+2}, \ldots)$ are independent for any fixed n since the A_k are independent (Theorem 23.3.4). So if $A \in \mathcal{T}$ then $A \in \sigma(A_{n+1}, A_{n+2}, \ldots)$. Thus A is independent of $\sigma(A_1), \ldots, \sigma(A_n)$ for any n, which shows that $A, A_1, A_2, \ldots$ is an independent sequence of events. By Theorem 23.3.4, $\sigma(A)$ and $\sigma(A_1, A_2, \ldots)$ are independent, but $A \in \mathcal{T} \subseteq \sigma(A_1, A_2, \ldots)$ so we have shown that A is independent of itself. Finally,

$$\begin{aligned} A \text{ is independent of itself} &\iff P(A) = P(A \cap A) = P(A)P(A) \\ &\iff P(A) = 0 \text{ or } 1. \end{aligned}$$

□

Chapter 24

Simple Random Variables

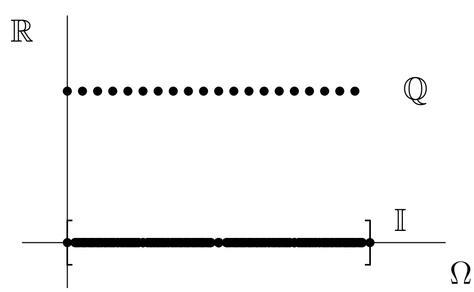
Definition. *Given a probability space $(\Omega, \mathcal{F}, P)$, we say $X : \Omega \to \mathbb{R}$ is a* **simple random variable** *if*

(1) $X(\Omega)$ is finite. This is the simple *condition.*

(2) For any $x \in \mathbb{R}$, the set $\{\omega : X(\omega) = x\}$ is an $\mathcal{F}$-set. This is sometimes called the measurable condition.

The measurability condition (2) will allow us to integrate simple random variables over P-measurable sets. Of course, the sets $\{\omega : X(\omega) = x\}$ need not be intervals. For example, consider the rational indicator function on $\Omega = [0, 1]$:

$$X(\omega) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$



Example 24.0.1. Recall the functions d_n, r_n and s_n introduced in Chapter 22 for the space of infinite sequences of coin flips (equivalently, dyadic representations of $\omega \in (0, 1]$). Each of these is a simple random variable. On the other hand, the length function ℓ_n is a random variable because it's measurable, but ℓ_n is not simple since its range is infinite.

Proposition 24.0.2. *X is a simple random variable $\iff$ there exists a finite collection of $A_i \in \mathcal{F}$ such that $\{A_i\}_{i=1}^r$ is a finite partition of Ω and there exist $x_i \in \mathbb{R}$ such that for any $\omega \in \Omega$, $X(\omega)$ may be expressed in the form*

$$X(\omega) = \sum_{i=1}^{r} x_i \chi_{A_i}(\omega).$$

Proof sketch. The backward direction is easy. For the forward direction, suppose X is a simple random variable. Let $\{x_1, \dots, x_r\} = X(\Omega)$. For $i = 1, \dots, r$, let $A_i = X^{-1}(x_i)$. Then the A_i partition Ω. $\square$

Notice that the partition $\{A_i\}$ need not be unique. Even more importantly, the x_i may not be unique – they may even repeat so that in some cases A_i and A_j, $i \neq j$, have the same value under X. This is useful if we are comparing simple random variables X and Y and want to use the same partition of Ω for each.

Definition. *For a subfield $\mathcal{G} \subset \mathcal{F}$, we say a simple random variable X is* $\mathcal{G}$**-measurable** *if $\{\omega : X(\omega) = x\} \in \mathcal{G}$ for every $x \in \mathbb{R}$.*

Proposition 24.0.3. *If X is $\mathcal{G}$-measurable and $H \subset \mathbb{R}$ then $\{\omega : X(\omega) \in H\}$ lies in $\mathcal{G}$.*

Proof. Notice that $\{\omega : X(\omega) = H\} = \bigcup\limits_{x \in H} \{\omega : X(\omega) = x\}$ is a finite union. $\square$

Proposition 24.0.3 can also be stated: if X is $\mathcal{G}$-measurable then $X^{-1}(H)$ is measurable for every $H \subset \mathbb{R}$. Define $\sigma(X)$ to be the smallest σ-field $\mathcal{G}$ (equivalently, the intersection of all $\mathcal{G}$) for which X is $\mathcal{G}$-measurable. The next theorem characterizes $\sigma(X)$ for collections of simple random variables.

Theorem 24.0.4. *Suppose $X_1, \dots, X_n$ and Y are simple random variables on a probability space $(\Omega, \mathcal{F}, P)$. Write $\overline{X} = (X_1, \dots, X_n)$ so that for any $\omega \in \Omega$, $\overline{X}(\omega) \in \mathbb{R}^n$.*

(1) $\sigma(\overline{X}) = \{\{(X_1(\omega), \dots, X_n(\omega)) \in H\} \mid \omega \in \Omega, H \subset \mathbb{R}^n\} = \{\overline{X}^{-1}(H) \mid H \subset \mathbb{R}^n\}$. Moreover, the H may be taken as finite subsets of $\mathbb{R}^n$.

(2) Y is $\sigma(\overline{X})$-measurable if and only if there exists a function $f : \mathbb{R}^n \to \mathbb{R}$ such that $Y = f(\overline{X}) = f(X_1, \dots, X_n)$.

Proof. (1) Let $\mathcal{M} = \{\overline{X}^{-1}(H) \mid H \subset \mathbb{R}^n\}$. We will show that $\mathcal{M} = \sigma(\overline{X})$. Consider a set $\overline{X}^{-1}(H) \in \mathcal{M}$. Clearly

$$\overline{X}^{-1}(H) = \bigcup_{i=1}^{r} \overline{X}^{-1}(\vec{x}) = \bigcup_{i=1}^{r} \left(\bigcap_{j=1}^{n} X_j^{-1}(x_j) \right),$$

where $\vec{x} = (x_1, \dots, x_n)$, and each $\bigcap_{j=1}^n X_j^{-1}(x_j)$ lies in $\sigma(\overline{X})$, so $\overline{X}^{-1}(H) \in \sigma(\overline{X})$. This shows $\mathcal{M} \subset \sigma(\overline{X})$. Additionally, $\mathcal{M}$ is a σ-field since

(i) $\Omega = X^{-1}(\mathbb{R}^n) \in \mathcal{M}$.

(ii) If $A \in \mathcal{M}$ then $A = \overline{X}^{-1}(H) \implies A^C = \overline{X}^{-1}(H^C) \in \mathcal{M}$.

(iii) For $A_i \in \mathcal{M}$, $\bigcup\limits_{i=1}^{\infty} A_i = \bigcup\limits_{i=1}^{\infty} \overline{X}^{-1}(H_i) = \overline{X}^{-1}\left(\bigcup\limits_{i=1}^{\infty} H_i\right)$ which lies in $\mathcal{M}$.

Finally, for fixed i, $\{\omega : X_i(\omega) = x_i\} \in \mathcal{M}$ but this is precisely the set

$$\{\omega : \overline{X}(\omega) \in \mathbb{R} \times \mathbb{R} \times \cdots \times \{x_i\} \times \cdots \times \mathbb{R}\}$$

which lies in $\mathcal{M}$. We have shown that $\mathcal{M}$ is a σ-field contained in $\sigma(\overline{X})$ on which the X_i are measurable, so it follows that $\mathcal{M} = \sigma(\overline{X})$.

(2) On one hand, suppose $Y = f(\overline{X})$. Then $\{\omega : Y(\omega) = y\} = \{\omega : \overline{X}(\omega) \in f^{-1}(y)\} \in \mathcal{M}$ which lies in $\sigma(\overline{X})$, so Y is $\sigma(\overline{X})$-measurable. On the other hand, if Y is $\sigma(\overline{X})$-measurable then let $Y(\Omega) = \{y_1, \ldots, y_r\}$. By (1), there exist subsets $H_1, \ldots, H_r \subset \mathbb{R}^n$ such that for all $1 \le i \le r$, $\{\omega : Y(\omega) = y_i\} = \{\omega : \overline{X}(\omega) \in H_i\}$. By construction the H_i are disjoint, and we can define $f(\overline{X}) = \displaystyle\sum_{i=1}^{r} y_i \chi_{H_i}(x_i)$. This completes the proof. □

As a result of this theorem, functions of simple random variables are simple random variables, so in probability theory we can take simple random variables X and Y and form new simple random variables: $X^2, e^{tX}, X + Y, \log X$, etc.

Examples.

1. Consider $d_1, \ldots, d_n$ for sequences ω of coin flips. Then $d_n \notin \sigma(d_1, \ldots, d_{n-1})$ which implies that d_n is independent of the other d_i. These d_i are defined in this way so as to make different coin flips independent.

2. The Rademacher functions $r_1, \ldots, r_n$ and the cumulative Rademacher functions $s_1, \ldots, s_n$ generate the same σ-field:

$$\sigma(r_1, \ldots, r_n) = \sigma(s_1, \ldots, s_n).$$

   This is because for each k, $s_k = \sum_{i=1}^{k} r_i$ and $r_k = s_k - s_{k-1}$ so one can apply (2) of Theorem 24.0.4.

3. The characteristic function χ_A is $\mathcal{G}$-measurable if and only if $A \in \mathcal{G}$.

24.1 Convergence in Measure

Suppose we have a sequence $\{X_n\}_{n=1}^{\infty}$ of simple random variables and X a simple random variable to which the X_n might converge. We are interested in pointwise convergence *almost everywhere.*

Definition. *The sequence (X_n) is said to* **converge to X pointwise almost everywhere** *if $P\left[\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right] = 1$, that is, $\lim X_n \equiv X$ on all of Ω except possibly on a set of measure zero.*

In analytic terms, $\lim X_n(\omega) = X(\omega) \iff$ for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, $|X_n(\omega) - X(\omega)| < \varepsilon$. Thus for a given $\omega \in \Omega$, X_n does not converge to $X(\omega)$ if and only if there is some $\varepsilon > 0$ such that $|X_n(\omega) - X(\omega)| \geq \varepsilon$ *infinitely often.*

Because we want to utilize countable unions, we replace ε with $\frac{1}{m}$ (these are equivalent by the Archimedean Principle). Then the set $\{\omega : \lim X_n(\omega) = X(\omega)\}$ has complement $\bigcup_{m=1}^{\infty} \left\{\omega : |X_n(\omega) - X(\omega)| \geq \frac{1}{m} \text{ i.o.}\right\}$ and its probability may be expressed as

$$\begin{aligned} P\left[\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right]^C &= P\left(\bigcup_{m=1}^{\infty} \left\{\omega : |X_n(\omega) - X(\omega)| \geq \tfrac{1}{m} \text{ i.o.}\right\}\right) \\ &= P\left(\bigcup_{m=1}^{\infty} \limsup \left\{\omega : |X_n(\omega) - X(\omega)| \geq \tfrac{1}{m}\right\}\right) \\ &= \sum_{m=1}^{\infty} 0 = 0. \end{aligned}$$

This suggests a different notion of convergence.

Definition. *We say a sequence of simple random variables (X_n)* **converges in measure** *to X, denoted $(X_n) \to_p X$, if for every $\varepsilon > 0$,*

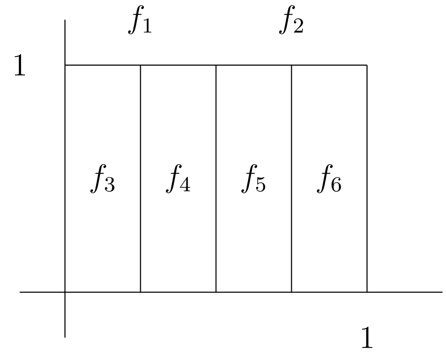
$$\lim_{n\to\infty} P[\omega : |X_n(\omega) - X(\omega)| \geq \varepsilon] = 0.$$

We can see from the work above that pointwise convergence (a.e.) implies convergence in measure, so the latter is a weaker form of convergence. In several sections we will see that the notions of pointwise convergence (a.e.) and convergence in measure can be translated into generalized laws of large numbers, with the weaker convergence corresponding to the weak law.

Examples.

1. Define $A_n = \{\omega : \ell_n(\omega) \geq \log_2 n\}$. We proved that $P(A_n) = \frac{1}{n}$ so this tends to 0 as $n \to \infty$, showing (A_n) converges in measure to $\varnothing$. However, we proved in Example 4 of Section 23.4 that $P[\omega : \omega \in A_n \text{ i.o.}] = 1$ using the Borel-Cantelli lemmas. So (A_n) clearly *does not converge* to $\varnothing$ pointwise.

2 The following sequence of simple random variables is sometimes called 'the typewriter' in measure theory.



Define the sequence beginning with

$$f_1 = \chi_{[0,\frac{1}{2}]} \qquad f_2 = \chi_{[\frac{1}{2},1]}$$

$$f_3 = \chi_{[0,\frac{1}{4}]} \qquad f_4 = \chi_{[\frac{1}{4},\frac{1}{2}]} \qquad f_5 = \chi_{[\frac{1}{2},\frac{3}{4}]} \qquad f_6 = \chi_{[\frac{3}{4},1]}$$

etc.

Then $(f_n) \to_P 0$, but for any $x \in [0,1]$, $f_n(x)$ diverges so we see that convergence in measure *does not imply* pointwise convergence.

24.2 Independent Variables

Definition. *A sequence* $\{X_1, X_2, \ldots\}$ *of simple random variables is said to be* **independent** *if* $\sigma(X_1), \sigma(X_2), \ldots$ *are independent, that is, if whenever* $H_1, H_2, \ldots \subset \mathbb{R}^n$,

$$P[\omega : X_1(\omega) \in H_1, \ldots, X_n(\omega) \in H_n] = P[\omega : X_1(\omega) \in H_1] \cdot \ldots \cdot P[\omega : X_n(\omega) \in H_n].$$

Example 24.2.1. Let $\Omega = S_n$, the set of permutations of a set of n elements. Assign the discrete probability $P(\omega) = \frac{1}{n!}$ for all $\omega \in S_n$. Define a simple random variable

$$X_k(\omega) = \begin{cases} 1 & \text{if position } k \text{ is the last position in a cycle in } \omega \\ 0 & \text{otherwise.} \end{cases}$$

For example, if $\omega = (1\ 4\ 3)(2\ 5)$ then $X_3 = X_5 = 1$ and $X_1 = X_2 = X_4 = 0$. For any $\omega \in S_n$, define $S(\omega) = \sum_{k=1}^{n} X_k(\omega)$. Then S represents the number of disjoint cycles in a cycle decomposition of ω.

Claim. *The* X_k *are independent with* $P[\omega : X_k(\omega) = 1] = \frac{1}{n-k+1}$.

This is easy to justify heuristically, but the details can be tricky. The idea is that $X_1 = 1$ if and only if ω is a permutation fixing 1; the probability of this happening is $\frac{1}{n}$. If $\omega(1) = 1$ then $\omega(2)$ is one of the remaining values $2, \ldots, n$ and thus $X_2(\omega) = 1$ if and only if $\omega(2) = 2$; this happens with probability $\frac{1}{n-1}$. On the other hand, if $X_1(\omega) = 0$, then $\omega(1) = i \neq 1$, so that $\omega(i)$ is one of the values $1, \ldots, i-1, i+1, \ldots, n$. Then $X_2(\omega) = 1$ if and only if $\omega(i) = 1$ which happens with probability $\frac{1}{n-1}$. This argument can be continued to show that $P[\omega : X_k(\omega) = 1] = \frac{1}{n-k+1}$, showing the X_k are indeed independent.

Definition. *Let* X *be a simple random variable, the* **distribution** *of* X *is the probability measure* μ *defined for all subsets* $A \subset \mathbb{R}$ *by*

$$\mu(A) = P[\omega : X(\omega) \in A].$$

The distribution μ is a discrete probability measure. If $\{x_1, \ldots, x_n\}$ are the distinct values in the range of X then $p_i := \mu(\{x_i\}) = P[\omega : X(\omega) = x_i]$ so for any $A \subset \mathbb{R}$, $\mu(A) = \sum_{x_i \in A} p_i$. In particular, this shows that $\mu(\mathbb{R}) = 1$, but even better, if B is the range of X, then $\mu(B) = 1$ so μ has finite support.

Theorem 24.2.2. *If* $\{\mu_n\}$ *is a sequence of probability measures on the class of all subsets of* $\mathbb{R}$ *such that each* μ_n *has finite support, then there exists an independent sequence* $\{X_n\}$ *of simple random variables on some probability space* $(\Omega, \mathcal{F}, P)$ *such that* X_n *has distribution* μ_n.

Proof. Let $\Omega = (0, 1]$, let $\mathcal{F} = \mathcal{B}$ the Borel σ-field and let $P = \lambda$, the Lebesgue measure on $\mathcal{F}$. We first consider the case where each μ_n has range $\{0, 1\}$. Set

$$p_n = \mu_n(\{0\}) \quad \text{and} \quad q_n = 1 - p_n = \mu_n(\{1\}).$$

Divide $\Omega = (0, 1]$ into two intervals I_0 and I_1, where $|I_0| = p_1$ and $|I_1| = q_1$. Define X_1 by

$$X_1(\omega) = \begin{cases} 0 & \omega \in I_0 \\ 1 & \omega \in I_1. \end{cases}$$

Since P is the Lebesgue measure, we have $P[\omega : X_1(\omega) = 0] = p_1$ and $P[\omega : X_1(\omega) = 1] = q_1$, so X_1 has been constructed so that its distribution is μ_1. Next, split I_0 into intervals I_{00} and I_{01} of lengths p_1p_2 and p_1q_2; likewise split I_1 into intervals I_{10} and I_{11} of lengths q_1p_2 and q_1q_2. Define the second simple random variable X_2 by

$$X_2(\omega) = \begin{cases} 0 & \omega \in I_{00} \cup I_{10} \\ 1 & \omega \in I_{01} \cup I_{11}. \end{cases}$$

By construction, $P[\omega : X_1 = X_2 = 0] = p_1p_2$ and similarly for the other three choices, so X_1 and X_2 are independent and X_2 has distribution μ_2. Repeat this process to define a sequence $\{X_n\}$ of independent s.r.v.'s such that each X_n has distribution μ_n.

In the general case, μ_1 has finite support so instead of dividing Ω into two intervals, we divide the space into the number of intervals corresponding to the size of the range of μ_1. The above proof is easily adapted to this setup. □

Example 24.2.3. A special case of the above construction is called a sequence of **Bernoulli trials**. Explicitly, Bernoulli trials are a sequence $\{X_n\}$ of independent random variables satisfying $P[\omega : X_n(\omega) = 1] = p$ and $P[\omega : X_n(\omega) = 0] = 1 - p$ for all n. The dyadic interval construction introduced in Chapter 22 are the intervals used to construct $\{X_n\}$ for $p = \frac{1}{2}$.

24.3 Expected Value and Variance

Definition. *Consider a simple random variable* $X = \sum_{i=1}^{n} x_i \chi_{A_i}$ *for* $x_i \in \mathbb{R}$ *and* $A_i \subset \mathbb{R}$. *The* **expected value** *or* **mean value** *of* X *is*

$$E[X] := \sum_{i=1}^{n} x_i P(A_i).$$

Let $\{x_i\}_{i=1}^{n}$ be the range of X. Then the expected value can be written

$$E[X] = \sum_{i=1}^{n} x_i P[\omega : X(\omega) = x_i].$$

This formulation shows that $E[X]$ only depends on the distribution of X, so if for two simple random variables X, Y, $P[\omega : X(\omega) = Y(\omega)] = 1$ then $E[X] = E[Y]$.

Examples.

1. Suppose $X = \frac{1}{4}\chi_{(0,1/2]} + \frac{-1}{4}\chi_{(1/2,1]}$. Another distribution for X is $X = \frac{1}{4}\chi_{(0,1/4]} + \frac{1}{2}\chi_{(1/4,1/2]} - \frac{1}{4}\chi_{(1/4,1]}$.

2. For $A \subseteq \mathbb{R}$, recall that χ_A is a simple random variable. Then $E[\chi_A] = P(A)$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Recall from Theorem 24.0.4 that if X is a simple random variable, so is $f(X)$. Then

$$E[f(X)] = \sum_{i=1}^{n} f(x_i)P(A_i) = \sum_{x} f(x)P[\omega : X(\omega) = x]$$

where the last sum is over all x in the range of $f(X)$.

Definition. *The* **kth moment** *of a simple random variable* X *is*

$$E[X^k] = \sum_{y} yP[\omega : X^k(\omega) = y]$$

where the sum is over all y *in the range of* X^k.

As above, an alternate expression for the kth moment of X is

$$E[X^k] = \sum_{i=1}^{n} x_i P[\omega : X^k(\omega) = x_i]$$

where $\{x_i\}_{i=1}^{n}$ is the range of X.

Proposition 24.3.1. *Let* X *and* Y *be two simple random variables given by* $X = \sum_{i=1}^{n} x_i \chi_{A_i}$ *and* $Y = \sum_{j=1}^{m} y_j \chi_{B_j}$.

(a) *(Linearity) For $\alpha, \beta \in \mathbb{R}$, $\alpha X + \beta Y$ is a simple random variable with expected value*

$$E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y].$$

(b) *If $X(\omega) \leq Y(\omega)$ for all $\omega \in S$ where S is a support of P, then $E[X] \leq E[Y]$.*

(c) *$|E[X - Y]| \leq E[|X - Y|]$.*

(d) *If X and Y are independent then $E[XY] = E[X]E[Y]$.*

Proof. (a) We create a mutual refinement of the A_i and B_j by $C_{ij} = A_i \cap B_j$. Then $\alpha X + \beta Y = \sum_{i,j} (\alpha x_i + \beta y_j) \chi_{C_{ij}}$. Therefore expected value is given by

$$\begin{aligned} E[\alpha X + \beta Y] &= \sum_{i,j} (\alpha x_i + \beta y_j) P(C_{ij}) = \sum_{i,j} (\alpha x_i + \beta y_j) P(A_i \cap B_j) \\ &= \alpha \sum_{i=1}^{n} x_i P(A_i) + \beta \sum_{j=1}^{m} y_j P(B_j) = \alpha E[X] + \beta E[Y]. \end{aligned}$$

(b) If $X(\omega) \leq Y(\omega)$ for all $\omega \in S$, then $x_i \leq y_j$ whenever $A_i \cap B_j$ is nonempty and thus

$$E[X] = \sum_{i,j} x_i P(A_i \cap B_j) \leq \sum_{i,j} y_i P(A_i \cap B_j) = E[Y].$$

(c) Using (b), we have $E[-|X|] \leq E[X] \leq E[|X|]$ so $|E[X]| \leq E[|X|]$. Moreover, linearity from part (a) gives us $|E[X - Y]| \leq E[|X - Y|]$.

(d) Note that $XY = \sum_{i,j} x_i y_j \chi_{C_{ij}}$, so if $A_i = [\omega : X(\omega) = x_i]$ and $B_j = [\omega : Y(\omega) = y_j]$ then $P(A_i \cap B_j) = P(A_i)P(B_j)$ by definition of independence. Thus $E[XY] = E[X]E[Y]$. □

Theorem 24.3.2 (Bounded Convergence)**.** *Suppose $\{X_n\}$ is a sequence of simple random variables that is uniformly bounded, i.e. there is some $K > 0$ such that $|X_n(\omega)| \leq K$ for all ω and n. If (X_n) converges pointwise a.e. to X, then $\lim E[X_n] = E[X]$.*

Proof. We know pointwise convergence a.e. implies convergence in measure, so $(X_n) \to_P X$. Choose K large enough so that it bounds $|X(\omega)|$ as well as $|X_n(\omega)|$, which is possible since X takes on finitely many values. Then for any n, $|X_n - X| \leq 2K$. If $A = [\omega : |X_n(\omega) - X(\omega)| \geq \varepsilon]$ then for all ω,

$$|X_n(\omega) - X(\omega)| \leq 2K\chi_A(\omega) + \varepsilon \chi_{A^C}(\omega) \leq 2K\chi_A(\omega) + \varepsilon.$$

Then by properties of expected value, $E[|X_n - X|] \leq 2KP[\omega : |X_n(\omega) - X(\omega)| \geq \varepsilon] + \varepsilon$. Since (X_n) converges in measure to X, $P[\omega : |X_n(\omega) - X(\omega)| \geq \varepsilon] \longrightarrow 0$. Therefore $E[|X_n - X|] < \varepsilon$ for any arbitrary ε, which shows $\lim E[|X_n - X|] = 0$. Linearity implies the result. □

Definition. *For a simple random variable X, the* **variance** *of X is*

$$\operatorname{Var}[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2.$$

Proposition 24.3.3. (a) *If X is a simple random variable and $\alpha, \beta \in \mathbb{R}$ then $\alpha X + \beta$ is a simple random variable with variance $\operatorname{Var}[\alpha X + \beta] = \alpha^2 \operatorname{Var}[X]$.*

(b) If $X_1, \dots, X_n$ are independent, simple random variables then

$$\operatorname{Var}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \operatorname{Var}[X_i].$$

Proof. (a) We have already seen that functions of simple random variables are simple random variables, so by properties of expected value,

$$\operatorname{Var}[\alpha X + \beta] = E[((\alpha X + \beta) - (\alpha E[X] + \beta))^2] = E[\alpha^2(X - E[X])^2] = \alpha^2 \operatorname{Var}[X].$$

(b) For each i, let $m_i = E[X_i]$. Then by linearity, $E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n m_i =: m$ and

$$\begin{aligned} E\left[\left(\sum_{i=1}^n X_i - m\right)^2\right] &= E\left[\left(\sum_{i=1}^n (X_i - m_i)\right)^2\right] \\ &= \sum_{i=1}^n E[(X_i - m_i)^2] + 2\sum_{1\le i<j\le n} E[(X_i - m_i)(X_j - m_j)]. \end{aligned}$$

If the X_i are independent, each $E[(X_i - m_i)(X_j - m_j)]$ splits into $(E[X_i] - m_i)(E[X_j] - m_j) = 0$ so the second sum vanishes. This implies the result. ☐

Definition. *A function $\varphi : I \to \mathbb{R}$ is* **convex** *if for all $0 \le p \le 1$ and $x, y \in I$,*

$$\varphi(px + (1-p)y) \le p\varphi(x) + (1-p)\varphi(y).$$

Notice that a sufficient condition for convexity is that $\varphi''(x) \ge 0$ for all $x \in I$.

Theorem 24.3.4. *Let X be a simple random variable with $E[X] = m$.*

(1) (Chebyshev's Inequality) For any $\alpha > 0$, $P[\omega : |X(\omega)| \ge \alpha] \le \dfrac{1}{\alpha}E[|X|]$.

(2) (Markov's Inequality) For any $\alpha > 0$, $P[\omega : |X(\omega)| \ge \alpha] \le \dfrac{1}{\alpha^k}E[|X|^k]$.

(3) (Chebyshev-Bienaymé Inequality) For $\alpha > 0$, $P[\omega : |X - m| \ge \alpha] \le \dfrac{1}{\alpha^2}\operatorname{Var}[X]$.

(4) (Jensen's Inequality) Suppose φ is a convex function on an interval containing the range of X. Then $\varphi(E[X]) \le E[\varphi(X)]$.

(5) (Hölder's Inequality) Suppose $p, q > 1$ are numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then $E[|XY|] \le E[|X|^p]^{1/p} \cdot E[|Y|^q]^{1/q}$.

The $p, q = 2$ case of Hölder's inequality is called *Schwartz's inequality:*

$$E[|XY|] \le E[X^2]^{1/2} \cdot E[Y^2]^{1/2}.$$

Moreover, setting $p = \frac{\beta}{\alpha}$, $q = \frac{\beta}{\beta - \alpha}$ for some $0 < \alpha \le \beta$, taking $Y \equiv 1$ and replacing X with $|X|^\alpha$ gives us *Lyapounov's inequality:*

$$E[|X|^\alpha]^{1/\alpha} \le E[|X|^\beta]^{1/\beta}.$$

24.4 Abstract Laws of Large Numbers

In this section we generalize the strong and weak laws of large numbers (Chapter 22) to a more general setting involving independent, simple random variables.

Definition. *A sequence $X_1, X_2, \ldots$ of simple random variables on a probability space $(\Omega, \mathcal{F}, P)$ is said to be* **identically distributed** *if their distributions are all the same. In the case that the X_n are also independent, this is abbreviated i.i.d.*

Theorem 24.4.1 (Strong Law)**.** *Suppose $\{X_n\}$ is a sequence of i.i.d. simple random variables on $(\Omega, \mathcal{F}, P)$. For each n, set $E[X_n] = m$ and $S_n = X_1 + \ldots + X_n$. Then*

$$P\left[\omega : \lim_{n\to\infty} \frac{1}{n} S_n(\omega) = m\right] = 1.$$

Proof. Without loss of generality, we may assume $m = 0$ by shifting all the X_n. Since they are identically distributed, it makes sense to set $E[X_n] = m$ for each n. First we show that $\left[\omega : \lim \frac{1}{n} S_n(\omega) = 0\right]$ is an $\mathcal{F}$-set so that we can define its probability. Note that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} S_n = 0 \iff & \text{ for every } m \in \mathbb{N} \text{ there is an } N \in \mathbb{N} \text{ such that} \\
& \text{ for all } n > N, \left|\frac{1}{n} S_n - 0\right| < \frac{1}{m} \\
\iff & \left|\sum_{k=1}^{n} X_k\right| < \frac{n}{m}.
\end{aligned}$$

Clearly $\left[\omega : |\sum_{k=1}^n X_k(\omega)| < \frac{n}{m}\right] \in \mathcal{F}$ and we can construct the desired set out of these:

$$\left[\omega : \lim_{n\to\infty} \frac{1}{n} S_n(\omega) = 0\right] = \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N+1}^{\infty} \left[\omega : \left|\sum_{k=1}^{n} X_k(\omega)\right| < \frac{n}{m}\right].$$

We have shown that $P\left[\lim \frac{1}{n} S_n = 0\right] = 1 \iff P\left[\lim \frac{1}{n} S_n \geq \varepsilon \text{ i.o.}\right] = 0$ for any arbitrary ε so we will prove this equivalent condition. Denote the 2nd and 4th moments of each X_i by $E[X_i^2] = \sigma^2$ and $E[X_i^4] = \xi^4$. Then $E[S_n^4] = n\xi^4 + 3n(n-1)\sigma^4 \leq kn^2$ for some k (see the table in Section 22.2 to see where these values come from). By Markov's inequality,

$$P[|S_n| \geq n\varepsilon] \leq \frac{k}{n^4\varepsilon^4} E[|S_n|^4] \leq \frac{kn^2}{n^4\varepsilon^4} = \frac{k}{n^2\varepsilon^4}.$$

As $n \to \infty$, this approaches 0 so by the first Borel-Cantelli lemma, $P[|S_n| \geq n\varepsilon \text{ i.o.}] = 0$. As discussed above, this implies the result. $\square$

Example 24.4.2. Consider the Bernoulli trials

$$X_n = \begin{cases} 0 & \text{with probability } p \\ 1 & \text{with probability } 1 - p. \end{cases}$$

It doesn't matter on which subset of the reals the X_n are defined, since by Theorem 24.2.2, there exists a sequence of independent simple random variables having the prescribed distribution. Clearly $E[X] = m$, so by the Strong Law, $\frac{1}{n}S_n \to p$ with probability 1. Moreover, since Bernoulli trials are independent, variance is given by $\sigma = \operatorname{Var}[X_n] = p(1-p)$.

The main limitation of the Strong Law is that we must have control over the 4th moments of the X_i, via the i.i.d. condition. The Weak Law is weaker than the Strong Law in its conclusion but it is useful when we only have control over lower moments.

Theorem 24.4.3 (Weak Law). *Let $\{X_n\}$, m and S_n be defined as in the Strong Law. Then for any $\varepsilon > 0$,*

$$\lim_{n\to\infty} P\left[\omega : \left|\frac{1}{n}S_n - m\right| \geq \varepsilon\right] = 0.$$

Proof. By Chebyshev-Bienaymé,

$$\begin{aligned} P\left[\omega : \left|\frac{1}{n}S_n - m\right| \geq \varepsilon\right] &\leq \frac{\operatorname{Var}[S_n]}{n^2\varepsilon^2} \\ &= \frac{n\operatorname{Var}[X]}{n^2\varepsilon^2} \quad \text{by independence} \\ &\longrightarrow 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence the Weak Law is proved. ☐

In general, $\operatorname{Var}[X]$ is proportional to the 2nd moment $E[X^2]$ which is easier to control than higher moments such as $E[X^4]$.

Example 24.4.4. Recall the probability space $\Omega_n = S_n$, the set of permutations of n symbols, equipped with the cycle completion variables:

$$X_{nk} = \begin{cases} 1 & \text{if the } k\text{th position finishes a cycle} \\ 0 & \text{otherwise.} \end{cases}$$

Then for any n, $\{X_{n1}, \ldots, X_{nn}\}$ are independent and $P[X_{nk} = 1] = \frac{1}{n-k+1} =: m_{nk}$. Variance in this case is $\sigma_{nk}^2 = m_{nk}(1 - m_{nk})$ as with Bernoulli trials. Let $S_n = X_{n1} + \ldots + X_{nn}$ so that for a permutation $\omega \in \Omega_n$, $S_n(\omega)$ represents the number of cycles of ω. Define $L_n = \sum_{k=1}^n \frac{1}{k}$. Then expected value and variance are calculated by

$$E[S_n] = \sum_{k=1}^n E[X_{nk}] = \sum_{k=1}^n \frac{1}{n-k+1} = L_n;$$

$$\begin{aligned} \text{and} \quad \operatorname{Var}[S_n] &= \sum_{k=1}^n \operatorname{Var}[X_{nk}] \quad \text{by independence} \\ &= \sum_{k=1}^n m_{nk}(1 - m_{nk}) < m_{nk}. \end{aligned}$$

So for any $\varepsilon > 0$,

$$\begin{aligned} P\left[\omega : \left|\frac{S_n(\omega) - L_n(\omega)}{L_n(\omega)}\right| \geq \varepsilon\right] &= P[\omega : |S_n(\omega) - L_n(\omega)| \geq \varepsilon L_n(\omega)] \\ &\leq \frac{1}{L_n^2 \varepsilon 2} \operatorname{Var}[S_n] \quad \text{by Chebyshev-Bienaymé} \\ &= \frac{1}{L_n^2 \varepsilon^2} L_n = \frac{1}{L_n \varepsilon^2} \end{aligned}$$

but L_n diverges as $n \to \infty$, so this fraction approaches 0. Thus the conclusion of the Weak Law holds and moreover we can see that $S_n \sim L_n \sim \log n$. So there are approximately (in a weak sense) $\log n$ cycles in an average permutation of n symbols. The Strong Law cannot be applied in this case since Ω_n is a different space for each n.

24.5 Second Borel-Cantelli Lemma Revisited

The independence condition in the second Borel-Cantelli lemma (23.4.2) is sometimes too restrictive. For example, recall simple random variables ℓ_n defined as the length of the run of heads beginning on the nth coin flip in an infinite sequence of flips. The problem is that the ℓ_n are not independent, but in some sense they are independent if the runs are far enough apart. Although we cannot apply BC2, we will prove a weaker theorem that will apply to the ℓ_n and other non-independent examples.

Let $A_1, A_2, \ldots \in \mathcal{F}$ and for each $n \in \mathbb{N}$, define $N_n(\omega) = \chi_{A_1}(\omega) + \ldots + \chi_{A_n}(\omega)$. Then $N_n(\omega)$ represents the number of occurrences of ω in the first n sets in the sequence $\{A_i\}$ and more importantly, $[\omega : \omega \in A_n \text{ i.o.}] = [\omega : \sup N_n(\omega) = +\infty]$. Suppose the A_n are independent and $p_n = P(A_n)$; set $m_n = p_1 + \ldots + p_n$. Then

$$E[N_n] = m_n \quad \text{and} \quad \operatorname{Var}[N_n] = \sum_{k=1}^{n} p_k(1-p_k) < m_n.$$

If $x < m_n$ then

$$\begin{aligned} P[\omega : N_n(\omega) \leq x] &\leq P[\omega : |N_n(\omega) - m_n| \geq |x - m_n|] \\ &\leq \frac{\operatorname{Var}[N_n]}{|x-m_n|^2} < \frac{m_n}{(m_n - x)^2}. \end{aligned}$$

If $\sum p_k$ diverges then $m_n \to \infty$ so for every $x \in \mathbb{R}$, $P[\omega : N_n(\omega) \leq x] \longrightarrow 0$. Moreover, since $P[\omega : \sup N_n(\omega) \leq x] \leq P[\omega : N_n(\omega) \leq x]$ for all n and the right term goes to 0, we conclude

$$P[\omega : \sup N_n(\omega) < \infty] = 0 \qquad \Longrightarrow \qquad P[\omega : \omega \in A_n \text{ i.o.}] = 1.$$

This is an alternate way to view the second Borel-Cantelli lemma. Notice that the conclusion still holds, even if the A_n are not independent, as long as $\dfrac{\operatorname{Var}[N_n]}{(m_n - x)^2} \longrightarrow 0$. It turns out that this happens when

$$\liminf_{n\to\infty} \frac{\sum_{i,j\leq n} P(A_i \cap A_j)}{\left(\sum_{k\leq n} P(A_k)\right)^2} = 1,$$

as we will see in the proof of the next theorem. In general, this liminf is greater than or equal to 1, so the variance condition will hold as long as the liminf is less than or equal to 1.

The following generalizes the second Borel-Cantelli lemma.

Theorem 24.5.1. *Let $\{A_n\}$ be a sequence of (not-necessarily) independent events and suppose $\sum_{n=1}^{\infty} P(A_n)$ diverges and*

$$\liminf_{n\to\infty} \frac{\sum_{i,j\leq n} P(A_i \cap A_j)}{\left(\sum_{k\leq n} P(A_k)\right)^2} \leq 1.$$

Then $P[\omega \in A_n \text{ i.o.}] = 1$.

Proof. As above, let $N_n = \chi_{A_1}(\omega) + \ldots + \chi_{A_n}(\omega)$. Set

$$\theta_n = \frac{\sum_{i,j\le n} P(A_i \cap A_j)}{\left(\sum_{k\le n} P(A_k)\right)^2}$$

so that the hypothesis reads $\liminf \theta_n \le 1$. By the work in the preceding paragraph, it is enough to show that

$$\frac{\mathrm{Var}[N_n]}{(m_n - x)^2} \longrightarrow 0 \quad \text{as } n \to \infty.$$

First, we estimate variance by

$$\begin{aligned}
\mathrm{Var}[N_n] &= E[N_n^2] - E[N_n]^2 \\
&= \left(\sum_{i,j\le n} E[\chi_{A_i}\chi_{A_j}]\right) - m_n^2 \\
&= \left(\sum_{i,j\le n} P(A_i \cap A_j)\right) - m_n^2 \\
&= \frac{\sum_{i,j\le n} P(A_i \cap A_j)}{\left(\sum_{k\le n} P(A_k)\right)^2}\left(\sum_{k\le n} P(A_k)\right)^2 - \left(\sum_{k\le n} P(A_k\right)^2 \\
&= (\theta_n - 1)\left(\sum_{k\le n} P(A_k)\right)^2 = (\theta_n - 1)m_n^2.
\end{aligned}$$

For a fixed $x \in \mathbb{R}$,

$$P[\omega : N_n(\omega) \le x] \le \frac{\mathrm{Var}[N_n]}{(m_n - x)^2} \le \frac{(\theta_n - 1)m_n^2}{(m_n - x)^2}$$

and as $n \to \infty$, $m_n \to \infty$, so if $\liminf \theta_n \le 1$ then the term on the right approaches 0. This implies $P[\omega : N_n(\omega) \to \infty] = 1$, i.e. $P[A_n \text{ i.o.}] = 1$. □

If the A_n are independent, it turns out that

$$\theta_n = 1 + \frac{\sum_{k\le n}(p_k - p_k^2)}{\left(\sum_{k\le n} p_k\right)^2}.$$

This ratio of series goes to 0, so $\liminf \theta_n = 1$ which implies the original BC2.

Example 24.5.2. Although the sequence $\{\ell_n\}$ of run-length simple random variables is not independent, we can use this generalized BC2 to prove the following result.

Claim. *$P[\omega : \ell_n(\omega) \ge r_n$ i.o.$] = 1$ if and only if $\sum_{n=1}^{\infty} 2^{-r_n}$ diverges.*

Proof. As before, without loss of generality we can assume the r_n are integers. Define $A_n = [\omega : \ell_n(\omega) \ge r_n] = [d_n(\omega) = d_{n+1}(\omega) = \ldots = d_{n+r_n-1}(\omega) = 0]$. As previously noted, if $j + r_j \le k$ then A_j and A_k are independent so $P(A_j \cap A_k) = P(A_j)P(A_k)$ in that case. On the other hand, if $j < k < j + r_j$ then $P(A_j \cap A_k) \leq P[d_j = d_{j+1} = \ldots = d_{k-1} = 0 \mid A_k]$. These events are now independent, so

$$P(A_j \cap A_k) \leq P[d_j = d_{j+1} = \ldots = d_{k-1} = 0]P(A_k) = \frac{1}{2^{k-j}} P(A_k).$$

Putting all the cases together, we have

$$\begin{aligned}
\sum_{j,k \leq n} P(A_j \cap A_k) &\leq \sum_{k \leq n} P(A_k) + 2 \sum_{\substack{j<k\leq n \\ j+r_j<k}} P(A_j)P(A_k) + 2 \sum_{j<k<j+r_j} \frac{1}{2^{k-j}} P(A_k) \\
&< 3 \sum_{k \leq n} P(A_k) + \left(\sum_{k \leq n} P(A_k) \right)^2 \quad \text{since } \sum \frac{1}{2^{k-j}} < 1.
\end{aligned}$$

Therefore if $\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} 2^{-r_n}$ diverges, we have

$$\theta_n = \frac{3 \sum_{k \leq n} P(A_k) + \left(\sum_{k \leq n} P(A_k) \right)^2}{\left(\sum_{k \leq n} P(A_k) \right)^2} \longrightarrow 1.$$

So by Theorem 24.5.1 we get the desired conclusion, i.e. $P[\omega : \ell_n(\omega) \geq r_n \text{ i.o.}] = 1$. On the other hand, the first Borel-Cantelli lemma applies even in the non-independent case to give us the converse. Hence $P[\omega : \ell_n(\omega) \geq r_n \text{ i.o.}] = 1 \iff \sum 2^{-r_n}$ diverges. $\square$

24.6 Bernstein's Theorem

Definition. *Suppose f is a real-valued function on $[0,1]$. The nth* **Bernstein polynomial** *for f is defined by*

$$B_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}.$$

Theorem 24.6.1 (Bernstein)**.** *If f is continuous on $[0,1]$ then the sequence (B_n) converges to f uniformly on $[0,1]$.*

Notice that Bernstein's Theorem is just a restatement of the Weierstrass Approximation Theorem. In functional analysis we typically prove Weierstrass's theorem using function convolution. Here, we instead have defined an interpolation of f for each n. We will explicitly prove that the B_n converge to f uniformly.

Proof. Since f is continuous on $[0,1]$, f is bounded and uniformly continuous. Set $M = \sup\limits_{x\in[0,1]} |f(x)|$ and $\delta(\varepsilon) = \sup|f(x)-f(y)|$ over all $x, y \in [0,1]$ such that $|x-y| < \varepsilon$. This value $\delta(\varepsilon)$ is sometimes called the *modulus of continuity* of f. We will show that

$$\sup_{x\in[0,1]} |f(x) - B_n(x)| \le \delta(\varepsilon) + \frac{2M}{n\varepsilon^2}$$

which will imply the result since if $\varepsilon = n^{-1/3}$, $\sup|f(x)-B_n(x)| \le \delta(n^{-1/3}) + \frac{2M}{n^{1/3}}$ which approaches 0 as $n \to \infty$.

Fix $n \in \mathbb{N}$, $x \in [0,1]$ and let $X_1, X_2, \ldots$ be simple random variables such that for each i, $P[X_i = 1] = x$ and $P[X_i = 0] = 1-x$. Set $S_n = X_1 + \ldots + X_n$. Then for any k, $P[S_n = k] = \binom{n}{k}x^k(1-x)^{n-k}$ so the probabilities are the coefficients in the Bernstein polynomials. This implies $E[f(S_n)] = \sum\limits_{k=0}^{n} f(k)P[S_n = k]$ so

$$\begin{aligned} E\left[f\left(\frac{S_n}{n}\right)\right] &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)P[S_n = k] \\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} = B_n(x). \end{aligned}$$

This implies $|f(x) - B_n(x)| = \left|f(x) - E\left[f\left(\frac{S_n}{n}\right)\right]\right| \le E\left[\left|f(x) - f\left(\frac{S_n}{n}\right)\right|\right]$ so we will estimate $E\left[\left|f(x) - f\left(\frac{S_n}{n}\right)\right|\right]$. Note that when $\left|\frac{S_n}{n} - x\right| < \varepsilon$, $\left|f\left(\frac{S_n}{n}\right) - f(x)\right| < \delta(\varepsilon)$, and when $\left|\frac{S_n}{n} - x\right| \ge \varepsilon$, $\left|f\left(\frac{S_n}{n}\right) - f(x)\right| < 2M$. Then

$$\begin{aligned} E\left[\left|f(x) - f\left(\tfrac{S_n}{n}\right)\right|\right] &\le \delta(\varepsilon)P\left[\left|\tfrac{S_n}{n} - x\right| < \varepsilon\right] + 2M \cdot P\left[\left|\tfrac{S_n}{n} - x\right| \ge \varepsilon\right] \\ &\le \delta(\varepsilon) + 2M \cdot P\left[\left|\tfrac{S_n}{n} - m\right| \ge \varepsilon\right] \quad \text{where } m = E\left[\tfrac{S_n}{n}\right] \\ &= \delta(\varepsilon) + 2M \cdot P[|S_n - nx| \ge n\varepsilon] \\ &\le \delta(\varepsilon) + 2M \cdot \frac{\mathrm{Var}[S_n]}{n^2\varepsilon^2} \quad \text{by Chebyshev's inequality.} \end{aligned}$$

By independence,

$$\operatorname{Var}[S_n] = \sum_{k=1}^{n} \operatorname{Var}[X_k] = n \operatorname{Var}[X_k] = nx(1-x) \leq n,$$

so our estimate for the above expected value is

$$E\left[\left|f(x) - f\left(\tfrac{S_n}{n}\right)\right|\right] \leq \delta(\varepsilon) + \frac{2M}{n\varepsilon^2}.$$

Finally, apply the comments from above and the fact that $|f(x)-B_n(x)| \leq E\left[\left|f(x) - f\left(\frac{S_n}{n}\right)\right|\right]$ for all $x \in [0,1]$ to finish the proof. $\square$

24.7 Gambling

In this section, our goal is to convince the reader that, in simple terms, gambling is a bad idea. We will show that, under some fallacious assumptions often made by gamblers, the standard casino game roulette is heavily biased in the casino's favor. The techniques developed in this section are easily adapted to any game involving a 'unit bet', that is, a bet of a fixed sum which is either doubled with a win or lost with a loss.

Suppose $\{X_i\}_{i=1}^{\infty}$ is an independent sequence of simple random variables on a probability space $(\Omega, \mathcal{F}, P)$ where each variable is defined by

$$X_i(\omega) = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } q = 1 - p. \end{cases}$$

As usual, set $S_n = X_1 + \ldots + X_n$ and $S_0 = 0$ (i.e. you can't win if you don't play).

Definition. *If $p > \frac{1}{2}$, the game is said to be* **favorable***; if $p = \frac{1}{2}$, it is a* **fair game***; and if $p < \frac{1}{2}$, the game is said to be* **unfavorable**.

Example 24.7.1. The standard casino setup for the game of roulette is as follows. A wheel with 38 slots is spun and a ball is dropped in so that it lands in one of the slots by the time the wheel stops spinning. There are 18 red slots and 18 black slots which together are labelled with the numbers 1 through 36. In addition, there are two green slots labelled 0 and 00. Set $p = \frac{18}{38}$ and $q = 1 - p = \frac{20}{38}$. We will assume the player places a \$1 bet on a single red or black number, with a 2:1 payout. The odds against the player winning on a single spin are $\rho = \frac{q}{p} = \frac{10}{9}$.

Suppose the player starts with a capital of a dollars and plays \$1 per spin (or per trial for a general game). She plays until reaching a fixed goal of c dollars (success) or until she loses her money, i.e. reaches 0 dollars (ruin). After n plays, the player has a total of $a + S_n$ dollars. Set

$$\begin{aligned} A_{a,n} &= [\omega : a + S_n(\omega) = c] \cap [\omega : 0 < a + S_i(\omega) < c \text{ for all } i = 1, \ldots, n-1] \\ B_{a,n} &= [\omega : a + S_n(\omega) = 0] \cap [\omega : 0 < a + S_i(\omega) < c \text{ for all } i = 1, \ldots, n-1] \\ \text{and} \quad s_c(a) &= \text{the probability of success} = P\left(\bigcup_{n=1}^{\infty} A_{a,n}\right) = \sum_{n=1}^{\infty} P(A_{a,n}). \end{aligned}$$

Fixing c and allowing a to vary, we want to study the optimal starting capital for achieving success. Later we will investigate other strategies for success. By convention, set $A_{a,0} = \varnothing$ and $A_{c,0} = \Omega$ so that $s_c(0) = 0$ and $s_c(c) = 1$. Given a, $s_c(a) = ps_c(a+1) + qs_c(a-1)$ which is really a second order boundary value problem:

$$\begin{aligned} s_c(a) &= ps_c(a+1) + qs_c(a-1) \\ s_c(0) &= 0 \\ s_c(c) &= 1. \end{aligned}$$

Let $\rho = \frac{q}{p}$. It turns out (see Billingsley) that the solutions to this boundary value problem are of the form

$$s_c(a) = \begin{cases} A + B\rho^a & \text{if } \rho \neq 1 \\ A + Ba & \text{if } \rho = 1. \end{cases}$$

Suppose $\rho \neq 1$, i.e. the game is unfavorable. Given the first boundary condition, we see that $0 = A + B \implies A = -B$, so the second boundary condition gives us

$$1 = A + B\rho^c = -B + B\rho^c \implies B = \frac{1}{\rho^c - 1}.$$

Thus the solution for an unfavorable game is

$$s_c(a) = \frac{\rho^a - 1}{\rho^c - 1}.$$

On the other hand, if the game is favorable ($\rho = 1$), then $A = 0$ and $A + Bc = 1$, implying $B = \frac{1}{c}$. This gives the solution

$$s_c(a) = \frac{a}{c}.$$

Examples.

1. Suppose the player starts with $a = \$900$ and has a goal of reaching $c = \$1000$. In the fair game, $s_c(a) = .9$ which is reasonably high. However, when $\rho = \frac{10}{9}$ as in roulette, $s_c(a) = .00003$, an extremely low probability.

2. Things are worse when $c = \$20,000$. In the fair case, $s_c(a) = .005$ which is unsurprisingly low. But in the unfavorable case, $s_c(a) \approx 3 \times 10^{-911}$.

Remark. Ruin is symmetric with success, i.e. the boundary value problem

$$\begin{aligned} r_c(a) &= pr_c(a+1) + qr_c(a-1) \\ r_c(0) &= 1 \\ r_c(c) &= 0 \end{aligned}$$

has the same solution in terms of ruin:

$$r_c(a) = \begin{cases} \dfrac{\rho^{c-1} - 1}{\rho^c - 1} & \text{if } \rho \neq 1 \\ \dfrac{c-a}{c} & \text{if } \rho = 1. \end{cases}$$

Notice that for any choices of c and a, $r_c(a) + s_c(s) = 1$, to the probability of the game ending (in success or ruin) is always 1.

What if our player had infinite capital? What are the odds of ever achieving the goal? Suppose $a, b > 0$ and define $H_{a,b}$ to be the event of reaching $+b$ before $-a$. For any finite capital a, this can be written

$$H_{a,b} = \bigcup_{n=1}^{\infty} \left([\omega : S_n(\omega) = b] \cap \left(\bigcap_{k=1}^{n-1} [\omega : -a < S_k(\omega) < b] \right) \right).$$

We can write the probability of one of these events as $P(H_{a,b}) = s_{a+b}(a)$. Also set H_b to be the event of *ever* gaining $+b$, that is the event of success with infinite capital:

$$H_b = \bigcup_{a=1}^{\infty} H_{a,b}.$$

Notice that for a fixed b, $\{H_{a,b}\}$ is a monotone increasing sequence that converges to H_b so by continuity from below (Proposition 17.2.1),

$$\begin{aligned}
P(H_b) &= \lim_{a\to\infty} P(H_{a,b}) = \lim_{a\to\infty} s_{a+b}(a) \\
&= \begin{cases} \displaystyle\lim_{a\to\infty} \frac{a}{a+b} & \text{if } \rho = 1 \\ \displaystyle\lim_{a\to\infty} \frac{1-\rho^a}{1-\rho^{a+b}} & \text{if } \rho \neq 1 \end{cases} \\
&= \begin{cases} 1 & \text{if } \rho \leq 1 \\ \rho^{-b} & \text{if } \rho > 1. \end{cases}
\end{aligned}$$

Examples. Use the same setup as before.

1. If $c = \$1000$ and $\rho = 1$ then $\lim s_{a+c}(a) = 1$, but if $\rho = \frac{10}{9}$ then $\lim s_{a+c}(a) = .00003$ — the same as before.

2. If $c = \$20,000$ and $\rho = 1$ then $\lim s_{a+c}(a) = 1$, but on the other hand when $\rho = \frac{10}{9}$, $\lim s_{a+c}(a) \approx 3 \times 10^{-911}$. This shows that having infinite capital helps in the fair game (this is to be expected) but has no effect on the unfavorable game.

What if we have some sort of 'strategy' for when to place our bets? We will see that this does not change the unfavorable game's outcome either.

Suppose $\{X_n\}$ are i.i.d. simple random variables and define an additional random variable

$$B_n = \begin{cases} 1 & \text{if we bet on the } n\text{th trial} \\ 0 & \text{if we don't bet on the } n\text{th trial.} \end{cases}$$

The way B_n is defined can only depend on $X_1, \dots, X_{n-1}$, i.e. B_n cannot 'predict the future'. In other words, B_n is measurable with respect to $F_{n-1} = \sigma(X_1, \dots, X_{n-1})$. Recall that this means $B_n = f(X_1, \dots, X_{n-1})$ for a function f. Set N_n to be the time the nth bet is placed – notice N_n is *not simple random*. We will assume $P[B_n = 1 \text{ i.o.}] = 1$ so that the game always terminates.

Definition. *Given $\{X_n\}$ a sequence of i.i.d. simple random variables, a choice of B_n that satisfies the above conditions is called a* **selection scheme**.

Theorem 24.7.2. *The sequence $\{Y_n\}$, where $Y_n = X_{N_n}$, is independent with $P[Y_n = 1] = p$ and $P[Y_n = -1] = q$. In other words, selection schemes do not change the outcome of a game.*

Proof omitted. □

Set $F_0 = a$, the 'initial fortune' of the player and for each n, let F_n be her/his fortune after the nth trial. We next define a way of altering the player's wagers for each trial in order to potentially optimize the odds of winning.

Definition. *Define the random variable W_n to represent the player's wager (in dollars, e.g.) on the nth trial of a game. If $W_n = g_n(F_0, X_1, \ldots, X_{n-1})$, that is, $W_n \in \sigma(X_1, \ldots, X_{n-1})$ and W_n depends on initial fortune, and in addition $W_n \geq 0$ for all n, then the choice of W_n is called a* **betting system**.

Our player's fortune can thus be written $F_n = F_{n-1} + W_n X_n$.

Example 24.7.3. If $X_1 = \ldots = X_{n-1} = -1$ then an example of a betting system is the double-or-nothing approach, $W_n = 2^{n-1}$ for all $n \geq 1$.

As we have it defined, W_n is not a simple random variable since it depends on F_0, which can take on any finite positive value. However, fixing F_0 makes W_n a simple random variable. Also notice that W_n is independent of X_n, so we can write

$$E[W_n X_n] = E[W_n]E[X_n] = (p - q)E[W_n].$$

If $p = q$ then $E[F_n] = E[F_{n-1} + W_n X_n] = E[F_{n-1}] + E[W_n] \cdot 0 = E[F_{n-1}]$. In other words, if the game is fair then $E[F_n] = F_0$ for every $n \geq 1$. If the game is unfavorable, then $E[F_n] \leq E[F_{n-1}]$ and so $E[F_n] \leq F_0$ for all n. This shows that a betting scheme cannot make an unfavorable game fair (or favorable) but it *can increase one's odds.*

Definition. *Suppose $\tau(F_0, \omega)$ is a function assigning values in $\mathbb{N} \cup \{0\}$ for all $\omega \in \Omega$ and $F_0 \geq 0$ by $\tau = n$ when the player bets on the first n games and then stops. If for all n, $[\omega : \tau(F_0, \omega) = n] \in \sigma(X_1, \ldots, X_n)$ and τ is finite with probability 1, then τ is called a* **stopping time**.

The dependence condition requires that τ does not depend on future information, only the previous trials of the game. The finite with probability 1 condition allows for some set of measure 0 on which τ has infinite values. Therefore τ is not a simple random variable because a game can be arbitrarily long.

Definition. *A* **gambling policy***, denoted π, is a betting system W_n for a particular initial capital F_0 together with a stopping time τ.*

Example 24.7.4. A selection scheme is a betting system defined by $W_n = B_n \in \{0, 1\}$. A stopping time for this W_n is $\tau = n$ where n is the first loss. To see this, note that $P[\omega : \tau(\omega) > n] = p^n$ and since $0 < p < 1$ (assuming an unfavorable game),

$$P[\omega : \tau(\omega) \text{ is not finite}] = P\left(\bigcap_{n=1}^{\infty}[\omega : \tau(\omega) > n]\right) = \lim_{n\to\infty} P[\omega : \tau(\omega) > n] = \lim_{n\to\infty} p^n = 0.$$

Thus τ is finite on a set of probability 1. Now consider

$$[\omega : \tau(\omega) = n] = \left(\bigcap_{k=1}^{n-1}[\omega : X_k(\omega) = 1]\right) \cap [\omega : X_n(\omega) = -1]$$

which lies in $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$. Hence τ is a stopping time so a selection scheme is a gambling policy. As Theorem 24.7.2 above shows, a selection scheme is an example of a policy that *does not* increase one's odds of winning.

Suppose $\pi = (F_0, W_n, \tau)$ is a gambling policy. Define

$$F_n^* = \begin{cases} F_n & n \leq \tau \\ F_\tau & n > \tau \end{cases} \qquad \text{and} \qquad W_n^* = \chi_{[\omega : n \leq \tau(\omega)]} W_n.$$

Then $F_n^* = F_{n-1}^* + W_n^* X_n$ so we can embed the stopping time into our betting system. Explicitly, this system tells the player to bet $W_n = 0$ when n is later than the stopping time. It still follows that $E[F_n] = E[F_{n-1}] = \ldots = F_0$ for any n in the fair scenario and $E[F_n] \leq E[F_{n-1}] \leq F_0$ in the unfavorable scenario.

In a finite capital scenario, stopping times cannot make an unfavorable game profitable. However, things are different in the infinite capital world. Assuming the game terminates, F_n^* converges to F_τ with probability 1 so by the bounded convergence theorem (24.3.2), $\lim E[F_n] = E[F_\tau]$ if the F_n^* are uniformly bounded. In this case, $E[F_\tau] = F_0$ if the game is fair and $E[F_\tau] \leq F_0$ if the game is unfavorable. If the F_n^* are not uniformly bounded, this means either the gambler or casino started with access to infinite capital, so the uniform bound condition is realistic.

Suppose $p \leq \frac{1}{2}$ and $F_0 = a$. If the player stops when either $F_n = c$ or $F_n = 0$ then

$$s_c(a) < (1 - s_c)(0) + s_c(c) = \tfrac{a}{c}.$$

On the other hand, if the player plays until $F_n = c$, possibly allowing for $F_n < 0$, then $F_\tau = c > a \implies E[F_\tau] > F_0$ — wait what? There is a possibility of raising the expected fortune above that of starting levels even in an unfair game! Of course this is only possible with access to infinite capital, so in the real world the probabilities behave closer to expectation.

There is a strategy that optimizes winning conditions. First, rescale the fortune scale so that $F_n \in [0, 1]$ for $n = 0, 1, 2, \ldots$ and set τ to be the stopping time when one of 0 or 1 is reached for the first time. The gambling policy π_b with this stopping time and

$$W_n = \begin{cases} F_{n-1} & \text{if } 0 \leq F_{n-1} \leq \frac{1}{2} \\ 1 - F_{n-1} & \text{if } \frac{1}{2} \leq F_{n-1} \leq 1 \end{cases}$$

is called **bold play**. Informally, it says that if the player can't reach her goal on the next trial, she wagers everything in a 'double-or-nothing' strategy. If she can reach her goal, she wagers exactly the amount that would guarantee her to have fortune 1 with a win. To see that π_b is a valid policy, we just need to check that τ is a stopping time. It's clear that τ only depends on $X_1, \ldots, X_n$ so it suffices to check the finite property. Consider

$$P[\omega : \tau(\omega) > n \mid \tau(\omega) > n - 1] = P[\omega : F_n(\omega) = 0, 1 \mid F_{n-1}\omega) \neq 0, 1].$$

If $F_{n-1} \leq \frac{1}{2}$, the outcomes can be split into

$$\begin{aligned} &F_n = F_{n-1} + F_{n-1} = 2F_{n-1} && \text{if } X_n = 1\text{; happens with probability } p \\ &F_n = F_{n-1} - F_{n-1} = 0 && \text{if } X_n = -1\text{; happens with probability } q. \end{aligned}$$

On the other hand, if $F_{n-1} \geq \frac{1}{2}$ then

$$\begin{aligned} F_n &= F_{n-1} + (1 - F_{n-1}) = 1 && \text{if } X_n = 1\text{; happens with probability } p \\ F_n &= F_{n-1} - (1 - F_{n-1}) = 2F_{n-1} - 1 && \text{if } X_n = -1\text{; happens with probability } q. \end{aligned}$$

The game terminates in the second and third cases, so the conditional probability is

$$P[\omega : \tau(\omega) > n \mid \tau(\omega) > n-1] \leq \max\{p, q\}.$$

Setting $m = \max\{p, q\}$, we have $P[\omega : \tau(\omega) > n] \leq m^n$ which tends to 0 as $n \to \infty$. Hence τ is a valid stopping time.

Theorem 24.7.5 (Dubins-Savage)**.** *Bold play is optimal in the unfavorable case.*

Proof sketch. Assume $F_\tau = 0$ or 1 so that τ is a simple random variable. Write $Q_\pi(x) = P[F_\tau = 1 \mid F_0 = x]$ for any gambling policy π and for each $x \in [0,1]$. Also set $Q(x) = Q_{\pi_b}(x)$ for bold play. One can check that for any π, $Q_\pi(0) = 0$, $Q_\pi(1) = 1$ and $0 \leq Q_\pi(x) \leq 1$. The theorem then reduces to proving that for every π on a game with $p \leq \frac{1}{2}$, $Q_\pi(x) \leq Q(x)$. The bold play function has the following properties:

- Q is increasing on $[0,1]$.
- Q is a continuous function of x.
- $Q(x) = \begin{cases} pQ(2x) & 0 \leq x \leq \frac{1}{2} \\ p + qQ(2x-1) & \frac{1}{2} \leq x \leq 1 \end{cases}$

These can be used to show $Q(x) \geq pQ(x+t) + qQ(x-t)$ whenever $0 \leq x - t < x < x + t \leq 1$ which implies the result. □

Remark. This optimal policy is not unique; however the Dubins-Savage theorem shows that bold play is one optimal policy. It is also known that the optimal probability of success $s_c(a)$ *is computable.*

Examples.

1. Set $a = \$900$ and $c = \$1000$ as before. In the fair case, $s_c(a) = .9$ and for any policy π, $Q_\pi(x) = x$ for every x. In the roulette odds, i.e. $p = \frac{10}{9}$, recall that unit stakes give us $s_c(a) = .00003$. However, bold play yields substantially better odds: $Q(.9) = .88$.

2. Set $a = \$100$ and $c = \$20,000$. This generates the following probabilities of success:

$$s_c(a) = \begin{cases} .005 & \text{fair game} \\ 3 \times 10^{-911} & \text{roulette odds} \\ .003 & \text{bold play.} \end{cases}$$

Notice that in both examples, bold play gives us odds that are on the same order of magnitude as the fair game odds.

24.8 Markov Chains

Definition. *Let S be a countable set and consider a doubly-indexed sequence $\{p_{ij}\}_{i,j\in S}$ such that $p_{ij} \geq 0$ and for each i, $\sum_{j\in S} p_{ij} = 1$. A* **Markov chain** *on a probability space $(\Omega, \mathcal{F}, P)$ is a sequence $\{X_n\}_{n=0}^{\infty}$ of random variables on $(\Omega, \mathcal{F}, P)$ such that*

(i) For each n, $X_n(\Omega) \subset S$.

(ii) For every subset $\{i_0, i_1, \ldots, i_n\} \subset S$ such that $P[X_0 = i_0, \ldots, X_n = i_n] > 0$,

$$P[X_{n+1} = j \mid X_0 = i_0, \ldots, X_n = i_n] = P[X_{n+1} = j \mid X_n = i_n] = p_{i_n j}.$$

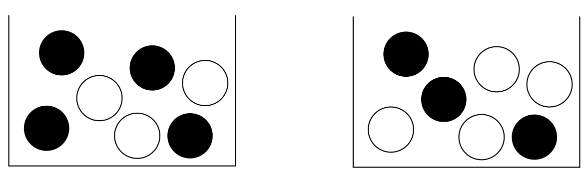
The p_{ij} are called the **transition probabilities** *for the Markov chain, each X_n is called the nth* **state** *and S is referred to as the* **state space**.

The Markov property (ii) implies independence of history, that is, each state only depends on the previous state and 'forgets' what has happened before that. Also, property (ii) also implies that the states are *autonomous*, i.e. they don't depend on n – if n is thought of as a time variable, then the outcome has the same dependence on the previous outcome no matter the current time.

We will denote the initial probabilities by $P[X_0 = i] = \alpha_i$ for each $i \in S$. Notice that $\sum_{i\in S} \alpha_i = 1$ and $\alpha_i \geq 0$ for every $i \in S$. The transition probabilities p_{ij} correspond to a matrix $P = (p_{ij})$ called a **transition matrix**. The conditions on the p_{ij} mean that the transition matrix $P = (p_{ij})$ is **stochastic**.

Examples.

1. Markov chains are useful for describing change in physical models. Consider the following (oversimplified) representation of diffusion. Suppose we have 2 buckets, a left and a right one, which are each filled with r balls that are either black or white. We know that there are k white balls, and therefore $r - k$ black balls, in the left bucket and k black balls, and therefore $r - k$ white balls, in the right bucket.



r balls	r balls
k white	$r - k$ white
$r - k$ black	k black

Consider the state space $S = \{0, 1, \ldots, r\}$ where a state $k \in S$ corresponds to there being k white balls in the left bucket. The example above is at state $k = 3$. The Markov process will be to draw one ball from the left bucket and one ball from the right bucket simultaneously, and to swap them and place each ball in the opposite bucket. We will compute the transition probabilities. Given $X_n = k$, the possible states for X_{n+1} are:

left	right	probability	X_{n+1}
white	white	$\left(\frac{k}{r}\right)\left(\frac{r-k}{r}\right)$	k
white	black	$\left(\frac{k}{r}\right)\left(\frac{k}{r}\right)$	$k-1$
black	white	$\left(\frac{r-k}{r}\right)\left(\frac{r-k}{r}\right)$	$k+1$
black	black	$\left(\frac{r-k}{r}\right)\left(\frac{k}{r}\right)$	k

Thus the transition probabilities are given by

$$\begin{aligned} p_{k(k+1)} &= \frac{(r-k)^2}{r^2} \\ p_{kk} &= \frac{2k(r-k)}{r^2} \\ p_{k(k-1)} &= \frac{k^2}{r^2} \\ p_{kj} &= 0 \quad \text{if } |j-k| \geq 2. \end{aligned}$$

Notice that for each k, $p_{kj} \geq 0$ for all j and $\sum_j p_{kj} = 1$.

(2) Random walks are another classic example of Markov chains. Suppose a person, dubbed the 'walker', is standing on the number line consisting of integer points. First consider the finite state space $S = \{0, 1, \ldots, r\}$ and assume the walker starts at one of the points in S. We also assume in the finite random walk scenario that the endpoints are absorbing, i.e. if $X_n = 0$ or r then $X_{n+j} = X_n$ for all $j \geq 1$. At time n, the walker moves to the right with probability p and moves left with probability $q = 1 - p$. Thus each X_n is a simple random variable. The transition probabilities are as follows:

$$p_{ij} = \begin{cases} p & \text{if } j = i+1, 0 < i < r \\ q & \text{if } j = i-1, 0 < i < r \\ 0 & \text{if } |j-i| \geq 2, 0 < i < r \\ 1 & \text{if } i = j = 0 \text{ or } r \\ 0 & \text{if } i \neq j = 0 \text{ or } r. \end{cases}$$

We typically assume $\alpha_i = 1$ for some $i \in S$ and $\alpha_j = 0$ for all $j \neq i$. That is, there is a positive probability that the walker starts on one of the endpoints.

(3) Most of the time we consider an unrestricted random walk, where the state space is

$S = \mathbb{Z}$. Here the transition probabilities are

$$p_{ij} = \begin{cases} p & \text{if } j = i+1 \\ q & \text{if } j = i-1 \\ 0 & \text{otherwise.} \end{cases}$$

If $p = q$, the random walk is said to be *symmetric*. In the symmetric case, the probability is 1 that the walker returns to her starting point.

4 For higher dimensional random walks, the state space is $S = \mathbb{Z}^k$, $k \geq 2$. Each point $y \in \mathbb{Z}^k$ has $2k$ 'neighbors' so the notation would be pretty ugly to write down explicitly. Things are even worse if the probabilities of the walker moving to a particular neighbor are not uniform, so let's assume the random walk is symmetric. Then

$$p_{yx} = \begin{cases} \frac{1}{2k} & \text{if } x \text{ is a neighbor of } y \\ 0 & \text{otherwise.} \end{cases}$$

It turns out that for $k = 2$, the probability that the walker returns to her starting location is still 1, but for any $k > 2$, the probability is 0. This suggests something subtle about the geometry of random walks. For this reason, random walks are an active area of modern research.

5 The following is known as either the Princess Problem or the Secretary Problem (in the latter, the princess is replaced by a businessperson trying to hire a secretary). Suppose a princess is trying to find a suitor. The rules are: the suitors appear in random order (e.g. they do not appear in order of increasing wealth or attractiveness); they appear one at a time; after meeting each suitor, the princess must decide whether to accept the suitor, at which point the process ends, or reject the suitor and continue the process. We also assume that the princess has some way of determining how the current suitor relates to every previous suitor in terms of desirability. If a suitor is more desirable than every previous suitor, we will say this suitor is *dominant*.

Let $S_1, \ldots, S_r$ be the suitors in the order they appear, so that $S = \{S_1, \ldots, S_r\}$ is the state space. Set $X_1 = 1$ and for each $n \geq 2$, set X_n to be the position of the nth dominant suitor, or $r + 1$ if the last dominant suitor has already occurred.

For example, suppose $S = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ and these are ranked in the order $S_3 < S_6 < S_2 < S_1 < S_4$. The dominant suitors are S_1 (which will always occur) and S_4, so $X_1 = S_1$, $X_2 = S_4$ and $X_3 = X_4 = \ldots = r + 1 = 7$.

In general,

$$\begin{aligned} P[S_k \text{ is dominant}] &= \frac{(k-1)!}{k!} = \frac{1}{k} \\ P[S_j \text{ is the next dominant suitor}] &= \frac{(j-2)!}{j!} = \frac{1}{j(j-1)} \quad \text{if } j > i \text{ and } 0 \text{ otherwise} \\ P[X_{n+1} = r+1 \mid X_n = k] &= \frac{\frac{(r-1)!}{r!}}{\frac{1}{k}} = \frac{k}{r}. \end{aligned}$$

Thus the transition probabilities are

$$p_{kj} = \begin{cases} \dfrac{k}{j(j-1)} & \text{if } k < j \le r \\ 0 & \text{if } j \le k < r \\ \frac{k}{r} & \text{if } j = r+1 \\ 1 & \text{if } j = i = r \\ 0 & \text{otherwise.} \end{cases}$$

The princess' strategy will be to pick a dominant suitor according to some stopping time τ. If she stops at X_τ then we want to know the probability that she picked the overall best suitor; this is expressed by $f(X_\tau)$ where $f(k) = \frac{k}{r}$. Given r the number of suitors, we also want to compute $E[f(X_\tau)]$ for different choices of τ but first we need to learn about expected values for Markov chains. We will return to the Princess Problem.

The Markov condition of independence of history allows us to calculate higher order transitions by stepping through one state at a time:

$$P[X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n] = \alpha_{i_0} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n}$$

and in general,

$$P[X_{m+t} = j_t, t = 0, \ldots, n \mid X_s = i_s \text{ for } 0 \le s \le m] = p i_m j_1 p_{j_1 j_2} \cdots p_{j_{n-1} j_n}.$$

We will denote a transition of degree n by $p_{ij}^{(n)}$. These can be written

$$p_{ij}^{(n)} = P[X_{m+n} = j \mid X_m = i] = \sum_{k_1, \ldots, k_{n-1} \in S} p_{ik_1} p_{k_1 k_2} \cdots p_{k_{n-1} j}.$$

If S is a finite state space then the transition matrix is $(p_{ij}^{(n)}) = P^n$ where $P = (p_{ij})$. Notice that $P_0 = I$ and $p_{ij}^{(0)} = \delta_{ij}$, the Kronecker delta. If S is countably infinite, the transition probabilities correspond to an infinite matrix which really lives in a Hilbert space.

Theorem 24.8.1. *Suppose (p_{ij}) is a doubly-indexed sequence of nonnegative real numbers such that for all i, $\sum_j p_{ij} = 1$ and suppose $\alpha_i \ge 0$ satisfy $\sum_i \alpha_i = 1$. Then there exists a probability space $(\Omega, \mathcal{F}, P)$ and a Markov chain $\{X_n\}_{n=0}^\infty$ on $(\Omega, \mathcal{F}, P)$ with the p_{ij} as its transition probabilities and the α_i as its initial probabilities.*

Proof sketch. Let $\Omega = (0, 1]$; $\mathcal{F} = \mathcal{B}$, the Borel σ-field; and $P = \lambda$, the Lebesgue measure on $\mathcal{B}$. First we want X_0 to equal i with probability α_i for each i. By Theorem 24.2.2, this is possible in theory but we want to explicitly construct the random variable X_0 so that we may continue the process in the next steps. Construct a collection of intervals $I_i^{(0)}$ with length α_i for each i by the following process: set $I_1^{(0)} = (0, \alpha_1]$, $I_2^{(0)} = (\alpha_1, \alpha_1 + \alpha_2]$, etc. It is evident that X_0 satisfies the desired properties. Next, we want X_1 to satisfy

$$P[X_1 = j, X_0 = i] = P[X_0 = i] P[X_1 = j \mid X_0 = i].$$

Subdivide each $I_i^{(0)}$ into $I_{ij}^{(1)}$ by a similar process as above, so that each $I_{ij}^{(0)}$ has length $\alpha_i p_{ij}$. Repeating this process of subdivision constructs a collection of intervals $I_{i_0 i_1 \cdots i_n}^{(n)}$ with length $\alpha_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}$. Finally, set

$$X_n(\omega) = \begin{cases} i & \text{if } \omega \in \bigcup_{i_0,\ldots,i_{n-1}} I_{i_0 i_1 \cdots i_{n-1} i}^{(n)} \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $\{X_n\}$ is a Markov chain with the given initial and transition probabilities.

$\square$

24.9 Transience and Persistence

Let $\{X_n\}_{n=0}^{\infty}$ be a Markov chain. First, let's set up some notation. If a probability is *conditioned on* $X_0 = i$, we will denote this by P_i. Define

$$f_{ij}^{(n)} = P_i[X_1 \neq j, X_2 \neq j, \ldots, X_{n-1} \neq j, X_n = j]$$

which represents the probability that the first occurrence of state j is at time n, given $X_0 = i$. Also set $f_{ij} = P_i\left(\bigcup_{n=1}^{\infty}[X_n = j]\right) = \sum_{n=1}^{\infty} f_{ij}^{(n)}$.

Definition. *For a Markov chain $\{X_n\}_{n=0}^{\infty}$ on state space S, a state i is* **persistent** *if $f_{ij} = 1$ and* **transient** *if $f_{ij} < 1$.*

Suppose $n_1 < n_2 < \ldots < n_k$ and consider

$$P_i[X_1 \neq j, X_2 \neq j, \ldots, X_{n_1} = j, \ldots, X_{n_k} = j] = f_{ij}^{(n_1)} f_{jj}^{(n_2-n_1)} \cdots f_{jj}^{(n_k-n_{k-1})}.$$

Then

$$\begin{aligned} P_i[X_n = j \text{ at least } k \text{ times}] &\geq \sum_{n_1,\ldots,n_k} f_{ij}^{(n_1)} f_{jj}^{(n_2-n_1)} \cdots f_{jj}^{(n_k-n_{k-1})} \\ &= f_{ij} f_{jj} \cdots f_{jj} = f_{ij} f_{jj}^{k-1}. \end{aligned}$$

Therefore

$$P_i[X_n = j \text{ i.o.}] = \begin{cases} 0 & \text{if } f_{jj} < 1 \\ 1 & \text{if } f_{jj} = 1. \end{cases}$$

Theorem 24.9.1. *A state i is transient $\iff$ $P_i[X_n = i$ i.o.$] = 0$ $\iff$ $\sum_n p_{ii}^{(n)}$ converges. Similarly, a state i is persistent $\iff$ $P_i[X_n = i$ i.o.$] = 1$*

Proof. By the first Borel-Cantelli lemma, if $\sum_n p_{ij}^{(n)} < \infty$ then $P_i[X_n = i \text{ i.o.}] = 0$. Based on the calculation above, $f_{ii} < 1$ so by definition i is transient. This proves $\sum_n p_{ii}^{(n)} < \infty \implies P_i[X_n = i \text{ i.o.}] = 0 \implies i$ transient. To close the logic loop, we must show $f_{ii} < 1 \implies \sum_n p_{ii}^{(n)} < \infty$. For any i, j, consider

$$\begin{aligned} p_{ij}^{(n)} &= P_i[X_n = j] \\ &= \sum_{s=0}^{n-1} P_i[X_1 \neq j, \ldots, X_{n-s-1} \neq j, X_{n-s} = j, X_n = j] \\ &= \sum_{s=0}^{n-1} P_i[X_1 \neq j, \ldots, X_{n-s-1} \neq j, X_{n-s} = j] P_j[X_s = j] \quad \text{by autonomy} \\ &= \sum_{s=0}^{n-1} f_{ij}^{(n-s)} p_{jj}^{(s)}. \end{aligned}$$

Next, we compute the sum of the $p_{ii}^{(n)}$:

$$\begin{aligned}
\sum_{t=1}^{n} p_{ii}^{(t)} &= \sum_{t=1}^{n} \sum_{s=0}^{t-1} f_{ii}^{(t-s)} p_{ii}^{(s)} \\
&= \sum_{s=0}^{n-1} p_{ii}^{(s)} \sum_{t=s+1}^{n} f_{ii}^{(t-s)} \quad \text{switching order of summation} \\
&\leq \sum_{s=0}^{n-1} p_{ii}^{(s)} f_{ii} \leq \sum_{s=0}^{n} p_{ii}^{(s)} f_{ii} \\
&= \sum_{t=1}^{n} p_{ii}^{(t)} f_{ii} + f_{ii} \quad \text{since } p_{ii}^{(0)} = 1 \text{ by a previous calculation.}
\end{aligned}$$

Rearranging this gives us $(1 - f_{ii}) \sum_{t=1}^{n} p_{ii}^{(t)} \leq f_{ii}$ so if $0 < f_{ii} < 1$,

$$\sum_{t=1}^{n} p_{ii}^{(t)} \leq \frac{f_{ii}}{1 - f_{ii}} \implies \sum_{t=1}^{\infty} p_{ii}^{(t)} \leq \frac{f_{ii}}{1 - f_{ii}}$$

and hence the sum converges. The statement for persistence is proven in a similar fashion. □

Example 24.9.2. We will prove Pólya's Theorem for symmetric k-dimensional random walks, which we state below. First, in order to employ Theorem 24.9.1 we want to know if $\sum_n a_n$ converges or not. If $k = 1$ then the only way to return to one's starting position is after an even number of moves, so $a_{2n+1} = 0$ for all n. On the other hand, if the walker returns to the start after $2n$ moves then she had to move left an equal number of times as she moved right. This means $a_{2n} = \binom{2n}{n} 2^{-2n}$. A well-known result called Stirling's Formula says that

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

Using this on a_{2n}, we have

$$a_{2n} = \frac{(2n)!}{(n!)^2 2^{2n}} = \frac{\sqrt{4\pi n} \left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n} \left(\frac{n}{e}\right)^n\right)^2 2^{2n}} = \frac{2\sqrt{\pi n}}{2\pi n} \frac{2^{2n} n^{2n}}{2^{2n} n^{2n}} \frac{\frac{1}{e^{2n}}}{\frac{1}{e^{2n}}} = \frac{1}{\sqrt{\pi n}}.$$

Then clearly $\sum_n a_n$ diverges (e.g. by a comparison test) so Theorem 24.9.1 tells us that each state in the state space is persistent.

Next, suppose $k = 2$. By similar logic as above,

$$a_{2n} = \sum_{u=0}^{n} \frac{(2n)!}{u!u!(n-u)!(n-u)!} 4^{-2n}$$

and another application of Stirling's Formula yields

$$a_{2n} \sim \frac{1}{\pi n}.$$

So $\sum_n a_n$ diverges, and thus the states are persistent in the $k = 2$ case.

It turns out that for $k \geq 3$, $a_{2n} \sim \frac{1}{n^{k/2}}$ which corresponds to a *convergent* series, so in these cases the states are all transient. Pólya's Theorem states this formally:

Theorem 24.9.3 (Pólya)**.** *A symmetric, k-dimensional random walk is persistent when $k = 1, 2$ and transient otherwise.*

Definition. *A Markov chain is said to be* **irreducible** *if for every i, j there exists an n such that $p_{ij}^{(n)} > 0$. In other words, in an irreducible chain there is always a finite sequence of transitions between any pair of states.*

Theorem 24.9.4. *If a chain is irreducible then either every state is transient or every state is persistent. Furthermore,*

(1) Transience is equivalent to $P_i\left(\bigcup_j [X_n = j \text{ i.o.}]\right) = 0$ and also to $\sum_n p_{ij}^{(n)} < \infty$ for all states $i, j \in S$.

(2) Persistence is equivalent to $P_i\left(\bigcup_j [X_n = j \text{ i.o.}]\right) = 1$ and also to $\sum_n p_{ij}^{(n)} = \infty$ for all $i, j \in S$.

Proof. Irreducibility implies for all $i, j \in S$ there exist r, s such that $p_{ij}^{(r)} > 0$ and $p_{ji}^{(s)} > 0$. Then $p_{ii}^{(r+s+n)} \geq p_{ij}^{(r)} p_{jj}^{(n)} p_{ji}^{(s)}$ so if $\sum_n p_{ii}^{(n)}$ converges then $\sum_n p_{jj}^{(n)}$ converges as well. By Theorem 24.9.4, this shows that if any one state is transient then they all are. If this is the case, then $f_{jj} < 1$ so

$$P_i\left(\bigcup_j [X_n = j \text{ i.o.}]\right) \leq \sum_{n=1}^{\infty} P_i[X_n = j \text{ i.o.}] = \sum_{j=1}^{\infty} f_{jj} = \sum_{j=1}^{\infty} 0 = 0.$$

Hence $P_i\left(\bigcup_j [X_n = j \text{ i.o.}]\right) = 0$. In addition,

$$\begin{aligned}
\sum_n p_{ij}^{(n)} &= \sum_n \sum_{v=1}^{n} f_{ij}^{(v)} p_{jj}^{(n-v)} \\
&= \sum_{v=1}^{\infty} f_{ij}^{(v)} \sum_{n=v}^{\infty} p_{jj}^{(n)} \quad \text{switching the summation} \\
&\leq \sum_{v=1}^{\infty} f_{ij}^{(v)} \sum_{n=0}^{\infty} p_{jj}^{(n)} \\
&\leq \sum_{n=0}^{\infty} p_{ij}^{(n)} < \infty \quad \text{since } f_{ij} \leq 1 \text{ for all } i, j.
\end{aligned}$$

Hence $\sum_n p_{ij}^{(n)}$ converges.

On the other hand, if every state is persistent then $P_j[X_n = j \text{ i.o.}] = 1$ by Theorem 24.9.4. Then

$$\begin{aligned} p_{ii}^{(m)} &= P_j[X_m = i] = P_j([X_m = i] \cap [X_n = j \text{ i.o.}]) \\ &\le \sum_{n>m} P_j[X_m = i, X_{m+1} \ne i, \ldots, X_n = j] \\ &= \sum_{n>m} p_{ji}^{(m)} f_{ij}^{(n-m)} = p_{ji}^{(m)} f_{ij}. \end{aligned}$$

So $p_{ji}^{(m)} \le p_{ji}^{(m)} f_{ij}$ which shows that $f_{ij} \ge 1$. But f_{ij} is a probability so $f_{ij} = 1$. Then by definition $P_i[X_n = j \text{ i.o.}] = 1$. Finally, by the contrapositive to the first Borel-Cantelli lemma, $\sum_n p_{ij}^{(n)}$ must diverges. □

Examples.

1. Suppose we have an irreducible Markov chain modeling a restricted random walk. Theorem 24.9.4 can be used to show that the probability of the random walker returning to her initial state infinitely often is 1. In other words, there are no transient states in a finite state space – if transient states exist (in an irreducible chain) then they imply the random walk will exit any finite subset of the state space.

2. Consider an *asymmetric random walk*, i.e. one where $p < \frac{1}{2}$. Suppose the state space is unrestricted, e.g. $S = \mathbb{Z}$. Then $f_{01} = \frac{p}{q} < 1$ so every state is transient. Notice in this case that $f_{10} = 1$ so it's not true that $f_{ij} < 1$ for every $i, j \in S$. The previous results only guarantee that $f_{ii} < 1$ for all i. When $p < \frac{1}{2}$ (if p is the probability of the walker moving right), it appears that the chain of right movements is persistent while the left movements are transient.

3. If the unrestricted walk is symmetric, i.e. $p = q = \frac{1}{2}$, and $2 \mid (n + j - i)$ then

$$p_{ij}^{(n)} = \binom{n}{\frac{n+j-i}{2}} \frac{1}{2^n} \sim \frac{1}{\sqrt{n}}.$$

If $|j - i| = -1, 0, 1$ then $\lim p_{ij}^{(n)} = 0$ even though the chain is persistent.

Definition. *A matrix $Q = (q_{ij})$ is said to be* **substochastic** *if $q_{ij} \ge 0$ for every i and the row sums satisfy $\sum_j q_{ij} \le 1$ for every i.*

Write $Q^n = (q_{ij}^{(n)})$ and $\sigma_i^{(n)} = \sum_j q_{ij}^{(n)}$ so that

$$\sigma_i^{(n+1)} = \sum_j q_{ij} \sigma_j^{(n)} \le \sum_j \sigma_j^{(n)}.$$

This implies that $\sigma_i^{(1)} \le 1$ and $\sigma_i^{(n+1)} \le \sigma_i^{(n)}$ for all i, n. So $(\sigma_i^{(n)})$ is a bounded, monotone sequence and hence $\sigma_i = \lim_n \sigma_i^{(n)}$ exists. Each σ_i satisfies a difference equation:

$$\sigma_i = \sum_j q_{ij} \sigma_j.$$

As it turns out, the σ_i are the maximal solutions to the boundary value problem

$$\begin{aligned} x_i &= \sum_j q_{ij} x_j, \quad 1 \le i \le n \\ 0 &\le x_i \le 1. \end{aligned}$$

(This is easily shown using the fact that $\sigma_i^{(n+1)} \le \sigma_i^{(n)}$ for all n.) Now if $U \subset S$ is a subset of the state space then $(p_{ij})_U$ is a substochastic matrix and $\sigma_i^{(n)} = P_i[X_t \in U, t \le n]$. Therefore by the above computations,

$$\sigma_i = \lim_{n\to\infty} \sigma_i^{(n)} = P_i[X_t \in U \text{ for all } t \ge 1].$$

Example.

4. Consider a *half-random walk* where the state space is $U = \mathbb{N} \cup \{0\}$, the right half of $S = \mathbb{Z}$. The difference equation from before is now a boundary value problem:

$$\begin{aligned} x_0 &= p x_1 \\ x_i &= p x_{i+1} + q x_{i-1}, \quad i \ge 1 \\ 0 &\le x_i \le 1. \end{aligned}$$

If $\rho = \frac{q}{p}$ then the solutions are of the form $x_n = A + Bn$ if $p = q$ or $x_n = A + B\rho^{n-1}$ if $p \ne q$. Notice that when $p \le q$, the solution is necessarily unbounded. However, when $p > q$, the solution is bounded. We want $0 \le x \le 1$ so we must have $A = 0$ in the case when $p = q$, or $B = -A$ in the case when $p \ne q$. Thus the solutions are

$$x_n = \begin{cases} A - An & \rho = 1 \\ A - A\rho^{n-1} & \rho \ne 1. \end{cases}$$

Theorem 24.9.5. *A state i_0 is transient $\iff$ there exists a nontrivial solution to the system*

$$x_i = \sum_{j \ne i_0} p_{ij} x_j, \quad 0 \le x_i \le 1 \text{ for all } i \ne i_0.$$

Proof. On one hand, suppose i_0 is persistent. By the discussion above, $P_i[X_n \ne i_0 \text{ for all } n]$ is a maximal solution to this system. But $P_i[X_n \ne i_0 \text{ for all } n] = 1 - f_{ii_0}$ so there is a nontrivial solution $\iff$ $f_{ii_0} < 1$ for some $i \ne i_0$ but this is impossible in the persistent case.

On the other hand, we proved that transience implies $f_{i_0 i_0} < 1$, but

$$\begin{aligned} f_{i_0 i_0} &= P_{i_0}[X_1 = i_0] + \sum_{n=2}^{\infty} \sum_{i \ne i_0} P_{i_0}[X_1 = i, X_2 \ne i_0, \ldots, X_n = i_0] \\ &= p_{i_0 i_0} + \sum_{i \ne i_0} p_{i_0 i} f_{i i_0}. \end{aligned}$$

If the f_{ii_0} were all 1, this would add up to 1 but $f_{i_0 i_0} < 1$ so the above shows that $f_{ii_0} < 1$ for some $i \ne i_0$. Hence there is a nontrivial solution. ☐

Example.

5 Queueing is used to model physical situations, such as customers standing in line, as well as computer processing. Suppose we have a state space $S = \mathbb{N} \cup \{0\}$ which represents the number of people currently in line. At each time k, one person at the front of the line is helped and then leaves, and simultaneously, 0, 1 or 2 people enter the line with probabilities t_0, t_1 and t_2, respectively. These satisfy $t_0 + t_1 + t_2 = 1$ and we assume $t_0, t_2 > 0$ so that the chain is irreducible. The queueing 'matrix' here is infinite:

$$P = \begin{bmatrix} t_0 & t_1 & t_2 & 0 & 0 & 0 & \cdots \\ t_0 & t_1 & t_2 & 0 & 0 & 0 & \cdots \\ 0 & t_0 & t_1 & t_2 & 0 & 0 & \cdots \\ 0 & 0 & t_0 & t_1 & t_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Notice that the first row is different than the random walk's transition matrix: if $i = 0$, no one is served until someone enters the line. Fix $i_0 = 0$ – since the chain is irreducible, either every state is persistent or every state is transient so no generality is lost. The system here is

$$\begin{aligned} x_1 &= t_1x_1 + t_2x_2 \\ x_k &= t_0x_{k-1} + t_1x_k + t_2x_{k+1}, \quad k \geq 2. \end{aligned}$$

This is essentially the same as the system for a half-random walk (see Example 4). So there is a nontrivial solution, i.e. i_0 is transient, if and only if $t_2 > t_0$ and conversely i_0 is persistent if and only if $t_2 \leq t_0$.

Definition. *A* **distribution** *is a sequence π_i satisfying $0 \leq \pi_i \leq 1$, $\sum_{i\in S} \pi_i = 1$ and $\sum_{i\in S} \pi_i p_{ij} = \pi_j$ for all $i, j \in S$. Additionally, the distribution is* **stationary** *if $P[X_0 = j] = \pi_j$ implies $P[X_n = j] = \pi_j$ for all n.*

Definition. *A state $j \in S$ has* **period** *t if whenever $p_{ij}^{(n)} > 0$ for any i, $t \mid n$. If 1 is a period for j then we say j is* **aperiodic**.

Example 24.9.6. A 1-dimensional random walk has period 2 since the walker must return to her starting position after an even number of moves.

Remark. In an irreducible chain, all periods are equal. We will usually assume that $\{X_n\}$ is an aperiodic, irreducible chain, such as in the next lemma.

Lemma 24.9.7. *Suppose a chain $\{X_n\}$ is an aperiodic, irreducible Markov chain. Then for every $i, j \in S$, there exists an $n_0 \in \mathbb{N}$ such that $p_{ij}^{(n)} > 0$ for all $n \geq n_0$.*

Proof. Let $M_j = \{n \in \mathbb{N} \mid p_{ij}^{(n)} > 0\}$. Then $p_{ij}^{(m+n)} \geq p_{ij}^{(m)}p_{ij}^{(n)}$ for all n so M_j is closed under addition. Since the chain is aperiodic, $\gcd(M_j) = 1$ so by elementary number theory there exists an n_0 such that $n \in M_j$ for all $n \geq n_0$. Let $i, j \in S$. Since the chain is irreducible, there exists an r such that $p_{ij}^{(r)} > 0$. If we let $n_{ij} = n_j + r$, then every $n \geq n_{ij}$ satisfies

$$p_{ij}^{(n)} \geq p_{ij}^{(r)} p_{jj}^{(n-r)} > 0 \cdot 0 = 0.$$

□

Theorem 24.9.8. *Let $\{X_n\}$ be an aperiodic, irreducible Markov chain and suppose a stationary distribution π_j exists. Then the chain is persistent, $\lim_n p_{ij}^{(n)} = \pi_j$, $\pi_i > 0$ for all i and the distribution is unique.*

Proof. First suppose the chain is transient. Then $p_{ij}^{(n)} \to 0$ as $n \to \infty$ for any $j \in S$. By the Weierstrass M-test, $\sum_j p_{ij}^{(n)} \pi_j$ converges absolutely and uniformly in n, so

$$\pi_i = \lim_n \left(\sum_j p_{ij}^{(n)} \pi_j \right) = \sum_j \left(\lim_n p_{ij}^{(n)} \right) \pi_j = 0.$$

Hence $\pi_i \equiv 0$ so $\pi_i \pi_i \neq 1$ and therefore no stationary distribution exists, contradicting the hypotheses. Therefore the chain is persistent.

Consider the double-indexed state space $S \times S$. Define $p(ij, kl) = p_{ik} p_{jl}$ to form a *direct product of Markov chains* $X_n \times Y_n$. One can prove that this is still irreducible and aperiodic given the assumptions on X_n. Then for all $i, j, i_0 \in S$, $P_{ij}[(X_n, Y_n) = (i_0, i_0) \text{ i.o.}] = 1$; that is, the two chains meet in finite time with probability 1. Let $\tau = \inf_n[(X_n, Y_n) = (i_0, i_0)]$. Then another way of saying the previous statement is that $\tau < \infty$ with probability 1. This implies

$$|p_{ik}^{(n)} - p_{jk}^{(n)}| \leq P_{ij}[\tau > n] \to 0$$

by the M-test. So i and j really don't affect the outcome after time τ. Note that

$$\begin{aligned} \pi_j - p_{jk}^{(n)} &= \sum_i \pi_i p_{ik}^{(n)} - \sum_j \pi_j p_{jk}^{(n)} \\ &= \sum_i \pi_i (p_{ik}^{(n)} - p_{jk}^{(n)}) \end{aligned}$$

and the combined sum approaches 0 by the M-test. Thus $\pi_k = \lim_n p_{ij}^{(n)}$ for any $j \in S$ and by uniqueness of limits, π_j is unique. Also, for n sufficiently large, $\pi_k = \sum_i \pi_i p_{ik}^{(n)} > 0$. This finishes everything we needed to check. □

Example.

5 continued. For the queueing model described before, we can plug in the row sums to obtain:

$$\begin{aligned} \pi_0 &= \pi_0 t_0 + \pi_1 t_1 \\ \pi_k &= \pi_{k-1} t_0 + \pi_k t_1 + \pi_{k+1} t_2, \quad k \geq 1. \end{aligned}$$

This has a nontrivial solution if the chain is persistent:

$$\pi_k = \begin{cases} A - Ak & t_0 = t_2 \\ A - A \left(\dfrac{t_0}{t_2} \right)^{k-1} & t_0 \neq t_2. \end{cases}$$

Of course the system is persistent if $t_0 \geq t_2$ and in particular there is a stationary distribution if $t_0 > t_2$, in which case the solution $\sum_k \left(A - A \left(\dfrac{t_0}{t_2} \right)^{k-1} \right)$ is a geometric series which we may evaluate. On the other hand, if $t_0 = t_2$ there is no stationary distribution even though the chain is persistent.

The examples illustrate our three possibilities so far for a Markov chain $\{X_n\}$:

- The chain is transient. In this case, $p_{ij}^{(n)} \to 0$ and the mean return time for a state $j \in S$ is $\mu_j = \sum_n n f_{jj}^{(n)} = \infty$.
- The chain is persistent but has no stationary distribution; this is called *nullpersistence*. In this case for all $j \in S$, $p_{ij}^{(n)} \to \pi_j$ and $\mu_j = \infty$.
- The chain is *positive persistent*, i.e. persistent with a stationary distribution. Here $p_{ij}^{(n)} \to \pi_j$ and $\mu_j = \frac{1}{\pi_j} < \infty$.

Our k-dimensional random walks for different k values illustrate all three scenarios. For $k = 1$, the chain is positive persistent; for $k = 2$, the chain is nullpersistent; and finally for $k \geq 3$, the chain is transient.

Part VI

Fourier Analysis

Chapter 25

Locally Compact Groups

25.1 Topological Vector Spaces

Definition. *A* **topological field** *is a field k with a topology with respect to which the addition, multiplication and inversion maps $+ : k \times k \to k, \cdot : k \times k \to k$ and $(-)^{-1} : k \to k$ are continuous, where $k \times k$ has the product topology.*

Definition. *For a topological field k, a* **topological vector space** *over k is a k-vector space V with a topology such that V is a topological abelian group and the structure map $k \times V \to V$ is continuous.*

Example 25.1.1. Let k be a topological field. Then any abstract k-vector space V is isomorphic to a direct sum of copies of k, $\varphi : V \xrightarrow{\sim} \bigoplus_\Omega k$, indexed by some set Ω. Then V inherits a topology by pulling back the subspace topology on $\bigoplus_\Omega k \subseteq \prod_\Omega k$ along φ and this makes V into a topological vector space.

Example 25.1.2. If V is a Banach space over $\mathbb{R}$ or $\mathbb{C}$, then V is a topological vector space with respect to the norm topology.

We will assume for the rest of these notes that all topological vector spaces are T_1 (and therefore Hausdorff by homogeneity). For a topological vector space V/k, let $\mathrm{Aut}(V)$ denote the k-automorphisms of V and let $\mathrm{Aut}_{top}(V)$ denote the subspace of continuous k-automorphisms of V having continuous inverses.

For a real or complex vector space V and a subset $S \subseteq V$, we say S is *convex* if for all $x, y \in S$, $tx + (1-t)y \in S$ for every value $t \in [0,1]$. We say V is *locally convex* if there exists a topological basis of V consisting of convex sets.

Example 25.1.3. When V is a Banach space, the metric balls $\{B(0,\varepsilon) \mid \varepsilon > 0\}$ form a system of convex neighborhoods around 0, so by homogeneity V is locally convex.

Definition. *Suppose G is a locally compact topological group and V is a locally convex topological vector space over $\mathbb{C}$. A* **topological representation** *of G is a group representation $\rho : G \to \mathrm{Aut}(V)$ such that the associated map*

$$\begin{aligned} G \times V &\longrightarrow V \\ (g, v) &\longmapsto \rho_g(v) \end{aligned}$$

is continuous (with respect to the product topology on $G \times V$).

Note that if ρ is a topological representation of G, then $\rho(G) \subseteq \mathrm{Aut}_{top}(V)$. The converse is not immediately true, but in a moment we will give conditions under which this does hold.

Definition. *Let X be a topological space, V a topological vector space and let $\mathrm{Map}(X, V)$ be the space of set maps $X \to V$. A set $F \in \mathrm{Map}(X, V)$ is said to be* **equicontinuous** *if for all $x \in X$ and every neighborhood $U \subseteq V$ of 0, there exists a neighborhood $W \subseteq X$ such that $f(y) \in U + f(x)$ for every $y \in W$ and $f \in F$.*

Proposition 25.1.4. *Suppose $\rho : G \to \mathrm{Aut}(V)$ is a representation of a locally compact group. Then ρ is a topological representation if and only if the following conditions are met:*

(1) For every compact set $K \subseteq G$, $\rho(K)$ is equicontinuous.

(2) For all $v \in V$, the map $G \to V, g \mapsto \rho_g(v)$ is continuous.

Proof. ($\implies$) Suppose ρ is a topological representation. Then for all $v \in V$, the map $G \to V, g \mapsto \rho_g(v)$ factors as a composition $G \to G \times V \to V$, where $G \to G \times V$ is the first coordinate inclusion (hence continuous), and $G \times V \to V$ is $(g, x) \mapsto \rho_g(x)$, which is continuous by hypothesis. Hence (2) holds.

For (1), fix a compact set $K \subseteq G$. It will suffice to show equicontinuity about $0 \in V$, i.e. for all neighborhoods $U \subseteq V$ of 0, there exists a neighborhood $W \subseteq V$ of 0 such that for all $y \in W$ and $g \in K$, $\rho_g(y) \in U$. We know $G \times V \to V$ is continuous, so for each $g \in G$, there exists a neighborhood $H_g \subseteq G$ of g and a neighborhood $W_g \subseteq V$ of 0 for which $\rho_h(W_g) \subseteq U$ for all $h \in H_g$. Since K is compact and covered by the H_g, there exist $g_1, \ldots, g_n$ such that $K \subseteq \bigcup_{i=1}^n H_{g_i}$. Set $W = \bigcap_{i=1}^n W_{g_i}$, which is then a neighborhood of 0 in V. Then for all $g \in K$ and $w \in W$, we have $\rho_g(w) \in W$ by construction. Hence $\rho(K)$ is equicontinuous.

($\impliedby$) Given (1) and (2), we want to show that $G \times V \to V$ is continuous, i.e. for fixed $(g, x) \in G \times V$ and for any neighborhood $U \subseteq V$ of 0, there exist neighborhoods $H \subseteq G$ of g and $W \subseteq V$ of 0 such that $\rho_h(x + w) - \rho_g(x) \in U$ for all $h \in H, w \in W$. Since V is locally convex, we can find a convex neighborhood of 0 contained in U, so we may assume U itself is convex. Also, since G is locally compact, there exists a compact neighborhood of g, say $K \subseteq G$. Now by (1), $\rho(K)$ is equicontinuous so there exists a neighborhood $W \subseteq V$ of 0 such that $\rho_h(w) \in \frac{1}{2}U$ for all $h \in K, w \in W$. And by (2), there exists a neighborhood $H \subseteq G$ of g such that $\rho_h(x) - \rho_g(x) \in \frac{1}{2}U$ for all $h \in H$. We may assume that $H \subseteq K$. Now we have that for all $h \in H, w \in W$,

$$\rho_h(x + w) - \rho_g(x) = \rho_h(w) + \rho_h(x) - \rho_g(x) \in \tfrac{1}{2}U + \tfrac{1}{2}U$$

but since U is convex, $\frac{1}{2}U + \frac{1}{2}U = U$ and hence $\rho_h(x + w) - \rho_g(x) \in U$. Hence ρ is a topological representation. $\square$

Example 25.1.5. If V is a Banach space, we may topologize $\mathrm{Aut}(V)$ as follows. Note that $\mathrm{Map}(V, V) \cong \prod_{v \in V} V$ so the product topology on $\prod_{v \in V} V$ induces a topology on $\mathrm{Map}(V, V)$ and in turn a subspace topology on $\mathrm{Aut}(V) \subseteq \mathrm{Map}(V, V)$ (this also induces a topology on $\mathrm{Aut}_{top}(V)$). In fact, this topology on $\mathrm{Aut}(V)$ is equivalent to the topology of pointwise convergence. Under this topology, *every* abstract representation $\rho : G \to \mathrm{Aut}(V)$ of a locally compact group is continuous. In particular, if $K \subseteq G$ is a compact set then $\rho(K)$ is always compact in $\mathrm{Aut}(V)$. This allows us to cut down the conditions in Proposition 25.1.4.

Corollary 25.1.6. *Suppose V is a Banach space and G is a locally compact group. Then a group representation $\rho : G \to \mathrm{Aut}(V)$ is a topological representation if and only if for all $x \in V$, the map $G \to V, g \mapsto \rho_g(x)$ is continuous.*

Let $\rho : G \to \mathrm{Aut}(V)$ be a representation.

Definition. *A* ***G-invariant subspace*** *of V is a subspace $W \subseteq V$ such that $\rho_g(W) \subseteq W$ for all $g \in G$.*

Definition. *A representation $\rho : G \to \mathrm{Aut}(V)$ is said to be* **algebraically irreducible** *if V has no proper G-invariant subspaces, i.e. V is simple as a $\mathbb{C}[G]$-module. We say ρ is* **topologically irreducible** *if V has no proper, closed G-invariant subspaces.*

Definition. *An* **equivalence of G-representations** *$(\rho, V) \sim (\rho', V')$ is a homeomorphism $T : V \to V'$ such that the diagram*

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\rho_g"'] & V' \arrow[d, "\rho'_g"] \\ V \arrow[r, "T"] & V' \end{tikzcd}$$

commutes for every $g \in G$, or equivalently T is a $\mathbb{C}[G]$-module homomorphism.

25.2 Banach Algebras

Suppose A and B are complex vector spaces and $\mathrm{Hom}(A, B)$ is the set of *continuous* (or equivalently, bounded) linear maps $A \to B$. Then by Theorem 15.2.5, $\mathrm{Hom}(A, B)$ is a Banach space with respect to the operator norm

$$||T||_{op} = \sup_{a \in A} \frac{||Ta||_B}{||a||_A}.$$

When $A = B$, we write $\mathrm{End}(A) = \mathrm{Hom}(A, A)$.

Definition. *A* **Banach algebra** *is a $\mathbb{C}$-algebra A with $1_A \in A$ (and possibly noncommutative) that admits the structure of a complex Banach space which is* **submultiplicative**, *i.e. $||ab|| \leq ||a||\,||b||$ for all $a, b \in A$, and is normalized so that $||1_A|| = 1$.*

Let A be a Banach algebra. Each $a \in A$ defines a linear map

$$\begin{aligned} \rho_a : A &\longrightarrow A \\ b &\longmapsto ab. \end{aligned}$$

Then $\rho_a \in \mathrm{End}(A)$ and it follows from $||1_A|| = 1$ that $||\rho_a||_{op} = ||a||$ for all $a \in A$. This determines an embedding $\rho : A \hookrightarrow \mathrm{End}(A)$. Let $A^\times$ be the units of A and observe that, by submultiplicativity, if $a \in A$ such that $||a|| < 1$, then $1 - a \in A^\times$ (this follows from the fact that $\sum_{n=1}^\infty a^n$ converges in A).

Proposition 25.2.1. *Let A be a Banach algebra. Then $A^\times \subseteq A$ is an open subset and $A^\times \to A^\times, a \mapsto a^{-1}$ is a homeomorphism.*

Proof. Let $a \in A^\times$ and take $b \in B(a, ||a^{-1}||^{-1})$. (Since $||\cdot||$ is only submultiplicative, $||a^{-1}||^{-1} \leq ||a||$ but not necessarily equal.) Then $||a - b|| < ||a^{-1}||^{-1}$ so multiplying by a^{-1}, we get

$$||a^{-1}(a - b)|| \leq ||a^{-1}||\,||a - b|| < 1$$

which by the remark above implies $1 - a^{-1}(a - b) \in A^\times$. Multiplying by a gives $b = a - (a - b) \in A^\times$, so we have an open neighborhood around a in $A^\times$. The second statement is an easy consequence. □

Definition. *Let A be a Banach algebra and $a \in A$. The* **spectrum** *of a is*

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{C} \mid \lambda 1_A - a \notin A^\times\}.$$

The **spectral radius** *of a is $r(a) = \sup\{|\lambda| : \lambda \in \mathrm{sp}(a)\}$ and the complement $\mathbb{C} \smallsetminus \mathrm{sp}(a)$ is called the* **resolvent set** *of a.*

Lemma 25.2.2. *For all $a \in A$, $r(a) \leq ||a||$.*

Proof. Suppose $\lambda \in \mathbb{C} \smallsetminus \{0\}$ such that $|\lambda| > ||a||$. Then

$$||\lambda^{-1} a|| < 1 \implies 1_A - \lambda^{-1} a \in A^\times \implies \lambda 1_A - a \in A^\times$$

so $\lambda \notin \mathrm{sp}(A)$. □

Theorem 25.2.3. *Let A be a Banach algebra and $a \in A$. Then*

(1) $\mathrm{sp}(a)$ *is a nonempty, compact subset of* $\mathbb{C}$.

(2) $\lim_{n\to\infty} ||a^n||^{1/n} = r(a)$.

Proof. (1) Define $\varphi_a : \mathbb{C} \to A$ by $\lambda \mapsto \lambda 1_A - a$. Then φ_a is continuous and $\varphi_a^{-1}(A^\times) = \mathbb{C} \setminus \mathrm{sp}(a)$, so the resolvent set is open by Proposition 25.2.1, so $\mathrm{sp}(a)$ is closed. Since $\mathrm{sp}(a)$ is also bounded, it is compact.

(2) omitted. □

Corollary 25.2.4 (Gelfand-Mazur Theorem)**.** *If A is a Banach algebra which is a division ring, then $A \cong \mathbb{C}$.*

Proof. Take $a \in A$. By assumption $A^\times = A \setminus \{0\}$, so if $\lambda 1_A - a \notin A^\times$ for some $\lambda \in \mathbb{C}$ then $a = \lambda 1_A$. By (1) of Theorem 25.2.3, $\mathrm{sp}(a) \neq \varnothing$ so such a $\lambda \in \mathbb{C}$ exists. Define $A \to \mathbb{C}$ by mapping $a \mapsto \lambda$. This gives the desired isomorphism. □

Suppose $J \subseteq A$ is a two-sided ideal. Then A/J is an algebra admitting a seminorm

$$||a + J|| = \inf_{x\in J} ||a - x||.$$

Proposition 25.2.5. *Suppose $J \subseteq A$ is a closed, two-sided ideal. Then*

(1) $||\cdot||$ *is a norm on* A/J.

(2) A/J *is a Banach space with respect to this norm.*

Proof. (1) If (x_n) is a sequence in J converging to $a \in A$, then $a \in J$ since J is closed. Hence whenever $||a+J|| = 0$, we have $a \in J$, so $||\cdot||$ is a nondegenerate. Further, suppose $a, b \in A$. Then

$$\begin{aligned}
||a + J||\,||b + J|| &= \inf_{x\in J} ||a - x|| \inf_{y\in J} ||b - y|| \\
&\geq \inf_{x,y\in J} ||a - x||\,||b - y|| \\
&\geq \inf_{x,y\in J} ||(a - x)(b - y)|| \quad \text{by submultiplicativity} \\
&= \inf_{x,y\in J} ||ab - xb - ay + xy|| \\
&= \inf_{x,y\in J} ||ab - (xb + ay - xy)|| \\
&\geq ||ab + J|| \quad \text{since } xb + ay - xy \in J.
\end{aligned}$$

Hence $||\cdot||$ is a norm.

(2) is straightforward. □

Remark. It is useful to note that for any two-sided ideal of A, the topological closure $\overline{J}$ is also a two-sided ideal of A, by submultiplicativity.

25.3 The Gelfand Transform

Suppose A is a commutative Banach algebra.

Definition. *A* **character** *of A is a $\mathbb{C}$-algebra homomorphism $\chi : A \to \mathbb{C}$. The set of characters of A is denoted $\widehat{A}$.*

Note that any character $\chi : A \to \mathbb{C}$ is surjective.

Proposition 25.3.1. *Let A be a commutative Banach algebra. Then*

(1) If $J \subseteq A$ is a maximal ideal, then J is closed.

(2) The map

$$\begin{aligned} \widehat{A} &\longrightarrow \mathrm{MaxSpec}(A) \\ \chi &\longmapsto \ker\chi \end{aligned}$$

is a bijection.

(3) Every character $\chi \in \widehat{A}$ is continuous.

(4) For all $a \in A$, $\mathrm{sp}(a) = \{\chi(a) \mid \chi \in \widehat{A}\}$.

Proof. (1) By Proposition 25.2.1, $A^\times$ is open in A so an ideal J is proper if and only if $\overline{J}$ is proper. This implies that maximal ideals are closed.

(2) Given a character $\chi \in \widehat{A}$, there is a factorization through the quotient:

$$\begin{tikzcd} A \arrow[rr, "\chi"] \arrow[dr, "p"'] & & \mathbb{C} \\ & A/\ker\chi \arrow[ur, "\overline{\chi}"'] & \end{tikzcd}$$

Since χ is surjective, $\overline{\chi}$ is surjective, so $A/\ker\chi$ is a field and thus $\ker\chi$ is a maximal ideal. On the other hand, for any $\mathfrak{m} \in \mathrm{MaxSpec}(A)$, the Gelfand-Mazur theorem (Corollary 25.2.4) implies

$$\begin{aligned} \overline{\chi}_{\mathfrak{m}} : A/\mathfrak{m} &\longrightarrow \mathbb{C} \\ \lambda 1_A &\longmapsto \lambda \end{aligned}$$

is the unique $\mathbb{C}$-algebra isomorphism $A/\mathfrak{m} \cong \mathbb{C}$. Hence $\mathfrak{m}$ defines a character $\chi_{\mathfrak{m}} := \overline{\chi}_{\mathfrak{m}} \circ p \in \widehat{A}$:

$$\begin{tikzcd} A \arrow[rr, "\chi_{\mathfrak{m}}"] \arrow[dr, "p"'] & & \mathbb{C} \\ & A/\mathfrak{m} \arrow[ur, "\overline{\chi}_{\mathfrak{m}}"'] & \end{tikzcd}$$

(3) Any $\chi \in \widehat{A}$ factors as $\chi : A \xrightarrow{p} A/\ker\chi \xrightarrow{\overline{\chi}} \mathbb{C}$ as above, and both maps are continuous.
(4) Let $a \in A$. Then

$$\begin{aligned}\lambda \in \mathrm{sp}(a) &\iff \lambda 1_A - a \notin A^\times \\ &\iff \lambda 1_A - a \in \mathfrak{m} \text{ for some maximal ideal } \mathfrak{m} \\ &\iff \chi(\lambda 1_A - a) = 0 \text{ for some } \chi \in \widehat{A} \text{ by (2)} \\ &\iff \lambda = \chi(a) \text{ for some } \chi \in \widehat{A}.\end{aligned}$$

Thus $\mathrm{sp}(a) = \{\chi(a) \mid \chi \in \widehat{A}\}$. $\square$

This allows us to view $\widehat{A}$ as a subring of $A^* = \mathrm{Hom}_{top}(A, \mathbb{C})$, the topological dual of A. We could equip A^* with the norm topology, but this turns out to be too strong of a topology for our purposes.

Definition. *The* **weak topology** *on A^* is the topology generated by all maps $A^* \to \mathbb{C}$ in A^{**}. The* **weak* topology** *on A^* is the toplogy generated by all of the evaluation maps $\mathrm{ev}_a \in A^{**}$ for $a \in A$, defined by*

$$\begin{aligned}\mathrm{ev}_a : A^* &\longrightarrow \mathbb{C} \\ \varphi &\longmapsto \varphi(a).\end{aligned}$$

We endow $\widehat{A}$ with the subspace topology induced by the weak* topology on A^*; this is called the *Gelfand topology* on $\widehat{A}$.

Lemma 25.3.2. *The weak* topology makes A^* into a locally convex topological vector space.*

Theorem 25.3.3 (Alaoglu)**.** *Let $B^* = \{f \in A^* : ||f||_{op} \leq 1\}$ be the unit ball in A^*. Then B^* is compact in the weak* topology.*

Lemma 25.3.4. *For any commutative Banach algebra A,*

(1) $\widehat{A} \subseteq B^*$.

(2) $\widehat{A}$ is compact and Hausdorff in the Gelfand topology.

Proof. (1) For all $a \in A$ and $\chi \in \widehat{A}$, $\chi(a) \in \mathrm{sp}(a)$ by (4) of Theorem 25.3.1, so

$$|\chi(a)| \leq r(a) \leq ||a||$$

by Lemma 25.2.2. Hence $||\chi|| \leq 1$.

(2) Since A^* is Hausdorff (this is easy to prove), the subspace $\widehat{A}$ is Hausdorff. To show $\widehat{A}$ is compact, it suffices by (1) to show that $\widehat{A}$ is closed in A^*. Suppose (χ_n) is a sequence in $\widehat{A}$ converging to $\chi \in A^*$. Convergence in the weak* topology means that for all $a \in A$, the sequence $(\chi_n(a))$ converges, say to $\chi(a)$. This defines $\chi : A \to \mathbb{C}$. Further, since each χ_n is a $\mathbb{C}$-algebra homomorphism, so is χ. Hence $\chi \in \widehat{A}$, so $\widehat{A}$ is closed. $\square$

For all $a \in A$, let the evaluation map $\widehat{A} \to \mathbb{C}, \chi \mapsto \chi(a)$ be denoted by $\hat{a}$. Let $\mathcal{C}(\widehat{A})$ be the $\mathbb{C}$-algebra of continuous maps $\widehat{A} \to \mathbb{C}$, which is a Banach space with respect to the sup norm $||f||_\infty \sup_{\chi \in \widehat{A}} |f(\chi)|$.

Definition. *The* **Gelfand transform** *of a commutative Banach algebra A is the map*

$$\begin{aligned} \Gamma : A &\longrightarrow \mathcal{C}(\widehat{A}) \\ a &\longmapsto \hat{a}. \end{aligned}$$

Theorem 25.3.5. *For any commutative Banach algebra A,*

(1) Γ *is a $\mathbb{C}$-algebra homomorphism which decreases in norm.*

(2) The image $\Gamma(A) \subseteq \mathcal{C}(\widehat{A})$ separates points.

(3) For all $a \in A$, $\hat{a}(\widehat{A}) = \mathrm{sp}(a)$ and $||\hat{a}||_\infty = r(a)$.

(4) $\ker \Gamma = r(A)$, *the Jacobson radical of A.*

(5) Γ *is injective if and only if A is semisimple as a ring.*

Proof. The proofs of all five properties are straightforward from the definitions. □

25.4 Spectral Theorems

Suppose A is a complex vector space of complex-valued functions on some space X.

Definition. *A complex function space A is* **self-adjoint** *if A is closed under complex conjugation, that is, for all $T \in A$, the function $\overline{T} : X \to \mathbb{C}, x \mapsto \overline{T}x := \overline{Tx}$ is also in A.*

Remark. Let $A_{\mathbb{R}} = A \cap \mathcal{C}(X, \mathbb{R})$ be the subspace of real-valued functions in A. Then A is self-adjoint if and only A can be written $A = A_{\mathbb{R}} + iA_{\mathbb{R}}$.

Now suppose X is a compact Hausdorff space. Set $\mathcal{C}(X) = \mathcal{C}(X, \mathbb{C})$ to distinguish from $\mathcal{C}(X, \mathbb{R})$. The Stone-Weierstrass theorem is an important result from functional analysis which in some ways gives a function space analogue of Hilbert's Nullstellensatz.

Theorem 25.4.1 (Stone-Weierstrass)**.** *If $A \subseteq \mathcal{C}(X, \mathbb{R})$ is a closed subalgebra that separates points in X, then either*

(1) $A = \mathcal{C}(X, \mathbb{R})$, or

(2) $A = \{f \in \mathcal{C}(X, \mathbb{R}) \mid f(x) = 0\}$ for some $x \in X$.

Further, if A is a unital algebra, then only (1) is possible.

The following is a complex analogue of the Stone-Weierstrass theorem.

Corollary 25.4.2. *Let A be a self-adjoint, unital subalgebra of $\mathcal{C}(X)$ that separates points in X. Then A is dense in $\mathcal{C}(X)$.*

Proof. By the remark, we may write $A = A_{\mathbb{R}} + iA_{\mathbb{R}}$. Since A separates points, so does $A_{\mathbb{R}}$, so by the Stone-Weierstrass theorem for this real function space, we get $\overline{A_{\mathbb{R}}} = \mathcal{C}(X, \mathbb{R})$. Hence $\overline{A} = \overline{A_{\mathbb{R}}} + \overline{iA_{\mathbb{R}}} = \mathcal{C}(X, \mathbb{R}) + i\mathcal{C}(X, \mathbb{R}) = \mathcal{C}(X)$. □

Let H be a Hilbert space (see Section 20.2) and consider $\mathrm{End}(H)$, the space of continuous (bounded) linear maps $H \to H$. Then $\mathrm{End}(H)$ is a Banach algebra. By Proposition 20.2.18, for each $T \in \mathrm{End}(H)$, there is a unique adjoint operator $T^* \in \mathrm{End}(H)$ satisfying

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \quad \text{for all } x, y \in H.$$

This defines an involution $\mathrm{End}(H) \to \mathrm{End}(H), T \mapsto T^*$. Recall that an operator $T \in \mathrm{End}(H)$ is

- **self-adjoint** if $T = T^*$;
- **unitary** if $T^{-1} = T^*$;
- **normal** if $TT^* = T^*T$.

Proposition 25.4.3. *If $T \in \mathrm{End}(H)$ is normal then $||T|| = r(T)$, the spectral radius of T.*

Proof. On one hand, we have $r(T) \leq ||T||$ by Lemma 25.2.2. Note that when T is normal, the operator TT^* is self-adjoint. This allows us to write the following for any $n \geq 1$:

$$\begin{aligned}
||T||^{2^n} &= \left(||T||^2\right)^{2^{n-1}} \\
&= ||TT^*||^{2^{n-1}} \quad \text{by Lemma 20.2.19(ii)} \\
&= ||(TT^*)^{2^n}||^{1/2} \quad \text{since } TT^* \text{ is self-adjoint} \\
&= ||T^{2^n}(T^*)^{2^n}||^{1/2} \quad \text{since } T \text{ is normal} \\
&= ||T^{2^n}(T^{2^n})^*||^{1/2} \\
&= \left(||T^{2^n}||^2\right)^{1/2} \quad \text{by Lemma 20.2.19(ii) again} \\
&= ||T^{2^n}||.
\end{aligned}$$

Recall from (2) of Theorem 25.2.3 that $r(T) = \lim_{n\to\infty} ||T^n||^{1/n}$. Then the above shows that $r(T) \geq \lim_{n\to\infty} ||T^{2^n}||^{2^{-n}} = \lim_{n\to\infty} ||T|| = ||T||$ so we conclude that $r(T) = ||T||$. □

Proposition 25.4.4. *Let $T \in \mathrm{End}(H)$. Then*

(a) If T is unitary, then $\mathrm{sp}(T) \subseteq S^1$.

(b) If T is self-adjoint, then $\mathrm{sp}(T) \subseteq \mathbb{R}$.

Proof. (a) Note that in general, $\lambda \in \mathrm{sp}(T)$ if and only if $\lambda^{-1} \in \mathrm{sp}(T^{-1})$. So if T is unitary, meaning $TT^* = 1$, then it follows from Lemma 20.2.19(i) that $||T|| = ||T^{-1}|| = 1$. Thus if $\lambda \in \mathrm{sp}(T)$, then $|\lambda| \leq 1$, but at the same time $\lambda^{-1} \in \mathrm{sp}(T^{-1})$ implies $|\lambda^{-1}| \leq 1$. Hence $|\lambda| = 1$, or $\lambda \in S^1$.

(b) The operator

$$\exp(iT) = \sum_{n=0}^{\infty} \frac{(iT)^n}{n!}$$

is well-defined (the sum converges) and we have

$$(\exp(iT))^* = \sum_{n=0}^{\infty} \frac{((iT)^*)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-iT)^n}{n!} = \exp(-iT).$$

Therefore $\exp(iT)$ is unitary, so for $\lambda \in \mathrm{sp}(T)$, $\exp(i\lambda) \in \mathrm{sp}(\exp(iT)) \subseteq S^1$ by part (a), so we must have $|\exp(i\lambda)| = 1$ and therefore $\lambda \in \mathbb{R}$. □

Suppose A and B are complex Banach algebras, each with an involution $*$ that is conjugate-linear, anti-multiplicative and satisfies $||xx^*|| = ||x||^2$ for all $x \in A$ (resp. $x \in B$). Such an algebra is called a C^*-*algebra* and a $*$-morphism is an algebra homomorphism $\varphi : A \to B$ such that $\varphi(x^*) = (\varphi(x))^*$ for all $x \in A$.

Proposition 25.4.5. *Let A be a self-adjoint, unital, closed, commutative subalgebra of $\mathrm{End}(H)$. Then the Gelfand transform $\Gamma : A \to \mathcal{C}(\widehat{A})$ is an isometry and a $*$-isomorphism of $\mathbb{C}$-algebras with respect to the adjoint on A and complex conjutation on $\mathcal{C}(\widehat{A})$.*

Proof. Since A is commutative and self-adjoint, any $T \in A$ is normal. Thus by Proposition 25.4.3 and Theorem 25.3.5, $||T|| = r(T) = ||\widehat{T}||$, so Γ is an isometry. It remains to show Γ is surjective and is a $*$-morphism.

Notice that if $T \in A$ is self-adjoint, then for any $\gamma \in \widehat{A}$, $\widehat{T}(\gamma) = \gamma(T) \in \mathrm{sp}(T) \subseteq \mathbb{R}$ by Proposition 25.4.4(b). More generally, any $T \in A$ can be written $T = T_0 + iT_1$ for the self-adjoint operators $T_0 = \frac{T+T^*}{2}$ and $T_1 = \frac{T-T^*}{2i}$. Then $\Gamma(T_0), \Gamma(T_1) \in \mathcal{C}(A, \mathbb{R})$, so

$$\begin{aligned}
\Gamma(T^*) &= \Gamma((T_0 + iT_1)^*) \\
&= \Gamma(T_0^* - iT_1^*) \quad \text{by Lemma 20.2.19(iii)} \\
&= \Gamma(T_0 - iT_1) \quad \text{by self-adjointness} \\
&= \Gamma(T_0) - i\Gamma(T_1) \quad \text{by Theorem 25.3.5} \\
&= \overline{\Gamma(T_0) + i\Gamma(T_1)} \quad \text{since } \Gamma(T_0), \Gamma(T_1) \in \mathbb{R} \\
&= \overline{\Gamma(T)}.
\end{aligned}$$

Hence Γ respects the involutions on A and $\mathcal{C}(\widehat{A})$.

For surjectivity, recall from Theorem 25.3.5 that $\Gamma(A)$ separates points and is unital. Further, $\Gamma(A)$ is self-adjoint since Γ is a $*$-morphism. Finally, A is isometric and isomorphic as a complex algebra to $\Gamma(A) \subseteq \mathcal{C}(\widehat{A})$, but $A \subseteq \mathrm{End}(H)$ is closed which implies that $\Gamma(A) \subseteq \mathcal{C}(\widehat{A})$ is also closed. Hence by Corollary 25.4.2, $\Gamma(A) = \mathcal{C}(\widehat{A})$ so Γ is surjective. □

For a normal operator $T \in \mathrm{End}(H)$, let A_T denote the smallest subalgebra of $\mathrm{End}(H)$ containing T which is self-adjoint, unital, closed and commutative. Equivalently, A_T is the subalgebra of $\mathrm{End}(H)$ generated by $\{1, T, T^*\}$.

Theorem 25.4.6 (First Spectral Theorem)**.** *Let $T \in \mathrm{End}(H)$ be a normal operator. Then there is a map*

$$\Phi : \mathcal{C}(\mathrm{sp}(T)) \longrightarrow A_T$$

which is an isometry and a $*$-isomorphism of unitary $\mathbb{C}$-algebras. Further, if $i_T : \mathrm{sp}(T) \hookrightarrow \mathbb{C}$ is the natural inclusion, then $\Phi(i_T) = T$.

Proof. Consider the map $\Psi : \mathcal{C}(\mathrm{sp}(T)) \to \mathcal{C}(\widehat{A}_T)$ which sends $f \mapsto f \circ \widehat{T}$, which is well-defined since $\mathrm{im}\,\widehat{T} = \mathrm{sp}_{A_T}(T)$, the spectrum of T *in the subalgebra* A_T. Then to prove the theorem, we will show Ψ is an isometry and a $*$-isomorphism and $\mathrm{sp}_{A_T}(T) = \mathrm{sp}(T)$, so that we can define Φ by

$$\begin{tikzcd}
\mathcal{C}(\mathrm{sp}(T)) \arrow[r, "\Psi"] \arrow[dr, "\Phi"'] & \mathcal{C}(\widehat{A}_T) \arrow[d, "\Gamma"] \\
& A_T
\end{tikzcd}$$

since Γ is an isometry and a $*$-isomorphism by Proposition 25.4.5.

To show $\mathrm{sp}_{A_T}(T) = \mathrm{sp}(T)$, note that $\mathrm{sp}(T) \subseteq \mathrm{sp}_{A_T}(T)$ always holds. On the other hand, for $\lambda \in \mathrm{sp}_{A_T}(T)$, the Hahn-Banach theorem (in the form of Corollary 20.1.7) implies that there exists a function $f \in \mathcal{C}(\mathrm{sp}_{A_T}(T))$ satisfying $f(\lambda) = 1$, $||f|| = 1$ and $f \equiv 0$ outside an ε-neighborhood of λ, i.e. for some $\varepsilon > 0$, $f(\mu) = 0$ whenever $|\mu - \lambda| \geq \varepsilon$. Set $P = \Phi(f)$. Then for the inclusion $i : \mathrm{sp}_{A_T}(T) \hookrightarrow \mathbb{C}$, we have

$$||(T - \lambda 1_H)P|| = ||\Phi^{-1}((T - \lambda 1_H)P)|| = ||(i - \lambda)f|| \leq \varepsilon$$

since for any μ, $((i-\lambda)f)(\mu) = (\mu - \lambda)f(\mu)$. If $T - \lambda 1_H$ had an inverse in $\mathrm{End}(H)$, we would have

$$1 = ||P|| = ||(T - \lambda 1_H)^{-1}(T - \lambda 1_H)P|| \leq ||(T - \lambda 1_H)^{-1}||\varepsilon$$

by submultiplicativity of $||\cdot||$, but this would imply

$$\frac{1}{\varepsilon} \leq ||(T - \lambda 1_H)^{-1}||$$

for *all* $\varepsilon > 0$, which is impossible. Hence $T - \lambda 1_H$ is not a unit in $\mathrm{End}(H)$, so $\lambda \in \mathrm{sp}(T)$, which proves $\mathrm{sp}_{A_T}(T) \subseteq \mathrm{sp}(T)$.

Now to show Ψ is an isometry and a $*$-isomorphism, note that $\widehat{T} : \widehat{A}_T \to \mathrm{sp}_{A_T}(T) = \mathrm{sp}(T)$ is surjective and continuous by Proposition 25.3.1. Moreover, if $\widehat{T}(\gamma_1) = \widehat{T}(\gamma_2)$ for $\gamma_1, \gamma_2 \in \widehat{A}_T$, then $\gamma_1(T) = \gamma_2(T)$, which is equivalent to

$$\gamma_1(T^*) = \overline{\gamma_1(T)} = \overline{\gamma_2(T)} = \gamma_2(T^*)$$

since Γ is a $*$-morphism. By definition A_T is generated by $\{1, T, T^*\}$, so this implies that $\gamma_1 = \gamma_2$ on A_T, but since A_T is closed, $\gamma_1 = \gamma_2$ identically. Thus $\widehat{T}$ is injective, hence a continuous bijection. By Lemma 25.3.4, $\widehat{A}_T$ is compact, so $\widehat{T}$ is also a closed map and hence a homeomorphism. We have thus proven that Ψ is an isomorphism (and it's not to hard to show it preserves adjoints), so finally, notice that f and $f \circ \widehat{T}$ each take on the same values in $\mathbb{C}$. Therefore $||f|| = ||f \circ \widehat{T}||$, so Ψ is an isometry. □

25.5 Unitary Representations

Definition. *Let G be a locally compact group and $\rho : G \to \mathrm{Aut}(H)$ be a topological representation, where H is a Hilbert space. Then ρ is* **unitary** *if for all $g \in G$, ρ_g is unitary, i.e. $\rho_g^* = \rho_g^{-1}$.*

Notice that when ρ is a unitary representation, we have $\langle x, y\rangle = \langle \rho_g(x), \rho_g(y)\rangle$ for all $g \in G$ and $x, y \in H$.

Proposition 25.5.1. *Let H be a Hilbert space and $T \in \mathrm{End}(H)$ be a normal operator. Then the following are equivalent:*

(1) $\mathrm{sp}(T)$ is a singleton.

(2) $A_T \cong \mathbb{C}$ as C^*-algebras.

(3) $T = \lambda 1_H$ for some $\lambda \in \mathbb{C}$.

Proof. (1) $\implies$ (2) If $\mathrm{sp}(T) = *$, then $\mathcal{C}(\mathrm{sp}(T)) \cong \mathbb{C}$ so the spectral theorem (25.4.6) implies that $A_T \cong \mathbb{C}$.

(2) $\implies$ (3) If $A_T \cong \mathbb{C}$, then T may be viewed as $\lambda 1_H \in A_T$ for some $\lambda \in \mathbb{C}$.

(3) $\implies$ (1) For any $\mu \in \mathrm{sp}(T)$, $(\mu - \lambda)1_H = \mu 1_H - \lambda 1_H \notin \mathrm{End}(H)^\times$, but this is only possible when $\mu - \lambda = 0$, i.e. $\mu = \lambda$. Therefore λ is the only element of $\mathrm{sp}(T)$. □

Recall Schur's Lemma from representation theory.

Theorem 25.5.2 (Schur's Lemma)**.** *Let G be an abstract group and suppose $\rho : G \to \mathrm{Aut}(V)$ and $\rho' : G \to \mathrm{Aut}(V')$ are irreducible representations. Then any $T \in \mathrm{Hom}_G(V, V')$ is either trivial or a k-vector space isomorphism.*

This generalizes to the case of topological representations of locally compact groups as follows.

Theorem 25.5.3. *Suppose G is a locally compact group, H is a Hilbert space and $\rho : G \to \mathrm{Aut}(H)$ is a topological representation that is topologically irreducible and unitary. Then any normal operator $T \in \mathrm{End}_G(H)$ is of the form $T = \lambda 1_H$ for some $\lambda \in \mathbb{C}$. In particular, for every operator T, $TT^* = \lambda 1_H$ for some $\lambda \in \mathbb{C}$.*

Proof. For any $T \in \mathrm{End}_G(H)$, let T^* be the adjoint. Then for all $g \in G$ and $x, y \in H$,

$$\begin{aligned}
\langle \rho_g(x), T^*\rho_g(y)\rangle &= \langle T\rho_g(x), \rho_g(y)\rangle \\
&= \langle \rho_g(Tx), \rho_g(y)\rangle \quad \text{since } T \text{ is } G\text{-equivariant} \\
&= \langle Tx, y\rangle \quad \text{since } \rho \text{ is unitary} \\
&= \langle x, T^*y\rangle \quad \text{by adjunction} \\
&= \langle \rho_g(x), \rho_g(T^*y)\rangle \quad \text{by unitary again.}
\end{aligned}$$

In particular, for $x = 1_H$, this gives $\langle 1, T^*\rho_g(y)\rangle = \langle 1, \rho_g(Ty)\rangle$, but $\langle 1, \cdot\rangle$ is injective, so this implies $T^*\rho_g = \rho_g T^*$ for all $g \in G$. Hence T^* is G-equivariant. Since A_T is generated as a subalgebra of $\mathrm{End}(H)$ by $\{1, T, T^*\}$ and all of these are now G-equivariant, it follows that $A_T \subseteq \mathrm{End}_G(H)$.

Now take T to be normal and suppose $\lambda_1, \lambda_2 \in \mathrm{sp}(T)$ are distinct. Since $\mathrm{sp}(T)$ is Hausdorff, there are disjoint neighborhoods $U_1, U_2 \subseteq \mathrm{sp}(T)$ of λ_1 and λ_2, respectively. Choose functions $f_1, f_2 \in \mathcal{C}(\mathrm{sp}(T))$ such that for $i = 1, 2$, $f_i(\mathrm{sp}(T) \smallsetminus \{U_i\}) = 0$ and $f_i(\lambda_i) = 1$, again using Corollary 20.1.7 for example. Then $f_1, f_2 \neq 0$ but since $U_1 \cap U_2 = \varnothing$, $f_1 f_2 = 0$. Let $\Phi : \mathcal{C}(\mathrm{sp}(T)) \to A_T$ be the isomorphism from the spectral theorem (25.4.6). Then since $f_1 \neq 0$, $\Phi(f_1)(H)$ is nonzero. On the other hand, $\Phi(f_1) \in A_T \subseteq \mathrm{End}_G(H)$ by the first paragraph, so $\Phi(f_1)(H)$ is a nonzero, G-equivariant subspace of $\mathrm{End}_G(H)$ and by the same argument, so is its closure. Since ρ is topologically irreducible, this means $\overline{\Phi(f_1)(H)} = H$. Applying this again for $\Phi(f_2)$, we conclude that $\overline{\Phi(f_2)\Phi(f_1)(H)} = H$, but $\Phi(f_2 f_1)(H) = \Phi(0)(H) = \{0\}$, contradicting the fact that Φ is an algebra homomorphism. Hence $\mathrm{sp}(T)$ can only consist of one point, so Proposition 25.5.1 shows that $T = \lambda 1_H$ for some $\lambda \in \mathbb{C}$. □

Corollary 25.5.4. *Suppose G is a locally compact abelian group, H is a Hilbert space and $\rho : G \to \mathrm{Aut}(H)$ is a unitary, irreducible topological representation. Then $\dim_{\mathbb{C}}(H) = 1$.*

Proof. Because ρ is unitary, every $g \in G$ acts by a unitary normal operator $\rho_g \in \mathrm{End}(H)$, so Theorem 25.5.3 shows that $\rho_g = \chi(g) 1_H$ for some $\chi(g) \in \mathbb{C}$. In fact, $\chi(g) \in S^1$ by Proposition 25.4.4(a). Then for any $x \in H$, $\mathbb{C}x$ is a G-invariant, closed subspace of H so by irreducibility of ρ, $H = \mathbb{C}x$. □

Chapter 26

Duality

Let G be a topological abelian group and let S^1 be the unit circle in $\mathbb{C}$. The multiplicative group of characters

$$\widehat{G} = \{f : G \to S^1 \mid f \text{ is a continuous homomorphism}\}$$

is called the *Pontrjagin dual* of G. Endowed with the compact-open topology, $\widehat{G}$ becomes a topological group and one can prove the following properties:

Proposition 26.0.1. *For a topological abelian group G with Pontrjagin dual $\widehat{G}$,*

(1) If G is discrete, $\widehat{G}$ is compact.

(2) If G is compact, $\widehat{G}$ is discrete.

(3) If G is locally compact then so is $\widehat{G}$.

The Pontrjagin dual is the key ingredient in establishing the Fourier transform and proving the Pontrjagin duality theorem for locally compact groups.

26.1 Functions of Positive Type

Assume G is a locally compact abelian group with (left) Haar measure ds and set

$$\mathcal{C}_c(G) = \{f : G \to \mathbb{C} \mid f \text{ is continuous with compact support}\}.$$

Then $\mathcal{C}_c(G)$ is dense in $L^p(G)$ for all $1 \le p \le \infty$.

Definition. *A Haar measurable function $\varphi \in L^\infty(G)$ is of* **positive type** *if for all $f \in \mathcal{C}_c(G)$,*

$$\iint_{G\times G} \varphi(s^{-1}t) f(s)\, ds\, \overline{f(t)}\, dt \ge 0.$$

Let φ be a function of positive type. Then

$$\langle f_1, f_2\rangle_\varphi = \iint_{G\times G} \varphi(s^{-1}t) f_1(s)\, ds\, \overline{f_2(t)}\, dt$$

defines a sesquilinear form on $\mathcal{C}_c(G)$. Set $W_\varphi = \{f \in \mathcal{C}_c(G) \mid \langle f, f\rangle_\varphi = 0\}$.

Lemma 26.1.1. *For all functions φ of positive type on G, W_φ is a vector subspace of $\mathcal{C}_c(G)$ and $\langle \cdot, \cdot\rangle_\varphi$ descends to a positive definite, Hermitian form on the the quotient $\mathcal{C}_c(G)/W_\varphi$.*

Let V_φ be the completion of the normed space $(\mathcal{C}_c(G)/W_\varphi, \langle \cdot, \cdot\rangle_\varphi)$. By abuse of notation we will also denote the extension of $\langle \cdot, \cdot\rangle_\varphi$ to this completion by $\langle \cdot, \cdot\rangle_\varphi$.

Proposition 26.1.2. *For every function φ of positive type on G, V_φ is a Hilbert space.*

Now for $f : G \to \mathbb{C}$ and $s \in G$, define the function $L_s f : G \to \mathbb{C}$ by $L_s f(t) = f(s^{-1}t)$.

Lemma 26.1.3. *For any $f : G \to \mathbb{C}$ and $s \in G$,*

(a) If $f \in \mathcal{C}_c(G)$ then $L_s f \in \mathcal{C}_c(G)$.

(b) If φ is a function of positive type and $f \in \mathcal{C}_c(G)$, then $\langle L_s f, L_s f\rangle_\varphi = \langle f, f\rangle_\varphi$.

(c) The assignment $G \to \mathcal{C}_c(G), s \mapsto L_s f$ is continuous for each $f \in \mathcal{C}_c(G)$.

Proof. (a) and (c) are routine. For (b), we have

$$\begin{aligned}
\langle L_s f, L_s f\rangle_\varphi &= \iint_{G\times G} \varphi(t^{-1}u) f(s^{-1}t)\, dt\, \overline{f(s^{-1}u)}\, du \\
&= \iint_{G\times G} \varphi((s^{-1}t)^{-1}(s^{-1}u)) f(s^{-1}t)\, dt\, \overline{f(s^{-1}u)}\, du \\
&= \iint_{G\times G} \varphi(t^{-1}u) f(t)\, dt\, \overline{f(u)}\, du \quad \text{by left-invariance of Haar measure} \\
&= \langle f, f\rangle_\varphi.
\end{aligned}$$

□

Theorem 26.1.4. *Let G be a locally compact group and φ a function of positive type on G. Then $s \mapsto L_s$ induces a unitary representation of G on V_φ.*

Proof. Lemma 26.1.3 implies that $s \mapsto L_s$ is a unitary representation of G abstractly, so it will suffice to show it is also a topological representation. By Corollary 25.1.6, it's even enough to show that for each $f \in \mathcal{C}_c(G)$, $s \mapsto L_s f$ is continuous, but this can be shown by normal analytical methods (see Ramakrishnan-Valenza for the proof). $\square$

Definition. *Let f and g be complex-valued Borel functions on a locally compact topological group G, equipped with a (left) Haar measure ds. Then the* **convolution** *of f and g is the function*

$$f * g(t) := \int_G g(s^{-1}t)f(s)\,ds = \int_g g(s^{-1})f(ts)\,ds.$$

Proposition 26.1.5. *Let G be a locally compact abelian group. Then*

(i) If $f * g(x)$ exists for some $x \in G$, then $g * f(x)$ exists and $f * g(x) = g * f(x)$.

(ii) If $f, g \in L^1(G)$ then $f * g(x)$ exists for almost all $x \in G$. Moreover, $||f * g||_1 \le ||f||_1||g||_1$ so in particular $f * g \in L^1(G)$.

(iii) For $f, g, h \in L^1(G)$, $(f * g) * h = f * (g * h)$.

Proof. Straightforward from the definitions. $\square$

Corollary 26.1.6. *$L^1(G)$ is a Banach algebra with respect to $*$.*

We will mainly be interested in convolutions of functions $f \in \mathcal{C}_c(G)$ and $\varphi \in L^\infty(G)$ of positive type. In this case, $f * \varphi$ exists everywhere and is continuous.

Theorem 26.1.7. *Let φ be a function of positive type on G. Then there exists $x_\varphi \in V_\varphi$ such that $\varphi(s) = \langle x_\varphi, L_s x_\varphi\rangle_\varphi$ for almost all $s \in G$.*

Proof. Let $\{U_\alpha\}$ be a system of open neighborhoods of $e \in G$. Since G is Hausdorff, $\bigcap_\alpha U_\alpha = \{e\}$. The index set $\{\alpha\}$ is a directed set under the partial ordering defined by $\alpha \le \beta$ if $U_\beta \subseteq U_\alpha$. By Urysohn's lemma for locally compact spaces, for each α there exists a continuous function $g_\alpha : G \to \mathbb{R}_+$ such that the support of g_α is a compact subset of U_α and $\int_G g_\alpha(s)\,ds = 1$. This defines a net $\{g_\alpha\,ds\}_\alpha$ of positive linear functionals on $\mathcal{C}_c(G)$; explicitly, $f \mapsto \int_G f(s)g_\alpha(s)\,ds$. These functionals weakly converge to the Dirac measure $\delta_e : f \mapsto f(e)$. Let $f \in \mathcal{C}_c(G)$. Then for any α, Fubini's theorem (18.4.3) gives

$$\iint_{G\times G} \varphi(s^{-1}t)f(s)\,ds\,g_\alpha(t)\,dt = \int_G (f * \varphi)(t)g_\alpha(t)\,dt$$

which exists because $f * \varphi$ is continuous and g_α has compact support. Define

$$\Phi(f) := \lim_\alpha \langle f, g_\alpha\rangle_\varphi = \lim_\alpha \int_G (f * \varphi)(t)g_\alpha(t)\,dt.$$

This determines a linear form Φ on V_φ which, after replacing $f * \varphi$ by $(f * \varphi)h$ for a function h with compact support and such that $h \equiv 1$ on a neighborhood eventually containing the support of g_α, is of the form

$$\Phi(f) = (f * \varphi)(e) = \int_G \varphi(s^{-1})f(s)\,ds. \tag{26.1}$$

Since V_φ is a Hilbert space, it is reflexive by the Riesz representation theorem (20.2.13), meaning there is some $x_\varphi \in V_\varphi$ such that $\Phi(\xi) = \langle \xi, x_\varphi \rangle_\varphi$ for all $\xi \in V_\varphi$. Then $\{g_\alpha\}$ converges weakly to x_φ in V_α, so for any $\xi \in V_\varphi$ and $s \in G$ we have

$$\begin{aligned}\langle \xi, L_s x_\varphi \rangle_\varphi &= \lim_\alpha \langle \xi, L_s x_\varphi \rangle_\varphi \\ &= \lim_\alpha \iint_{G\times G} \varphi(t^{-1}u)\xi(t)\,dt\, g_\alpha(s^{-1}u)\,du \\ &= \int_G \varphi(t^{-1}s)\xi(t)\,dt \quad \text{by (1).}\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle L_s x_\varphi, \xi \rangle &= \lim_\alpha \langle L_s g_\alpha, \xi \rangle_\varphi \\ &= \lim_\alpha \iint_{G\times G} \varphi(t^{-1}u) g_\alpha(s^{-1}t)\,dt\,\overline{\xi(u)}\,du \\ &= \int_G \varphi(s^{-1}u)\overline{\xi(u)}\,du \quad \text{by (1).}\end{aligned}$$

Combining these we get

$$\langle \xi, L_s x_\varphi \rangle_\varphi = \int_G \varphi(t^{-1}s)\xi(t)\,dt = \int_G \overline{\varphi(s^{-1}t)}\xi(t)\,dt \tag{26.2}$$

and in particular for $s = e$,

$$\langle \xi, x_\varphi \rangle_\varphi = \int_G \overline{\varphi(t)}\xi(t)\,dt. \tag{26.3}$$

Now for any $h \in \mathcal{C}_c(G)$, consider

$$\begin{aligned}\langle \xi, h \rangle_\varphi &= \iint_{G\times G} \varphi(s^{-1}t)\xi(s)\,ds\,\overline{h(t)}\,dt \\ &= \int_G \langle \xi, L_t x_\varphi \rangle_\varphi \overline{h(t)}\,dt \quad \text{by (2).}\end{aligned}$$

Extend this by continuity to all of V_φ and consider the $\mathbb{C}G$-submodule V' of V_φ generated by x_φ. If $\xi \in V'$ for some $\xi \in V_\varphi$, then the above shows $\langle \xi, L_t x_\varphi \rangle_\varphi = 0$ for all $t \in G$, so $\xi \equiv 0$. Hence $V' = V_\varphi$. Now taking $\xi = x_\varphi$ in (3) shows that for all $\psi \in V_\varphi$,

$$\int_G \varphi(s)\overline{\psi(s)}\,ds = \langle x_\varphi, \psi \rangle_\varphi = \int_G \langle x_\varphi, L_s x_\varphi \rangle_\varphi \overline{\psi(s)}\,ds.$$

Hence $\varphi(s) = \langle x_\varphi, L_s x_\varphi \rangle_\varphi$ for almost all $s \in G$. $\square$

Corollary 26.1.8. *Let φ be a function of positive type on G. Then φ is equal almost everywhere to a continuous function of positive type on G. If, moreover, φ is itself continuous, then*

(i) $\varphi(e) \geq 0$*, where* $e \in G$ *is the identity.*

(ii) $\varphi(e) = \sup_{s\in G} |\varphi(s)|$.

(iii) For all $s \in G$, $\varphi(s^{-1}) = \overline{\varphi(s)}$.

Proof. By Theorem 26.1.7, $\varphi(s) = \langle x_\varphi, L_s x_\varphi\rangle_\varphi$ a.e. for some $x_\varphi \in V_\varphi$, but the latter is continuous. Now assume φ is continuous.

(i) Since $\langle \cdot, \cdot\rangle_\varphi$ is positive definite on V_φ, $\varphi(e) = \langle x_\varphi, L_e x_\varphi\rangle_\varphi = \langle x_\varphi, x_\varphi\rangle_\varphi \geq 0$.

(ii) For any $s \in G$, consider

$$\begin{aligned}
|\varphi(s)|^2 &= |\langle x_\varphi, L_s x_\varphi\rangle_\varphi|^2 \\
&\leq |\langle x_\varphi, x_\varphi\rangle_\varphi|\,|\langle L_s x_\varphi, L_s x_\varphi\rangle_\varphi| \quad \text{by Cauchy-Schwarz (20.2.4)} \\
&= \langle x_\varphi, x_\varphi\rangle_\varphi \langle x_\varphi, x_\varphi\rangle_\varphi \quad \text{by Lemma 26.1.3(b)} \\
&= (\langle x_\varphi, x_\varphi\rangle_\varphi)^2 = \varphi(e)^2.
\end{aligned}$$

Taking the square root of both sides, we get $\varphi(e) = \sup_{s\in G} |\varphi(s)|$.

(iii) For $s \in G$,

$$\begin{aligned}
\varphi(s^{-1}) &= \langle x_\varphi, L_{s^{-1}} x_\varphi\rangle_\varphi \\
&= \langle L_s x_\varphi, x_\varphi\rangle_\varphi \quad \text{by Theorem 26.1.4} \\
&= \overline{\langle x_\varphi, L_s x_\varphi\rangle_\varphi} \quad \text{by Hermitian property} \\
&= \overline{\varphi(s)}.
\end{aligned}$$

□

Set $\mathcal{P}(G) = \{\varphi : G \to \mathbb{C} \mid \varphi \text{ is continuous, of positive type and } ||\varphi||_\infty \leq 1\}$. Observe that for any φ of positive type, if $||\varphi||_\infty \leq 1$ then $\varphi(e) \leq 1$ by Corollary 26.1.8(ii).

Definition. *We say a function $\varphi \in \mathcal{P}(G)$ is* **elementary** *if $\varphi(e) = 1$ and for any decomposition $\varphi = \varphi_1 + \varphi_2$, with $\varphi_1, \varphi_2 \in \mathcal{P}(G)$, there exist scalars $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$ satisfying $\lambda_1 + \lambda_2 = 1$, $\varphi_1 = \lambda_1\varphi$ and $\varphi_2 = \lambda_2\varphi$. Let $\mathcal{E}(G)$ be the set of all elementary functions on G, together with the zero map.*

Theorem 26.1.9. *Let φ be a continuous function of positive type on G satisfying $\varphi(e) = 1$. Then $\varphi \in \mathcal{E}(G)$ if and only if the unitary representation $s \mapsto L_s$ of G into V_φ is irreducible.*

Theorem 26.1.10. *Let G be a locally compact abelian group. Then the elementary functions of positive type on G are precisely the continuous characters of G, i.e. $\mathcal{E}(G) \smallsetminus \{0\} = \widehat{G}$.*

Proof. Given φ of positive type on G such that $\varphi(e) = 1$, consider the following two conditions:

(i) The unitary representation of G on V_φ given by $s \mapsto L_s$ is irreducible.

(ii) φ is a character of G.

By Theorem 26.1.10, showing that (i) and (ii) are equivalent will imply the statement of this theorem.

(ii) $\implies$ (i) Take $\varphi \in \widehat{G}$ and $f \in \mathcal{C}_c(G)$. Then

$$\begin{aligned}\langle f, f\rangle_\varphi &= \iint_{G\times G} \varphi(s^{-1}t) f(s)\, ds\, \overline{f(t)}\, dt \\ &= \left| \int_G \overline{\varphi(s)} f(s)\, ds \right|^2\end{aligned}$$

by Fubini's theorem (18.4.3), which shows that W_φ has codimension 1 in $\mathcal{C}_c(G)$ and hence $\dim V_\varphi = 1$. Since G is abelian, V_φ is an irreducible G-module.

(i) $\implies$ (ii) By Corollary 25.5.4, if the unitary representation $s \mapsto L_s$ is irreducible, it is one-dimensional. So for all $\xi \in V_\varphi$, $L_s(\xi) = \lambda(s)\xi$ for λ a continuous function of s. Since L_s is unitary, Proposition 25.4.4 shows that $||L_s|| = 1$, which implies $|\lambda(s)| = 1$, and thus λ is a character of G. Finally, for all $s \in G$,

$$\begin{aligned}\varphi(s) &= \langle x_\varphi, L_s x_\varphi\rangle_\varphi \\ &= \overline{\lambda(s)} \langle x_\varphi, x_\varphi\rangle_\varphi \\ &= \overline{\lambda(s)}\varphi(e) = \overline{\lambda(s)}.\end{aligned}$$

Hence $\varphi(s)$ is a character of G. $\square$

26.2 Fourier Inversion

Let G be a locally compact abelian group with (bi-invariant) Haar measure dx and character group $\widehat{G}$.

Definition. *The* **Fourier transform** *of a function $f \in L^1(G)$ is the function $\hat{f} : \widehat{G} \to \mathbb{C}$ defined by*

$$\hat{f}(\chi) = \int_G f(y)\overline{\chi}(y)\, dy$$

for all $\chi \in \widehat{G}$.

Note that $|\hat{f}(\chi)| \leq ||f||_1$ for all $\chi \in \widehat{G}$.

Example 26.2.1. Let $G = \mathbb{R}$. Then each $t \in \mathbb{R}$ may be identified with a group character $s \mapsto e^{ist}$. Then the Fourier transform of any $f \in L^1(\mathbb{R})$ is the standard Fourier transform:

$$\hat{f}(t) = \int_{\mathbb{R}} f(s)e^{-ist}\, ds.$$

Let $V(G)$ denote the space of continuous functions of positive type in $\mathcal{C}_c(G)$ and set $V^1(G) = V(G) \cap L^1(G)$. The goal of this section is to prove the Fourier inversion formula:

Theorem 26.2.2 (Fourier Inversion Formula)**.** *Let G be a locally compact abelian group with Haar measure dx. Then there exists a Haar measure $d\chi$ on $\widehat{G}$ which satisfies*

$$f(y) = \int_{\widehat{G}} \hat{f}(\chi)\chi(y)\, d\chi$$

for all $f \in V^1(G)$. Moreover, the assignment $f \mapsto \hat{f}$ defines a bijection $V^1(G) \cong V^1(\widehat{G})$.

Definition. *The measure $d\chi$ on $\widehat{G}$ is called the* **dual measure** *to dx.*

To prepare for the proof of the Fourier inversion formula, we relate the Fourier and Gelfand transforms by the following result. Let $B = L^1(G)$ and let $\widehat{B} = \mathrm{Hom}_{\mathbb{C}}(B, \mathbb{C})^\times$ be the space of complex characters of B. For $\chi \in \widehat{G}$ and $f \in L^1(G)$, define

$$\hat{\nu}_\chi(f) := \hat{f}(\chi) = \int_G f(y)\overline{\chi(y)}\, dy.$$

Proposition 26.2.3. *For each $\chi \in \widehat{G}$, $\hat{\nu}_\chi \in \widehat{B}$ and the assignment*

$$\begin{aligned} \widehat{G} &\longrightarrow \widehat{B} \\ \chi &\longmapsto \hat{\nu}_\chi \end{aligned}$$

is a bijection.

Definition. *The* **ring of Fourier transforms** *of G is $\widehat{A}(G) = \{\hat{f} \mid f \in L^1(G)\}$.*

By Proposition 26.2.3, each Fourier transform $\hat{f} \in \widehat{A} = \widehat{A}(G)$ can be viewed as the Gelfand transform of f. Explicitly,

$$\hat{f}(\hat{\nu}_\chi) := \hat{f}(\chi) = \hat{\nu}_\chi(f).$$

Let $\widehat{G}$ have the transform topology induced by $\widehat{A}$, i.e. the weakest topology with respect to which each $\hat{f} \in \widehat{A}$ is continuous. Also, let $\mathcal{C}_0(\widehat{G})$ denote the $\mathbb{C}$-algebra of rapidly-decaying maps on G, or equivalently, the space of continuous functions on the one-point compactification of $\widehat{G}$ which are 0 at the point at infinity.

Proposition 26.2.4. *The ring of Fourier transforms $\widehat{A} = \widehat{A}(G)$ separates points and is a self-adjoint, dense subalgebra of $\mathcal{C}_0(\widehat{G})$.*

Moving towards the proof of Theorem 26.2.2, we now discuss Fourier transforms of character measures. For a locally compact group G with character group $\widehat{G}$, let $\hat{\mu}$ be a Radon measure on $\widehat{G}$ with finite total mass, that is, $\hat{\mu}(\widehat{G}) < \infty$. A standard analysis result is:

Lemma 26.2.5. *On a locally compact, Hausdorff space X, there is a bijective correspondence between finite Radon measures μ and linear functionals $f \mapsto \int_X f\,d\mu$ on $\mathcal{C}_0(X)$.*

Definition. *For a finite Radon measure $\hat{\mu}$ on $\widehat{G}$, the* **Fourier transform** *of $\hat{\mu}$ is the function $T_{\hat{\mu}} : G \to \mathbb{C}$ defined for each $y \in G$ by*

$$T_{\hat{\mu}}(y) := \int_{\widehat{G}} \chi(y)\,d\hat{\mu}(\chi).$$

Lemma 26.2.6. *For any finite Radon measure $\hat{\mu}$,*

(a) The Fourier transform $T_{\hat{\mu}}$ is continuous and bounded on G.

(b) For all $f \in L^1(G)$,

$$\int_{\widehat{G}} \overline{\hat{f}(\chi)}\,d\hat{\mu}(\chi) = \int_G \overline{f(y)}T_{\hat{\mu}}(y)\,dy.$$

Proof. (a) Continuity is clear. Boundedness follows from the fact that $T_{\hat{\mu}}(y) \le \hat{\mu}(\widehat{G})$ for all $y \in G$.

(b) By Fubini's theorem (18.4.3) and the definitions of $\hat{f}$ and $T_{\hat{\mu}}$,

$$\begin{aligned}\int_{\widehat{G}} \overline{\hat{f}(\chi)}\,d\hat{\mu}(\chi) &= \iint_{G\times\widehat{G}} \overline{f(y)}\chi(y)\,dy\,d\hat{\mu}(\chi)\\ &= \iint_{\widehat{G}\times G} \overline{f(y)}\chi(y)\,d\hat{\mu}(\chi)\,dy\\ &= \int_G \overline{f(y)}T_{\hat{\mu}}(y)\,dy.\end{aligned}$$

□

Proposition 26.2.7. *Let $\hat{\mu}$ be a finite Radon measure on $\widehat{G}$. If $T_{\hat{\mu}}(y) = 0$ for all $y \in G$, then $\hat{\mu} = 0$. That is, $\hat{\mu}$ is completely determined by its Fourier transform.*

Proof. Suppose $T_{\hat{\mu}}(y) = 0$ for all $y \in G$. Then by Lemma 26.2.6(b),

$$\int_{\widehat{G}} \overline{\hat{f}(\chi)}\, d\hat{\mu}(\chi) = \int_G \overline{f(y)} T_{\hat{\mu}}(y)\, dy = 0$$

for all $f \in L^1(G)$. Since the ring of Fourier transforms $\widehat{A} = \widehat{A}(G)$ is dense in $\mathcal{C}_0(G)$ by Proposition 26.2.4, this implies that

$$\int_{\widehat{G}} g(\chi) d\hat{\mu}(\chi) = 0$$

for all continuous functions $g : \widehat{G} \to \mathbb{C}$ with compact support. Finally, Lemma 26.2.5 shows that $\hat{\mu} = 0$. □

As in Section 26.1, let $\mathcal{P}(G)$ be the set of continuous functions of positive type on G with norm at most 1.

Theorem 26.2.8 (Bochner)**.** *Let G be a locally compact abelian group. The functions in $\mathcal{P}(G)$ are precisely the Fourier transforms of Radon measures $\hat{\mu}$ on $\widehat{G}$ with finite total mass $\hat{\mu}(\widehat{G}) \leq 1$.*

Proof. (Sketch) Let $\widehat{M} = \{\hat{\mu} \mid \hat{\mu} \text{ is a Radon measure on } \widehat{G}, \hat{\mu}(\widehat{G}) \leq 1\}$. If $\hat{\mu} \in \widehat{M}$ is a point-measure of total mass 1 concentrated at some $\chi \in \widehat{G}$, then for any $y \in G$, the Fourier transform of $\hat{\mu}$ can be written

$$T_{\hat{\mu}}(y) = \int_G \chi(y)\, d\hat{\mu}(\chi) = \chi(y).$$

Thus the Fourier transform of $\hat{\mu}$ is the character χ which is a function of positive type on G such that $||\chi||_\infty \leq 1$, by Theorem 26.1.10. The general case is obtained by taking weakly convergent limits of point-measures of total mass 1.

Conversely, by Lemma 26.2.6(a), the Fourier transform is a continuous map $\widehat{M} \to \mathcal{P}(G)$. Then the same argument using weakly convergent limits of point-measures can be used to show that the image of $\widehat{M}$ is (weakly) compact, hence closed in $\mathcal{P}(G)$. Finally, one observes that the image of $\widehat{M}$ in $\mathcal{P}(G)$ is convex and contains $\widehat{G} \cup \{0\}$, and then the characterization of elementary functions as extreme points of $\mathcal{P}(G)$, together with Theorem 26.1.10, will imply that this image is all of $\mathcal{P}(G)$. □

Let G be a locally compact abelian group and set $V = V(G)$, the complex vector space of continuous functions of positive type on G. Then Corollary 26.1.8(ii) implies the functions of V are bounded. Put $V^1 = V^1(G) = V \cap L^1(G)$.

Corollary 26.2.9. *Each function $f \in V$ uniquely determines a Radon measure $\hat{\mu}_f$ of finite total mass on $\widehat{G}$ such that f is the Fourier transform of $\hat{\mu}_f$.*

Proof. Existence is given by Bochner's theorem, while uniqueness is guaranteed by Proposition 26.2.7. □

As a result, we may view any function $f \in V$ as $f(y) = \int_{\widehat{G}} \chi(y)\, d\hat{\mu}_f(\chi)$.

Lemma 26.2.10. *There exists a net of functions $\{f\}$ on $V^1 = V^1(G)$ such that the associated sequence of Fourier transforms $\{\hat{f}\}$ converges uniformly to the constant function 1 on all compact subsets of $\widehat{G}$.*

Lemma 26.2.11. *Let $f, g \in V^1$. Then $\hat{g}\, d\hat{\mu}_f = \hat{f}\, d\hat{\mu}_g$ as measures on $\widehat{G}$.*

Proof. By Proposition 26.2.7, it's enough to show the equality on the corresponding Fourier transforms. For any $y \in G$, consider

$$\begin{aligned}
T_{\hat{g}\, d\hat{\mu}_f}(y) = \int_{\widehat{G}} \chi(y)\hat{g}(\chi)\, d\hat{\mu}_f(\chi) &= \iint_{G\times\widehat{G}} \chi(y)g(z)\overline{\chi(z)}\, dz\, d\hat{\mu}_f(\chi) \quad \text{by definition of } \hat{g} \\
&= \iint_{\widehat{G}\times G} \chi(y)g(z)\overline{\chi(z)}\, d\hat{\mu}_f(\chi)\, dz \quad \text{by Fubini's theorem (18.4.3)} \\
&= \iint_{\widehat{G}\times G} \chi(z^{-1}y)g(z)\, d\hat{\mu}_f(\chi)\, dz \quad \text{after a change of variables} \\
&= \int_G f(z^{-1}y)g(z)\, dz \quad \text{by Corollary 26.2.9}
\end{aligned}$$

but this equals $f * g$, the convolution of f and g. Since $f * g$ is symmetric with respect to f and g, this implies $T_{\hat{g}\, d\hat{\mu}_f} = T_{\hat{f}\, d\hat{\mu}_g}$. □

Let $\mathcal{F}$ be the set of bounded continuous functions $\varphi : \widehat{G} \to \mathbb{C}$ for which there exists a Radon measure $\hat{\nu}_\varphi$ on $\widehat{G}$ with finite total mass that satisfies $\varphi\, d\hat{\mu}_f = \hat{f}\, d\hat{\nu}_\varphi$ for all $f \in V^1$. Then Lemma 26.2.11 shows that the Fourier transforms of the functions in V^1 lie in $\mathcal{F}$. In particular, $\mathcal{F}$ is nonempty.

Lemma 26.2.12. *Let $\varphi \in \mathcal{F}$. Then*

(i) The associated measure $\hat{\nu}_\varphi$ is unique.

(ii) If $\varphi = \hat{f}$ for some $f \in L^1(G)$, then $\hat{\nu}_\varphi = \hat{\mu}_f$, where $\hat{\mu}_f$ is the unique Radon measure corresponding to f in Corollary 26.2.9.

(iii) If φ is positive, then $\hat{\nu}_\varphi$ is positive.

(iv) Let $\mathcal{C}_B(\widehat{G})$ be the ring of bounded continuous functions on $\widehat{G}$. Then $\mathcal{F}$ is a $\mathcal{C}_B(\widehat{G})$-module and the map $\varphi \mapsto \hat{\nu}_\varphi$ gives a module homomorphism of $\mathcal{F}$ into the space of complex Radon measures on $\widehat{G}$ of finite total mass.

(v) Every translation of φ lies in $\mathcal{F}$.

Proof. (i) Let $\{f\}$ be as in Lemma 26.2.10. Then

$$d\hat{\nu}_\varphi = \lim_f \varphi\, d\hat{\mu}_f$$

and the $\hat{\mu}_f$ are unique by Corollary 26.2.9, so this implies $\hat{\nu}_\varphi$ is unique.

(ii) This already holds for $f \in V^1$ by the paragraph proceeding this lemma, and now (i) implies the property for all $f \in L^1(G)$.

(iii) This uses the same argument as in (i).

(iv) Again, use Lemma 26.2.10 and the fact that limits are linear.

(v) For any measure μ, element $z \in \widehat{G}$ and subset $E \subseteq \widehat{G}$, set $\mu^z(E) = \mu(z^{-1}E)$. To prove the statement fix $\chi_0 \in \widehat{G}$ and suppose $\psi(\chi) = \varphi(\chi_0^{-1}\chi)$. Then for all $h \in \mathcal{C}_c(G)$ and $f \in L^1(G)$,

$$\int_{\widehat{G}} h(\chi)\psi(\chi)\, d\hat{\mu}_f(\chi) = \int_{\widehat{G}} h(\chi)\varphi(\chi_0^{-1}\chi)\, d\hat{\mu}_f(\chi) = \int_{\widehat{G}} h(\chi_0\chi)\varphi(\chi)\, d\hat{\mu}_f^{\chi_0^{-1}}(\chi)$$

by a change of variables. We claim that $d\hat{\mu}_f^{\chi_0^{-1}} = d\hat{\mu}_{\chi_0^{-1}f}$. Indeed, by Bochner's theorem (26.2.8),

$$\begin{aligned} f(y) &= \int_{\widehat{G}} \chi(y)\, d\hat{\mu}_f(\chi) = \int_{\widehat{G}} (\chi_0\chi)(y)\, d\hat{\mu}_f(\chi_0\chi) \\ \text{so} \quad \chi_0^{-1}f(y) &= \int_{\widehat{G}} \chi(y)\, d\hat{\mu}_f^{\chi_0^{-1}}(\chi) \end{aligned}$$

but by uniqueness of $\hat{\mu}_{\chi_0^{-1}f}$, this proves $d\hat{\mu}_{\chi_0^{-1}f} = d\hat{\mu}_f^{\chi_0^{-1}}$. Now continuing with the above computation, we have

$$\begin{aligned} \int_{\widehat{G}} h(\chi)\psi(\chi)\, d\hat{\mu}_f(\chi) &= \int_{\widehat{G}} h(\chi_0\chi)\varphi(\chi)\, d\hat{\mu}_{\chi_0^{-1}f}(\chi) \\ &= \int_{\widehat{G}} h(\chi_0\chi)(\chi_0\hat{f})(\chi) d\hat{\nu}_\varphi(\chi) \quad \text{by } \varphi \in \mathcal{F} \\ &= \int_{\widehat{G}} h(\chi_0\chi)\hat{f}(\chi_0\chi)\, d\hat{\nu}_\varphi(\chi) \quad \text{by definition of } \hat{f} \\ &= \int_{\widehat{G}} h(\chi)\hat{f}(\chi)\, d\hat{\nu}_\varphi^{\chi_0}(\chi) \quad \text{by a change of variables.} \end{aligned}$$

Hence $\psi\, d\hat{\mu}_f = \hat{f}\, d\hat{\nu}_\varphi^{\chi_0}$ for all $f \in L^1(G)$, but $d\hat{\nu}_\varphi^{\chi_0} = d\hat{\nu}_\psi$, so we get $\psi \in \mathcal{F}$ as desired. □

We now prove the main statement in the Fourier inversion formula (Theorem 26.2.2).

Theorem 26.2.13. *Let G be a locally compact abelian group. Then there exists a Haar measure $d\chi$ on $\widehat{G}$ such that for all $f \in V^1(G)$,*

$$f(y) = \int_{\widehat{G}} \hat{f}(y)\chi(y)\, d\chi.$$

Proof. By Corollary 26.2.9, any $f \in V^1 = V^1(G)$ can be written

$$f(y) = \int_{\widehat{G}} \chi(y)\, d\hat{\mu}_f(\chi)$$

so it will suffice to show $d\hat{\mu}_f = \hat{f}\, d\chi$ as measures on $\widehat{G}$.

We first show that $\mathcal{C}_c(\widehat{G}) \subseteq \mathcal{F}$. Take $\psi \in \mathcal{C}_c(\widehat{G})$ and let $K \subseteq \widehat{G}$ be a compact set containing the support of ψ. Using Lemma 26.2.10, one can construct a function $f \in V^1$ such that $\hat{f}$ is bounded away from 0 on K. Then $a = \frac{\psi}{\hat{f}}$ is a bounded, continuous function on K. Extend a to all of $\widehat{G}$ by setting $a \equiv 0$ on the complement of K. Then $a \in \mathcal{C}_B(\widehat{G})$, and $\hat{f} \in \mathcal{F}$ from before, so by Lemma 26.2.12(iv), $\psi = \hat{f}a \in \mathcal{F}$. Thus $\mathcal{C}_c(\widehat{G}) \subseteq \mathcal{F}$.

Next, define a map

$$\begin{aligned} \eta : \mathcal{C}_c(\widehat{G}) &\longrightarrow \mathbb{C} \\ \varphi &\longmapsto \int_{\widehat{G}} 1 \, d\hat{\nu}_\varphi(\chi). \end{aligned}$$

Since any $\varphi \in \mathcal{C}_c(\widehat{G})$ is also in $\mathcal{F}$, this is well-defined. We claim η is a nonzero linear functional. If $f \in V^1$ is not identically zero, then Corollary 26.2.9 implies $\hat{\mu}_f$ is a nonzero measure. Thus there exists some $a \in \mathcal{C}_B(\widehat{G})$ such that $a \, d\hat{\mu}_f \neq 0$. Take $\psi = a\hat{f}$, so that by the Radon-Nikodym derivative formula (Theorem 19.2.6), $d\hat{\nu}_\psi = a \, d\hat{\mu}_f$. Then by the preceding observation, $d\hat{\nu}_\psi \neq 0$, so η is nonzero. Linearity of η is given by Lemma 26.2.12(iv).

Now, the correspondence between Radon measures and linear functionals in Lemma 26.2.5 shows that η determines a Radon measure $d\chi$ of finite total mass on $\widehat{G}$. Moreover, since $\hat{\nu}_\varphi$ is positive for all functions φ of positive type (by Lemma 26.2.12(iii)), it follows that $d\chi$ is a positive Radon measure. To show $d\chi$ is in fact a Haar measure on $\widehat{G}$, it will suffice to show η is left-invariant. For any $\chi_0 \in \widehat{G}$, let L_{χ_0} be the left-translation operator $\psi \mapsto \chi_0\psi$. Then we have

$$\begin{aligned} \eta(L_{\chi_0}\psi) &= \int_{\widehat{G}} 1 \, d\hat{\nu}_{L_{\chi_0}\psi}(\chi) \\ &= \int_{\widehat{G}} 1 \, d\hat{\nu}_\psi^{\chi_0}(\chi) \quad \text{by Lemma 26.2.12(v)} \\ &= \int_{\widehat{G}} L_{\chi_0^{-1}} \, d\hat{\nu}_\psi(\chi) \quad \text{by a change of variables} \\ &= \int_{\widehat{G}} 1 \, d\hat{\nu}_\psi(\chi) \quad \text{since } L_{\chi_0^{-1}} \text{ is a homeomorphism} \\ &= \eta(\psi). \end{aligned}$$

Hence η is left-invariant, so it follows that $d\chi$ is a Haar measure. Explicitly, this satisfies

$$\int_{\widehat{G}} \psi(\chi) \, d\chi = \int_{\widehat{G}} 1 \, d\hat{\nu}_\psi(\chi)$$

for all $\psi \in \mathcal{C}_c(\widehat{G})$.

Finally, we show the Fourier inversion formula. For $\varphi \in \mathcal{F}$ and $a \in \mathcal{C}_c(\widehat{G})$, Lemma 26.2.12(iv) shows that

$$\int_{\widehat{G}} a(\chi)\varphi(\chi) \, d\chi = \int_{\widehat{G}} 1 \, d\hat{\nu}_{a\varphi}(\chi) = \int_{\widehat{G}} a(\chi) \, d\hat{\nu}_\varphi(\chi).$$

Hence $\varphi \, d\chi = d\hat{\nu}_\varphi$ for all $\varphi \in \mathcal{F}$. In particular, for $f \in V^1$ we know $\hat{f} \in \mathcal{F}$ from before, and $\hat{f} \, d\chi = d\hat{\mu}_f$ by Lemma 26.2.12(ii), so we get

$$f(y) = \int_{\widehat{G}} \chi(y) \, d\hat{\mu}_f(\chi),$$

proving the formula. □

Corollary 26.2.14. *For $f \in L^1(G)$,*

(1) If f is continuous and of positive type, then $\hat{f}$ is nonnegative.

(2) $\int_G f(y)\,dy$ is nonnegative.

(3) If f is nonnegative then $\hat{f}$ is a function of positive type on $\widehat{G}$.

Finally, we obtain half of the second statement in Theorem 26.2.2, namely, that any function in V^1 can be recovered from its Fourier transform.

Corollary 26.2.15. *The map $V^1(G) \to V^1(\widehat{G}), f \mapsto \hat{f}$, is injective.*

Proof. Suppose $\hat{f} = \hat{g}$. Then by Theorem 26.2.13,

$$f(y) = \int_{\widehat{G}} \hat{f}(y)\chi(y)\,d\chi = \int_{\widehat{G}} \hat{g}(y)\chi(y)\,d\chi = g(y).$$

□

It remains to show $f \mapsto \hat{f}$ is surjective. This will be proven using Pontrjagin duality in the next section.

26.3 Pontrjagin Duality

Let G be a topological abelian group, $S^1 \subseteq \mathbb{C}$ the complex unit circle and $\widehat{G} = \mathrm{Hom}_{cts}(G, S^1)$ the Pontrjagin dual of G. An element $\chi \in \widehat{G}$ is called a (complex) character of G. We endow $\widehat{G}$ with the *compact-open topology*, namely the topology generated by open sets of the form $W(K, V)$ where $K \subseteq G$ is compact, $V \subseteq S^1$ is open and $W(K, V)$ contains the trivial character $\mathbf{1} : G \to S^1, g \mapsto 1$.

Lemma 26.3.1. *$\widehat{G}$ is a topological abelian group with respect to the compact-open topology.*

Our goal in this section is to prove:

Theorem 26.3.2 (Pontrjagin Duality)**.** *Let G be a locally compact, Hausdorff abelian group. Then the map*

$$\begin{aligned} \alpha : G &\longrightarrow \widehat{\widehat{G}} \\ y &\longmapsto (e_y : \chi \mapsto \chi(y)) \end{aligned}$$

is an isomorphism of topological abelian groups.

For each $y \in G$, the map $\alpha(y) = e_y$ is called the *evaluation map* at y. Fix $\chi \in \widehat{G}, y \in G$ and take an open neighborhood $U \subseteq S^1$ of $\chi(y)$. Since G is locally compact, we can choose a sufficiently small compact neighborhood $K \subseteq G$ of y such that $\chi \in W(K, U)$ and $e_y(W(K, U)) \subseteq U$. This shows that α is continuous at $e_y(\chi) = \chi(y)$, so e_y is continuous and hence α is well-defined. Now let us show that α is injective.

Lemma 26.3.3. *Let G be a locally compact, Hausdorff abelian group. For $f \in \mathcal{C}_c(G)$, set $\tilde{f}(y) := \overline{f(y^{-1})}$. Then*

(i) For every $f \in \mathcal{C}_c(G)$, $f * \tilde{f}$ is a continuous function of positive type on G.

(ii) For any neighborhood $V \subseteq G$ containing the identity e, there is a continuous function of positive type g on G such that V contains the support of g and $g(e) = 1$.

Lemma 26.3.4. *The map $\alpha : G \to \widehat{\widehat{G}}, y \mapsto e_y$ is injective.*

Proof. This amounts to saying that $\widehat{G}$ separates points in G. Suppose $z \in G \smallsetminus \{e\}$. We must produce a character $\chi \in \widehat{G}$ for which $\chi(z) \neq \chi(e)$. Assume to the contrary that $\chi(z) = 1$ for all $\chi \in \widehat{G}$. Then for all $f \in L^1(G)$,

$$\widehat{L_z f}(\chi) = \int_G f(zy)\overline{\chi(y)}\, dy = \int_G f(zy)\overline{\chi(zy)}\, dy = \hat{f}(\chi)$$

so $\hat{f} = \overline{L_z f}$. By Corollary 26.2.15, we know the Fourier transform is injective, so $f = L_z f$ holds for all $f \in V^1(G)$. Now since G is Hausdorff, there exists a neighborhood $U \subseteq G$ of e such that $z^{-1}U \cap U = \varnothing$. By Lemma 26.3.3(ii), there exists a continuous, nonzero function f of positive type, with compact support contained in U, such that $f(e) = 1$. Now $f = L_z f$ is impossible since $z^{-1}U$ is disjoint from U and therefore cannot intersect the support of f. Hence $\chi(z) \neq 1$ for some character χ. $\square$

Let $\mathbf{1} \in \widehat{G}$ be the trivial character. Then the sets

$$W(\widehat{K}, V) = \{\psi \in \widehat{\widehat{G}} \mid \psi(\chi) \in V \text{ for all } \chi \in \widehat{K}\},$$

where $\widehat{K}$ is a compact neighborhood of $\mathbf{1} \in \widehat{G}$ and V is an open neighborhood of $1 \in S^1$, form a neighborhood basis of the trivial element $\widehat{\mathbf{1}} \in \widehat{\widehat{G}}$. Define

$$W_G(\widehat{K}, V) = \alpha^{-1}(W(\widehat{K}, V)) = \{y \in G \mid \chi(y) \in V \text{ for all } \chi \in \widehat{K}\}.$$

Proposition 26.3.5. *The subsets $W_G(\widehat{K}, V)$, where $\widehat{K}$ ranges over all compact neighborhoods of $\mathbf{1} \in \widehat{G}$ and V ranges over all open neighborhoods of $1 \in S^1$, form a neighborhood basis for the topology on G.*

Proof. Let $U \subseteq G$ be an open neighborhood of the identity e. By Lemma 26.3.3(ii), there exists a continuous function g of positive type on G, with compact support contained in U, satisfying $g(e) = 1$. Then by Corollary 26.2.14, $\hat{g} \geq 0$, so Fourier inversion (Theorem 26.2.13) gives us

$$1 = g(e) = \int_{\widehat{G}} \hat{g}(\chi)\, d\chi.$$

Note that $\hat{g}\, d\chi$ is a finite, positive Radon measure so in particular it is inner regular. Thus for all $\varepsilon > 0$, there exists a compact set $\widehat{K} \subseteq \widehat{G}$ such that $\int_{\widehat{K}} \hat{g}(\chi)\, d\chi \geq 1 - \varepsilon$. By Fourier inversion again, we can write $g(y)$ for any $y \in G$ as

$$g(y) = \int_{\widehat{K}} \hat{g}(\chi)\chi(y)\, d\chi + \int_{\widehat{K}^c} \hat{g}(\chi)\chi(y)\, d\chi.$$

Taking V to be a sufficiently small open neighborhood of $1 \in S^1$, we get

$$\left| 1 - \int_{\widehat{K}} \hat{g}(\chi)\chi(y)\, d\chi \right| < \varepsilon$$

for all $y \in W_G(\widehat{K}, V)$. On the other hand,

$$\left| \int_{\widehat{K}^c} \hat{g}(\chi)\chi(y)\, d\chi \right| < \varepsilon$$

always holds. Thus $|g(y)| \geq 1 - 2\varepsilon$ for all $y \in W_G(\widehat{K}, V)$ so in particular $W_G(\widehat{K}, V)$ is contained in the support of g, hence $W_G(\widehat{K}, V) \subseteq U$. □

Corollary 26.3.6. *$\alpha : G \to \widehat{\widehat{G}}$ is a homeomorphism onto its image.*

Proof. According to Proposition 26.3.5, α induces a bijection on neighborhood bases of G and $\alpha(G) \subseteq \widehat{\widehat{G}}$. □

Corollary 26.3.7. *$\alpha(G)$ is closed in $\widehat{\widehat{G}}$.*

Proof. Since $\alpha(G)$ is a locally compact, dense subset of $\overline{\alpha(G)}$, general topology says that it is also open in $\overline{\alpha(G)}$. But in a topological group, open subgroups are also closed, so this implies $\alpha(G)$ is closed in $\overline{\alpha(G)}$, hence $\alpha(G) = \overline{\alpha(G)}$. □

Thus to prove Pontrjagin duality, we only need to show that $\alpha(G)$ is dense in $\widehat{\widehat{G}}$. This requires an important sequence of results culminating in Plancherel's theorem.

For $f \in L^1(G)$, let $\tilde{f}(y) = \overline{f(y^{-1})}$ as in Lemma 26.3.3.

Lemma 26.3.8. *For any $f \in L^1(G)$ and $\chi \in \widehat{G}$, $\hat{\tilde{f}}(\chi) = \overline{\hat{f}(\chi)}$.*

Proof. By Lemma 26.3.3, we have

$$\begin{aligned}\hat{\tilde{f}}(\chi) &= \int_G \tilde{f}(y)\overline{\chi(y)}\,dy = \int_G \overline{f(y^{-1})}\chi(y^{-1})\,dy \\ &= \int_G \overline{f(y)}\chi(y)\,dy = \overline{\int_G f(y)\overline{\chi(y)}\,dy} = \overline{\hat{f}(\chi)}.\end{aligned}$$

□

Lemma 26.3.9. *If $f \in L^1(G) \cap L^2(G)$, then $||f||_2 = ||\hat{f}||_2$.*

Proof. For any $f \in L^1(G) \cap L^2(G)$, set $g = f * \tilde{f}$. Then by the same logic as in Lemma 26.3.3(i), g is of positive type. Consider

$$\begin{aligned}\int_G |f(y)|^2\,dy &= \int_G f(y)\,\overline{f(y)}\,dy = \int_G f(y^{-1})\,\overline{f(y^{-1})}\,dy \quad \text{by a change of variables} \\ &= \int_G f(y^{-1})\tilde{f}(y)\,dy = g(e) = \int_{\widehat{G}} \hat{g}(\chi)\,d\chi \quad \text{by Fourier inversion} \\ &= \int_{\widehat{G}} \hat{f}(\chi)\overline{\hat{f}(\chi)}\,d\chi = \int_{\widehat{G}} \hat{f}(\chi)\hat{\tilde{f}}(\chi)\,d\chi \quad \text{by Lemma 26.3.8} \\ &= \int_{\widehat{G}} \hat{f}(\chi)\overline{\hat{f}(\chi)}\,d\chi = \int_{\widehat{G}} |\hat{f}(\chi)|^2\,d\chi.\end{aligned}$$

Taking the square root of both sides, we get $||f||_2 = ||\hat{f}||_2$. □

Corollary 26.3.10. *The Fourier transform defines an isometric embedding*

$$L^1(G) \cap L^2(G) \hookrightarrow L^2(\widehat{G}).$$

Let $\widehat{A} = \widehat{A}(G)$ be the ring of Fourier transforms of $L^1(G)$ and set

$$\widehat{A}_1 = \{\hat{f} \mid f \in L^1(G) \cap L^2(G)\} \subseteq \widehat{A}.$$

Lemma 26.3.11. *$\widehat{A}_1$ is an $\alpha(G)$-invariant subspace of $\widehat{A}$.*

Proof. For any $y_0 \in G, f \in L^1(G) \cap L^2(G)$ and $\chi \in \widehat{G}$,

$$\begin{aligned}(\alpha(y_0)\hat{f})(\chi) &= \chi(y_0) \int_G f(y)\overline{\chi(y)}\, dy \\ &= \int_G f(y)\overline{\chi(y_0^{-1})}\,\overline{\chi(y)}\, dy \\ &= \int_G f(y)\overline{\chi(y_0^{-1}y)}\, dy \quad \text{since } \chi \text{ is a character} \\ &= \int_G f(y_0 y)\overline{\chi(y)}\, dy \quad \text{by a change of variables} \\ &= \widehat{L_{y_0}f}(\chi).\end{aligned}$$

Clearly $L_{y_0}f \in L^1(G) \cap L^2(G)$, so we see that $\alpha(y_0)f \in \widehat{A}_1$. □

Lemma 26.3.12. *$\widehat{A}_1$ is dense in $L^2(\widehat{G})$.*

Proof. First, $L^2(\widehat{G})$ is a Hilbert space, so by the Riesz representation theorem (20.2.13), $L^2(\widehat{G})$ can be identified with its dual space of linear functionals $L^2(\widehat{G})^* = \{\langle \cdot, \chi\rangle \mid \chi \in L^2(\widehat{G})\}$. By Corollary 20.2.12, if $\widehat{A}_1$ is not dense in $L^2(\widehat{G})$ then there exists a nonzero $g \in L^2(\widehat{G})$ that is orthogonal to all of $\widehat{A}_1$. Since $\alpha(G)\widehat{A}_1 \subseteq \widehat{A}_1$ by Lemma 26.3.11, we see that for all $f \in \widehat{A}_1$ and $y \in G$, $\alpha(y^{-1})f \in \widehat{A}_1$ and so

$$\int_{\widehat{G}} g(\chi)f(\chi)\overline{\chi(y)}\, d\chi = \int_G g(\chi)(\alpha(y^{-1})f)(\chi)\, d\chi = \langle g, \alpha(y^{-1}f)\rangle = 0.$$

Thus the Fourier transform of the measure $g\bar{f}\, d\chi$ is trivial. Moreover, $g\bar{f} \in L^1(\widehat{G})$ and $d\chi$ is a finite Radon measure, which means $g\bar{f}\, d\chi$ is also a finite Radon measure, so that $g\bar{f}\, d\chi = 0$ implies $g\bar{f} = 0$ a.e. by Proposition 26.2.7. Note that for any $\chi \in \widehat{G}$ and $h \in L^1(G)$, $\widehat{\chi h} = L_\chi \hat{h}$. Therefore if $f \in \widehat{A}_1$ is nonzero and continuous, then for every $\chi \in \widehat{G}$, there exist a continuous element of $\widehat{A}_1$, namely a translate of f, that is nonzero at χ. By Lemma 26.3.11, such an f is guaranteed to exist, so $g\bar{f} = 0$ a.e. then implies that $g = 0$ a.e., that is, $g = 0$ in $L^2(\widehat{G})$. This contradicts our initial assumption, so $\widehat{A}_1$ is dense in $L^2(\widehat{G})$. □

This proves:

Theorem 26.3.13 (Plancherel)**.** *Let G be a locally compact, Hausdorff abelian group. Then the Fourier transform $L^1(G) \cap L^2(G) \to L^2(\widehat{G}), f \mapsto \hat{f}$ extends by continuity to a map*

$$F : L^2(G) \longrightarrow L^2(\widehat{G})$$

which is an isomorphism of Hilbert spaces – in particular, an isometry.

The map F is called the *Plancherel transform* of G. We will denote the Plancherel transform of a function $f \in L^2(G)$ by $\hat{f}$, even though technically this is an extension of the Fourier transform.

Corollary 26.3.14 (Parseval's Identity)**.** *For all $f, g \in L^2(G)$,*

$$\int_G f(y)\overline{g(y)}\,dy = \int_{\widehat{G}} \hat{f}(\chi)\overline{\hat{g}(\chi)}\,d\chi.$$

The Plancherel transform also gives us a converse to the reciprocity formula of Lemma 26.2.11.

Corollary 26.3.15. *Let $f, g \in L^2(G), h \in L^1(G)$ and suppose $h = fg$ pointwise. Then $\hat{h} = \hat{f} * \hat{g}$.*

Proof. For any $\chi_0 \in \widehat{G}$, we have

$$\begin{aligned}
\hat{h}(\chi_0) &= \int_G f(y)g(y)\overline{\chi_0(y)}\,dy \\
&= \int_G f(y)\overline{\overline{g(y)}\chi_0(y)}\,dy \\
&= \int_{\widehat{G}} \hat{f}(\chi)\overline{(\widehat{\bar{g}\chi_0})(\chi)}\,d\chi \\
&= \int_{\widehat{G}} \hat{f}(\chi)\hat{g}(\chi^{-1}\chi_0)\,d\chi \\
&= (\hat{f} * \hat{g})(\chi_0).
\end{aligned}$$

Therefore $\hat{h} = \hat{f} * \hat{g}$. □

Corollary 26.3.16. *Set $\mathcal{C}_2(\widehat{G}) = \{f * g \mid f, g \in L^2(\widehat{G})\}$. Then $\widehat{A} = \mathcal{C}_2(\widehat{G})$.*

Proof. Take $h \in L^1(G)$. Then h can be written as a product of $L^2(G)$ functions, e.g. as $h = r \cdot |r|$ where

$$r(y) = \begin{cases} \dfrac{h(y)}{\sqrt{|h(y)|}}, & h(y) \neq 0 \\ 0, & h(y) = 0. \end{cases}$$

Then $\hat{h} = \hat{f} * \hat{g}$ by Corollary 26.3.15, so $\widehat{A} \subseteq \mathcal{C}_2(\widehat{G})$. Conversely, Plancherel's theorem gives a bijection $L^2(G) \leftrightarrow L^2(\widehat{G})$ so any element $f * g \in \mathcal{C}_2(\widehat{G})$ corresponds to $\hat{f} * \hat{g} = \widehat{fg} \in \widehat{A}$. This shows that $\mathcal{C}_2(\widehat{G}) \subseteq \widehat{A}$. □

Proposition 26.3.17. *If $U \subseteq \widehat{G}$ is a nonempty open set, then there exists a nonzero Fourier transform $\hat{f} \in \widehat{A}$ with support contained in U.*

Proof. Since U is nonempty and open, it has (finite) positive measure so by inner regularity, there exists a compact set $K \subseteq U$ with $\mathrm{vol}(K) > 0$. For all $x \in K$, we can find an open neighborhood $V_x \subseteq \widehat{G}$ containing $\mathbf{1}$ and an open neighborhood $U_x \subseteq \widehat{G}$ containing x such that $U_xV_x \subseteq U$. Since K is compact, there is a compact neighborhood $V \subseteq \widehat{G}$ containing $\mathbf{1}$ such that $\mathrm{vol}(V) > 0$ and $KV \subseteq U$. Define $\hat{f} = \chi_K * \chi_V$ where $\chi_K, \chi_V \in L^2(\widehat{G})$ are the characteristic functions on K, V, respectively. Then by Corollary 26.3.16, $\hat{f} \in \widehat{A}$. Finally, the support of $\hat{f}$ by definition is $KV \subseteq U$, and we have

$$\int_{\widehat{G}} \hat{f}(\chi)\,d\chi = \mathrm{vol}(K)\,\mathrm{vol}(V) > 0,$$

so $\hat{f}$ is nonzero. □

We are now prepared to prove Pontrjagin duality.

Proof of Theorem 26.3.2. In light of Corollaries 26.3.6 and 26.3.7, it remains to show that $\alpha(G)$ is dense in $\widehat{\widehat{G}}$. Suppose to the contrary that $\alpha(G)$ is not dense. Then $\overline{\alpha(G)}^c$ is a nonempty open set in $\widehat{\widehat{G}}$, so by Proposition 26.3.17, there exists a nonzero function $\varphi \in L^1(\widehat{G})$ such that $\hat{\varphi}|_{\alpha(G)} = 0$. This implies that for any $y \in G$,

$$\int_{\widehat{G}} \varphi(\chi)\chi(y^{-1})\,d\chi = \hat{\varphi}(\alpha(y)) = 0,$$

so $\varphi\,d\chi = 0$. By Lemma 26.2.6, $\varphi = 0$ a.e., contradicting our assumption that φ was nonzero in $L^1(\widehat{G})$. Hence $\alpha(G)$ is dense in G as claimed. $\square$

Corollary 26.3.18. *For any locally compact abelian group G, the Fourier transform induces a bijection $V^1(G) \leftrightarrow V^1(\widehat{G})$.*

Proof. By Corollary 26.2.15, the map is injective so it remains to show surjectivity. Take $F \in V^1(\widehat{G})$ and define a function $f : G \to \mathbb{C}$ by

$$f(y) = \int_{\widehat{G}} F(\chi)\chi(y)\,d\chi = \int_{\widehat{G}} F(\chi)\overline{\alpha(y^{-1})(\chi)}\,d\chi = \widehat{F}(\alpha(y^{-1})).$$

By Pontrjagin duality (Theorem 26.3.2), we can identify $\widehat{F}(\alpha(y^{-1})) = \widehat{F}(y^{-1})$, which is a continuous function of positive type on $G = \widehat{\widehat{G}}$. Then Corollary 26.2.14 says that $f \in V^1(G)$. Finally, by Theorem 26.2.13, we have

$$\begin{aligned} F(\chi) &= \int_G \widehat{F}(y)\chi(y)\,dy = \int_G f(y^{-1})\chi(y)\,dy \\ &= \int_G f(y)\overline{\chi(y)}\,dy = \hat{f}(\chi). \end{aligned}$$

Hence the Fourier transform $V^1(G) \to V^1(\widehat{G})$ is surjective, so it is a bijection. $\square$