Sheaf Cohomology

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Contents

0 Introduction

These notes follow a course on sheaf theory in topology and algebraic geometry taught by Dr. Andrei Rapinchuk at the University of Virginia in Fall 2018. Topics include:

- Basic presheaf and sheaf theory on a topological space
- Some algebraic geometry
- Sheaf cohomology
- Applications to topology and algebraic geometry.

The main references are Godement's original 1958 text *Topologie Algébrique et Théorie des Faisceaux*, Wedhorn's *Manifolds, Sheaves and Cohomology*, Bredon's *Sheaf Theory* and Hartshorne's *Algebraic Geometry*.

1 Presheaves and Sheaves on a Topological Space

In this chapter, we describe the theory of presheaves and sheaves on topological spaces. The basic intuition is that for a topological space X, one wishes to study a given property defined on open sets $U \subseteq X$ that behaves well under union and intersection of open sets.

1.1 Presheaves

Definition. *For a topological space X, a* **presheaf** *on X is a contravariant functor $F : \mathtt{Top}_X \to \mathcal{C}$, where $\mathcal{C}$ is any category. For each inclusion of open sets $V \hookrightarrow U$, the induced morphism $F_{UV} : F(U) \to F(V)$ is called* **restriction***, written $F_{UV}(\sigma) = \sigma|_V$.*

We will typically take $\mathcal{C}$ to be a 'set category' such as `Set`, `Ab`, `Ring`, `Alg`, etc.

Example 1.1.1. The classic example of a presheaf on a topological space X assigns to each open $U \subseteq X$ the set $F(U)$ of all continuous (or differentiable, smooth, holomorphic, etc.) functions on U, with restriction maps given by restriction of functions. Each of these is a presheaf of rings. More generally, if Y is any topological space, the functor $U \mapsto C(U, Y)$, the set of continuous functions $U \to Y$, is a presheaf of sets on X.

Example 1.1.2. Let S be a set and define a functor $F : \mathtt{Top}_X \to \mathtt{Set}$ by $F(U) = S$ for all $U \subseteq X$, with the restriction maps given by $\sigma|_V = \sigma \in S$ for any inclusion $V \hookrightarrow U$. Then F is a presheaf, called the *constant presheaf on S*.

Example 1.1.3. For a map $\pi : Y \to X$, the *presheaf of sections* is the functor $\Gamma(\pi, -)$ sending $U \mapsto \Gamma(\pi, U) := \{s : U \to Y \mid \pi \circ s = id_U\}$. An element of $\Gamma(\pi, U)$ is called a *section of π over U* and an element of $\Gamma(\pi, X)$ is called a *global section*.

Example 1.1.4. For a presheaf F on X and an open set $U \subseteq X$, the functor $F|_U : V \mapsto F(V)$ is a presheaf on U. The operation $F \mapsto F|_U$ is called restriction of a presheaf.

Example 1.1.5. Suppose A is a set (or an abelian group, ring, etc.). The *skyscraper presheaf* at $x \in X$ with coefficients in A is the presheaf A_x defined by

$$x_*A(U) = \begin{cases} A, & x \in U \\ *, & x \notin U \end{cases}$$

where $*$ denotes a point set. One can think of a skyscraper presheaf as a sort of constant presheaf concentrated at the point x.

Example 1.1.6. Let K be an algebraically closed field, $A = K[x_1, \ldots, x_n]$ the polynomial ring in n variables and $V \subseteq K^n$ an algebraic set defined by an ideal $I \subset A$. The *structure presheaf* on V is the presheaf $\mathcal{O}_V$ defined on Zariski-open sets $U \subseteq V$ by

$$U \longmapsto \mathcal{O}_V(U) = \{f : U \to K \mid f \text{ is a regular function}\}.$$

Note that regular functions over an open set form a ring, so $\mathcal{O}_V$ is a presheaf of rings (even K-algebras). Let $K[V] = \mathcal{O}_V(V)$, so that there is a surjective ring homomorphism

$$\begin{aligned} \varphi : K[x_1, \ldots, x_n] &\longrightarrow K[V] \\ p &\longmapsto p|_V. \end{aligned}$$

Then $K[V] \cong K[x_1, \ldots, x_n]/I(V)$ where $I(V) = \ker \varphi$ is the vanishing ideal of V.

When V is a variety (i.e. an irreducible algebraic set), $K[V]$ is an integral domain so it has a field of fractions $K(V)$, called the *field of rational functions* of V. In this case, for each open $U \subseteq V$,

$$\mathcal{O}_V(U) = \left\{ f \in K(V) \mid \text{for all } x \in V, \text{ there exist } g_x, h_x \in K[V] \text{ such that } f = \frac{g_x}{h_x}, h_x(x) \neq 0 \right\}.$$

So $\mathcal{O}_V$ is really a sub-presheaf of the constant presheaf defined by $K(V)$.

1.2 Sheaves

Sheaves are a tool for encoding the local and global data of a topological space: in many situations, the topological properties that one can define on a collection of open sets (local data) can be glued together to give properties on their union (global data). The following definition makes this rigorous.

Definition. *A presheaf $F : \mathtt{Top}_X \to \mathcal{C}$ is called a* **sheaf** *on X provided it satisfies the following 'descent conditions':*

(1) For any open covering $U = \bigcup U_i$, if there exist sections $s, t \in F(U)$ such that $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$.

(2) For any open covering $U = \bigcup U_i$ admitting sections $s_i \in F(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, there exists a unique section $s \in F(U)$ such that $s|_{U_i} = s_i$ for all i.

Example 1.2.1. The presheaf $F : U \mapsto C(U, Y)$ of continuous functions on X in Example 1.1.1 is a sheaf. Indeed, if $U = \bigcup U_i$ is an open cover in X and $f, g : U \to Y$ are continuous maps such that $f|_{U_i} = g|_{U_i}$, this means $f(x) = g(x)$ for all $x \in U_i$. Since the U_i cover U, this means $f(x) = g(x)$ for all $x \in U$ and hence $f = g$. If instead we have continuous functions $f_i : U_i \to Y$ for each i with $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then $f_i(x) = f_j(x)$ for all $x \in U_i \cap U_j$. For $x \in U$, define $f : U \to Y$ by $f(x) = f_i(x)$ if $x \in U_i$. Then the previous statement implies f is well-defined and continuous on U. Hence $f \in F(U)$ and $f|_{U_i} = f_i$ for all i.

Example 1.2.2. Let $F(U)$ be the set of bounded functions $U \to \mathbb{R}$. Then $F : U \mapsto F(U)$ is a presheaf on X, with the usual restriction maps, but F is not a sheaf. For example, when $X = \mathbb{R}$, consider the open cover of X given by $U_n = \left(n - \frac{1}{3}, n + \frac{4}{3}\right)$. Then $f_n(x) = x$ defines a sequence of bounded functions on each U_n which agree on intersections, but the only possible lift of $\{f_n\}$ to X is $f(x) = x$, an unbounded function. Hence axiom (2) fails.

Example 1.2.3. The skyscraper presheaf x_*A of Example 1.1.5 at a point $x \in X$ is a sheaf on X: suppose $U = \bigcup U_i$ is an open cover and $s, t \in x_*A(U)$ such that $s|_{U_i} = t|_{U_i}$ for all U_i. If $x \notin U$, then $x_*A(U) = *$ so $s = t$ is immediate. If $x \in U$, then $x_*A(U) = A$ and $x \in U_i$ for some i, but $s|_{U_i} = s \in A$ and $t|_{U_i} = t \in A$, so $s = t$. On the other hand, if $s_i \in x_*A(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $U_i \cap U_j \neq \varnothing$, then there are two cases again. If $x \notin U$, there is a trivial lift of the s_i to $s \in x_*A(U) = *$. If $x \in U$, x lies in some U_i so we may take $s = s_i \in x_*A(U_i) = A = x_*A(U)$. If x also lies in U_j, then $U_i \cap U_j \neq \varnothing$ so $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ and in particular $s_i = s_j$. Therefore s is well-defined, so x_*A is a sheaf.

Example 1.2.4. For an affine algebraic variety $V \subseteq K^n$, the structure presheaf $\mathcal{O}_V$ of Example 1.1.6 is a sheaf. The proof is similar to the argument in Example 1.2.1.

Lemma 1.2.5. *If F is a sheaf on X and $U = \bigcup U_i$ is an open cover, then $F(U) = \bigcap F(U_i)$.*

Proof. For any $U_i \cap U_j \neq \varnothing$, the pushforward diagram

$$\begin{array}{ccc} U_i \cap U_j & \longrightarrow & U_i \\ \downarrow & & \downarrow \\ U_j & \longrightarrow & U \end{array}$$

induces a pullback diagram of sets (or abelian groups, rings, etc.)

$$\begin{array}{ccc} F(U) & \longrightarrow & F(U_i) \\ \downarrow & & \downarrow \\ F(U_j) & \longrightarrow & F(U_i \cap U_j) \end{array}$$

So $F(U) = F(U_i) \cap F(U_j)$ by definition. This extends to arbitrary unions. □

Proposition 1.2.6. *For an affine algebraic variety V over K and a regular function $f \in K[V]$, let $D(f) = \{x \in V \mid f(x) \neq 0\}$ be the principal open subset defined by f. Then $\mathcal{O}_V(D(f)) = K[V]_f$, the localization at all powers of f.*

Proof. It is clear that $K[V]_f \subseteq \mathcal{O}_V(D(f))$. Going the other direction, suppose $g \in \mathcal{O}_V(D(f))$. Then for any $x \in D(f)$, $g = \frac{a_x}{b_x}$ in $K(V)$ with $b_x(x) \neq 0$. Let I be the ideal in $K[V]$ generated by $\{b_x\}_{x \in D(f)}$. Then $V(I) \cap D(f) = \varnothing$ so $V(I) \subseteq V((f))$. By Hilbert's Nullstellensatz, this implies $f^m \in I$ for some $m \geq 1$, so $f^m = \sum_{i=1}^r h_{x_i} b_{x_i}$ for some $x_1, \ldots, x_r$ and $h_{x_i} \in K[V]$. Thus $gf^m = \sum_{i=1}^r h_{x_i} a_{x_i}$ which shows $g = \sum_{i=1}^r \frac{h_{x_i} a_{x_i}}{f^m} \in K[V]_f$. □

1.3 The Category of Sheaves

Let $\texttt{Presh}_X$ be the category of presheaves of abelian groups on X, with morphisms given by natural transformations of functors, and define $\texttt{Sh}_X$ to be the full subcategory of sheaves in $\texttt{Presh}_X$. (That is, a morphism of sheaves on X is a morphism of the underlying presheaves.)

Many categorical constructions in $\texttt{Presh}_X$ can be constructed locally, i.e. over open sets $U \subseteq X$. For example:

Definition. *For two presheaves F and G on X, the* **presheaf product** *$F \times G$ is defined by $(F \times G)(U) = F(U) \times G(U)$. Likewise, the* **presheaf coproduct** *$F \coprod G$ is defined by $(F \coprod G)(U) = F(U) \coprod G(U)$.*

Definition. *Let $\varphi : F \to G$ be a morphism of presheaves.*

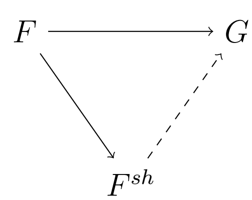
- *The* **presheaf image** *$\operatorname{im}\varphi$ is defined by $(\operatorname{im}\varphi)(U) = \operatorname{im}(F(U) \to G(U))$.*
- *The* **presheaf kernel** *$\ker\varphi$ is defined by $(\ker\varphi)(U) = \ker(F(U) \to G(U))$.*
- *The* **presheaf cokernel** *$\operatorname{coker}\varphi$ is defined by $(\operatorname{coker}\varphi)(U) = \operatorname{coker}(F(U) \to G(U))$.*

Lemma 1.3.1. *If $\varphi : F \to G$ is a morphism of sheaves, then $\ker\varphi$ is a sheaf.*

Proof. Suppose $U = \bigcup_i U_i$ is an open cover in X and $s, t \in (\ker\varphi)(U)$ such that $s|_{U_i} = t|_{U_i}$ for all U_i. Then since $(\ker\varphi)(U) \subseteq F(U)$, the first sheaf axiom for F implies directly that $s = t$. Now suppose there are sections $s_i \in (\ker\varphi)(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all overlapping U_i, U_j. Since $(\ker\varphi)(U_i) \subseteq F(U_i)$, the second sheaf axiom for F implies there exists a section $s \in F(U)$ such that $s|_{U_i} = s_i$ for all i. We must show $s \in (\ker\varphi)(U) \subseteq F(U)$. Note that $\varphi_U(s)|_{U_i} = \varphi_{U_i}(s|_{U_i}) = \varphi_{U_i}(s_i) = 0$ for each U_i, so by the first sheaf axiom for G, we must have $\varphi_U(s) = 0$. □

However, not all of the constructions above yield sheaves. For example, the cokernel of a morphism of sheaves is not a sheaf in general. To remedy this, we introduce a sheaf F^{sh} associated to any presheaf F, called the *sheafification* of F.

Theorem (Sheafification)**.** *For any presheaf F on X, there exists a sheaf F^{sh} on X together with a morphism $F \to F^{sh}$ such that for any sheaf G and morphism of presheaves $F \to G$, there is a unique morphism of sheaves $F^{sh} \to G$ making the diagram*



commute. Further, $F \mapsto F^{sh}$ is a functor $\texttt{Presh}_X \to \texttt{Sh}_X$ which is left adjoint to the forgetful functor $\texttt{Sh}_X \hookrightarrow \texttt{Presh}_X$.

To prove this, we need the following definition.

Definition. *For a presheaf F on X and a point $x \in X$, the* **stalk** *of F at x is the direct limit*

$$F_x := \varinjlim F(U)$$

taken over all open sets $U \subseteq X$ containing x.

Example 1.3.2. For the constant presheaf F_S, the stalks are all the same: $F_{S,x} = S$. The same is true if F is the sheaf of *locally constant functions* on X, i.e. $F(U)$ consists of functions $f : U \to S$ which are constant on some open subset $V \subseteq U$.

Example 1.3.3. Suppose X is T1 and let x_*A be the skyscraper sheaf on X with value A on neighborhoods of $x \in X$ (see Example 1.1.5). Then for a point $y \in X$, the stalk $(x_*A)_y$ is A if $y = x$ and 0 otherwise. Therefore we can think of a skyscraper sheaf as a sort of constant sheaf concentrated at the point x.

When X is not T1 and x is not a closed point of X, then the stalks are a little 'fuzzier': for $y \in X$,

$$(x_*A)_y = \begin{cases} A, & y \in \overline{\{x\}} \\ 0, & y \notin \overline{\{x\}}. \end{cases}$$

Example 1.3.4. For an affine algebraic variety $V \subseteq \mathbb{A}^n$, the stalk of the structure sheaf $\mathcal{O}_V$ (see Example 1.1.6) at a point $x \in V$ is

$$\mathcal{O}_{V,x} = K[V]_{\mathfrak{m}_x},$$

the localization of $K[V]$ at the maximal ideal $\mathfrak{m}_x = \{f \in K[V] \mid f(x) = 0\}$. Equivalently, $\mathcal{O}_{V,x}$ consists of all rational functions $f \in K(V)$ which are defined at x, meaning on some neighborhood U of x, $f = \frac{g_x}{h_x}$ for some $g_x, h_x \in \mathcal{O}_V(U)$ with $h_x(x) \neq 0$.

Let $\varphi : F \to G$ be a morphism of presheaves on X. Then for any $x \in X$, since the stalks F_x and G_x are defined via direct limits, there is an induced morphism $\varphi_x : F_x \to G_x$.

Lemma 1.3.5. *Let F be a sheaf on X and $U \subseteq X$ an open set with sections $s, t \in F(U)$. Then $s = t$ if and only if for all $x \in U$, $s|_x = t|_x$ in F_x. In other words, the morphism $F(U) \to \prod_{x \in U} F_x$ is injective.*

Proof. If $s|_x = t|_x$, then there is some neighborhood $U_x \subseteq U$ of x such that $s|_{U_x} = t|_{U_x}$. Since this holds for all $x \in U$ and $U = \bigcup_{x \in U} U_x$, we have $s|_{U_x} = s|_{U_x}$ for all U_x, but by the sheaf axioms this implies $s = t$. □

Corollary 1.3.6. *Let F be a presheaf, G a sheaf and $\varphi, \psi : F \to G$ two morphisms of presheaves. Then $\varphi = \psi$ if and only if $\varphi_x = \psi_x : F_x \to G_x$ for all $x \in X$.*

Proof. The ($\implies$) implication is clear, so suppose $\varphi_x = \psi_x$ for all x. Let $U \subseteq X$ be an open set and take $s \in F(U)$. Then the elements $\varphi_U(s), \psi_U(s) \in G(U)$ restrict on stalks to:

$$\begin{aligned} \varphi_U(s)|_x &= \varphi_x(s|_x) && \text{since } \varphi \text{ is a morphism} \\ &= \psi_x(s|_x) && \text{by hypothesis} \\ &= \psi_U(s)|_x && \text{since } \psi \text{ is a morphism.} \end{aligned}$$

Therefore by Lemma 1.3.5, $\varphi_U(s) = \psi_U(s)$. Since this holds for all U, we get $\varphi = \psi$. □

Definition. *Let F be a sheaf of abelian groups on X and $s \in \Gamma(X, F)$ a global section. Then the* **support** *of s is the set*

$$\text{supp}(s) = \{x \in X \mid s|_x \neq 0 \text{ in } F_x\}.$$

Lemma 1.3.7. *For any $s \in \Gamma(X, F)$, $\operatorname{supp}(s)$ is a closed set in X.*

Proposition 1.3.8. *Let $\varphi : F \to G$ be a morphism of presheaves, with F a sheaf. Then*

(1) The morphisms $\varphi_x : F_x \to G_x$ are injective for all $x \in X$ if and only if the morphisms $\varphi_U : F(U) \to G(U)$ are injective for all open sets $U \subseteq X$.

(2) The morphisms $\varphi_x : F_x \to G_x$ are bijective for all $x \in X$ if and only if the morphisms $\varphi_U : F(U) \to G(U)$ are bijective for all open sets $U \subseteq X$.

Proof. (1) Assume φ_x are injective for all $x \in X$ and suppose $s, t \in F(U)$ are sections such that $\varphi_U(s) = \varphi_U(t)$ in $G(U)$. Then for all $x \in X$,

$$\varphi_x(s|_x) = \varphi_U(s)|_x = \varphi_U(t)|_x = \varphi_x(t|_x)$$

since φ is a morphism, so injectivity implies $s|_x = t|_x$. Finally, Lemma 1.3.5 shows that $s = t$ in $F(U)$.

Conversely, assume that each φ_U is injective. For a fixed $x \in X$, take $s_x, t_x \in F_x$ such that $\varphi_x(s_x) = \varphi_x(t_x)$ in G_x. Then there is a neighborhood U of x such that $s_x = s|_x$ and $t_x = t|_x$ for some $s, t \in F(U)$. Since $G_x = \varinjlim G(U)$ over all neighborhoods U of x, the condition that $\varphi_x(s_x) = \varphi_x(t_x)$ in G_x means there is some smaller neighborhood $V \subseteq U$ containing x on which $\varphi_U(s)|_V = \varphi_U(t)|_V$. Since φ is a morphism, this is equivalent to $\varphi_V(s|_V) = \varphi_V(t|_V)$, but by injectivity of φ_V, we get $s|_V = t|_V$. Passing to the direct limit, we have $s_x = t_x$ in F_x. Hence φ_x is injective.

(2) The ($\Longleftarrow$) implication is trivial. For ($\Longrightarrow$), suppose each φ_x is bijective. Fix $U \subseteq X$ and $t \in G(U)$; we must find $s \in F(U)$ so that $\varphi_U(s) = t$. For any $x \in U$, consider the image t_x of t in the stalk G_x. By hypothesis, there is some $s_x \in F_x$ such that $\varphi_x(s_x) = t_x$. Then there is a neighborhood $U_x \subseteq U$ of x on which $s_x = s_{U_x}|_x$ for some $s_{U_x} \in F(U_x)$. By construction, $\varphi_{U_x}(s_{U_x}) = t$ in G_x, so there is a neighborhood $V_x \subseteq U_x$ of x with $\varphi_{V_x}(s_{U_x}|_{V_x}) = \varphi_{U_x}(s_{U_x})|_{V_x} = t_{V_x}$. Set $s_{V_x} = s_{U_x}|_{V_x}$. Letting x range over the points of U, we get a cover $U = \bigcup_{x \in U} V_x$ and moreover, if $x, y \in U$, then

$$\begin{aligned}
\varphi_{V_x \cap V_y}(s_{V_x}|_{V_x \cap V_y}) &= \varphi_{V_x}(s_{V_x})|_{V_x \cap V_y} \quad \text{since } \varphi \text{ is a morphism} \\
&= t|_{V_x \cap V_y} \\
&= \varphi_{V_y}(s_{V_y})|_{V_x \cap V_y} \\
&= \varphi_{V_x \cap V_y}(s_{V_y}|_{V_x \cap V_y}) \quad \text{for the same reason.}
\end{aligned}$$

We are assuming each φ_x is bijective, so in particular (1) implies each $\varphi_{V_x \cap V_y}$ is injective, and thus $s_{V_x}|_{V_x \cap V_y} = s_{V_y}|_{V_x \cap V_y}$. Now the second sheaf axiom guarantees that there is a section $s \in F(U)$ such that $s|_{V_x} = s_{V_x}$ for all $x \in U$. Finally, observe that

$$\varphi_U(s)|_{V_x} = \varphi_{V_x}(s|_{V_x}) = \varphi_{V_x}(s_{V_x}) = t|_{V_x}$$

for all $x \in U$. Thus the first sheaf axiom implies $\varphi_U(s) = t$. □

Remark. Notice that the proof of surjectivity in (2) crucially relies on the fact that the φ_x and φ_U are already *injective*. It is not true in general that the φ_x are all surjective if and only if the φ_U are all surjective.

Proposition 1.3.9. *For any $x \in X$, the stalk functor* $\texttt{Presh}_X \to \texttt{Ab}, F \mapsto F_x$, *is exact.*

Proof. Suppose $F' \xrightarrow{\varphi} F \xrightarrow{\psi} F''$ is an exact sequence of presheaves on X. We must show the induced sequence of stalks $F'_x \xrightarrow{\varphi_x} F_x \xrightarrow{\psi_x} F''_x$ is exact. First, since $\psi \circ \varphi = 0$ globally, on each stalk we get $\psi_x \circ \varphi_x = 0$ so $\operatorname{im} \varphi_x \subseteq \ker \psi_x$. On the other hand if $s_x \in \ker \psi_x$, then for some neighborhood U of x, there is a section $s \in F(U)$ such that $s|_x = s_x$ and $\psi_U(s)|_x = \psi_x(s_x) = 0$. Thus on some smaller neighborhood $V \subseteq U$ of x, $\psi_V(s|_V) = \psi_U(s)|_V = 0$, so $s|_V \in \ker \psi_V$. By exactness, this means $s|_V = \varphi_V(s')$ for some $s' \in F'(V)$. Then we have $s_x = \varphi_V(s')_x = \varphi_x(s'_x)$ for $s'_x = s'|_x$ in F_x. Hence $\ker \psi_x \subseteq \operatorname{im} \varphi_x$ so the sequence is exact. □

This gives us a good definition for what it means for a sequence of *sheaves* to be exact.

Definition. *A sequence of sheaves $F' \to F \to F''$ is* **exact** *if for all $x \in X$, the sequence of stalks $F'_x \to F_x \to F''_x$ is exact.*

Theorem 1.3.10. *Let $0 \to F' \xrightarrow{\varphi} F \xrightarrow{\psi} F'' \to 0$ be a short exact sequence of sheaves on X. Then for every open set $U \subseteq X$, the sequence*

$$0 \to F'(U) \to F(U) \to F''(U)$$

is exact. That is, the forgetful functor $\texttt{Sh}_X \to \texttt{Presh}_X$ *is left exact.*

Proof. By definition, $0 \to F'_x \xrightarrow{\varphi_x} F_x \xrightarrow{\psi_x} F''_x \to 0$ is exact for all $x \in X$, so Proposition 1.3.8(1) implies $0 \to F'(U) \to F(U)$ is exact for all open sets $U \subseteq X$. Set $K(U) = \ker(F(U) \to F''(U))$. Then by Lemma 1.3.1, $K : U \mapsto K(U)$ is a sheaf and it is easy to see that the stalks are $K_x = \ker(F_x \to F''_x)$. Now we have $(\psi \circ \varphi)_x = \psi_x \circ \varphi_x = 0$ for all $x \in X$ by exactness of the sequence of stalks, so $(\psi \circ \varphi)_U = \psi_U \circ \varphi_U = 0$ for all U, and thus $\operatorname{im} \varphi_U \subseteq K(U)$. Finally, φ can be written as a morphism $F \to K$, but on the level of stalks this is a bijection $\varphi_x : F_x \xrightarrow{\sim} K_x = \operatorname{im} \varphi_x$, so by Proposition 1.3.8(2), $\varphi_U : F(U) \to K(U)$ is also bijective. Hence $F \cong \operatorname{im} \varphi = K$ so the sequence $0 \to F'(U) \to F(U) \to F''(U)$ is exact. □

In general, if $F \to F''$ is surjective, it need not be true that $F(U) \to F''(U)$ is surjective. By definition, surjectivity of sheaves means surjectivity of stalks, so an element $s''_x \in F''_x$ is the image of some $s_x \in F_x$ but this s_x need only be defined on *some* neighborhood of x, not necessarily on arbitrary U.

Example 1.3.11. Take $X = \mathbb{C}$ and let $\mathcal{O}$ be the sheaf of holomorphic functions from Example 1.1.1. If $\mathcal{O}^\times$ denotes the sheaf of *nonvanishing* holomorphic functions on $\mathbb{C}$, then there is a morphism of sheaves $\varphi : \mathcal{O} \to \mathcal{O}^\times$ given locally by $\varphi_U : \mathcal{O}(U) \to \mathcal{O}(U)^\times, f \mapsto e^{2\pi i f}$. In fact, one can see that φ is an isomorphism on stalks (using the complex logarithm). However, on $U = \mathbb{C}^\times$, the map $\mathcal{O}(\mathbb{C}^\times) \to \mathcal{O}(\mathbb{C}^\times)^\times$ is not surjective, since $f(z) = z$ is not in the image. Thus on the level of stalks we have exact sequences

$$0 \to \mathbb{Z}_x \xrightarrow{\psi_x} \mathcal{O}_x \xrightarrow{\varphi_x} \mathcal{O}_x^\times \to 0$$

where $\mathbb{Z}$ denotes the constant sheaf, but $0 \to \mathbb{Z}(\mathbb{C}^\times) \to \mathcal{O}(\mathbb{C}^\times) \to \mathcal{O}(\mathbb{C}^\times)^\times \to 0$ is not exact.

Definition. *A sheaf F on X is called* **flasque** *(or* **flabby***) if for every open inclusion $V \subseteq U$, the restriction map $F(U) \to F(V)$ is surjective.*

Theorem 1.3.12. *If $0 \to F' \xrightarrow{\varphi} F \xrightarrow{\psi} F'' \to 0$ is a short exact sequence of sheaves with F' flasque, then the sequence*

$$0 \to F'(U) \to F(U) \to F''(U) \to 0$$

is exact for any open $U \subseteq X$. If in addition F is flasque, then so is F''.

Proof. By Theorem 1.3.10, it's enough to show $\psi_U : F(U) \to F''(U)$ is surjective. Further, after restriction of sheaves, it's enough to consider the case $U = X$. Let $t \in F''(X)$. Then for any $x \in X$, there is a neighborhood $U_x \subseteq X$ of x and a section $s_{U_x} \in F(U_x)$ such that $\psi_{U_x}(s_{U_x}) = t|_{U_x}$, by surjectivity of stalks. Consider the set of pairs

$$\mathcal{C} = \{(U, s) \mid U \subseteq X, s \in F(U), \psi_U(s) = t|_U\}$$

of open sets which admit an extension of t. Then $\mathcal{C}$ is a partially ordered set via $(U_1, s_1) \preceq (U_2, s_2)$ if and only if $U_1 \subseteq U_2$ and $s_2|_{U_1} = s_1$. For a linearly ordered subset (U_α, s_α) in $\mathcal{C}$, the pair $(\bigcup U_\alpha, s^*)$ is an upper bound, where $s^* \in F\left(\bigcup U_\alpha\right)$ is defined using the sheaf conditions on F. Thus by Zorn's Lemma, $\mathcal{C}$ contains a maximal element, say (U, s). Assume $x \in X \smallsetminus U$. Then there is a neighborhood V of x and a section $s_{V,0} \in F(U)$ with $\psi_V(s_{V,0}) = t|_V$ by surjectivity on stalks. Note that

$$\psi_{U\cap V}(s|_{U\cap V} - s_{V,0}|_{U\cap V}) = \psi_{U\cap V}(s|_{U\cap V}) - \psi_{U\cap V}(s_{V,0}|_{U\cap V}) = t|_{U\cap V} - t|_{U\cap V} = 0$$

so $s|_{U\cap V} - s_{V,0}|_{U\cap V} \in \psi_{U\cap V} = \operatorname{im}\varphi_{U\cap V}$ by exactness at $F(U\cap V)$. That is, $s|_{U\cap V} - s_{V,0}|_{U\cap V} = \varphi_{U\cap V}(s_0')$ for some $s_0' \in F'(U \cap V)$. Now since F' is flasque, $s_0' = s'|_{U\cap V}$ for some $s' \in F'(V)$. Set $s_V = s_{V,0} + \varphi_V(s')$. By construction, $s|_{U\cap V} = s_V|_{U\cap V}$ so by the sheaf axioms on F over $U \cup V$, there is a section $s_{U\cup V} \in F(U \cup V)$ with $s_{U\cup V}|_U = s$ and $s_{U\cup V}|_V = s_V$. This shows that $(U, s) \prec (U \cup V, s_{U\cup V})$, contradicting the fact that (U, s) is a maximal element in $\mathcal{C}$. Hence $X \smallsetminus U = \varnothing$, or $U = X$, and $s \in F(X)$ satisfies $\psi_X(s) = t$ by definition.

For the last statement, consider the following diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F'(U) & \longrightarrow & F(U) & \longrightarrow & F''(U) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F'(V) & \longrightarrow & F(V) & \longrightarrow & F''(V) & \longrightarrow & 0
\end{array}$$

If F is flasque, then the middle column is surjective, so it follows that the right column is also surjective. Hence F'' is also flasque. □

Proposition 1.3.13. *Let $\varphi : F \to G$ be a morphism of presheaves on X. Then*

(1) φ is surjective over all open sets $U \subseteq X$ if and only if φ is an epimorphism in the category $\texttt{Presh}_X$.

(2) If F and G are sheaves, then φ is surjective on stalks if and only if φ is an epimorphism in the category $\mathtt{Sh}_X$.

Proof. (1) Let $\operatorname{coker}\varphi$ be the presheaf cokernel of φ. Then φ is an epimorphism if and only if $\operatorname{coker}\varphi = 0$. For any open $U \subseteq X$, $(\operatorname{coker}\varphi)(U)$ is the cokernel of $\varphi_U : F(U) \to G(U)$ in the category of abelian groups, so it is 0 if and only if φ_U is surjective. On the level of sheaves, $\operatorname{coker}\varphi = 0$ if and only if $(\operatorname{coker}\varphi)(U) = 0$ for all U, thus φ is surjective on open sets if and only if it's an epimorphism.

(2) First suppose φ is surjective on stalks. If

$$F \xrightarrow{\varphi} G \underset{g}{\overset{f}{\rightrightarrows}} H$$

is a diagram of sheaves such that $f \circ \varphi = g \circ \varphi$, then for all $x \in X$, we have

$$f_x \circ \varphi_x = (f \circ \varphi)_x = (g \circ \varphi)_x = g_x \circ \varphi_x$$

but since φ_x is surjective, this implies $f_x = g_x$. Applying Corollary 1.3.6 gives us $f = g$, so φ is an epimorphism.

Conversely, if φ is an epimorphism, let $C = \operatorname{coker}\varphi$ be the presheaf cokernel of φ. If $p : G \to G/\varphi(F) \cong C$ is the natural projection of presheaves, consider the diagram

$$F \xrightarrow{\varphi} G \underset{0}{\overset{p}{\rightrightarrows}} C$$

Then $p \circ \varphi = 0 = 0 \circ \varphi$ so by assumption, $p = 0$. That is, $C = 0$, which means $C_x = 0$ for all $x \in X$. But $C_x = \operatorname{coker}(\varphi_x : F_x \to G_x)$, so $C_x = 0$ implies φ_x is surjective for all $x \in X$. ☐

1.4 Étale Space and Sheafification

Definition. *Let $p : E \to X$ be a morphism in a category $\mathcal{C}$. A* **section** *of p is a morphism $\sigma : X \to E$ such that $p \circ \sigma = id_X$.*

Definition. *When $p : E \to X$ is a continuous map of topological spaces, a* **local section** *of p over an open set $U \subseteq X$ is a map $\sigma : U \to E$ such that $p \circ \sigma = id_X|_U$. The set of all sections of p over U is denoted $\Gamma_p(U)$. A* **global section** *of p is a section over the open set $U = X$.*

Let $p : E \to X$ be a *local homeomorphism* of topological spaces, i.e. a map such that for each point $e \in E$, there is a neighborhood $V_e \subseteq E$ which is mapped homeomorphically by π to a neighborhood U_x of $x = p(e)$ in X. For each $x \in X$, we will write $E_x = p^{-1}(x) \subseteq E$.

Proposition 1.4.1. *Let $p : E \to X$ be a local homeomorphism. Then*

(1) The functor $\Gamma_p(-) : \mathtt{Top}_X \to \mathtt{Set}, U \mapsto \Gamma_p(U)$, is a sheaf on X with restriction maps $\Gamma_p(U) \to \Gamma_p(V)$ for $V \subseteq U$ given by restriction of sections $s \mapsto s|_V$.

(2) Sets of the form $s(U)$, where $U \subseteq X$ is open and $s \in \Gamma_p(U)$, form a basis for the topology on E.

(3) The stalk of $\Gamma_p(-)$ at $x \in X$ is the fibre E_x.

Proof. (1) and (2) are routine. Here's a proof of (3):

For each neighborhood $U \subseteq X$ of x, define a map

$$\begin{aligned} \nu_U : \Gamma_p(U) &\longrightarrow E_x \\ s &\longmapsto s(x). \end{aligned}$$

This is compatible with restriction, i.e. if $V \subseteq U$ is a smaller neighborhood of x, the diagram

$$\begin{array}{ccc} \Gamma_p(U) & & \\ \downarrow & \overset{\nu_U}{\searrow} & \\ & & E_x \\ & \underset{\nu_V}{\nearrow} & \\ \Gamma_p(V) & & \end{array}$$

commutes. Therefore the ν_U induce a morphism $\nu : \Gamma_p(-)_x = \varinjlim \Gamma_p(U) \to E_x$. For $e \in E_x$, take a neighborhood $V_e \subseteq E$ such that $p|_{V_e} : V_e \to \pi(V_e) = U$ is a homeomorphism. Then the map $s = p|_{V_e}^{-1}$ is a section of p over U and $s(x) = s(p(e)) = e$, so ν is onto. Finally, if s and t are sections of p over some neighborhoods U_1 and U_2 of x, respectively, such that $s(x) = t(x)$ in E_x, then we may choose neighborhoods $V_1, V_2 \subseteq E$ such that $p|_{V_1} : V_1 \xrightarrow{\sim} U_1$ and $p|_{V_2} : V_2 \to U_2$ are homeomorphisms. Set $V = V_1 \cap V_2$ and $U = \pi(V)$. Then p is a homeomorphism on V, so $p \circ s|_U = p \circ t|_U$ implies $s|_U = t|_U$ so $s = t$ in $\Gamma_p(-)_x$. Thus ν is a bijection. □

Definition. *The* **éspace étale** *of a sheaf F on X is the space $E_F = \coprod_{x \in X} F_x$ with the topology induced by the open sets*

$$V_{U,s} := \{(x, s|_{F_x}) \mid x \in U, s \in F(U)\}$$

where U runs over all open sets in X. The **étale cover** *$p_F : E_F \to X$ is naturally defined as $(x, f|_{F_x}) \mapsto x$.*

Example 1.4.2. For the sheaf of holomorphic functions $\mathcal{O}$ on $\mathbb{C}$, the open sets $V_{U,f}$ (for holomorphic functions $f : U \to \mathbb{C}$) are given by

$$V_{U,f} = \{(z_0, p(z)) \mid p(z) \text{ is the Taylor series expansion of } f(z) \text{ at } z_0\}.$$

Theorem 1.4.3. *There is an equivalence of categories between maps $p : E \to X$ and sheaves F on X given by $(p : E \to X) \mapsto \Gamma_p$ and $F \mapsto (p_F : E_F \to X)$.*

Proof. We have seen, courtesy of Proposition 1.4.1, that Γ_p is a sheaf on X. On the other hand, it follows from the definition of the topology on E_F that the map $p_F : E_F \to X$ is continuous. Therefore it remains to show that the assignments $p \mapsto \Gamma_p$ and $F \mapsto p_F$ define an equivalence of categories. That is, we must show that for a sheaf F on X, there is an isomorphism $\Gamma_{p_F} \cong F$, and for a map $p : E \to X$, there is a homeomorphism $E_{\Gamma_p} \cong E$ making the diagram

$$\begin{array}{ccc} E_{\Gamma_p} & \xrightarrow{\sim} & E \\ & \searrow \quad \swarrow_{p} & \\ & X & \end{array}$$

commute. First let F be a sheaf, let $U \subseteq X$ be open, take $\sigma \in F(U)$ and define $s \in \Gamma_{p_F}(U)$ by $s(x) = (x, \sigma|_x)$. Consider $s^{-1}(V_{U,f}) \subseteq U$ for some $f \in F(U)$. Then

$$\begin{aligned} x \in s^{-1}(V_{U,f}) &\iff s(x) = f(x) \\ &\iff s|_W = f|_W \text{ for some neighborhood } W \text{ of } x. \end{aligned}$$

For such an x we have $W \subseteq s^{-1}(V_{U,f})$, so the morphism $F \to \Gamma_{p_F}$ sending $\sigma \mapsto s$ is well-defined and continuous. Suppose $s' \in \Gamma_{p_F}(U)$ such that $s(x) = s'(x)$ for all $x \in U'$. Then there exists an open cover $U = \bigcup U_i$ with $s|_{U_i} = s'|_{U_i}$ for all U_i, so by the sheaf axioms, $s = s'$. Therefore $F \to \Gamma_{p_F}$ is one-to-one.

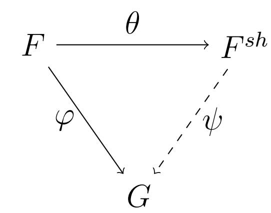
On the other hand, suppose $s : U \to E_{\Gamma_p}$ is a section of p_F. Then

$$s^{-1}(V_{U',s'}) = \{x' \in U' \mid s(x') = s'(u')\}.$$

On the open set $s^{-1}(V_{U',s'}) \subseteq U$, s comes from s' uniquely. Therefore if U' is a neighborhood of x, then $s^{-1}(V_{U',s'})$, where $s'(x) = s(x)$, is a neighborhood of x. This means there is an open cover $U = \bigcup U_i$ with $\sigma_i \in F(U_i)$ such that $\sigma_i(x) = s(x)$ for all $x \in U$. Moreover, since $F \to \Gamma_{p_F}$ is one-to-one, we have $\sigma_i|_{U_i \cap U_j} = \sigma_j|_{U_i \cap U_j}$ for all i, j. Therefore by the sheaf axioms, there is a unique section $\sigma \in F(U)$ with $\sigma|_{U_i} = \sigma_i$ for all i, which then satisfies $\sigma(x) = s(x)$ for all $x \in U$. Thus $F \to \Gamma_{p_F}$ is onto, hence an isomorphism.

Now let $p : E \to X$ be a continuous map. Define $E \to E_{\Gamma_p}$ as follows. For $y \in E$, set $x = p(y) \in X$. Then for all open sets $U \subseteq X$ containing X, $y \in p^{-1}(U)$, so y in fact lies in the stalk $\Gamma_{p,x} = \varinjlim \Gamma_p(U)$. Thus the assignment $E \to E_{\Gamma_p}, y \mapsto y$ is well-defined. It is clearly one-to-one and onto, and continuity follows easily from the definition of the topology on E_{Γ_p}. □

Theorem 1.4.4 (Sheafification)**.** *For every presheaf (of sets) F on X, there exists a presheaf F^{sh} together with a morphism of presheaves $\theta : F \to F^{sh}$ such that for every $x \in X$, $\theta_x : F_x \to F^{sh}_x$ is a bijection. Moreover, for any morphism $\varphi : F \to G$ where G is a sheaf, there is a unique morphism of sheaves $\psi : F^{sh} \to G$ such that the diagram*



commutes.

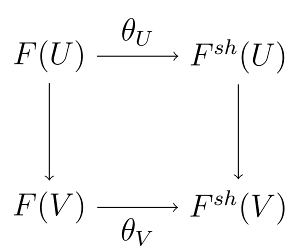
Corollary 1.4.5. *If F is a sheaf, then $F^{sh} = F$.*

Proof. This follows from Proposition 1.3.8(2) and the fact that $F_x \to F_x^{sh}$ is a bijection for all $x \in X$. □

Proof of 1.4.4. Let $E = E_F \xrightarrow{\pi} X$ be the étale space of F and define $F = \Gamma_\pi(-)$, the sheaf of sections of π. Then the morphism $\theta : F \to F^{sh}$ is defined by

$$\begin{aligned} \theta_U : F(U) &\longrightarrow F^{sh}(U) \\ s &\longmapsto (\tilde{s} : U \to E, x \mapsto s|_x). \end{aligned}$$

Suppose $V \subseteq U$ is an inclusion of open sets. It is immediate that the diagram

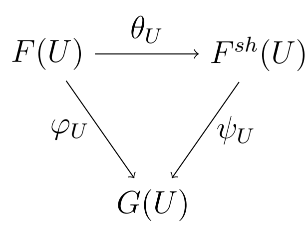


commutes, so the θ_U indeed give a morphism $\theta : F \to F^{sh}$. On stalks, we have that $F_x^{sh} = \Gamma(-, \pi)_x = E_x = F_x$ by Proposition 1.4.1(3), and it is clear that $\theta_x : F_x \to F_x$ is just the identity morphism $s_x \mapsto s_x$.

To prove the universal property, note that Corollary 1.4.5 already follows from the paragraph above, so for any sheaf G, $\theta : G \to G^{sh}$ is the identity on G. If $\varphi : F \to G$ is a morphism (of presheaves), then by Theorem 1.4.3 there is a continuous map $E_\varphi : E_F \to E_G$ and thus a morphism $\psi : F^{sh} = \Gamma_{\pi_F} \to \Gamma_{\pi_G} = G^{sh} = G$. Now for all open $U \subseteq X$ and sections $s \in F(U)$,

$$\psi_U \circ \theta_U(s) = \psi_U(\tilde{s}) = E_\varphi \circ \widetilde{s} = \varphi_U(s)$$

by construction, so the diagram



commutes, and once again this is compatible with restriction of sections along $V \subseteq U$. The fact that such a ψ is unique follows from the bijection $F_x \to F_x^{sh}$ and Proposition 1.3.8. □

Definition. *For a presheaf F on X, we call F^{sh} the* **sheafification** *of F, or the* **sheaf associated to** *F.*

Remark. Abstractly, we have an adjoint pair of functors (see Section A.2)

$$\texttt{Presh}(\mathcal{C}) \underset{\text{forget}}{\overset{\text{sh}}{\rightleftarrows}} \texttt{Sh}(\mathcal{C})$$

which means sheafification is right exact and the forgetful functor is left exact (we already knew the latter from Theorem 1.3.10). In a more general setting, such as when we consider the categories of presheaves and sheaves of abelian groups, rings, algebras, etc., we can take this as our *definition* of the sheafification functor: it is the left adjoint to the (left exact) forgetful functor $\texttt{Sh}_X \hookrightarrow \texttt{Presh}_X$.

Alternatively, we can equip an associated sheaf F^{sh} with the structure of a sheaf of abelian groups (or rings, algebras, etc.) as follows. Let $E \xrightarrow{\pi} X$ be the étale cover for the presheaf F and consider the fibre product

$$E \times_X E := \{(e_1, e_2) \in E \times E \mid \pi(e_1) = \pi(e_2)\}.$$

Then $E \times_X E \to E, (e_1, e_2) \mapsto e_1 + e_2$ is a well-defined, continuous map of topological spaces since the fibres of E (i.e. the stalks of F) are abelian groups. The induced map

$$F^{sh}(U) \times F^{sh}(U) = \Gamma_\pi(U) \times \Gamma_\pi(U) \cong \Gamma_{\pi \times_X \pi}(U) \longrightarrow \Gamma_\pi(U) = F^{sh}(U)$$

gives an additive structure on each $F^{sh}(U)$, making F^{sh} into a sheaf of abelian groups.

2 Čech Cohomology

2.1 The Mittag-Leffler Problem

We motivate the definition of Čech cohomology with a classical problem originally studied by Mittag-Leffler. Let X be a Riemann surface, that is, a 2-manifold admitting a complex structure with holomorphic transition functions. We do not assume X to be closed. Suppose E is a closed, discrete subset of X and for each point $a \in E$, we are given a function $z_a : U_a \to \mathbb{C}$ on some neighborhood $U_a \subseteq X$ of a such that $z_a(a) = 0$. Consider a function

$$p_a(z_a) = \alpha_{-m} z_a^{-m} + \alpha_{-m+1} z_a^{-m+1} + \ldots + \alpha_{-1} z_a^{-1}, \quad \alpha_j \in \mathbb{C}.$$

That is, p_a is a polynomial in z_a^{-1}, or a Laurent polynomial on U_a centered at a. The *Mittag-Leffler problem* is to find a meromorphic function $f : X \to \mathbb{C}$ such that f is holomorphic on $X \smallsetminus E$ and for all $a \in E$, the function $f - p_a$ has a removable discontinuity at a. Then p_a will be the *principal part* of f on U_a.

In short, the Mittag-Leffler problem asks us to extend meromorphic functions defined on open sets in X to a meromorphic function on the whole Riemann surface. The following restatement will be useful for later generalization. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X and suppose $\{f_i : U_i \to \mathbb{C}\}$ is a collection of meromorphic functions such that each f_i is either holomorphic on U_i or has a single pole $a_i \in U_i$, with $a_i \notin U_j$ for any $j \neq i$. The Mittag-Leffler problem is then to find a meromorphic function $f : X \to \mathbb{C}$ such that for each $i \in I$, $f|_{U_i} - f_i$ is holomorphic.

First notice that if the f_i agree on all overlaps $U_i \cap U_j$, then the sheaf condition on $\mathcal{M}$ (the sheaf of meromorphic functions on X) guarantees that there is a global meromorphic function $f \in \mathcal{M}(X)$ so that $f|_{U_i} - f_i = 0$ for all i, a much stronger conclusion than the Mittag-Leffler problem asks for. Thus in some sense, the problem is to study how far one can lift local meromorphic functions that do not glue together on overlaps. In any case, one way to find such a function $f : X \to \mathbb{C}$ in the previous paragraph is to find a family $\{h_i : U_i \to \mathbb{C}\}$ of holomorphic functions on each U_i such that $(f_i + h_i)|_{U_i \cap U_j} = (f_j + h_j)|_{U_i \cap U_j}$ for all i, j. This can be rewritten as

$$f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j} = h_j|_{U_i \cap U_j} - h_i|_{U_i \cap U_j}.$$

Set $t_{ij} = f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j}$. Then if the above equation is satisfied, we have $t_{ij} \in \mathcal{O}(U_i \cap U_j)$, where $\mathcal{O}$ is the sheaf of holomorphic functions on X (see Example 1.1.1) and

$$t_{jk} - t_{ik} + t_{ij} = 0.$$

Thus we want to find holomorphic functions $h_i \in \mathcal{O}(U_i)$ satisfying:

(1) $t_{ij} = h_j - h_i$ on $U_i \cap U_j$ (the boundary condition); and

(2) $t_{jk} - t_{ik} + t_{ij} = 0$ on $U_i \cap U_j \cap U_k$ (the cycle condition).

This generalizes to any space X with open cover $\mathcal{U} = \{U_i\}_{i \in I}$.

Definition. *Let F be a presheaf on X. A family of sections $(t_{ij}) \in \prod_{i,j} F(U_i \cap U_j)$ is called a* **Čech 1-cocycle** *if for all i, j, k, $t_{jk} - t_{ik} + t_{ij} = 0$ on $U_i \cap U_j \cap U_k$. The group of Čech 1-cocycles is written $\check{Z}^1(\mathcal{U}, F)$.*

Definition. *A family $(t_{ij}) \in \prod_{i,j} F(U_i \cap U_j)$ is called a* **Čech 1-coboundary** *if there is a family $(h_i) \in \prod_i F(U_i)$ such that $t_{ij} = h_j - h_i$ on each $U_i \cap U_j$. This forms a subgroup of $\check{Z}^1(\mathcal{U}, F)$ which is denoted $\check{B}^1(\mathcal{U}, F)$.*

Definition. *The* **first Čech cohomology group of the cover** $\mathcal{U}$ *with coefficients in F is the quotient group*

$$\check{H}^1(\mathcal{U}, F) := \check{Z}^1(\mathcal{U}, F)/\check{B}^1(\mathcal{U}, F).$$

Now to solve the Mittag-Leffler problem, it's enough to show that for a Riemann surface X with cover $\mathcal{U}$, $\check{H}^1(\mathcal{U}, \mathcal{O}) = 0$ where $\mathcal{O}$ is the sheaf of holomorphic functions on X. Note that these Čech cohomology groups depend on the cover $\mathcal{U}$. To package together all of the information about holomorphic functions on covers of X, we introduce the notion of a refinement of covers.

Definition. *Let $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{V} = \{V_j\}_{j\in J}$ be open covers of X. We say $\mathcal{V}$ is a* **refinement** *of $\mathcal{U}$ if there exists a function $\tau : J \to I$ such that for all $j \in J$, $V_j \subseteq U_{\tau(j)}$.*

Notice that such a refinement τ induces a map

$$\begin{aligned} \tau^1 : \check{Z}^1(\mathcal{U}, F) &\longrightarrow \check{Z}^1(\mathcal{V}, F) \\ (t_{ik})_{i,k\in I} &\ \left(t_{\tau(j)\tau(\ell)}|_{V_j \cap V_\ell}\right)_{j,\ell\in J}. \end{aligned}$$

Lemma 2.1.1. *If τ is a refinement from $\mathcal{U}$ to $\mathcal{V}$, then $\tau^1(\check{B}^1(\mathcal{U}, F)) \subseteq \check{B}^1(\mathcal{V}, F)$.*

Thus there is an induced map on Čech cohomology groups, $\check{H}^1(\mathcal{U}, F) \to \check{H}^1(\mathcal{V}, F)$ and one can check that this does not depend on the choice of map $\tau : J \to I$.

Definition. *The* **first Čech cohomology** *of X with coefficients in F is the direct limit*

$$\check{H}^1(X, F) = \varinjlim \check{H}^1(\mathcal{U}, F)$$

over all open covers $\mathcal{U}$ of X, ordered by refinement.

Lemma 2.1.2. *For any sheaf F on a Riemann surface X, the maps*

$$\check{H}^1(\mathcal{U}, F) \longrightarrow \check{H}^1(X, F)$$

are injective for every open cover $\mathcal{U}$ of X.

Proof. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. If $g = (g_{ik})_{i,k\in I} \in \check{H}^1(\mathcal{U}, F)$ becomes 0 in $\check{H}^1(X, F)$, this means there is some open cover $\mathcal{V} = \{V_j\}_{j\in J}$ which is a refinement of $\mathcal{U}$, say by $\tau : J \to I$, such that $g' = (g'_{j\ell})_{j,\ell} = \left(g_{\tau(j)\tau(\ell)}|_{V_j\cap V_\ell}\right)_{j,\ell}$ lies in $\check{B}^1(\mathcal{V}, F)$. Since g' is a

1-coboundary, there is a family $(h_j) \in \prod_j F(V_j)$ such that $g'_{j\ell} = h_j - h_\ell$ on $V_j \cap V_\ell$. For a fixed index $i \in I$, observe that

$$\begin{aligned} g'_{j\ell} &= g_{\tau(j)\tau(\ell)} \\ &= g_{\tau(j)i} + g_{i\tau(\ell)} \quad \text{by the cocycle condition} \\ &= g_{i\tau(\ell)} - g_{i\tau(j)} \quad \text{by the cocycle condition again} \\ &= h_j - h_\ell \quad \text{on } V_j \cap V_\ell. \end{aligned}$$

This can be rewritten as

$$g_{i\tau(j)} + h_j = g_{i\tau(\ell)} + h_\ell \quad \text{on } U_i \cap V_j \cap V_\ell.$$

Consider the cover $\{U_i \cap V_j\}_{j\in J}$ of U_i. By the sheaf axioms for F, there exists a section $t_i \in F(U_i)$ such that for each V_j, $t_i|_{U_i\cap V_j} = (g_{i\tau(j)} + h_j)|_{U_i\cap V_j}$. Now let $i, k \in I$ be arbitrary and fix some $j \in J$. Then

$$\begin{aligned} g_{ik} = g_{i\tau(j)} + g_{\tau(j)k} &= g_{i\tau(j)} - g_{k\tau(j)} \text{by the cocycle condition} \\ &= (t_i - h_j) - (t_k - h_j) \quad \text{by the above} \\ &= t_i - t_k \quad \text{on } U_i \cap U_k \cap V_j. \end{aligned}$$

Since j was arbitrary and $\{U_i \cap U_k \cap V_j\}_{j\in J}$ covers $U_i \cap U_k$, the sheaf axioms imply $g_{ik} - t_i - t_k$ on $U_i \cap U_k$. Hence $g_{ik} \in \check{B}^1(\mathcal{U}, F)$ so $g = 0$ in $\check{H}^1(\mathcal{U}, F)$. □

Theorem 2.1.3 (Mittag-Leffler). *For a non-compact Riemann surface X with sheaf of holomorphic functions $\mathcal{O}$, $\check{H}^1(U, \mathcal{O}) = 0$ for every open cover $\mathcal{U}$ of X.*

Proof. We will show that $\check{H}^1(X, \mathcal{O}) = 0$ in this situation, so Lemma 2.1.2 implies that $\check{H}^1(\mathcal{U}, \mathcal{O}) = 0$. □

This same strategy allows us to classify vector bundles on a manifold. Let X be a (smooth) real manifold and $\texttt{Vect}_k(X)$ the category of (smooth) vector bundles of rank k on X. For each open $U \subseteq X$, let $S(U)$ denote the algebra of smooth functions $U \to \mathbb{R}$ and set $F(U) = GL_k(S(U))$. Then F is a sheaf *of groups* on X. Let $\mathcal{U} = \{U_i\}$ be an open cover of X. Notice that for an element $\varphi = (\varphi_{ij}) \in \prod_{i,j} F(U_i \cap U_j)$, we have

$$\varphi_{jk} \circ \varphi_{ik}^{-1} \circ \varphi_{ij} = I_k$$

on $U_i \cap U_j \cap U_k$, where I_k is the $k \times k$ identity matrix. Using this as a jumping off point, one can define noncommutative versions of $\check{Z}^1(\mathcal{U}, F), \check{B}^1(\mathcal{U}, F)$ and $\check{H}^1(\mathcal{U}, F)$ – here, they are just pointed sets, with basepoint corresponding to the identity matrix. As before, set

$$\check{H}^1(X, F) = \varinjlim \check{H}^1(\mathcal{U}, F)$$

where the $\mathcal{U}$ are ordered by refinement. Then:

Proposition 2.1.4. *For a manifold X and $k \geq 1$, the set $\check{H}^1(X, GL_k(S))$ is the set of isomorphism classes of rank k vector bundles on X. Moreover, for each open cover $\mathcal{U}$ of X, $\check{H}^1(\mathcal{U}, GL_k(S))$ is the set of isomorphism classes of rank k vector bundles that are trivial over $\mathcal{U}$.*

Example 2.1.5. When $k = 1$, $\check{H}^1(X, S^\times)$ classifies line bundles (up to isomorphism) on X. This is a group under $\otimes$, called the *Picard group* of X.

2.2 The Čech Complex and Čech Cohomology

Definition. *Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X and let F be a sheaf of abelian groups on X. The* **Čech complex** *of F with respect to $\mathcal{U}$ is the cochain complex $C^\bullet(\mathcal{U}, F)$ defined by*

$$C^p(\mathcal{U}, F) = \prod_{i_0, \dots, i_p \in I} F(U_{i_0} \cap \cdots \cap U_{i_p})$$

with differential

$$\begin{aligned} d : C^p(\mathcal{U}, F) &\longrightarrow C^{p+1}(\mathcal{U}, F) \\ \alpha &\longmapsto \left(\sum_{k=0}^{p+1} (-1)^k \alpha_{i_0, \dots, i_{k-1}, i_{k+1}, \dots, i_{p+1}}|_{U_{i_0, \dots, i_{p+1}}} \right) \end{aligned}$$

where $U_{i_0, \dots, i_{p+1}} = U_{i_0} \cap \cdots \cap U_{i_{p+1}}$.

Lemma 2.2.1. $d^2 = 0$*; that is, $C^\bullet(\mathcal{U}, F)$ is a cochain complex.*

Definition. *The pth* **Čech cohomology with respect to an open cover** $\mathcal{U}$ *of a space X with coefficients in a sheaf F is the pth cohomology of the Čech complex:*

$$\check{H}^p(\mathcal{U}, F) := H^p(C^\bullet(\mathcal{U}, F)).$$

Lemma 2.2.2. *For any sheaf F and cover $\mathcal{U}$ of X, $\check{H}^0(\mathcal{U}, F) = H^0(X, F) = \Gamma(X, F)$.*

Proof. By definition, $\check{H}^0(\mathcal{U}, F) = \ker(d : C^0(\mathcal{U}, F) \to C^1(\mathcal{U}, F))$. For $\alpha = (\alpha_i) \in C^0(\mathcal{U}, F)$, we have $d\alpha = (\alpha_i - \alpha_j)_{i,j}$ which is zero if and only if $\alpha_i = \alpha_j$ on $U_i \cap U_j$ for all i, j. Thus $\ker d = \Gamma(X, F)$. □

Suppose $\mathcal{U}'$ is a refinement of $\mathcal{U}$, that is, $\mathcal{U}' = \{U'_j\}_{j \in J}$ is a cover of X and there is a function $\lambda : J \to I$ such that for all $j \in J$, $U'_j \subseteq U_{\lambda(j)}$. Then there is a chain map $C^\bullet(\mathcal{U}', F) \to C^\bullet(\mathcal{U}, F)$ given by

$$C^p(\mathcal{U}', F) \longrightarrow C^p(\mathcal{U}, F), \; (\alpha_{j_0, \dots, j_p}) \longmapsto (\alpha_{\lambda(j_0), \dots, \lambda(j_p)}|_{U_{j_0} \cap \cdots \cap U_{j_p}}).$$

This in turn induces maps on Čech cohomology:

$$\check{H}^p(\mathcal{U}', F) \longrightarrow \check{H}^p(\mathcal{U}, F)$$

for all p.

Definition. *The pth* **Čech cohomology** *of a space X with coefficients in a sheaf F is the direct limit*

$$\check{H}^p(X, F) := \varinjlim \check{H}^p(\mathcal{U}, F)$$

taken over all covers $\mathcal{U}$ ordered with respect to refinement.

Fix a space X, an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ and a sheaf F on X. We construct a *complex* of sheaves of abelian groups $\mathcal{C}^p(\mathcal{U}, F)$ on X as follows. For an open set $V \subseteq X$, define

$$\mathcal{C}^p(\mathcal{U}, F) : V \longmapsto \Gamma(V, \mathcal{C}^p(\mathcal{U}, F)) := \prod_{i_0,\ldots,i_p\in I} F(V \cap U_{i_0,\ldots,i_p})$$

where as usual $U_{i_0,\ldots,i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$.

Proposition 2.2.3. *If F is a sheaf on X, then $\mathcal{C}^p(\mathcal{U}, F)$ is a sheaf for all $p \geq 0$.*

We give $\mathcal{C}^\bullet(\mathcal{U}, F)$ the structure of a complex of sheaves by letting

$$d_V : \Gamma(V, \mathcal{C}^p(\mathcal{U}, F)) \longrightarrow \Gamma(V, \mathcal{C}^{p+1}(\mathcal{U}, F))$$

be the map induced from the Čech differential $d : C^p(\mathcal{U}, F) \to C^{p+1}(\mathcal{U}, F)$ restricted to each $V \cap U_{i_0,\ldots,i_p}$.

Proposition 2.2.4. *For an open cover $\mathcal{U}$ and a sheaf F, there is an exact sequence of sheaves*

$$0 \to F \to \mathcal{C}^0(\mathcal{U}, F) \to \mathcal{C}^1(\mathcal{U}, F) \to \cdots$$

Proof. The sheaf axioms on F imply $F \to \mathcal{C}^0(\mathcal{U}, F)$ is injective. To prove exactness at $\mathcal{C}^p(\mathcal{U}, F)$, we need to check that the sequence of stalks

$$\mathcal{C}^{p-1}(\mathcal{U}, F)_x \xrightarrow{\alpha} \mathcal{C}^p(\mathcal{U}, F)_x \xrightarrow{\beta} \mathcal{C}^{p+1}(\mathcal{U}, F)_x$$

is exact for all $x \in X$. Since $\mathcal{U}$ covers X, we know $x \in U_j$ for some U_j. Take $f_x \in \mathcal{C}^p(\mathcal{U}, F)_x$ and let V be a neighborhood of x and $f \in \Gamma(V, \mathcal{C}^p(\mathcal{U}, F))$ with $f|_x = f_x$. We may assume $V \subseteq U_j$. Then for each $p \geq 1$,

$$\begin{aligned} \theta^p : \mathcal{C}^p(\mathcal{U}, F)_x &\longrightarrow \mathcal{C}^{p-1}(\mathcal{U}, F)_x \\ f_x &\longmapsto f_{j,i_0,\ldots,i_{p-1}}|_x \end{aligned}$$

is well-defined and independent of the choice of V and U_j. Now with the same f_x and f as above, we have

$$\begin{aligned} \theta^{p+1}(df)_x &= (df)_{j,i_0,\ldots,i_p}|_x \\ &= \left.\left(f_{i_0,\ldots,i_p} - \sum_{k=0}^{p} (-1)^k f_{j,i_0,\ldots,i_{k-1},i_{k+1},\ldots,i_p} \right)\right|_x \\ &= f_x - d(\theta^p f_x). \end{aligned}$$

Hence $\theta^{p+1} d + d\theta^p$ is the identity on $\mathcal{C}^p(\mathcal{U}, F)_x$, which shows the identity map on this stalk is chain homotopic to the zero map and thus $\ker\beta = \operatorname{im}\alpha$, proving exactness. □

Definition. *For a sheaf F on X, the exact sequence*

$$0 \to F \to \mathcal{C}^0(\mathcal{U}, F) \to \mathcal{C}^1(\mathcal{U}, F) \to \cdots$$

is called the **Čech resolution** *of F with respect to the cover $\mathcal{U}$.*

The sheafification functor $\texttt{Presh}_X \to \texttt{Sh}_X$ (Theorem 1.4.4) can alternatively be constructed using the Čech resolution. In fact, this construction works in a much more general context known as Grothendieck topology. Let F be a presheaf on a space X, $\mathcal{U} = \{U_i\}$ an open cover of X and

$$\check{H}^0(\mathcal{U}, F) = \left\{ f = (f_i) \in \prod F(U_i) : f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \text{ for all } i, j \right\}.$$

For an open set $U \subseteq X$, let

$$\check{H}^0(U, F|_U) = \varinjlim \check{H}^0(\mathcal{U}', F|_U)$$

where the limit is over all open covers $\mathcal{U}'$ of U. Define the set $F^\#(U) = \check{H}^0(U, F|_U)$. For an inclusion of open sets $V \hookrightarrow U$ in X, there is a natural restriction map $\check{H}^0(U, F|_U) \to \check{H}^0(V, F|_V)$ coming from restriction of an open cover. This makes $U \mapsto F^\#(U)$ into a presheaf on X. Moreover, Proposition 2.2.4 shows that $F \to F^\#$ is injective.

Definition. *We call a presheaf F on X a* **separated presheaf** *if it satisfies the first sheaf axiom; that is, if $U = \bigcup U_i$ is an open cover and there are sections $s, t \in F(U)$ such that $s|_{U_i} = t|_{U_i}$ for all U_i, then $s = t$.*

Theorem 2.2.5. *Let F be a presheaf on X. Then*

(1) $F^\#$ is a separated presheaf on X.

(2) If F is a separated presheaf, then $F^\#$ is a sheaf. In particular, $F^{\#\#}$ is always a sheaf on X.

(3) $F^{\#\#}$ is a sheafification of F.

Proof. (1) Suppose $s, t \in F^\#(U)$ and there is an open cover $\{V_j\}$ of U such that $s|_{V_j} = t|_{V_j}$ for all V_j. We must show $s = t$ in $F^\#(U)$. Since they are elements of a direct limit, s and t have representatives

$$s_U = (s_i), t_U = (t_i) \in \check{H}^0(\mathcal{U}, F|_U)$$

on some open cover $\mathcal{U}$ of U – a priori there may be two different covers supporting s and t, but after a common refinement of the covers we may assume this takes place on just a single cover. By hypothesis, there is an open cover $\mathcal{W}_j = \{W_{jk}\}_k$ of each V_j that is a refinement of $\{V_j \cap U_i\}_i$ such that $s_i|_{W_{jk}} = t_i|_{W_{jk}}$ for all k. Then $\mathcal{W} = \{W_{jk}\}_{j,k}$ is an open cover of U on which $s_i|_{W_{jk}} = t_i|_{W_{jk}}$ on every element. Therefore the images of s_U and t_U in $\varinjlim \check{H}^0(\mathcal{U}, F|_U)$ are equal, so $s = t$ as required.

(2) Now assume F itself is separated; we must show that $F^\#$ satisfies the second sheaf axiom. As an intermediate step, we claim that for any open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of U, the map

$$\check{H}^0(\mathcal{U}, F|_U) \longrightarrow \check{H}^0(U, F|_U)$$

is injective (this is conditioned on F being separated). Notice that if $s, t \in \check{H}^0(\mathcal{U}, F|_U)$ become equal in the direct limit $\check{H}^0(U, F|_U)$, then there exists a cover $\mathcal{V} = \{V_j\}_{j \in J}$ of U

such that $s|_{V_j} = t|_{V_j}$ for all $j \in J$. We may assume $\mathcal{V}$ is a refinement of $\mathcal{U}$, say by a map $\tau : J \to I$. Then $s_{\tau(j)}|_{V_j} = t_{\tau(j)}|_{V_j}$ for all $j \in J$. Since s and t are Čech 0-cocycles, we have

$$s_i|_{U_i \cap U_{\tau(j)}} = s_{\tau(j)}|_{U_i \cap U_{\tau(j)}} \qquad \text{and} \qquad t_i|_{U_i \cap U_{\tau(j)}} = t_{\tau(j)}|_{U_i \cap U_{\tau(j)}}$$

for all i, j. Since $V_j \subseteq U_{\tau(j)}$ for all j, this means

$$s_i|_{U_i \cap V_j} = t_i|_{U_i \cap V_j}$$

for all i, j. Fixing $i \in I$, the collection $\{U_i \cap V_j\}_j$ is an open cover of U_i and $s_i|_{U_i \cap V_j} = t_i|_{U_i \cap V_j}$, so by the separated condition, $s_i = t_i$ on U_i. This implies $s = t$ in $\check{H}^0(\mathcal{U}, F|_U)$, proving the claim.

Now take $s_i \in F^\#(U_i)$ for some cover $\mathcal{U} = \{U_i\}_{i \in I}$ of U such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$. We must construct a section $s \in F^\#(U)$ restricting to s_i on each U_i. Each s_i is represented by some $s_{\mathcal{V}_i} = (s_{ik}) \in \check{H}^0(\mathcal{V}_i, F|_{U_i})$ for an open cover $\mathcal{V}_i = \{V_{ik}\}_k$ of U_i. Then $\mathcal{V} = \{V_{ik}\}_{i,k}$ is an open cover of U. Fixing $i, j \in I$, consider the cover $\mathcal{W} = \{V_{ik} \cap V_{j\ell}\}_{k,\ell}$ of $U_i \cap U_j$. The sections

$$\tilde{s}_i = (s_{ik}|_{V_{ik} \cap V_{j\ell}}), \tilde{s}_j = (s_{j\ell}|_{V_{ik} \cap V_{j\ell}}) \in \check{H}^0(\mathcal{W}, F|_{U_i \cap U_j})$$

become equal in $\check{H}^0(U_i \cap U_j, F|_{U_i \cap U_j})$ by assumption, so because F is separated, the previous paragraph shows $\tilde{s}_i = \tilde{s}_j$ in $\check{H}^0(\mathcal{W}, F|_{U_i \cap U_j})$. Therefore the $\tilde{s}_i$ give a well-defined element $s \in \check{H}^0(U, F|_U)$ restricting to s_i on each U_i. Hence $F^\#$ is a sheaf.

(3) Suppose $F \to G$ is a morphism of presheaves, where G is a sheaf. This determines a chain map

$$C^\bullet(\mathcal{U}, F) \longrightarrow C^\bullet(\mathcal{U}, G)$$

for any open cover $\mathcal{U}$ of X. Hence there is a map on Čech cohomology

$$\check{H}^p(\mathcal{U}, F) \longrightarrow \check{H}^p(\mathcal{U}, G)$$

for all $p \geq 0$, which is compatible with refinement. In particular, there is a map

$$\check{H}^0(X, F) \longrightarrow \check{H}^0(X, G).$$

Let $U \subseteq X$ be an open set. Then there is a map $\check{H}^0(U, F|_U) \to \check{H}^0(U, G|_U)$ which fits into a diagram

$$\begin{array}{ccc} F(U) & \longrightarrow & G(U) \\ \downarrow & \nearrow & \downarrow \cong \\ \check{H}^0(U, F|_U) & \longrightarrow & \check{H}^0(U, G|_U) \end{array}$$

The right column is an isomorphism since G is a sheaf. Thus we may defined the dashed arrow $\check{H}^0(U, F|_U) \to G(U)$ which induces a morphism of presheaves $F^\# \to G$ through which $F \to G$ factors. Repeating the argument shows that $F \to G$ further factors through a morphism $F^{\#\#} \to G$. Uniqueness follows from a similar argument as the one in the proof of Theorem 1.4.4. □

3 Sheaf Cohomology

3.1 Derived Functors and Cohomology

To construct sheaf cohomology, we first recall the basic theory of derived functors. Proofs can be found in Rotman, among many places.

Definition. *Suppose $T : \mathcal{A} \to \mathcal{C}$ is a covariant, additive, right exact functor between abelian categories, where $\mathcal{A}$ has enough projectives. The n***th left derived functor** *of T is the functor*

$$\begin{aligned} L_nT : \mathcal{A} &\longrightarrow \mathcal{C} \\ A &\longmapsto H_n(T(P_\bullet)) \end{aligned}$$

where $P_\bullet$ is a fixed projective resolution of A.

Derived functors in some sense measure the failure of exactness in the nth homological dimension of the functor T. The dual notion to a left derived functor is a right derived functor.

Definition. *For a covariant, additive, left exact functor $S : \mathcal{A} \to \mathcal{C}$ between abelian categories, where $\mathcal{A}$ has enough injectives, the n***th right derived functor** *of S is the functor*

$$\begin{aligned} R^nS : \mathcal{A} &\longrightarrow \mathcal{C} \\ A &\longmapsto H_{-n}(S(E_\bullet)) \end{aligned}$$

where $E_\bullet$ is a fixed injective resolution of A.

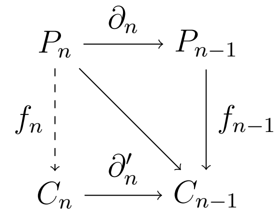
In defining left and right derived functors, we are implicitly choosing a particular projective (or injective) resolution of A. The Comparison Theorem says that unique chain maps exist between projective resolutions of M and N when $M \to N$ is a module homomorphism, so this choice does not matter when defining L_nT and R^nT.

Theorem 3.1.1 (Comparison). *Let $P_\bullet : \cdots \to P_2 \to P_1 \to P_0$ be a projective chain complex and suppose $C_\bullet : \cdots C_2 \to C_1 \to C_0$ is an acyclic chain complex. Then for any homomorphism $\varphi : H_0(P_\bullet) \to H_0(C_\bullet)$, there is a chain map $f : P_\bullet \to C_\bullet$ whose induced map on H_0 is φ, and φ is unique up to chain homotopy.*

Proof. Consider the diagram

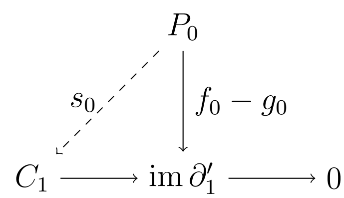
$$\begin{array}{ccccccccccc} \longrightarrow & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\partial_0} & H_0(P_\bullet) & \longrightarrow & 0 \\ & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle \varphi} & & \\ \longrightarrow & C_2 & \xrightarrow{\partial_2'} & C_1 & \xrightarrow{\partial_1'} & C_0 & \xrightarrow{\partial_0'} & H_0(C_\bullet) & \longrightarrow & 0 \end{array}$$

Since P_0 is projective, there exists an f_0 lifting φ to $P_0 \to C_0$. Inductively, given f_{n-1} we have a diagram



Note that $f_{n-1}\partial$ has image lying in $\ker \partial'_n \subseteq C_{n-1}$, but $C_\bullet$ is acyclic, so $\partial'_n(C_n) = \ker \partial'_{n-1}$. Since P_n is projective, we can lift $f_{n-1}\partial_n$ to the desired map $f_n : P_n \to C_n$. By construction, $f = \{f_n\}_{n=0}^{\infty}$ satisfies the desired properties.

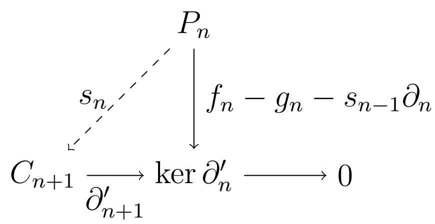
For uniqueness, suppose $g : P_\bullet \to C_\bullet$ is another chain map restricting to φ on H_0. Since $f_0 - g_0 = 0$ on H_0, it must be that $(f_0 - g_0)(P_0) \subseteq \ker \partial'_0 = \operatorname{im} \partial'_1$, so by projectivity of P_0 there exists $s_0 : P_0 \to C_1$ making the following diagram commute:



Inductively, given $s_0, \ldots, s_{n-1}$ satisfying $f_k - g_k = \partial'_{k+1}s_k + s_{k+1}\partial_k$ for all $0 \leq k \leq n-1$, we have

$$\begin{aligned} \partial'_n(f_n - g_n - s_{n-1}\partial_n) &= (f_{n-1} - g_{n-1})\partial_n - \partial'_n s_{n-1}\partial_n \quad \text{since } f, g \text{ are chain maps} \\ &= (\partial'_n s_{n-1} - s_{n-2}\partial_{n-1})\partial_n - \partial'_n s_{n-1}\partial_n = 0. \end{aligned}$$

Hence there is a commutative diagram



This establishes the chain homotopy $s : P_\bullet \to C_\bullet$ such that $f_n - g_n = \partial'_{n+1}s_n + s_{n+1}\partial_n$ for all $n \geq 0$. Hence f is unique up to chain homotopy. □

Corollary 3.1.2. *Let $g : M \to N$ be R-linear and pick projective resolutions $P_\bullet$ and $Q_\bullet$ of M and N, respectively. Then there exists a chain map $f : P_\bullet \to Q_\bullet$ such that $H_0(f) = g$ and f is unique up to chain homotopy.*

Proof. Given projective resolutions $P_\bullet, Q_\bullet \to M$, we have $M = H_0(P_\bullet) = H_0(Q_\bullet)$ so let $\varphi = id_M$. Since projective resolutions are acyclic, the comparison theorem gives us a chain map $f : P_\bullet \to Q_\bullet$. Reversing the roles of $P_\bullet$ and $Q_\bullet$ gives a chain map in the opposite direction, and uniqueness forces the composition of these maps to be the identity in either direction. □

Theorem 3.1.3. *If $T : \mathcal{A} \to \mathcal{C}$ is an additive covariant functor between abelian categories, and $\mathcal{A}$ has enough projectives, then each $L_nT : \mathcal{A} \to \mathcal{C}$ is an additive covariant functor.*

Proposition 3.1.4. *If $T : \mathcal{A} \to \mathcal{C}$ is an additive covariant functor of abelian categories, then $(L_nT)A = 0$ for all $A \in \operatorname{obj}(\mathcal{A})$ whenever n is negative.*

Proof. This is immediate from the definition since the nth term of a resolution P of A is 0 whenever n is negative. □

Analogous statements hold for right derived functors.

3.2 Sheaf Cohomology

Let X be a topological space and consider the category $\mathtt{Sh}_X$ of sheaves of abelian groups on X. Let $\Gamma : F \mapsto \Gamma(X, F) = F(X)$ be the global sections functor on X. By Theorem 1.3.10, Γ is left exact but in general it is not right exact.

Theorem 3.2.1. *For any topological space X, the category $\mathtt{Sh}_X$ has enough injectives.*

Proof. Given $x \in X$ and an abelian group A, let x_*A denote the skyscraper sheaf at x with coefficients A:

$$x_*A(U) = \begin{cases} A, & x \in U \\ 0, & x \notin U. \end{cases}$$

Then for a sheaf G, $\operatorname{Hom}(G, x_*A)$ consists of collections of homomorphisms $G(U) \to A$ for each U containing x which are compatible with the restrictions $G(U) \to G(V)$ for $V \hookrightarrow U$. By the universal property of stalks (they are limits), this is the same as a map $G_x \to A$. Thus

$$\operatorname{Hom}_{\mathtt{Sh}_X}(G, x_*A) \cong \operatorname{Hom}_{\mathtt{Ab}}(G_x, A).$$

If A is an injective abelian group, this shows that x_*A is an injective object of $\mathtt{Sh}_X$. Moreover, since Hom commutes with products, we have

$$\operatorname{Hom}_{\mathtt{Sh}_X}\left(G, \prod_{x\in X} x_*A_x\right) \cong \prod_{x\in X} \operatorname{Hom}_{\mathtt{Sh}_X}(G, x_*A_x) \cong \prod_{x\in X} \operatorname{Hom}_{\mathtt{Ab}}(G_x, A_x)$$

for any family of abelian groups $(A_x)_{x\in X}$. Since $\mathtt{Ab}$ has enough injectives, for each $x \in X$ we may choose a monic $\varphi_x : G_x \hookrightarrow A_x$ with A_x injective, so that the induced morphism

$$G \longrightarrow \prod_{x\in X} x_*A_x$$

is monic and $\prod_{x\in X} x_*A_x$ is injective. □

This allows us to define the cohomology groups of X with coefficients in a sheaf F to be the right derived functors of Γ.

Definition. *Let X be a topological space and F a sheaf on X. Then the* **sheaf cohomology** *of X with coefficients in F is the sequence of right derived functors of $\Gamma : \mathtt{Sh}_X \to \mathtt{Ab}$,*

$$H^i(X, F) := R^i\Gamma(X, F) = H^i(\Gamma(X, E^\bullet))$$

where $E^\bullet$ is an injective resolution of F in the category of sheaves.

One way to compute sheaf cohomology is by comparing it to Čech cohomology, which is much more amenable to calculation. To this end, we have the following consequence of the Čech resolution construction.

Theorem 3.2.2. *For any sheaf F on X and any open cover $\mathcal{U}$ of X, there is a natural map*

$$\check{H}^p(\mathcal{U}, F) \longrightarrow H^p(X, F)$$

for all $p \geq 0$ which is compatible with refinement. In particular, there is a natural map

$$\check{H}^p(X, F) \longrightarrow H^p(X, F)$$

for all $p \geq 0$.

Proof. Applying the global sections functor to the Čech resolution of F yields

$$0 \to F(X) \to \check{C}^0(\mathcal{U}, F) \to \check{C}^1(\mathcal{U}, F) \to \cdots$$

Clearly the cohomology groups of this sequence are precisely the Čech cohomology groups $\check{H}^p(\mathcal{U}, F)$. Therefore there is a chain map

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \longrightarrow & \mathcal{C}^0(\mathcal{U}, F) & \longrightarrow & \mathcal{C}^1(\mathcal{U}, F) & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & E^0 & \longrightarrow & E^1 & \longrightarrow & \cdots
\end{array}$$

for any injective resolution $E^\bullet$ of F. This induces the maps on cohomology, $\check{H}^p(\mathcal{U}, F) \to H^p(X, F)$, and one can check that they commute with the maps coming from refinements of open covers. The second statement follows. □

One of the main phenomena to exploit is that $\mathtt{Sh}_X$ has many acyclic objects which are not injective. This is important because of the following result.

Theorem 3.2.3. *Let F be a sheaf on X and suppose $L^\bullet$ is an acyclic resolution of F, i.e. a complex of sheaves for which each L^n is acyclic. Then for all $i \geq 0$,*

$$H^i(X, F) \cong H^i(\Gamma(X, L^\bullet)).$$

Proof. The statement is trivial for $i = 0$. Let

$$0 \to F \to L^0 \xrightarrow{f_0} L^1 \xrightarrow{f_1} L^2 \to \cdots$$

be our acyclic resolution of F. Set $K^0 = F$ and for each $i \geq 1$, $K^i = \operatorname{im} f_{i-1} \subseteq L^i$. Then there are short exact sequences of sheaves

$$0 \to K^i \xrightarrow{e_i} L^i \xrightarrow{g_i} K^{i+1} \to 0$$

where e_i is the inclusion of a subsheaf and the g_i satisfy $e_{i+1} \circ g_i = f_i$. Since Γ is left exact (Theorem 1.3.10), each sequence

$$0 \to \Gamma(X, K^i) \xrightarrow{\Gamma e_i} \Gamma(X, L^i) \xrightarrow{\Gamma g_i} \Gamma(X, K^{i+1})$$

is exact. Note that Γe_i identifies $\Gamma(X, K^i)$ with $\ker \Gamma g_i$, which is equal to $\ker \Gamma f_i$ since Γe_i is injective. Likewise, $\operatorname{im} \Gamma g_{i-1} = \operatorname{im} \Gamma f_{i-1}$. Therefore $\Gamma(X, K^i)/\operatorname{im} \Gamma g_{i-1} \cong \ker \Gamma f_i/\operatorname{im} \Gamma f_{i-1} = H^i(\Gamma(X, L^\bullet))$. On the other hand, there is a long exact sequence

$$0 \to \Gamma(X, K^i) \to \Gamma(X, L^\bullet) \xrightarrow{\Gamma g_i} \Gamma(X, K^{i+1}) \to H^1(X, K^i) \to H^1(X, L^i) = 0$$

with a zero on the end because L^i is acyclic. Hence

$$H^1(X, K^i) \cong \Gamma(X, K^{i+1})/\operatorname{im} \Gamma g_i = H^{i+1}(\Gamma(X, L^\bullet)).$$

The same argument shows that there is an isomorphism $H^{j-1}(X, K^{i+1}) \cong H^j(X, K^i)$ for each $j \geq 2$. In particular, $H^1(X, K^i) \cong H^{i+1}(X, K^0) = H^{i+1}(K, F)$ so we obtain $H^{i+1}(X, F) \cong H^{i+1}(\Gamma(X, L^\bullet))$ for all $i \geq 0$. □

In fact, Theorem 3.2.3 holds in much greater generality, namely for the derived functors of a left exact functor out of an abelian category with enough injectives, but we will only need it for sheaf cohomology computations in these notes.

In general, cohomology is hard to compute from the derived functors definition. However, we may begin by recovering ordinary (singular) cohomology with coefficients:

Theorem 3.2.4. *Let X be a topological space, A an abelian group and A_X the constant sheaf on X with stalks A. Then*

$$H^i(X, A_X) \cong H^i(X; A)$$

where $H^i(X; A)$ denotes the ordinary cohomology of X with coefficients in A.

Proof. For each open set $U \subseteq X$, let $C_n(U; A)$ denote the A-module spanned by singular simplices on U and let $\partial : C_n(U; A) \to C_{n-1}(U; A)$ denote the simplicial boundary map. Dualizing gives us the modules of singular cosimplices $C^n(U; A) := \operatorname{Hom}_A(C_n(U; A), A)$ with coboundary map given by the adjoint, $d = \partial^* : C^n(U; A) \to C^{n+1}(U; A)$. The assignment $U \mapsto C^n(U; A)$ defines a presheaf C^n on X for each $n \geq 0$. In fact by construction, for any inclusion $V \hookrightarrow U$, the restriction map $C^n(U; A) \to C^n(V; A)$ is surjective, so C^n is a flasque presheaf on X. Note that C^n satisfies the gluing axiom of a sheaf, but it is *not separated.* However, sheafifying a flasque presheaf which satisfies the gluing axiom yields a flasque sheaf in general, so setting $\mathcal{C}^n := (C^n)^{sh}$, we get a complex of flasque sheaves $\mathcal{C}^\bullet$. The differential $d : \mathcal{C}^n \to \mathcal{C}^{n+1}$ is induced by the coboundary maps $d_U : C^n(U; A) \to C^{n+1}(U; A)$. In fact, one can show that $\mathcal{C}^n$ is isomorphic to the quotient sheaf C^n/C_0^n, where

$$C_0^n(U) = \{\varphi \in C^n(U) \mid \text{there exists an open cover } \mathcal{U} = \{U_i\} \text{ of } U \text{ such that } \varphi|_{U_i} \equiv 0\}.$$

To verify that $\mathcal{C}^\bullet$ is an exact complex, it suffices to compute exactness on stalks. On a neighborhood of each point, however, this sequence becomes $C^\bullet(U)$ which is exact by construction in algebraic topology. Note that $\ker(d : \mathcal{C}^0 \to \mathcal{C}^1)$ is precisely the constant sheaf A_X, so $\mathcal{C}^\bullet$ is a flasque resolution of X. We will show (Theorem 3.4.3) that flasque sheaves are acyclic, so Theorem 3.2.3 implies that

$$H^i(X, A_X) \cong H^i(\Gamma(X, \mathcal{C}^\bullet))$$

for all $i \geq 0$.

Finally, singular cohomology is computed by the complex of abelian groups $C^\bullet(X)$ but we have a short exact sequence of complexes of presheaves

$$0 \to C_0^\bullet \to C^\bullet \to \mathcal{C}^\bullet \to 0$$

so to prove $H^i(X; A) = H^i(\Gamma(X, C^\bullet)) \cong H^i(\Gamma(X, \mathcal{C}^\bullet)) = H^i(X, A_X)$, it remains to show that $C_0^\bullet$ is acyclic. Fix an open cover $\mathcal{U}$ of X and let $C_n^{\mathcal{U}}(U; A)$ denote the submodule of $C_n(U; A)$ spanned by simplices lying in an element of $\mathcal{U}$. Set $C_{\mathcal{U}}^n(U; A) = \mathrm{Hom}_A(C_n^{\mathcal{U}}(U; A), A)$ and let $C_{\mathcal{U}}^\bullet$ be the resulting complex of presheaves on X. There is a natural morphism of presheaves $C^n \to C_{\mathcal{U}}^n$ which is surjective; call its kernel $C_{\mathcal{U},0}^n$. Then we can see that

$$C_0^n = \varinjlim C_{\mathcal{U},0}^n$$

where the limit is taken over all open covers $\mathcal{U}$ of X. This identification is compatible with differentials, so we get $C_0^\bullet = \varinjlim C_{\mathcal{U},0}^\bullet$. By a result in algebraic topology, there exists a cover $\mathcal{U}$ for which $C_{\mathcal{U},0}^\bullet$ is acyclic, which finishes the proof. □

Now the strategy for computing general sheaf cohomology is clear: we may take an injective resolution of A_X and compute the cohomology of the resulting complex. Of course, different ways of resolving a constant sheaf can give different ways of computing cohomology, and these observations will produce beautiful comparisons between different flavors of cohomology theory.

3.3 Direct and Inverse Image

Let $\mathtt{Sh}_X$ denote the category of sheaves (of abelian groups) on a topological space X. For a continuous map $f : X \to Y$, there are various functors between $\mathtt{Sh}_X$ and $\mathtt{Sh}_Y$ which may illuminate the structure of these categories. The easiest to define is the direct image.

Definition. *For a morphism $f : X \to Y$ and a presheaf F on $\mathtt{Top}_X$, define the* **direct image** *(or* **pushforward***) of F along f to be the presheaf $f_*F : \mathtt{Top}_Y^{op} \to \mathtt{Ab}$ sending $V \mapsto (f_*F)(V) := F(f^{-1}(V))$ for all open $V \subseteq Y$.*

Proposition 3.3.1. *For any continuous map $f : X \to Y$, f_* is a functor $\mathtt{Sh}_X \to \mathtt{Sh}_Y$.*

Proof. Suppose $\varphi : F \to G$ is a morphism of presheaves on X. Then for all open $V \subseteq Y$, define $(f_*\varphi)_V : (f_*F)(V) \to (f_*G)(V)$ to be the map $\varphi_{f^{-1}(V)} : F(f^{-1}(V)) \to G(f^{-1}(V))$. This shows that f_* is a functor. To finish, take a covering $V = \bigcup V_i$ in Y, set $U = f^{-1}(V)$ and consider the covering $U = \bigcup U_i$ where $U_i = f^{-1}(V_i)$. Suppose $s, t \in (f_*F)(V) = F(U)$ such that $s|_{V_i} = t|_{V_i}$ for all V_i. This means $s|_{V_i} = t|_{V_i}$ in $(f_*F)(V_i) = F(U_i)$, so by the first sheaf axiom for F, $s = t$ in $F(U) = (f_*F)(V)$. On the other hand, suppose $s_i \in (f_*F)(V_i) = F(U_i)$ for all i such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for all V_i, V_j. This means s_i and s_j are equal in each $(f_*F)(V_i \cap V_j) = F(f^{-1}(V_i \cap V_j)) = F(U_i \cap U_j)$ so by the second sheaf axiom on F, there is a section $s \in F(U)$ restricting to s_i on each U_i. That is, $s \in (f_*F)(V)$ such that $s|_{V_i} = s_i$ in each $(f_*F)(V_i)$. Hence f_*F is a sheaf. □

Example 3.3.2. If $i : \{x\} \to X$ is a point of X, then any sheaf on $\{x\}$ is an abelian group, say A, and $i_*A = A_x$ is the skyscraper sheaf on X supported at x with stalk A. This explains the usual notation x_*A for skyscraper sheaves.

Example 3.3.3. Consider the inclusion $j : D^2 \hookrightarrow \mathbb{R}^2$ and let F be the sheaf of locally constant functions on D^2 with values in S (as in Example 1.3.2). Then for all open sets $U \subseteq \mathbb{R}^2$, $(j_*F)(U) = F(U \cap D^2)$. If $U \cap D^2$ has n connected components, then $(j_*F)(U) = S^{\times n}$.

Example 3.3.4. Let $p : X \to Y$ be a finite-sheeted covering space of degree $n > 1$ and let $F = F_S$ be the sheaf of locally constant functions on Y with values in S, $|S| > 1$. Then for $V \subseteq Y$ sufficiently small, $f^{-1}(V) = \coprod_{i=1}^n U_i$ for disjoint neighborhoods $U_1, \ldots, U_n$ which are each homeomorphic to V. Thus $(p_*F)(V) = S^{\times n}$. This shows that for each $y \in Y$, the stalk at y of F is

$$(p_*F)_y = S^{\times n}.$$

Another sheaf on Y with stalk $S^{\times n}$ is the locally constant sheaf $G = F_{S^{\times n}}$. However, p_*F is not isomorphic to G, as we have

$$(p_*F)(Y) = S \neq S^{\times n} = G(Y).$$

Proposition 3.3.5. *For any map $f : X \to Y$, the functor $f_* : \mathtt{Sh}_X \to \mathtt{Sh}_Y$ is left exact.*

Proof. Suppose $0 \to F \to G \to H \to 0$ is a short exact sequence in $\mathtt{Sh}_X$. By Theorem 1.3.10, the sequence

$$0 \to F(U) \to G(U) \to H(U)$$

is exact for all open sets $U \subseteq X$. In particular, if $V \subseteq Y$ is open, then the above applied to $U = f^{-1}(V)$ shows that

$$0 \to (f_*F)(V) \to (f_*G)(V) \to (f_*H)(V)$$

is exact. Passing to stalks proves the statement. □

This allows us to define the right derived functors of f_*.

Definition. *Let $E^\bullet$ be an injective resolution of a sheaf $F \in \mathtt{Sh}_X$ and suppose $f : X \to Y$ is a continuous map. The* **higher direct images** *of f are the right derived functors of the direct image functor:*

$$R^n f_* F := \mathcal{H}^n(f_* E^\bullet)$$

where for a complex of sheaves $G^\bullet$ on Y and $n \geq 0$, $\mathcal{H}^n(G^\bullet)$ denotes the quotient sheaf $\ker(G^n \to G^{n+1})/\operatorname{im}(G^{n-1} \to G^n)$.

Proposition 3.3.6. *Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ are continuous maps. Then $(g \circ f)_* = g_* \circ f_*$.*

Lemma 3.3.7. *Let $f : X \to Y$ be a continuous map, $\mathcal{V} = \{V_j\}$ an open cover of Y and $\mathcal{U} = \{f^{-1}(V_j)\}$ the induced open cover of X. Then for all sheaves F on X and all $p \geq 0$,*

$$\check{H}^p(\mathcal{U}, F) = \check{H}^p(\mathcal{V}, f_* F).$$

Proof. Set $V_{j_0,\ldots,j_p} = V_{j_0} \cap \cdots \cap V_{j_p}$ and $U_{j_0,\ldots,j_p} = f^{-1}(V_{j_0,\ldots,j_p})$. Then $(f_*F)(V_{j_0,\ldots,j_p}) = F(U_{j_0,\ldots,j_p})$ so it follows that $C^p(\mathcal{U}, F) = C^p(\mathcal{V}, f_*F)$ for all $p \geq 0$. Hence the cohomologies of these complexes are the same. □

Example 3.3.8. Let $i : X \hookrightarrow Y$ be a closed embedding and let F be a sheaf on X. Then for any $y \in Y$, the stalk of i_*F at y is

$$(i_*F)_y = \begin{cases} F_x, & y = i(x) \in i(X) \\ 0, & y \notin i(X). \end{cases}$$

In particular, this proves:

Lemma 3.3.9. *If $i : X \hookrightarrow Y$ is a closed embedding, then $i_* : \mathtt{Sh}_X \to \mathtt{Sh}_Y$ is an exact functor.*

Therefore, if $E^\bullet$ is an injective resolution of a sheaf F on X, then for a closed embedding $i : X \hookrightarrow Y$ the cohomology of i_*F may be computed via the resolution $i_*E^\bullet$:

$$H^p(X, F) = H^p(Y, i_*F) = H^p(\Gamma(Y, i_*E^\bullet)).$$

Let $f : X \to Y$ be any continuous map. There is also an inverse image functor $f^* : \mathtt{Sh}_Y \to \mathtt{Sh}_X$ which is a little harder to construct, since $f(U)$ need not be open in Y for all open $U \subseteq X$.

Definition. *Let $f : X \to Y$ be a continuous map, F a sheaf on Y and define a presheaf $f^{-1}F$ by*

$$(f^{-1}F)(U) = \varinjlim F(V)$$

where the direct limit is taken over all open sets $V \subseteq Y$ containing $f(U)$. The **inverse image** *(or* **pullback***) of F along f, denoted f^*F, is the sheafification of $f^{-1}F$.*

Remark. The inverse image sheaf may also be defined on the level of étale spaces as follows. Let $E_F \to Y$ be the étale space of F. Then f^*F is the sheaf of sections of the map $E = E_{f^*F} \to X$ whose total space is the pullback in the diagram

$$\begin{array}{ccc} E & \longrightarrow & X \\ \downarrow & & \downarrow f \\ E_F & \longrightarrow & Y \end{array}$$

Example 3.3.10. If $f : X \to Y$ is an open map, then for all $U \subseteq X$, $(f^{-1}F)(U) = F(f(U))$ and this already defines a sheaf, so $f^{-1}F = f^*F$.

Example 3.3.11. Let $i : \{x\} \hookrightarrow X$ be the inclusion of a point in X and let F be a sheaf on X. Then $i^*F = F_x$, the (constant sheaf on $\{x\}$ given by the) stalk of F at x. More generally:

Proposition 3.3.12. *For any continuous map $f : X \to Y$, sheaf F on Y and point $x \in X$, there is a natural isomorphism $(f^*F)_x \cong F_{f(x)}$.*

Proof. Let $f^{-1}F$ be the presheaf $U \mapsto \varinjlim F(V)$ as in the definition of inverse image, set $y = f(x)$ and suppose $U \subseteq X$ is an open neighborhood of x and $V \subseteq Y$ is an open set in Y containing $f(U)$. Then y lies in every such V so we get morphisms $F(V) \to F_y$ which assemble into a morphism out of the direct limit,

$$\lambda_U : (f^{-1}F)(U) \longrightarrow F_y.$$

Taking the limit over all such neighborhoods U of x gives a morphism $\lambda : (f^{-1}F)_x \to F_y$. We claim this is an isomorphism. For $s_y \in F_y$, choose a representative $s_V \in F(V)$ on some open neighborhood V of y. Then $U = f^{-1}(V)$ is an open neighborhood of x in X such that $f(U) \subseteq V$. Hence there are maps

$$F(V) \to (f^{-1}F)(U) \xrightarrow{\lambda_U} F_y$$

whose composition is the stalk map $F(V) \to F_y$. Thus $\lambda_U(f^{-1}s_V)$ maps to s_y in the stalk at y, so λ is surjective.

On the other hand, if $t_1, t_2 \in (f^{-1}F)_x$ with $\lambda(t_1) = \lambda(t_2)$, then there is some neighborhood U of x such that $t_1 = t_{U,1}|_x$ and $t_2 = t_{U,2}|_x$ for sections $t_{U,1}, t_{U,2} \in (f^{-1}F)(U)$. (A priori, these happen on different open neighborhoods of x but we may pass to the intersection.) In turn these $t_{U,1}, t_{U,2}$ are represented by sections $s_{V,1}, s_{V,2} \in F(V)$ for some open set V containing $f(U)$ and the fact that $\lambda(t_1) = \lambda(t_2)$ ensures that $s_{V,1}|_y = s_{V,2}|_y$ in F_y. Thus by shrinking V, we have $s_{V,1}|_{V'} = s_{V,2}|_{V'}$ for some open neighborhood $V' \subseteq V$ containing y. Let $U' = f^{-1}(V') \subseteq U$. Then $x \in U'$ and $t_{U,1}|_{U'} = t_{U,2}|_{U'}$ so $t_1 = t_2$ in $(f^{-1}F)_x$. Thus λ is an isomorphism.

Finally, the sheafification map $f^{-1}F \to f^*F$ induces a map on stalks $(f^{-1}F)_x \to (f^*F)_x$ such that the composition $(f^{-1}F)_x \to (f^*F)_x \cong F_y$ is precisely the isomorphism λ. This shows $(f^*F)_x \cong F_y$ as desired. $\square$

Proposition 3.3.13. *For any continuous map $f : X \to Y$, f^* is a functor $\mathtt{Sh}_Y \to \mathtt{Sh}_X$.*

Proof. By definition, f^*F is a sheaf on X for all sheaves F on Y, so we need only prove naturality. Let $\varphi : F \to G$ be a morphism of sheaves on Y and fix an open set $U \subseteq X$. Then the maps $\varphi_V : F(V) \to G(V)$ over all open $V \subseteq Y$ containing U assemble into a map

$$(f^{-1}F)(U) = \varinjlim F(V) \longrightarrow \varinjlim G(V) = (f^{-1}G)(U).$$

Passing to the sheafification, we get a map $f^*\varphi : f^*F \to f^*G$. $\square$

Corollary 3.3.14. *For any $f : X \to Y$, the pullback functor $f^* : \mathtt{Sh}_Y \to \mathtt{Sh}_X$ is exact.*

Proof. Let $0 \to F \to G \to H \to 0$ be an exact sequence of sheaves on Y. Then for all $y \in Y$, the sequence on stalks $0 \to F_y \to G_y \to H_y \to 0$ is exact. Thus Proposition 3.3.12 says that $0 \to (f^*F)_x \to (f^*G)_x \to (f^*H)_x \to 0$ is exact for all $x \in X$, with $f(x) = y$. By definition this means $0 \to f^*F \to f^*G \to f^*H \to 0$ is exact. $\square$

Proposition 3.3.15. *For any $f : X \to Y$ and any sheaf F on Y, f^*F is the sheaf of sections of the map $E_F \times_Y X \to X$, where $E_F \to Y$ is the étale space of F.*

Proof. Set $E = E_F \times_Y X$ and let E_{f^*F} be the étale space of $f^*F = (f^{-1}F)^{sh}$. Then $E_{f^*F} = E_{f^{-1}F}$ so it's enough to show that $E_{f^{-1}F} = E$. The maps $\lambda : (f^{-1}F)_x \to F_{f(x)}$ for all $x \in X$ in the proof of Proposition 3.3.12 induce a map of sets

$$\Phi : E_{f^{-1}F} = \coprod_{x \in X} (f^{-1}F)_x \longrightarrow \coprod_{y \in Y} F_y = E_F.$$

It's easy to check that Φ is continuous with respect to the topology on the étale spaces. Moreover, the diagram

$$\begin{array}{ccc} E_{f^{-1}F} & \longrightarrow & E_F \\ \downarrow & & \downarrow \\ X & \xrightarrow{\;f\;} & Y \end{array}$$

commutes by construction, so by the universal property of pullbacks, we get a map $\Psi : E_{f^{-1}F} \to E = E_F \times_Y X$. Notice that the fibres of $E_{f^{-1}F}$ and E are the same and Ψ induces bijections on these fibres. Hence Ψ is a continuous bijection. Finally, the fact that $E_{f^{-1}F} \to X$ and $E \to X$ are both local homeomorphisms ensures that Ψ is a homeomorphism. $\square$

Example 3.3.16. Let $f : X \to Y$ be continuous and $F = F_S$ the sheaf of locally constant functions on Y with values in S. Then $E_F = S \times Y \to Y$ where S has the discrete topology. By Proposition 3.3.12, $E_{f^*F} = E_F \times_Y X = (S \times Y) \times_Y X = S \times X$, which by Theorem 1.4.3 has sheaf of sections $F_S \to X$, the sheaf of locally constant functions on X. So pullback of a locally constant sheaf is again locally constant.

Example 3.3.17. Let $i : X \hookrightarrow Y$ be the inclusion of a subspace. Then i^* is just the restriction functor $F \mapsto F|_X$.

We will show that (f^*, f_*) is an adjoint pair of functors. To do so, recall the following characterization of adjointness from homological algebra:

Definition. *For an adjoint pair (S, T), the natural transformations $\eta : id_{\mathcal{A}} \to TS$ and $\varepsilon : ST \to id_{\mathcal{B}}$ are called the* **unit** *and* **counit** *of the adjunction, respectively. The conditions $T\varepsilon \circ \eta T = id$ and $\varepsilon S \circ S\eta = id$ are called the* **triangle identities**.

Theorem 3.3.18. *For a continuous map $f : X \to Y$, (f^*, f_*) is an adjoint pair of functors.*

Proof. Let F be a sheaf on X and G be a sheaf on Y. Then since $f^*G = (f^{-1}G)^{sh}$, Theorem 1.4.4 gives a natural isomorphism

$$\mathrm{Hom}_{\mathtt{Sh}_X}(f^*G, F) \cong \mathrm{Hom}_{\mathtt{Presh}_X}(f^{-1}G, F)$$

so by Lemma A.2.2, we must construct a natural isomorphism

$$\mathrm{Hom}_{\mathtt{Presh}_X}(f^{-1}G, F) \cong \mathrm{Hom}_{\mathtt{Sh}_Y}(G, f_*F).$$

Notice that for any open $V \subseteq Y$,

$$(f_*f^{-1}G)(V) = (f^{-1}G)(f^{-1}(V)) = \varinjlim G(V')$$

over all open sets $V' \subseteq Y$ containing $f(f^{-1}(V))$. But since V itself is an open set containing $f(f^{-1}(V))$, we get a map $G(V) \to \varinjlim G(V')$ which is compatible with restriction. This defines a natural transformation

$$\eta : id_{\mathtt{Sh}_Y} \longrightarrow f_*f^{-1}$$

which will be the unit of our adjunction. Next, we construct a counit of the adjunction. For any open $U \subseteq X$, we have

$$(f^{-1}f_*F)(U) = \varinjlim (f_*F)(V) = \varinjlim F(f^{-1}(V))$$

where the limit is over all open $V \subseteq Y$ containing $f(U)$. Well for every such V, $f^{-1}(V) \supseteq U$ so the restriction maps $F(f^{-1}(V)) \to F(U)$ assemble to give a map $\varinjlim F(f^{-1}(V)) \to F(U)$. This defines the counit

$$\varepsilon : f^{-1}f_* \longrightarrow id_{\mathtt{Presh}_X}.$$

Finally, by Lemma A.2.2 it remains to show that η and ε satisfy the triangle identities. For an open set $V \subseteq Y$, we have

$$\begin{aligned} f_*\varepsilon_F \circ \eta_{f_*F}(V) &: f_*F(V) \to f_*f^{-1}f_*F(V) \to f_*F(V) \\ \text{which is} \quad & F(f^{-1}(V)) \to \varinjlim F(f^{-1}(V')) \to F(f^{-1}(V)) \end{aligned}$$

where the limit in the middle is over all open $V' \subseteq Y$ containing $f(f^{-1}(V))$. Since $V \supseteq f(f^{-1}(V))$, this composition is clearly the identity on $F(f^{-1}(V))$. Therefore $f_*\varepsilon_F \circ \eta_{f_*F}$ is the identity on f_*F. Likewise, for $U \subseteq X$ open,

$$\begin{aligned} \varepsilon_{f^{-1}G} \circ f^{-1}\eta_G(U) &: f^{-1}G(U) \to f^{-1}f_*f^{-1}G(U) \to f^{-1}G(U) \\ \text{which is} \quad & \varinjlim G(V) \to \varinjlim G(V') \to \varinjlim G(V) \end{aligned}$$

where the outer limits are over all open $V \subseteq Y$ containing $f(U)$ and the middle limit is over all open $V' \subseteq Y$ containing $f(f^{-1}(V))$ for any V containing $f(U)$. Any such V in the first limit also appears in the second limit, so the composition is given by the trivial restrictions $G(V) \to G(V)$ for all V. That is, the composition is the identity on $\varinjlim G(V) = f^{-1}G(U)$. □

Corollary 3.3.19. *For any maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, $(g \circ f)^* = f^* \circ g^*$.*

Proof. Each of $(g \circ f)^*$ and $f^* \circ g^*$ is a left adjoint of $(g \circ f)_* = g_* \circ f_*$. □

Corollary 3.3.20. *Let $f : X \to Y$ be a continuous map and E an injective sheaf on X. Then f_*E is an injective sheaf on Y.*

Proof. Suppose $0 \to F \to G$ is an exact sequence in $\mathtt{Sh}_Y$. Then since f^* is exact, $0 \to f^*F \to f^*G$ is an exact sequence in $\mathtt{Sh}_X$ and by adjointness of (f^*, f_*), we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}(G, f_*E) & \longrightarrow & \mathrm{Hom}(F, f_*E) & \longrightarrow & 0 \\ \cong \downarrow & & \downarrow \cong & & \\ \mathrm{Hom}(f^*G, E) & \longrightarrow & \mathrm{Hom}(f^*F, E) & \longrightarrow & 0 \end{array}$$

with isomorphisms on the columns. Since E is injective, the bottom row is exact. Therefore the top row is also exact, which proves f_*E is injective. □

Remark. More generally, any functor with an exact left adjoint preserves injectives.

Corollary 3.3.21. *For any continuous map $f : X \to Y$ and sheaf F on X, there is a homomorphism*

$$H^p(X, F) \longrightarrow H^p(Y, f_*F)$$

for all $p \geq 0$.

Proof. Take an injective resolution $E^\bullet$ of F in $\mathtt{Sh}_X$. Then by Corollary 3.3.20, $f_*E^\bullet$ is an injective resolution of f_*F in $\mathtt{Sh}_Y$, so we can compute cohomology using this resolution:

$$H^p(Y, f_*F) = H^p(\Gamma(Y, f_*E^\bullet)).$$

Since each f_*E^n is defined locally by a direct limit $\varinjlim E^n(f^{-1}(V))$, there is an induced map of global sections $\Gamma(X, E^n) \to \Gamma(Y, f_*E^n)$ for each $n \geq 0$. This induces the desired maps on cohomology: $H^p(X, F) \longrightarrow H^p(Y, f_*F)$. □

Remark. To compute $H^p(X, f^*G)$ for a sheaf G on Y is harder in general, since if $E^\bullet$ is an injective resolution of G in $\mathtt{Sh}_Y$, the inverse images $f^*E^\bullet$ need not form an injective resolution of f^*G on X. However, by the theory of derived categories, there exists an injective complex of sheaves $J^\bullet$ on X and an embedding $f^*E^\bullet \to J^\bullet$ which is a *quasi-isomorphism*, i.e. it induces isomorphisms on all cohomology groups. Therefore the cohomology of f^*G can be computed with this resolution:

$$H^p(X, f^*G) = H^p(\Gamma(X, J^\bullet)).$$

Further, there are chain maps

$$\Gamma(Y, E^\bullet) \to \Gamma(X, f^*E^\bullet) \to \Gamma(X, J^\bullet)$$

which induce a homomorphism on cohomology

$$H^p(Y, G) \longrightarrow H^p(X, f^*G)$$

for all $p \geq 0$. In principle these can be defined directly from the definitions of sheaf cohomology and f^*, but the derived categorical perspective makes this straightforward.

One thing to note about the direct image functor is that it does not preserve stalks. Indeed, if $f : X \to Y$ is an arbitrary continuous map and $y \in Y$, then for any sheaf F on X, $(f_*F)_y = F_x$ if $y = f(x) \in f(X)$, but if $y \notin f(X)$, then $(f_*F)_y$ need not be 0 (the value of F over the empty set). One way to remedy this is to define a new type of pushforward functor that respects stalks. We start by giving the definition for open embeddings.

Definition. *Let $j : U \hookrightarrow X$ be the inclusion of an open set. For a sheaf F on U, define the* **extension by zero** *of F along j to be the sheafification $j_!F$ of the presheaf P on X defined by*

$$P(V) = \begin{cases} F(V), & V \subseteq U \\ 0, & V \not\subseteq U. \end{cases}$$

Lemma 3.3.22. *For any open embedding $j : U \hookrightarrow X$, the assignment $F \mapsto j_!F$ is a functor $\mathtt{Sh}_U \to \mathtt{Sh}_X$.*

Proposition 3.3.23. *For any open embedding $j : U \hookrightarrow X$, $(j_!, j^*)$ is an adjoint pair of functors.*

Proof. Let F be a sheaf on U and G a sheaf on X. First recall that the inverse image functor j^* is simply restriction: $j^*G = G|_U$. Then for the presheaf P defined above, we have

$$\begin{aligned} \mathrm{Hom}_{\mathtt{Sh}_X}(j_!F, G) &\cong \mathrm{Hom}_{\mathtt{Presh}_X}(P, G) && \text{by the universal property of sheafification} \\ &\cong \mathrm{Hom}_{\mathtt{Sh}_U}(F, G|_U) && \text{by definition of } P \\ &= \mathrm{Hom}_{\mathtt{Sh}_U}(F, j^*G). \end{aligned}$$

These isomorphisms are all natural in F and G, so $(j_!, j^*)$ is an adjoint pair. □

Corollary 3.3.24. *For any open embedding $j : U \hookrightarrow X$, $j_!$ is exact and j^* preserves injectives.*

Proof. Exactness of $j_!$ can be checked on stalks, and the second statement is a consequence of the remark following Corollary 3.3.20. □

If $j : U \hookrightarrow X$ is an open embedding, the above gives us a triple of functors $(j_!, j^*, j_*)$ in which each consecutive pair is an adjunction and j^* is exact. In a similar fashion, we can extend the adjunction (i^*, i_*) for a *closed* embedding $i : Z \hookrightarrow X$.

Definition. *Let $i : Z \hookrightarrow X$ be the inclusion of a closed subset. For a sheaf F on X, define the* **inverse image of F supported on Z** *to be the sheaf $i^!F$ whose value on an open set $V \subseteq Z$ is*

$$(i^!F)(V) = \Gamma_Z(V, F) := \{s \in \Gamma(V, F) \mid \mathrm{supp}(s) \subseteq V\}.$$

Lemma 3.3.25. *For any closed embedding $i : Z \hookrightarrow X$, $F \mapsto i^! F$ is a functor* $\mathtt{Sh}_X \to \mathtt{Sh}_Z$.

Proposition 3.3.26. *For any closed embedding $i : Z \hookrightarrow X$, $(i_*, i^!)$ is an adjoint pair of functors.*

Corollary 3.3.27. *For any closed embedding $i : Z \hookrightarrow X$, i_* is exact and $i^!$ preserves injectives.*

Therefore we have a triple of functors $(i^*, i_*, i^!)$ in which each consecutive pair is an adjunction and i_* is exact. We remark that the functor $f^!$ does not exist in general – although a derived analogue does exist and agrees with the derived functor of $i^!$ for a closed embedding.

Let $i : Z \hookrightarrow X$ be a closed embedding, set $U = X \smallsetminus Z$ and let $j : U \hookrightarrow X$ be the corresponding open embedding. Define a category $\mathcal{C}$ whose objects are triples (G, H, φ) where G is a sheaf on Z, H is a sheaf on U and $\varphi : G \to i^* j_* H$ is a morphism of sheaves, and whose morphisms are pairs of morphisms $(\xi : G \to G', \chi : H \to H')$ making the diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & i^* j_* H \\ \xi \downarrow & & \downarrow i^* j_*(\chi) \\ G' & \xrightarrow[\varphi']{} & i^* j_* H' \end{array}$$

commute. Let $\eta : F \to j_* j^* F$ be the unit of the adjunction (j^*, j_*). Then this induces a morphism of sheaves $i^*(\eta) : i^* F \to i^* j_* j^* F$.

Theorem 3.3.28 (Recollment of Sheaves)**.** *For $i : Z \hookrightarrow X$ and $j : U = X \smallsetminus Z \hookrightarrow X$ as above,*

$$\begin{aligned} \mathtt{Sh}_X &\longrightarrow \mathcal{C} \\ F &\longmapsto (i^* F, j^* F, i^*(\eta)) \end{aligned}$$

is an equivalence of categories.

When $i : Z \hookrightarrow X$ is a closed embedding and G is a sheaf on Z, note that the stalks of the direct image sheaf $F = i_* G$ are precisely

$$F_x = \begin{cases} G_x, & x \in Z \\ 0, & x \notin Z. \end{cases}$$

Since i^* preserves stalks, this shows that the counit $\varepsilon : i^* i_* G \to G$ of the adjunction (i^*, i_*) is in fact an isomorphism. Hence $i_* G = F = i_! G$. On the other hand, for the open embedding $j : U = X \smallsetminus Z \hookrightarrow X$, we have

$$(j_* G)_x = \begin{cases} G_x, & x \in \overline{U} \\ 0, & x \notin \overline{U}. \end{cases}$$

Therefore $j_* G \neq j_! G$ in general. This presents an obstacle to constructing the adjunctions $(f_!, f^*, f_*, f^!)$ for more general maps f.

Definition. *A subset $W \subseteq X$ is* **locally closed** *if*

(1) W is open in $\overline{W}$.

(2) $W = U \cap C$ where $U \subseteq X$ is open and $C \subseteq X$ is closed.

(3) For all $x \in W$, there is a neighborhood $V \subseteq X$ of x such that $V \cap W$ is closed in V.

Proposition 3.3.29. *The adjunctions $(i_!, i^*, i_*, i^!)$ exist for any locally closed embedding $i : W \hookrightarrow X$.*

Proof. We have an open embedding $W \hookrightarrow \overline{W}$ and a closed embedding $\overline{W} \hookrightarrow X$, so apply Theorem 3.3.28. □

Conversely:

Proposition 3.3.30. *Let $i : W \hookrightarrow X$ be the inclusion of an arbitrary subset. If every sheaf on X is of the form $i_! G$ for some $G \in \mathtt{Sh}_W$, then W is locally closed in X.*

Theorem 3.3.31. *Let $i : Z \hookrightarrow X$ be a closed embedding, set $U = X \smallsetminus Z$ and let $j : U \hookrightarrow X$ be the corresponding open embedding. Then for any sheaf F on X, the sequence*

$$0 \to j_! j^* F \to F \to i_* i^* F \to 0$$

is exact.

Proof. By definition, it suffices to check this on stalks: for $x \in X$, the sequence is

$$\begin{cases} 0 \to F_x \xrightarrow{id} F_x \to 0 \to 0, & x \in U \\ 0 \to 0 \to F_x \xrightarrow{id} F_x \to 0, & x \in Z. \end{cases}$$

□

Remark. Note that when $j : Z \hookrightarrow X$ is a closed embedding, the morphism of sheaves $j_! G \to j_* G$ is injective for all $G \in \mathtt{Sh}_Z$, so for every open set $V \subseteq X$, we may view $\Gamma(V, j_! G)$ as a subset of $\Gamma(V, j_* G) = \Gamma(V \cap U, G)$, where $U = X \smallsetminus Z$. Explicitly,

$$\Gamma(V, j_! G) = \{s \in \Gamma(V \cap U, G) \mid \operatorname{supp}(s) \text{ is closed in } U\}.$$

This explains another common name, 'sections with compact support', for $j_!$. For a general map $f : X \to Y$, this suggests defining the *pushforward with compact support* $f_! : \mathtt{Sh}_Z \to \mathtt{Sh}_X$ by

$$(f_! F)(V) = \{s \in \Gamma(f^{-1}(V), F) \mid f|_{\operatorname{supp}(s)} : \operatorname{supp}(s) \to V \text{ is proper}\}.$$

(Recall that a map is proper if the preimage of every compact set is compact.)

3.4 Acyclic Sheaves

Injective resolutions are hard to come by in sheaf theory, but Theorem 3.2.3 shows that sheaf cohomology can be computed via acyclic resolutions. This gives us the following result, which will be useful in reducing cohomology computations to simpler subspaces.

Proposition 3.4.1. *Let $i : Z \hookrightarrow X$ be the inclusion of a closed subset. Then for any sheaf of abelian groups F on Z,*

$$H^i(Z, F) \cong H^i(X, i_*F)$$

for all $i \geq 0$.

Proof. Suppose $E^\bullet$ is an injective resolution of F on Z. Then by exactness of i_*, $i_*E^\bullet$ is an injective resolution of i_*F on X and $\Gamma(Z, E^\bullet) = \Gamma(X, i_*E^\bullet)$ so the cohomology groups agree. □

Recall (Section 1.3) that a sheaf F on X is flasque if for all open sets $V \subseteq U$, the restriction $F(U) \to F(V)$ is surjective.

Lemma 3.4.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then any injective $\mathcal{O}_X$-module is flasque.*

Proof. For any open set U, let $j : U \hookrightarrow X$ be the inclusion map and set $\mathcal{O}_U = j_!(\mathcal{O}_X|_U)$. Then $\mathcal{O}_U$ is an $\mathcal{O}_X$-module. Suppose F is an injective $\mathcal{O}_X$-module and $V \subseteq U$ are open sets in X. Then there is an induced map $\mathcal{O}_V \hookrightarrow \mathcal{O}_U$ which is an inclusion since $j_!$ is exact, so by the injective property for F, $\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, F) \to \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_V, F)$ is surjective. Finally, the adjunction $(j_!, j^*)$ gives us an isomorphism

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, F) \cong \text{Hom}_{\mathcal{O}_X|_U}(\mathcal{O}_X|_U, j^*F) = F(U).$$

Hence $F(U) \to F(V)$ is surjective, so F is flasque. □

Theorem 3.4.3. *Let F be a flasque sheaf on X. Then F is acyclic.*

Proof. Since $\texttt{Sh}_X$ has enough injectives by Theorem 1.3.10, there is an exact sequence of sheaves

$$0 \to F \to G \to H \to 0$$

where G is injective. By Theorem 1.3.12,

$$0 \to F(X) \to G(X) \to H(X) \to 0$$

is also exact, so in the long exact sequence in sheaf cohomology, we have

$$0 \to F(X) \to G(X) \twoheadrightarrow H(X) \to H^1(X, F) \to H^1(X, G) = 0$$

where the last term is zero since G is injective, and thus acyclic. Therefore exactness implies $H^1(X, F) = 0$. Now for $i \geq 2$, inductively assume that $H^{i-1}(X, Q) = 0$ for all flasque sheaves Q on X. Part of the long exact sequence looks like

$$0 = H^{i-1}(X, G) \to H^{i-1}(X, H) \to H^i(X, F) \to H^i(X, G) = 0,$$

once again using that G is injective. Thus $H^i(X, F) \cong H^{i-1}(X, H)$. Meanwhile, Lemma 3.4.2 shows that G is flasque, so the second statement in Theorem 1.3.12 implies H is also flasque and hence $H^{i-1}(X, H) = 0$ by the inductive hypothesis. This proves $H^i(X, F) = 0$ for all $i \geq 2$. □

Remark. If $(X, \mathcal{O}_X)$ is a ringed space, Theorem 3.4.3 shows that flasque sheaves are acyclic for the functor $\Gamma : \mathtt{Mod}_{\mathcal{O}_X} \to \mathtt{Ab}$, so the derived functors of Γ agree with the sheaf cohomology $H^i(X, -)$ for sheaves of abelian groups.

Theorem 3.4.4 (Grothendieck Vanishing)**.** *Let X be a noetherian topological space of dimension n. Then for any sheaf of abelian groups F on X,*

$$H^i(X, F) = 0$$

for all $i > n$.

Proof. First suppose X is reducible; let Z be one of its irreducible components and set $U = X \smallsetminus Z$. Following the notation above, let $j : U \hookrightarrow X$ be the open inclusion and set $F_U = j_!(F|_U)$. Likewise, let $i : Z \hookrightarrow X$ be the closed inclusion and set $F_Z = i_*(F|_Z)$. Then by Theorem 3.3.31, there is an exact sequence

$$0 \to F_U \to F \to F_Z \to 0.$$

By the long exact sequence in sheaf cohomology, it's enough to prove $H^i(X, F_U) = 0$ and $H^i(X, F_Z) = 0$ for all $i > n$. Note that F_U extends to a sheaf on $\overline{U}$, which is a closed subset with one less irreducible component than X. Applying Proposition 3.4.1 to $\overline{U} \hookrightarrow X$ and noting that Z is closed and irreducible, we may reduce to the case where X is itself irreducible.

We induct on $n = \dim X$. If $n = 0$, $\Gamma : \mathtt{Sh}_X \to \mathtt{Ab}$ is an equivalence of categories (X is irreducible) and in particular exact, so $R^i\Gamma$ is trivial for $i > 0$. For $n > 0$, note that we have

$$F = \varinjlim F_B$$

where the limit is over all *finite subsets* $B \subseteq \bigcup_{U \subseteq X} F(U)$, directed by inclusion, and F_B is the subsheaf of F generated by the sections in B. Since cohomology commutes with direct limits, it suffices to prove $H^i(X, F_B) = 0$ for all B. Suppose $B' \subseteq B$ and set $q = |B \smallsetminus B'|$. Then there is an exact sequence

$$0 \to F_{B'} \to F_B \to G \to 0$$

where G is a sheaf generated by q sections over some open subsets of X. Using the long exact sequence coming from this sequence, we may reduce to the case where $q = 1$. In this case we have a short exact sequence

$$0 \to K \to \mathbb{Z}_U \to F \to 0$$

for some open set $U \subseteq X$, where $\mathbb{Z}$ denotes the (locally) constant sheaf on X with values in $\mathbb{Z}$. Thus we may prove the theorem by showing that $H^i(X, \mathbb{Z}_U) = 0$ for any open U and $H^i(X, K) = 0$ for any nonzero subsheaf $K \subseteq \mathbb{Z}_U$. For such a subsheaf K, we have that K_x is a subgroup of $\mathbb{Z}_{U,x}$ for all $x \in U$, so there is some open set $V \subseteq U$ for which $K|_V \cong d\mathbb{Z}|_V$. Abstractly, $d\mathbb{Z}_V \cong \mathbb{Z}_V$ so there is a short exact sequence

$$0 \to \mathbb{Z}_V \to K \to Q \to 0.$$

The quotient sheaf Q is supported on $\overline{U \smallsetminus V}$ which has dimension $< n$ since X is irreducible. Therefore $H^i(X, Q) = 0$ for all $i \geq n$ by the inductive hypothesis. Thus it remains to show $H^i(X, \mathbb{Z}_U) = 0$ for $i > n$ and all U (which implies vanishing for $\mathbb{Z}_V$ if $V \subseteq U$). Set $Z = X \smallsetminus U$. Since X is irreducible, $\dim Z < n$ so by induction, $H^i(X, \mathbb{Z}_Z) = 0$ for $i \geq n$. On the other hand, $\mathbb{Z}$ is obviously flasque so Theorem 3.4.3 shows that $H^i(X, \mathbb{Z}) = 0$ for $i > 0$. Thus the long exact sequence induced by the short exact sequence

$$0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}_Z \to 0$$

gives us $H^i(X, \mathbb{Z}_U) \cong H^{i-1}(X, \mathbb{Z}_Z)$ which is 0 for all $i - 1 \geq n$. This finishes the proof. □

Suppose $\mathcal{U} = \{U_i\}$ is an open covering of a space X and F is a sheaf of abelian groups on X. By Theorem 3.2.2, there is a natural map $\check{H}^p(\mathcal{U}, F) \to H^p(X, F)$ for every $p \geq 0$.

Proposition 3.4.5. *If F is a flasque sheaf on X, then $\check{H}^p(\mathcal{U}, F) = 0$ for all open covers $\mathcal{U}$ of X and all $p > 0$.*

Proof. Let $\mathcal{C}^\bullet(\mathcal{U}, F)$ be the Čech resolution of F with respect to $\mathcal{U}$. Recall that $\mathcal{C}^p(\mathcal{U}, F)$ is defined for any open $V \subseteq X$ by

$$\Gamma(V, \mathcal{C}^p(\mathcal{U}, F)) = \prod_{i_0, \ldots, i_p} F(V \cap U_{i_0, \ldots, i_p})$$

where $U_{i_0, \ldots, i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$. For each of these $U_{i_0, \ldots, i_p}$, the sheaf $V \mapsto F(V \cap U_{i_0, \ldots, i_p})$ is flasque and since products of flasque sheaves are flasque, the entire sheaf $\mathcal{C}^p(\mathcal{U}, F)$ is flasque. Therefore $\mathcal{C}^p(\mathcal{U}, F)$ is acyclic by Theorem 3.4.3, so the Čech resolution can used to compute sheaf cohomology:

$$\check{H}^p(\mathcal{U}, F) = H^p(\mathcal{C}^\bullet(\mathcal{U}, F)) = H^p(X, F) = 0.$$

□

Definition. *A sheaf F on X is* **acyclic for an open cover** *$\mathcal{U} = \{U_i\}$ if for all $p > 0$, $H^p(U_{i_0, \ldots, i_p}, F|_{U_{i_0, \ldots, i_p}}) = 0$.*

Such covers exist in many common situations, e.g. for manifolds with a Riemannian metric. The following result allows for relatively easy computations of sheaf cohomology when an acyclic cover is available.

Theorem 3.4.6 (Leray)**.** *If F is a sheaf on X which is acyclic for an open cover $\mathcal{U}$, then the maps*

$$\check{H}^p(\mathcal{U}, F) \longrightarrow H^p(X, F)$$

are isomorphisms for all $p \geq 0$.

Proof. We prove this by induction on p. For $p = 0$, this is clear since both $\check{H}^0(\mathcal{U}, F)$ and $H^0(X, F)$ are $\Gamma(X, F)$. Now assume $\check{H}^j(\mathcal{U}, G) \to H^j(X, G)$ is an isomorphism for all $j \leq p$ and all acyclic sheaves G. For our given F, take a short exact sequence of sheaves

$$0 \to F \to E \to Q \to 0$$

where E is injective. Set $U = U_{i_0,\ldots,i_p}$. Then by the hypothesis on F, $H^i(U, F|_U) = 0$ for all $i > 0$. Since E is injective, and thus flasque by Lemma 3.4.2, we have $H^i(U, E|_U) = 0$ as well. Thus in the long exact sequence induced by the above sequence, we can see

$$0 = H^i(U, E|_U) \to H^i(U, Q) \to H^{i+1}(U, F) = 0.$$

This shows $H^i(U, Q) = 0$ for all $i \geq 1$, so by taking the product over all such $U_{i_0,\ldots,i_p}$, we conclude that Q is acyclic for $\mathcal{U}$. Therefore, the sequence of complexes of sheaves

$$0 \to \mathcal{C}^\bullet(\mathcal{U}, F) \to \mathcal{C}^\bullet(\mathcal{U}, E) \to \mathcal{C}^\bullet(\mathcal{U}, Q) \to 0$$

is exact and induces a long exact sequence

$$\cdots \to \check{H}^{p-1}(\mathcal{U}, E) \to \check{H}^{p-1}(\mathcal{U}, Q) \to \check{H}^p(\mathcal{U}, F) \to \check{H}^p(\mathcal{U}, E) \to \cdots$$

Now since E is flasque, Proposition 3.4.5 shows that $\check{H}^{p-1}(\mathcal{U}, E) = \check{H}^p(\mathcal{U}, E) = 0$. Therefore we have a diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \check{H}^{p-1}(\mathcal{U}, Q) & \longrightarrow & \check{H}^p(\mathcal{U}, F) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^{p-1}(X, Q) & \longrightarrow & H^p(X, F) & \longrightarrow & 0 \end{array}$$

By induction, the left column is an isomorphism, so it follows that $\check{H}^p(\mathcal{U}, F) \to H^p(X, F)$ is an isomorphism. □

Corollary 3.4.7. *For any sheaf F on X, the map $\check{H}^1(X, F) \to H^1(X, F)$ is an isomorphism.*

Proof. Let E be an injective sheaf and $0 \to F \to E \to Q \to 0$ a short exact sequence of sheaves on X. Fix an open cover $\mathcal{U}$ of X. Then the morphism of complexes of sheaves $\mathcal{C}^\bullet(\mathcal{U}, F) \to \mathcal{C}^\bullet(\mathcal{U}, E)$ is injective; let $\mathcal{K}^\bullet(\mathcal{U})$ be its cokernel, so that we have a short exact sequence of complexes

$$0 \to \mathcal{C}^\bullet(\mathcal{U}, F) \to \mathcal{C}^\bullet(\mathcal{U}, E) \to \mathcal{K}^\bullet(\mathcal{U}) \to 0.$$

Then by Proposition 3.4.5, $\check{H}^1(\mathcal{U}, E) = 0$ and there is a diagram with exact rows:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & F(X) & \longrightarrow & E(X) & \longrightarrow & H^0(\mathcal{K}^\bullet(\mathcal{U})) & \longrightarrow & \check{H}^1(\mathcal{U}, F) & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow \text{id} & & \vdots & & \downarrow & & \\ 0 & \longrightarrow & F(X) & \longrightarrow & E(X) & \longrightarrow & Q(X) & \longrightarrow & H^1(X, F) & \longrightarrow & 0 \end{array}$$

After passing to the direct limit over all $\mathcal{U}$, it's enough to show that

$$\alpha : \varinjlim H^0(\mathcal{K}^\bullet(\mathcal{U})) \to H^0(X, Q)$$

is an isomorphism. Injectivity is easy. For surjectivity, take $s \in H^0(X,Q) = \check{H}^0(X,Q)$. Then there exists an open cover $\mathcal{U} = \{U_i\}$ of X such that $s = (s_i) \in \prod E(U_i)$ and these s_i satisfy $s_i - s_j \in \ker(F(U_i \cap U_j) \to E(U_i \cap U_j)) = F(U_i \cap U_j)$ for all overlaps $U_i \cap U_j$. By definition this (s_i) defines a Čech 0-cocycle in $\check{H}^0(\mathcal{U}, E)$ whose image in $H^0(\mathcal{K}^\bullet(\mathcal{U}))$ maps to s along α. Hence α is an isomorphism. □

Definition. *The* **support** *of a morphism of sheaves $\varphi : F \to G$ is the closed set*

$$\mathrm{supp}(\varphi) = \overline{\{x \in X \mid \varphi_x \neq 0\}}$$

where $\varphi_x : F_x \to G_x$ is the map on stalks.

Definition. *An open cover $\mathcal{U}$ of X is* **locally finite** *if every $x \in X$ has a neighborhood which has nonempty intersection with only finitely many elements of $\mathcal{U}$. The space X is* **paracompact** *if every open cover has a locally finite refinement.*

Example 3.4.8. All metric spaces are paracompact.

Definition. *Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a locally finite cover of X and F a sheaf on X. A* **partition of unity** *for F subordinate to $\mathcal{U}$ is a collection of morphisms of sheaves $\{\eta_i : F \to F\}_{i \in I}$ such that for each $i \in I$, $\mathrm{supp}(\eta_i) \subseteq U_i$, and for all $x \in X$, $\sum_{i \in I} \eta_{i,x} = id_{F_x}$.*

Note that when $\mathcal{U}$ is locally finite, only finitely many of the maps $\eta_{i,x} : F_x \to F_x$ are nonzero for any given $x \in X$, so the sum $\sum_{i \in I} \eta_{i,x}$ is always defined.

Definition. *A sheaf F on X is* **fine** *if it admits partitions of unity subordinate to any locally finite cover of X.*

Proposition 3.4.9. *Let $0 \to F \to G \to H \to 0$ be a short exact sequence of sheaves on a paracompact space X. If F is fine, then $0 \to F(X) \to G(X) \to H(X) \to 0$ is exact. Moreover, if every open set $U \subseteq X$ is paracompact and G is flasque, then H is flasque as well.*

Proof. Similar to the proof of Theorem 1.3.12. □

Corollary 3.4.10. *Let F be a fine sheaf on X. Then F is acyclic.*

Proof. Similar to the proof of Theorem 3.4.3. □

Example 3.4.11. Let $(X, \mathcal{O}_X)$ be a ringed space. Assume $\mathcal{U} = \{U_i\}_{i \in I}$ is a locally finite open cover and $\mathcal{O}_X$ admits a partition of unity subordinate to $\mathcal{U}$; equivalently, assume there exist sections $\{s_i \in \mathcal{O}_X(X)\}_{i \in I}$ for which $\mathrm{supp}(s_i) \subseteq U_i$ for all $i \in I$ and $\sum_{i \in I} s_i = 1$ in $\mathcal{O}_X(X)$. Let F be an $\mathcal{O}_X$-module. Then the maps $\eta_{i,U} : F(U) \to F(U), t \mapsto s_i|_U \cdot t$ induce a partition of unity $\{\eta_i : F \to F\}$ for F subordinate to $\mathcal{U}$. Hence sheaves of modules over a fine sheaf of rings are also fine.

3.5 De Rham Cohomology

Let X be a smooth n-manifold with tangent bundle $\pi : TX \to X$. A smooth section of π is called a *vector field* on X, that is, a smooth assignment of a tangent vector $v_x \in T_xX$ for each $x \in X$. Let $U \subseteq X$ be a coordinate chart with local coordinates $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. Then if $x \in U$, a tangent vector v_x can be written

$$v_x = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i}$$

for some smooth functions $f_i : U \to \mathbb{R}$. Let $\mathcal{V}(U)$ denote the space of vector fields over U, which is a module over the ring of smooth functions on U. Let $\mathcal{O}_X$ be the sheaf of smooth functions on X, which makes $(X, \mathcal{O}_X)$ into a ringed space. Then the assignment $U \mapsto \mathcal{V}(U)$ defines a sheaf of $\mathcal{O}_X$-modules on X.

On the other hand, the cotangent bundle of X, $\pi^* : T^*X \to X$, defined by dualizing TX fibrewise, has as its sheaf of sections the sheaf of (smooth) *differential* 1*-forms* Ω^1_X, which is also an $\mathcal{O}_X$-module. An element of Ω^1_X is a smooth assignment $x \mapsto \omega_x \in C^\infty(T_xX, \mathbb{R})$. Such an element ω_x can be written in local coordinates (say over $U \subseteq X$ containing x) as

$$\omega_x = \varphi_1(x)\, dx_1 + \ldots + \varphi_n(x)\, dx_n$$

for smooth functions $\varphi_i : U \to \mathbb{R}$ and basis elements $\{dx_1, \ldots, dx_n\}$ of $C^\infty(T_xX, \mathbb{R})$ defined by $\left\langle dx_i, \frac{\partial}{\partial x_j} \right\rangle = \delta_{ij}$.

Let $\Omega^k_X = \bigwedge^k T^*X$ be the *sheaf of differential k-forms* on X. By convention $\Omega^0_X = \mathcal{O}_X$. This defines a sheaf of graded rings $\Omega^\bullet_X(U) = \bigoplus_{k=0}^\infty \Omega^k_X(U)$ for every open $U \subseteq X$. Explicitly, there is a complex

$$\mathcal{O}_X = \Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots$$

where d is the exterior derivative which is defined over $U \subseteq X$ by

$$d_U : \Omega^k_X(U) \longrightarrow \Omega^{k+1}_X(U)$$
$$\omega = \sum_{i_1,\ldots,i_k} \varphi_{i_1,\ldots,i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \longmapsto d\omega = \sum_{i_1,\ldots,i_k} \sum_{j=1}^{n} \frac{\partial \varphi_{i_1,\ldots,i_k}}{\partial x_j}\, dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Theorem 3.5.1. *Suppose X is a smooth manifold. Then*

(1) X is paracompact.

(2) For every locally finite cover $\mathcal{U}$ of X, there is a partition of unity for $\mathcal{O}_X$ subordinate to $\mathcal{U}$.

Corollary 3.5.2. *For each $k \geq 0$, Ω^k_X is a fine sheaf. In particular, Ω^k_X is acyclic.*

Proof. Theorem 3.5.1 says that $\mathcal{O}_X$ is a fine sheaf on X, and since each Ω^k_X is a module over $\mathcal{O}_X$, the first statement follows from Example 3.4.11. For the second statement, apply Corollary 3.4.10. □

Definition. *The sheaf cohomology of* $\Omega_X^\bullet$ *is called the* **de Rham cohomology** *of* X*, written*

$$H^i_{dR}(X) := H^i(\Gamma(X, \Omega_X^\bullet)).$$

Theorem 3.5.3. *For a smooth manifold* X*, the complex of sheaves*

$$0 \to \mathbb{R}_X \xrightarrow{i} \Omega_X^0 \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \cdots$$

is acyclic, where $\mathbb{R}_X$ *denotes the constant sheaf on* X *with stalks* $\mathbb{R}$*, and* i *is the inclusion of constant functions.*

Proof. It is enough to compute this on stalks, where we have for each $x \in X$ a coordinate chart $U \cong \mathbb{R}^n$. By Poincaré's Lemma in algebraic topology, $H^i_{dR}(\mathbb{R}^n) = 0$ for all $i > 0$, so $H^i_{dR}(U) = 0$ for all $i > 0$ as well. Taking direct limits commutes with cohomology, so

$$H^i(X; \Omega_{X,x}^\bullet) = \varinjlim H^i(U; \Omega_X^\bullet(U)) = \varinjlim H^i_{dR}(U) = 0.$$

□

Corollary 3.5.4. *For a smooth manifold* X*, there is an isomorphism*

$$H^\bullet_{dR}(X) \cong H^\bullet(X, \mathbb{R}_X)$$

where $\mathbb{R}_X$ *is the constant sheaf.*

Proof. Theorem 3.5.3 shows that $\Omega_X^0 \to \Omega_X^1 \to \Omega_X^2 \to \cdots$ is a deleted acyclic resolution of $\mathbb{R}_X$, so this follows from the definition of sheaf cohomology as the right derived functors of $\Gamma(X, \mathbb{R}_X)$. □

Corollary 3.5.5 (De Rham's Theorem)**.** *For a smooth manifold* X*, there is an isomorphism between de Rham cohomology and singular cohomology:*

$$H^\bullet_{dR}(X) \cong H^\bullet(X; \mathbb{R}).$$

Proof. Apply Corollary 3.5.4 and Theorem 3.2.4. □

Theorem 3.5.6. *For a smooth, compact manifold* X*,* $H^i_{dR}(X)$ *is finite dimensional for all* $i \geq 0$*.*

Proof. Endow X with a Riemannian metric. Then every point in X admits a neighborhood which is *geodesically convex*, meaning any two points in the neighborhood lie on a geodesic which is contained in the neighborhood. Since X is compact, we may choose a finite cover $\mathcal{U} = \{U_1, \ldots, U_m\}$ consisting of these geodesically convex open sets. Note that for any $i_1, \ldots, i_r \in \{1, \ldots, m\}$, the intersection $U_{i_1,\ldots,i_r} = U_{i_1} \cap \cdots \cap U_{i_r}$ is also geodesically convex. A geodesically convex set has vanishing de Rham cohomology, so de Rham's theorem says that $H^i(U_{i_1,\ldots,i_r}, \mathbb{R}_X|_{U_{i_1,\ldots,i_r}}) \cong H^i_{dR}(U_{i_1,\ldots,i_r}) = 0$ for all $i > 0$. Therefore $\mathbb{R}_X$ is acyclic for the cover $\mathcal{U}$, so by Leray's theorem (3.4.6), $H^i(X, \mathbb{R}_X) = \check{H}^i(\mathcal{U}, \mathbb{R}_X)$. Now observe that each $\mathcal{C}^i(\mathcal{U}, \mathbb{R}_X)$ in the Čech complex is a *finite* product of one-dimensional vector spaces, hence itself finite dimensional, so $\check{H}^i(\mathcal{U}, \mathbb{R}_X)$ is finite dimensional. □

4 Scheme Theory

4.1 Affine Schemes

Hilbert's Nullstellensatz is an important theorem in commutative algebra which is essentially the jumping off point for classical algebraic geometry (by which we mean the study of algebraic varieties in affine and projective space). We recall the statement here.

Theorem 4.1.1 (Hilbert's Nullstellensatz). *If k is an algebraically closed field, then there is a bijection*

$$\begin{aligned} \mathbb{A}^n_k &\longleftrightarrow \operatorname{MaxSpec} k[t_1, \dots, t_n] \\ P = (\alpha_1, \dots, \alpha_n) &\longmapsto \mathfrak{m}_P = (t_1 - \alpha_1, \dots, t_n - \alpha_n), \end{aligned}$$

where $\mathbb{A}^n_k = k^n$ is affine n-space over k and MaxSpec *denotes the set of all maximal ideals of a ring.*

Further, if $f : A \to B$ is a morphism of finitely generated k-algebras then we get a map $f^* : \operatorname{MaxSpec} B \to \operatorname{MaxSpec} A$ given by $f^*\mathfrak{m} = f^{-1}(\mathfrak{m})$ for any maximal ideal $\mathfrak{m} \subset B$. Note that if k is not algebraically closed, $f^{-1}(\mathfrak{m})$ need not be a maximal ideal of A.

Lemma 4.1.2. *Let $f : A \to B$ be a ring homomorphism and $\mathfrak{p} \subset B$ a prime ideal. Then $f^{-1}(\mathfrak{p})$ is a prime ideal of A.*

Proof. Exercise. □

This suggests a natural replacement for MaxSpec A, called the *prime spectrum*:

$$\operatorname{Spec} A = \{\mathfrak{p} \subset A \mid \mathfrak{p} \text{ is a prime ideal}\}.$$

Definition. *An* **affine scheme** *is a ringed space with underlying topological space $X = \operatorname{Spec} A$ for some ring A.*

In order to justify this definition, I will now tell you the topology on Spec A and the sheaf of rings making it into a ringed space. For any subset $E \subseteq A$, define

$$V(E) = \{\mathfrak{p} \in \operatorname{Spec} A \mid E \subseteq \mathfrak{p}\}.$$

Lemma 4.1.3. *Let A be a ring and $E \subseteq A$ any subset. Set $\mathfrak{a} = (E)$, the ideal generated by E. Then*

(a) $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$ where r denotes the radical of an ideal.

(b) $V(\{0\}) = \operatorname{Spec} A$ and $V(A) = \varnothing$.

(c) For a collection of subsets $\{E_i\}$ of A, $V\left(\bigcup E_i\right) = \bigcap V(E_i)$.

(d) For any ideals $\mathfrak{a}, \mathfrak{b} \subset A$, $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

As a result, the sets $V(E)$ for $E \subseteq A$ form the closed sets for a topology on $\operatorname{Spec} A$, called the *Zariski topology*.

Next, for any prime ideal $\mathfrak{p} \subset A$, let $A_\mathfrak{p}$ denote the localization at $\mathfrak{p}$. For any open set $U \subseteq \operatorname{Spec} A$, we define

$$\mathcal{O}(U) = \left\{ s : U \to \coprod_{\mathfrak{p} \in U} A_\mathfrak{p} \;\middle|\; s(\mathfrak{p}) \in A_\mathfrak{p}, \exists\, \mathfrak{p} \in V \subseteq U \text{ and } f, g \in A \text{ so that } s(\mathfrak{q}) = \frac{f}{g} \text{ for all } \mathfrak{q} \in V \right\}.$$

Theorem 4.1.4. $(\operatorname{Spec} A, \mathcal{O})$ *is a ringed space. Moreover,*

(1) For any $\mathfrak{p} \in \operatorname{Spec} A$, $\mathcal{O}_\mathfrak{p} \cong A_\mathfrak{p}$ *as rings.*

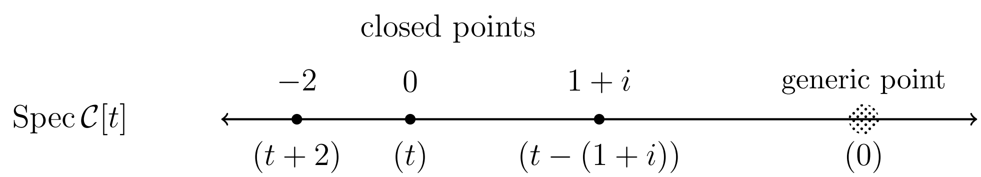
(2) $\Gamma(\operatorname{Spec} A, \mathcal{O}) \cong A$ *as rings.*

(3) For any $f \in A$, *define the open set* $D(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}$. *Then the* $D(f)$ *form a basis for the topology on* $\operatorname{Spec} A$ *and* $\mathcal{O}(D(f)) \cong A_f$ *as rings.*

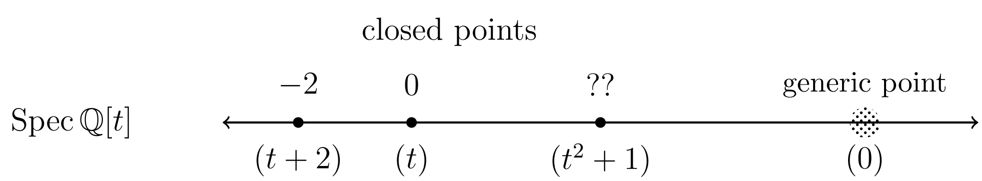
Example 4.1.5. For any field k, $\operatorname{Spec} k$ is a single point $*$ corresponding to the zero ideal, with sheaf $\mathcal{O}(*) \cong k$.

Example 4.1.6. Let $A = k[t_1, \ldots, t_n]$ be the polynomial ring in n variables over k. Then $\operatorname{Spec} A = \mathbb{A}^n_k$, the affine n-space over k. For example, when $A = k[t]$ is the polynomial ring in a single variable, $\operatorname{Spec} k[t] = \mathbb{A}^1_k$, the affine line.

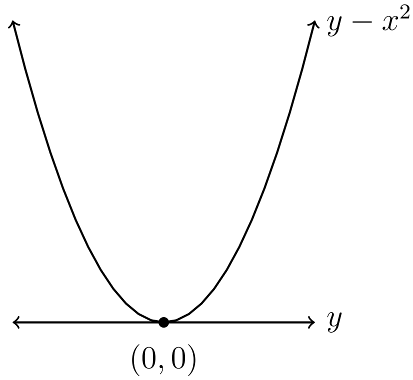
When $k = \mathcal{C}$, Hilbert's Nullstellensatz tells us that all the closed points of $\mathbb{A}^1_k$ correspond to maximal ideals of the form $(t - \alpha)$ for $\alpha \in \mathcal{C}$. But there is also a non-closed, 'generic point' corresponding to the zero ideal which was not detected before.



On the other hand, if $k = \mathbb{Q}$ or another non-algebraically closed field, the same closed points corresponding to linear ideals $(t - \alpha)$ show up, as well as the generic point corresponding to (0), but there are also points corrresponding to ideals generated by higher degree irreducible polynomials like $t^2 + 1$. Thus the structure of $\operatorname{Spec} \mathbb{Q}[t]$ is much different than the algebraically closed case.



Example 4.1.7. Let X be an algebraic variety over a field k, $x \in X$ a point and consider the affine scheme $Y = \operatorname{Spec}(k[\varepsilon]/(\varepsilon^2))$. We can think of Y as a "big point" with underlying space $*$ corresponding to the zero ideal, along with a "tangent vector" extending infinitesimally in every direction around $*$. Then any map $Y \to X$ determines a unique tangent vector in $T_x X$, the tangent space of X at x. This idea is useful in intersection theory. For example, consider the tangency of the x-axis and the parabola $y = x^2$ in $\mathbb{A}^2_k$:



As a variety, this point $(0,0)$ corresponds to the quotient of k-algebras $k[x,y]/r(y, y-x^2) = k[x]/r(x^2) = k[x]/(x) = k$. Thus the information of tangency is lost. However, as an affine scheme, $(0,0)$ corresponds to $\mathrm{Spec}(k[x,y]/(y, y-x^2)) = \mathrm{Spec}(k[x]/(x^2))$ so the intersection information is preserved.

4.2 Schemes

In this section we define a scheme and prove some basic properties resulting from this definition. Recall that a ringed space is a pair $(X, \mathcal{F})$ where X is a topological space and $\mathcal{F}$ is a sheaf of rings on X.

Definition. *A* **locally ringed space** *is a ringed space $(X, \mathcal{F})$ such that for all $P \in X$, there is a ring A such that $\mathcal{F}_P \cong A_\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset A$.*

Example 4.2.1. Any affine scheme $\mathrm{Spec}\, A$ is a locally ringed space by (1) of Theorem 4.1.4. We will sometimes denote the structure sheaf $\mathcal{O}$ by $\mathcal{O}_A$.

Definition. *The* **category of locally ringed spaces** *is the category whose objects are locally ringed spaces $(X, \mathcal{F})$ and whose morphisms are morphisms of ringed spaces $(X, \mathcal{F}) \to (Y, \mathcal{G})$ such that for each $P \in X$, the induced map $f_P^\# : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a morphism of local rings, i.e. $(f_P^\#)^{-1}(\mathfrak{m}_P) = \mathfrak{m}_{f(P)}$ where $\mathfrak{m}_P$ (resp. $\mathfrak{m}_{f(P)}$) is the maximal ideal of the local ring $\mathcal{O}_{X,P}$ (resp. $\mathcal{O}_{Y,f(P)}$).*

We are now able to define a scheme.

Definition. *A* **scheme** *is a locally ringed space $(X, \mathcal{O}_X)$ that admits an open covering $\{U_i\}$ such that each U_i is affine, i.e. there are rings A_i such that $(U_i, \mathcal{O}_X|_{U_i}) \cong (\mathrm{Spec}\, A_i, \mathcal{O}_{A_i})$ as locally ringed spaces.*

The category of schemes `Sch` is defined to be the full subcategory of schemes in the category of locally ringed spaces. Denote the subcategory of affine schemes by `AffSch`. Also let `CommRings` denote the category of commutative rings with unity.

Proposition 4.2.2. *There is an isomorphism of categories*

$$\begin{aligned} \texttt{AffSch} &\xrightarrow{\sim} \texttt{CommRings}^{op} \\ (X, \mathcal{O}_X) &\longmapsto \mathcal{O}_X(X) \\ (\mathrm{Spec}\, A, \mathcal{O}) &\longleftarrow\!\shortmid\; A. \end{aligned}$$

Proof. (Sketch) First suppose we have a homomorphism of rings $f : A \to B$. By Lemma 4.1.2 this induces a morphism $f^* : \operatorname{Spec} B \to \operatorname{Spec} A, \mathfrak{p} \mapsto f^{-1}(\mathfrak{p})$ which is continuous since $f^{-1}(V(\mathfrak{a})) = V(f(\mathfrak{a}))$ for any ideal $\mathfrak{a} \subset A$. Now for each $\mathfrak{p} \in \operatorname{Spec} B$, define the localization $f_\mathfrak{p} : A_{f^*\mathfrak{p}} \to B_\mathfrak{p}$ using the universal property of localization. Then for any open set $V \subseteq \operatorname{Spec} A$, we get a map

$$f^\# : \mathcal{O}_A(V) \longrightarrow \mathcal{O}_B((f^*)^{-1}(V)).$$

One checks that each is a homomorphism of rings and commutes with the restriction maps. Thus $f^\# : \mathcal{O}_A \to \mathcal{O}_B$ is defined. Moreover, the induced map on stalks is just each $f_\mathfrak{p}$, so the pair $(f^*, f^\#)$ gives a morphism $(\operatorname{Spec} B, \mathcal{O}_B) \to (\operatorname{Spec} A, \mathcal{O}_A)$ of locally ringed spaces, hence of schemes.

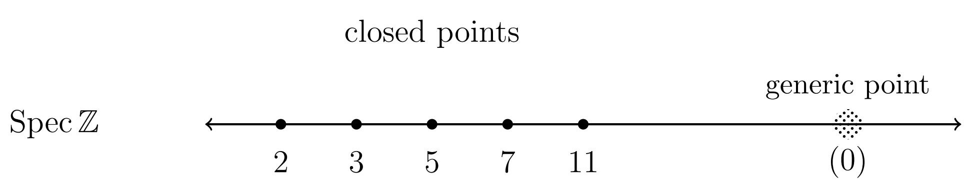
Conversely, take a morphism of schemes $(\varphi, \varphi^\#) : (\operatorname{Spec} B, \mathcal{O}_B) \to (\operatorname{Spec} A, \mathcal{O}_A)$. This induces a ring homomorphism $\Gamma(\operatorname{Spec} A, \mathcal{O}_A) \to \Gamma(\operatorname{Spec} B, \mathcal{O}_B)$ but by (2) of Theorem 4.1.4, $\Gamma(\operatorname{Spec} A, \mathcal{O}_A) \cong A$ and $\Gamma(\operatorname{Spec} B, \mathcal{O}_B) \cong B$ so we get a homomorphism $A \to B$. It's easy to see that the two functors described give the required isomorphism of categories. □

Example 4.2.3. We saw in Example 4.1.5 that for any field k, $\operatorname{Spec} k = *$ is a point with structure sheaf $\mathcal{O}(*) = k$. If $L_1, \ldots, L_r$ are finite separable field extensions of k, we call $A = L_1 \times \cdots \times L_r$ a *finite étale k-algebra*. Then $\operatorname{Spec} A = \operatorname{Spec} L_1 \coprod \cdots \coprod \operatorname{Spec} L_r$ is (schematically) a disjoint union of points.

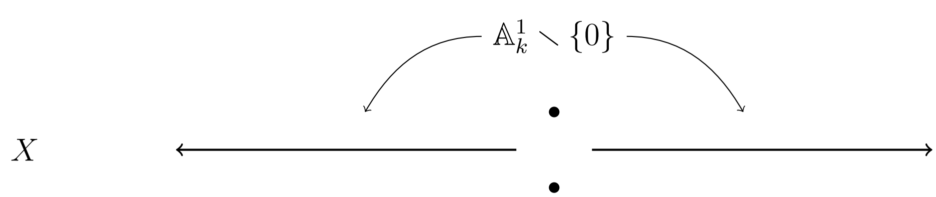
Example 4.2.4. Let A be a DVR with residue field k. Then $\operatorname{Spec} A = \{0, \mathfrak{m}_A\}$, a closed point for the maximal ideal $\mathfrak{m}$ and a generic point for the zero ideal. There are two open subsets here, $\{0\}$ and $\operatorname{Spec} A$, and we have $\mathcal{O}_A(\{0\}) = k$ and $\mathcal{O}_A(\operatorname{Spec} A) = A$.

Example 4.2.5. If k is a field and A is a finitely generated k-algebra, then the closed points of $X = \operatorname{Spec} A$ are in bijection with the closed points of an affine variety over k with coordinate ring A.

Example 4.2.6. Let $A = \mathbb{Z}$ (or any Dedekind domain). Then $\dim A = 1$ and it turns out that $\dim \operatorname{Spec} A = 1$ for some appropriate notion of dimension (see Section 4.3). Explicitly, $\operatorname{Spec} \mathbb{Z}$ has a closed point for every prime $p \in \mathbb{Z}$ and a generic point for (0):



Example 4.2.7. Let k be a field, $X_1 = X_2 = \mathbb{A}^1_k$ two copies of the affine line and $U_1 = U_2 = \mathbb{A}^1_k \smallsetminus \{0\}$, where 0 is the closed point of $\mathbb{A}^1_k$ corresponding to (x) in $k[x]$. Then we can glue together X_1 and X_2 along the identity map $U_1 \to U_2$ to get a scheme X which looks like the affine line with the origin "doubled". Note that X is not affine!



4.3 Properties of Schemes

Many definitions in ring theory can be rephrased for schemes. For example:

Definition. *A scheme X is* **reduced** *if for all open $U \subseteq X$, $\mathcal{O}_X(U)$ has no nilpotent elements.*

Definition. *A scheme X is* **integral** *if for all open $U \subseteq X$, $\mathcal{O}_X(U)$ has no zero divisors.*

Lemma 4.3.1. *X is integral if and only if X is reduced and irreducible as a topological space.*

Proof. ($\implies$) Clearly integral implies reduced, so we just need to prove X is irreducible. Suppose $X = U \cup V$ for open subsets $U, V \subseteq X$. Then $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \times \mathcal{O}_X(V)$ which is not a domain unless one of $\mathcal{O}_X(U), \mathcal{O}_X(V)$ is 0. In that case, U or V is empty, so this shows X is irreducible.

($\impliedby$) Suppose X is reduced and irreducible, but there exists an open set $U \subseteq X$ and $f, g \in \mathcal{O}_X(U)$ with $fg = 0$. Define closed sets

$$C = \{P \in U \mid f_P \in \mathfrak{m}_P \subset \mathcal{O}_{X,P}\}$$
$$D = \{P \in U \mid g_P \in \mathfrak{m}_P \subset \mathcal{O}_{X,P}\}.$$

Then by definition of $\mathcal{O}_X$, we must have $C \cup D = U$. By irreducibility, $C = U$ without loss of generality. Thus for any affine open set $U' \subseteq U$ with $U' = \operatorname{Spec} A$, we have $(\mathcal{O}_X|_{U'})(D(f)) = 0$ but by (3) of Theorem 4.1.4, $\mathcal{O}_{U'}(D(f)) \cong A_f$, the localization of A at powers of f. When $A_f = 0$, f is nilpotent but by assumption this means $f = 0$. Hence X is integral. □

Definition. *The* **dimension** *of a scheme X (or any topological space) is*

$$\dim X = \sup\{n \in \mathbb{N}_0 \mid \text{ there exists a chain of irreducible, closed sets } X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n \subseteq X\}.$$

Proposition 4.3.2. *Let A be a noetherian ring. Then $\dim \operatorname{Spec} A = \dim A$, the Krull dimension of A.*

Be warned that the converse to Proposition 4.3.2 is false in general.

Definition. *Let X be a scheme. Then*

- *X is* **locally noetherian** *if each stalk $\mathcal{O}_{X,P}$ is a local noetherian ring.*
- *X is* **noetherian** *if X is integral and locally noetherian.*
- *An integral scheme X is* **normal** *if each stalk $\mathcal{O}_{X,P}$ is integrally closed in its field of fractions.*
- *X is* **regular** *if each $\mathcal{O}_{X,P}$ is regular as a local ring, that is, $\dim \mathcal{O}_{X,P} = \dim \mathfrak{m}_P/\mathfrak{m}_P^2$ as $\mathcal{O}_{X,P}/\mathfrak{m}_P$-vector spaces.*

Definition. *Let $U \subseteq X$ be an open subset. Then $(U, \mathcal{O}_X|_U)$ is a scheme which we call an* **open subscheme** *of X. The natural morphism $j : U \hookrightarrow X$, $j^\# : \mathcal{O}_X \to j_*\mathcal{O}_X|_U$ is called an* **open immersion**.

Example 4.3.3. For $X = \operatorname{Spec} A$, let $f \in A$ and recall the open set $D(f)$ defined in Theorem 4.1.4. Then $D(f)$ is an open subscheme of X and the open immersion $D(f) \hookrightarrow X$ corresponds to the natural inclusion of prime ideals $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$ (this is a property of any localization).

Definition. *Let $A \to A/I$ be a quotient homomorphism of rings. Then the induced morphism $\operatorname{Spec}(A/I) \to \operatorname{Spec} A$ is called an* **affine closed immersion**. *For a general morphism of schemes $f : X \to Y$, f is called a* **closed immersion** *if f is injective, $f(X) \subseteq Y$ is closed and there exists a covering of X by affine open sets $\{U_i\}$ such that each $f|_{U_i} : U_i \to f(U_i)$ is an affine closed immersion. The set $f(X)$ is called a* **closed subscheme** *of Y.*

Definition. *A morphism $f : Y \to X$ is* **locally of finite type** *if there exists an affine covering $X = \bigcup U_i$, with $U_i = \operatorname{Spec} A_i$, such that each $f^{-1}(U_i)$ has an open covering $f^{-1}(U_i) = \bigcup_{j=1}^{n_i} \operatorname{Spec} B_{ij}$ for $n_i < \infty$ and B_{ij} a finitely generated A_i-algebra. Further, we say f is a* **finite morphism** *if each $n_i = 1$, i.e. $f^{-1}(U_i) = \operatorname{Spec} B_i$ for some finitely generated A_i-algebra B_i.*

4.4 Sheaves of Modules

Through Proposition 4.2.2, we are able to transfer commutative ring theory to the language of affine schemes. In this section, we define a suitable setting for transferring module theory to the language of sheaves and schemes.

Definition. *Let $(X, \mathcal{O}_X)$ be a ringed space. A* **sheaf of $\mathcal{O}_X$-modules**, *or an $\mathcal{O}_X$-module for short, is a sheaf of abelian groups $\mathcal{F}$ on X such that each $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and for each inclusion of open sets $V \subseteq U$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

If $\mathcal{F}(U) \subseteq \mathcal{O}_X(U)$ is an ideal for each open set U, then we call $\mathcal{F}$ a **sheaf of ideals** *on X.*

Example 4.4.1. Let $f : Y \to X$ be a morphism of ringed spaces. Then the pushforward sheaf $f_*\mathcal{O}_Y$ is naturally an $\mathcal{O}_X$-module on X via $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$. Additionally, the kernel sheaf of $f^\#$, defined on open sets by $(\ker f^\#)(U) = \ker(\mathcal{O}_X(U) \to f_*\mathcal{O}_Y(U))$, is a sheaf of ideals on X.

Most module terminology extends to sheaves of $\mathcal{O}_X$-modules. For example,

- A *morphism of $\mathcal{O}_X$-modules* is a morphism of sheaves $\mathcal{F} \to \mathcal{G}$ such that each $\mathcal{F}(U) \to \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-module map. We write $\operatorname{Hom}_X(\mathcal{F}, \mathcal{G}) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ for the set of morphisms $\mathcal{F} \to \mathcal{G}$ as $\mathcal{O}_X$-modules. This defines the category of $\mathcal{O}_X$-modules, written $\mathcal{O}_X$-`Mod`.

- Taking kernels, cokernels and images of morphisms of $\mathcal{O}_X$-modules again give $\mathcal{O}_X$-modules.
- Taking quotients of $\mathcal{O}_X$-modules by $\mathcal{O}_X$-submodules again give $\mathcal{O}_X$-modules.
- An *exact sequence of $\mathcal{O}_X$-modules* is a sequence $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ such that each $\mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$ is an exact sequence of $\mathcal{O}_X(U)$-modules.
- Basically any functor on modules over a ring generalizes to an operation on $\mathcal{O}_X$-modules, including Hom, written $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$; direct sum $\mathcal{F} \oplus \mathcal{G}$; tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$; and exterior powers $\bigwedge^n \mathcal{F}$.

The most important of these constructions for our purposes will be the direct sum operation.

Definition. *An $\mathcal{O}_X$-module $\mathcal{F}$ is* **free** *(of rank r) if $\mathcal{F} \cong \mathcal{O}_X^{\oplus r}$ as $\mathcal{O}_X$-modules. $\mathcal{F}$ is* **locally free** *if X has a covering $X = \bigcup U_i$ such that each $\mathcal{F}|_{U_i}$ is free as an $\mathcal{O}_X|_{U_i}$-module.*

Remark. The rank of a locally free sheaf of $\mathcal{O}_X$-modules is constant on connected components. In particular, the rank of a locally free $\mathcal{O}_X$-module is well-defined whenever X is connected.

Definition. *A locally free $\mathcal{O}_X$-module of rank 1 is called an* **invertible sheaf***.*

Let A be a ring, M an A-module and set $X = \operatorname{Spec} A$. To extend module theory to the language of schemes, we want to define an $\mathcal{O}_X$-module $\widetilde{M}$ on X. To start, for each $\mathfrak{p} \in \operatorname{Spec} A$, let $M_\mathfrak{p} = M \otimes_A A_\mathfrak{p}$ be the localization of the module M at $\mathfrak{p}$. Then $M_\mathfrak{p}$ is an $A_\mathfrak{p}$-module consisting of 'formal fractions' $\frac{m}{s}$ where $m \in M$ and $s \in S = A \setminus \mathfrak{p}$. For each open set $U \subseteq X$, define

$$\widetilde{M}(U) = \left\{ h : U \to \coprod_{\mathfrak{p} \in U} M_\mathfrak{p} \,\middle|\, s(\mathfrak{p}) \in M_\mathfrak{p}, \exists\, \mathfrak{p} \in V \subseteq U, m \in M, s \in A \text{ with } s(\mathfrak{q}) = \frac{m}{s} \text{ for all } \mathfrak{q} \in V \right\}.$$

(Compare this to the construction of the structure sheaf $\mathcal{O}_A$ on $\operatorname{Spec} A$ in Section 4.1. Also, note that necessarily the $s \in A$ in the definition above must lie outside of all $\mathfrak{q} \in V$.)

Proposition 4.4.2. *Let M be an A-module and $X = \operatorname{Spec} A$. Then $\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-modules on X, and moreover,*

(1) For any $\mathfrak{p} \in \operatorname{Spec} A$, $\widetilde{M}_\mathfrak{p} \cong M_\mathfrak{p}$ as rings.

(2) $\Gamma(X, \widetilde{M}) \cong M$ as A-modules.

(3) For any $f \in A$, $\widetilde{M}(D(f)) \cong M_f = M \otimes_A A_f$ as A-modules.

The proof is similar to the proof of Theorem 4.1.4; both can be found in Hartshorne.

Proposition 4.4.3. *Let $X = \operatorname{Spec} A$. Then the association*

$$\begin{aligned} A\text{-}\mathtt{Mod} &\longrightarrow \mathcal{O}_X\text{-}\mathtt{Mod} \\ M &\longmapsto \widetilde{M} \end{aligned}$$

defines an exact, fully faithful functor.

Proof. Similar to the proof of Proposition 4.2.2. □

These $\widetilde{M}$ will be our affine model for modules over a scheme X. We next define the general notion, along with an analogue of finitely generated modules over a ring.

Definition. *Let $(X, \mathcal{O}_X)$ be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is* **quasi-coherent** *if there is an affine covering $X = \bigcup X_i$, with $X_i = \operatorname{Spec} A_i$, and A_i-modules M_i such that $\mathcal{F}|_{X_i} \cong \widetilde{M}_i$ as $\mathcal{O}_X|_{X_i}$-modules. Further, we say $\mathcal{F}$ is* **coherent** *if each M_i is a finitely generated A_i-module.*

Example 4.4.4. For any scheme X, the structure sheaf $\mathcal{O}_X$ is obviously a coherent sheaf on X.

Let $\mathtt{QCoh}_X$ (resp. $\mathtt{Coh}_X$) be the category of quasi-coherent (resp. coherent) sheaves of $\mathcal{O}_X$-modules on X.

Theorem 4.4.5. *$\mathtt{QCoh}_X$ and $\mathtt{Coh}_X$ are abelian categories.*

Example 4.4.6. Let $X = \operatorname{Spec} A$, $I \subseteq A$ an ideal and $Y = \operatorname{Spec}(A/I)$. Then the natural inclusion $i : Y \hookrightarrow X$ is a closed immersion by definition, and it turns out that $i_*\mathcal{O}_Y \cong \widetilde{A/I}$ as $\mathcal{O}_X$-modules, so $i_*\mathcal{O}_Y$ is a quasi-coherent, even coherent, sheaf on X.

We next identify the image of the functor $M \mapsto \widetilde{M}$ from Proposition 4.4.3.

Theorem 4.4.7. *Let $X = \operatorname{Spec} A$. Then there is an equivalence of categories*

$$A\text{-}\mathtt{Mod} \xrightarrow{\sim} \mathtt{QCoh}_X.$$

Moreover, if A is noetherian, this restricts to an equivalence

$$A\text{-}\mathtt{mod} \xrightarrow{\sim} \mathtt{Coh}_X$$

where A-$\mathtt{mod}$ denotes the subcategory of finitely generated A-modules.

Proof. (Sketch) The association $M \mapsto \widetilde{M}$ sends an A-module to a quasi-coherent sheaf on $X = \operatorname{Spec} A$ by definition of quasi-coherence. Further, one can prove that a sheaf $\mathcal{F}$ on X is a quasi-coherent $\mathcal{O}_X$-module if and only if $\mathcal{F} \cong \widetilde{M}$ for an A-module M. The inverse functor $\mathtt{QCoh}_X \to A\text{-}\mathtt{Mod}$ is given by $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$.

When A is noetherian, the above extends to say that $\mathcal{F}$ is coherent if and only if $\mathcal{F} \cong \widetilde{M}$ for a finitely generated A-module M. The rest of the proof is identical. □

The following lemma generalizes Example 4.4.6.

Lemma 4.4.8. *Let $f : Y \to X$ be a morphism of schemes and let $\mathcal{G}$ be a quasi-coherent sheaf on Y. Then $f_*\mathcal{G}$ is a quasi-coherent sheaf on X. Further, if $\mathcal{G}$ is coherent and f is a finite morphism, then $f_*\mathcal{G}$ is also coherent.*

Note that the second statement is false in general.

4.5 The Proj Construction

A Category Theory

A.1 Representable Functors

Let $\mathcal{C}$ be a category and A, B two objects in $\mathcal{C}$.

Theorem A.1.1. *If* $A \cong B$ *then* $\mathrm{Hom}_{\mathcal{C}}(A, -)$ *and* $\mathrm{Hom}_{\mathcal{C}}(B, -)$ *are naturally isomorphic functors. Similarly,* $\mathrm{Hom}_{\mathcal{C}}(-, A) \cong \mathrm{Hom}_{\mathcal{C}}(-, B)$*.*

Proof. Let $A \cong B$ via the isomorphism $\alpha : A \to B$ and let α^{-1} be its inverse. Define a transformation $\tau : \mathrm{Hom}_{\mathcal{C}}(A, -) \to \mathrm{Hom}_{\mathcal{C}}(B, -)$ by

$$\begin{aligned} \tau_C = (\alpha^{-1})^* : \mathrm{Hom}_{\mathcal{C}}(A, C) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(B, C) \\ f &\longmapsto f\alpha^{-1} \end{aligned}$$

for each $C \in \mathrm{obj}(\mathcal{C})$. Note that $f\alpha^{-1} : B \to C$ so this map is well-defined.

Next we verify that for any morphism $f : C \to C'$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(A, C) & \xrightarrow{\tau_C} & \mathrm{Hom}(B, C) \\ {\scriptstyle f\square}\downarrow & & \downarrow{\scriptstyle f\square} \\ \mathrm{Hom}(A, C') & \xrightarrow[\tau_{C'}]{} & \mathrm{Hom}(B, C') \end{array}$$

If $h \in \mathrm{Hom}(A, C)$ then it maps $h \mapsto h\alpha^{-1} \mapsto fh\alpha^{-1}$ via the top map, and $h \mapsto fh \mapsto fh\alpha^{-1}$ via the bottom map, so indeed the diagram commutes. Thus τ is a natural transformation.

Finally we need to check that each τ_C is an isomorphism. To do this, construct a transformation $\sigma_C : \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$ taking $g \mapsto g\alpha$. Then for any $f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ we have

$$\tau_C(\sigma_C(g)) = \tau_C(g\alpha) = g\alpha\alpha^{-1} = g$$

and

$$\sigma_C(\tau_C(f)) = \sigma_C(f\alpha^{-1}) = f\alpha^{-1}\alpha = f.$$

Hence τ_C and σ_C are inverse morphisms, so τ_C is an isomorphism. This proves that τ is a natural isomorphism. The proof that $\mathrm{Hom}_{\mathcal{C}}(-, A) \cong \mathrm{Hom}_{\mathcal{C}}(-, B)$ is similar. □

We will see that the converse of this statement is also true. That is, $\mathrm{Hom}_{\mathcal{C}}(-, A)$ completely determines the object A itself (as does $\mathrm{Hom}_{\mathcal{C}}(A, -)$). But what's the meaning behind statements like this? For an object $A \in \mathrm{obj}(\mathcal{C})$, the functor $\mathrm{Hom}_{\mathcal{C}}(-, A) : \mathcal{C}^{op} \to \mathtt{Sets}$ gives us a lot of information about A. For example, if $\mathcal{C}$ is the category of groups and $\mathrm{Hom}_{\mathcal{C}}(\mathbb{Z}/2\mathbb{Z}, A)$ contains a nontrivial element, this tells us that A has at least one element of order 2. Likewise, in the category `Top` of topological spaces, elements of $\mathrm{Hom}(S^1, A)$ encode the information of closed loops in A. So "probing" A by different objects B, i.e. considering $\mathrm{Hom}_{\mathcal{C}}(B, A)$, tells us different sorts of information about A. By probing A with *all* objects in $\mathcal{C}$, would it be possible to know *all* information about A? The answer is yes, and the formal statement of this is called Yoneda's Lemma.

Lemma A.1.2 (Yoneda's Lemma)**.** *Let $\mathcal{C}$ be a category, $A \in \text{obj}(\mathcal{C})$ and $G : \mathcal{C} \to \texttt{Sets}$ a covariant functor. Then there is a bijection*

$$Nat(\text{Hom}_{\mathcal{C}}(-, A), G) \longleftrightarrow G(A).$$

Proof. Let $\tau : \text{Hom}_{\mathcal{C}}(-, A) \to G$ be a natural transformation. Then for each $B \in \text{obj}(\mathcal{C})$ and morphism $\varphi : A \to B$, the following diagram commutes:

$$\begin{array}{ccc} \text{Hom}(A, A) & \xrightarrow{\tau_A} & G(A) \\ {\scriptstyle \varphi_*}\big\downarrow & & \big\downarrow{\scriptstyle G(\varphi)} \\ \text{Hom}(B, A) & \xrightarrow[\tau_B]{} & G(B) \end{array}$$

where $\varphi_* : f \mapsto \varphi \circ f$. In particular, $1_A \in \text{Hom}_{\mathcal{C}}(A, A)$ maps to $\tau_A(1_A) \in G(A)$. Define the map $\text{Nat}(\text{Hom}_{\mathcal{C}}(-, A), G) \to G(A)$ by sending $\tau \mapsto \tau_A(1_A)$. If $\sigma : \text{Hom}_{\mathcal{C}}(-, A) \to G$ is another natural transformation such that $\sigma_A(1_A) = \tau_A(1_A)$, then the above diagram shows that for all $\varphi \in \text{Hom}_{\mathcal{C}}(B, A)$,

$$\sigma_B(\varphi) = \sigma_B\varphi_*(1_A) = G(\varphi)\tau_A(1_A) = G(\varphi)\sigma_A(1_A) = \tau_B\varphi_*(1_A) = \tau_B(\varphi).$$

Thus $\sigma = \tau$, so the assignment is one-to-one.

We next show every element $x \in G(A)$ is induced by such a natural transformation $\tau : \text{Hom}_{\mathcal{C}}(-, A) \to G$. For each $B \in \text{obj}(\mathcal{C})$ and $f \in \text{Hom}_{\mathcal{C}}(B, A)$, define $\tau_B(f) = G(f)(x)$. Then for any $B \to C$, the following diagram commutes:

$$\begin{array}{ccc} \text{Hom}(A, B) & \xrightarrow{\tau_B} & G(B) \\ \big\downarrow & & \big\downarrow \\ \text{Hom}(A, C) & \xrightarrow[\tau_C]{} & G(C) \end{array}$$

Hence τ is a natural transformation and by construction, $\tau_A(1_A) = G(1_A)(x) = 1_{G(A)}(x) = x$. This proves the bijection. □

Corollary A.1.3. *If $\text{Hom}_{\mathcal{C}}(-, A)$ and $\text{Hom}_{\mathcal{C}}(-, B)$ are naturally isomorphic, then $A \cong B$.*

Proof. Suppose $\tau : \text{Hom}_{\mathcal{C}}(-, A) \to \text{Hom}_{\mathcal{C}}(-, B)$ is a natural isomorphism. Applying Yoneda's Lemma to $G = \text{Hom}_{\mathcal{C}}(-, B)$, we get a unique element $x \in G(A) = \text{Hom}_{\mathcal{C}}(A, B)$. Applying Yoneda to the inverse of τ yields a unique inverse to x, proving $A \cong B$. □

For any category $\mathcal{C}$, there is a functor $\mathcal{C} \to \texttt{Fun}(\mathcal{C}^{op}, \texttt{Sets})$ given by $A \mapsto \text{Hom}_{\mathcal{C}}(-, A)$. The Yoneda Lemma shows that this functor is *fully faithful*, i.e. it embeds $\mathcal{C}$ as a full subcategory of $\texttt{Fun}(\mathcal{C}^{op}, \texttt{Sets})$. More precisely:

Corollary A.1.4 (Yoneda Embedding)**.** *There is a functor* $\mathcal{C} \to \mathtt{Fun}(\mathcal{C}^{op}, \mathtt{Sets}), A \mapsto \mathrm{Hom}_{\mathcal{C}}(-, A)$*, which is an isomorphism on* $\mathrm{Hom}_{\mathcal{C}}(B, C)$ *for all* $B, C \in \mathcal{C}$*.*

Lemma A.1.5 (Yoneda's Lemma, Covariant Version)**.** *Let* $\mathcal{C}$ *be a category,* $A \in \mathrm{obj}(\mathcal{C})$ *and* $G : \mathcal{C} \to \mathtt{Sets}$ *a covariant functor. Then there is a bijection*

$$\begin{aligned} Nat(\mathrm{Hom}_{\mathcal{C}}(A, -), G) &\longleftrightarrow G(A) \\ \tau &\longmapsto \tau_A(1_A). \end{aligned}$$

Proof. Similar to the proof of Lemma A.1.2. □

Corollary A.1.6 (Yoneda Embedding, Covariant Version)**.** *There is a fully faithful functor* $\mathcal{C} \to \mathtt{Fun}(\mathcal{C}, \mathtt{Sets}), A \mapsto \mathrm{Hom}_{\mathcal{C}}(A, -)$*.*

A.2 Adjoint Functors

Recall the following basic theorem in ring theory.

Theorem A.2.1 (Hom-Tensor Adjointness)**.** *Given rings* R *and* S *and modules* $A_R, {}_RB_S$ *and* C_S*, there is an isomorphism*

$$\begin{aligned} \tau_{A,B,C} : \mathrm{Hom}_S(A \otimes_R B, C) &\longrightarrow \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \\ f &\longmapsto [a \mapsto (b \mapsto f(a \otimes b))] \end{aligned}$$

i.e. given any $f : A \otimes_R B \to C$ *and an element* $a \in A$*, we can construct a linear map* $f_a : B \to C$ *sending* $b \mapsto f(a \otimes b)$*.*

The term adjointness comes from the idea that Hom may be viewed as an inner product on R-$\mathtt{Mod}$:

$$\langle \cdot, \cdot \rangle = \mathrm{Hom}_R(-, -) :_R \mathtt{Mod} \times_R \mathtt{Mod} \longrightarrow \mathtt{Ab}$$

which has $\otimes_R$ as its *adjoint functor.* In particular, we have functors

$$\begin{aligned} F_B &:= - \otimes_R B : \mathtt{Mod}_R \longrightarrow \mathtt{Mod}_S \\ G_B &:= \mathrm{Hom}_S(B, -) : \mathtt{Mod}_S \longrightarrow \mathtt{Mod}_R. \end{aligned}$$

Then Theorem A.2.1 says that F_B and G_B are adjoint functors. F_B is sometimes called the left adjoint and G_B the right adjoint, and together they are called an adjoint pair. This generalizes as follows.

Definition. *Let* $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ *and* $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ *be functors. Then* $(\mathcal{F}, \mathcal{G})$ *is called an* **adjoint pair** *if there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{B}}(\mathcal{F}-, -) \cong \mathrm{Hom}_{\mathcal{A}}(-, \mathcal{G}-).$$

In this case $\mathcal{F}$ *is called a* **left adjoint** *of* $\mathcal{G}$ *and* $\mathcal{G}$ *is a* **right adjoint** *of* $\mathcal{F}$*.*

Lemma A.2.2. *Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ be a pair of functors. Then $(\mathcal{F}, \mathcal{G})$ is an adjoint pair if and only if there exist natural transformations $\eta : id_{\mathcal{A}} \to \mathcal{GF}$ and $\varepsilon : \mathcal{FG} \to id_{\mathcal{B}}$ such that the compositions $\mathcal{G}\varepsilon \circ \eta\mathcal{G}$ and $\varepsilon\mathcal{F} \circ \mathcal{F}\eta$ are both the identity natural transformation.*

Proof. ($\Longrightarrow$) Suppose $(\mathcal{F}, \mathcal{G})$ is an adjoint pair. Then for any objects $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, there is an isomorphism

$$\Phi_{X,Y} : \mathrm{Hom}(\mathcal{F}X, Y) \xrightarrow{\sim} \mathrm{Hom}(X, \mathcal{G}Y).$$

Applying this to $Y = \mathcal{F}X$, we have $\Phi_{\mathcal{F}X,\mathcal{F}X} : \mathrm{Hom}(\mathcal{F}X, \mathcal{F}X) \xrightarrow{\sim} \mathrm{Hom}(X, \mathcal{GF}X)$ which takes the identity $id_{\mathcal{F}X}$ to some morphism $\eta_X = \Phi_{\mathcal{F}X,\mathcal{F}X}(id_{\mathcal{F}X}) : X \to \mathcal{GF}X$. This defines the natural transformation $\eta : id_{\mathcal{A}} \to \mathcal{GF}$. On the other hand, applying the natural isomorphism Φ to $X = \mathcal{G}Y$ gives an isomorphism $\Phi_{\mathcal{FG}Y,Y} : \mathrm{Hom}(\mathcal{FG}Y, Y) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{G}Y, \mathcal{G}Y)$ under which the identity $id_{\mathcal{G}Y}$ is mapped to by some $\varepsilon_Y = \Phi^{-1}_{\mathcal{FG}Y,Y}(id_{\mathcal{G}Y}) : \mathcal{FG}Y \to Y$. This defines the natural transformation $\varepsilon : \mathcal{FG} \to id_{\mathcal{B}}$. To check the identity conditions, apply Φ to the sequence $\varepsilon_Y : \mathcal{FG}Y \xrightarrow{id_{\mathcal{FG}Y}} \mathcal{FG}Y \xrightarrow{\varepsilon_Y} Y$ to get:

$$\mathcal{G}Y \xrightarrow{\eta_{\mathcal{G}Y}} \mathcal{GFG}Y \xrightarrow{\mathcal{G}\varepsilon} \mathcal{G}Y.$$

Thus $\mathcal{G}\varepsilon \circ \eta_{\mathcal{G}Y} = \Phi_{\mathcal{FG}Y,Y}(\varepsilon_Y)$ which by definition equals $id_{\mathcal{G}Y}$. The proof of the other identity is similar.

($\Longleftarrow$) Given natural transformations $\eta : id_{\mathcal{A}} \to \mathcal{GF}$ and $\varepsilon : \mathcal{FG} \to id_{\mathcal{B}}$, we define a natural transformation $\Phi_{X,Y} : \mathrm{Hom}(\mathcal{F}X, Y) \to \mathrm{Hom}(X, \mathcal{G}Y)$ by sending $\alpha : \mathcal{F}X \to Y$ to $\mathcal{G}(\alpha) \circ \eta : X \to \mathcal{GF}X \to \mathcal{G}Y$. Similarly, define its inverse $\Phi^{-1}_{X,Y} : \mathrm{Hom}(X, \mathcal{G}Y) \to \mathrm{Hom}(\mathcal{F}X, Y)$ by sending $\beta : X \to \mathcal{G}Y$ to $\varepsilon \circ \mathcal{F}(\beta) : \mathcal{F}X \to \mathcal{FG}Y \to Y$. To see that these are natural inverses, fix $X \in \mathcal{A}, Y \in \mathcal{B}$ and $\alpha : \mathcal{F}X \to Y$ and consider

$$\begin{aligned}\Phi^{-1}_{X,Y}\Phi_{X,Y}(\alpha) &= \Phi^{-1}_{X,Y}(\mathcal{G}(\alpha) \circ \eta) = \varepsilon \circ \mathcal{F}(\mathcal{G}(\alpha) \circ \eta)\\ &= \varepsilon \circ \mathcal{FG}(\alpha) \circ \mathcal{F}\eta \quad \text{since } \mathcal{F} \text{ is a functor}\\ &= \varepsilon\mathcal{F}(\mathcal{G}(\alpha))\mathcal{F}\eta = \varepsilon\mathcal{F} \circ \mathcal{F}\eta(\alpha) = \alpha\end{aligned}$$

by the identity $\varepsilon\mathcal{F} \circ \mathcal{F}\eta = id$. Similarly, for $\beta : X \to \mathcal{G}Y$, we have

$$\begin{aligned}\Phi_{X,Y}\Phi^{-1}_{X,Y}(\beta) &= \Phi_{X,Y}(\varepsilon \circ \mathcal{F}(\beta)) = \mathcal{G}(\varepsilon \circ \mathcal{F}(\beta)) \circ \eta\\ &= \mathcal{G}\varepsilon \circ \mathcal{GF}(\beta) \circ \eta \quad \text{since } \mathcal{G} \text{ is a functor}\\ &= \mathcal{G}\varepsilon \circ \eta\mathcal{G}(\beta) = \beta\end{aligned}$$

by the identity $\mathcal{G}\varepsilon \circ \eta\mathcal{G} = id$. Hence $\Phi_{X,Y}$ and $\Phi^{-1}_{X,Y}$ form a natural isomorphism. □

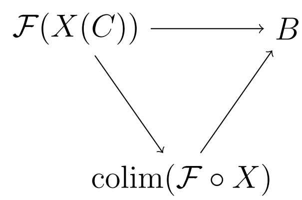
Definition. *For an adjoint pair $(\mathcal{F}, \mathcal{G})$, the natural transformations $\eta : id_{\mathcal{A}} \to \mathcal{GF}$ and $\varepsilon : \mathcal{FG} \to id_{\mathcal{B}}$ are called the* **unit** *and* **counit** *of the adjunction, respectively. The conditions $\mathcal{G}\varepsilon \circ \eta\mathcal{G} = id$ and $\varepsilon\mathcal{F} \circ \mathcal{F}\eta = id$ are called the* **triangle identities**.

Theorem A.2.3. *If $(\mathcal{F}, \mathcal{G})$ is an adjoint pair of functors, then $\mathcal{F}$ preserves colimits and $\mathcal{G}$ preserves limits.*

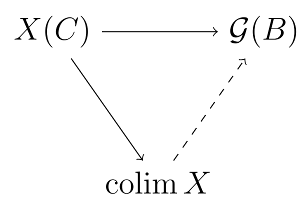
Proof. Assume $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$. Let $X : \mathcal{C} \to \mathcal{A}$ be a direct system and for each $C \in \mathcal{C}$, let $\varphi_C : X(C) \to \operatorname{colim} X$ be the induced morphism making the relevant diagrams commute. We must prove that $\mathcal{F}(\operatorname{colim} X)$ is the colimit of the direct system $\mathcal{F} \circ X : \mathcal{C} \to \mathcal{A} \to \mathcal{B}$. For any $C \in \mathcal{C}$ and $B \in \mathcal{B}$, there is an isomorphism

$$\operatorname{Hom}_{\mathcal{B}}(\mathcal{F}(X(C)), B) \cong \operatorname{Hom}_{\mathcal{A}}(X(C), \mathcal{G}(B)).$$

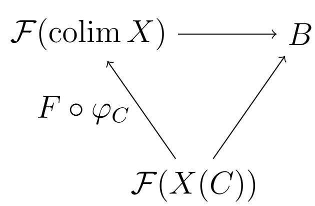
Thus for any diagram



the corresponding diagram can be completed:



On the other hand, applying $\mathcal{F}$ to $\operatorname{colim} X$ gives a diagram



In other words, every map $\mathcal{F}(X(C)) \to B$ which is compatible with the $F \circ \varphi_C$ will factor through $\mathcal{F}(\operatorname{colim} X)$. Hence by the universal property of colimits, $\mathcal{F}(\operatorname{colim} X) = \operatorname{colim}(\mathcal{F} \circ X)$. The proof that $\mathcal{G}$ preserves limits is dual. □

Corollary A.2.4. *For any (S, R)-bimodule L, $L \otimes_R -$ preserves colimits and $\operatorname{Hom}_S(L, -)$ preserves limits.*

Theorem A.2.5. *Let $\mathcal{F}$: $\texttt{Mod}_R \to \texttt{AbGps}$ be an additive functor. Then the following are equivalent:*

(1) $\mathcal{F} \cong L \otimes_R -$ for some R-module L.

(2) $\mathcal{F}$ preserves colimits.

(3) $\mathcal{F}$ is right exact and preserves direct sums.

(4) $\mathcal{F}$ has a right adjoint.

Likewise, we have:

Theorem A.2.6. *Let* $\mathcal{F} : \texttt{Mod}_R \to \texttt{AbGps}$ *be an additive functor. Then the following are equivalent:*

(1) $\mathcal{F} \cong \mathrm{Hom}_R(L, -)$ *for some R-module L.*

(2) $\mathcal{F}$ *preserves limits.*

(3) $\mathcal{F}$ *is left exact and preserves direct products.*

(4) $\mathcal{F}$ *has a left adjoint.*

A.3 Limits and Colimits

Products and coproducts are special cases of a more general notion in category theory. Let $\mathcal{C}$ be a *diagram category*, i.e. a category whose objects are the vertices of a directed graph and whose morphisms are in bijection with the directed edges of this graph (there may be multiple loops on a given vertex, but every vertex is assumed to possess a distinguished 'identity' loop).

Definition. *A* **direct** *(resp.* **inverse***)* **system** *in a category* $\mathcal{A}$ *is a covariant (resp. contravariant) functor* $\mathcal{C} \to \mathcal{A}$ *where* $\mathcal{C}$ *is a diagram category.*

Definition. *Let* $\mathcal{F} : \mathcal{C} \to \mathcal{A}$ *be an inverse system in* $\mathcal{A}$*. The* **limit** *(also called the* **projective** *or* **inverse limit***) of* $\mathcal{F}$ *is an object* $\lim \mathcal{F} \in \mathrm{obj}(\mathcal{A})$ *such that for all objects* $C \in \mathcal{C}$*, there are morphisms* $\varphi_C : \lim \mathcal{F} \to \mathcal{F}(C)$ *making the diagrams*

$$\begin{array}{ccc} & \lim \mathcal{F} & \\ {\scriptstyle\varphi_C}\swarrow & & \searrow{\scriptstyle\varphi_{C'}} \\ \mathcal{F}(C) & \longrightarrow & \mathcal{F}(C') \end{array}$$

commute whenever $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}(C), \mathcal{F}(C')) \neq \varnothing$*, and such that* $\lim \mathcal{F}$ *is universal among all such objects in* $\mathcal{A}$*.*

Limits are sometimes also written $\varprojlim \mathcal{F}$. By the universal property, limits are unique up to unique isomorphism.

Example A.3.1. Let $\mathcal{C}$ be the category consisting of two objects $\{1, 2\}$ and only identity morphisms. An inverse system $\mathcal{F} : \mathcal{C} \to \mathcal{A}$ is just defined by specifying two objects, $\mathcal{F}(1) = a_1$ and $\mathcal{F}(2) = a_2$. Then $\lim \mathcal{F}$ is the product of these elements, $a_1 \sqcap a_2$.

Definition. *Let* $\mathcal{F} : \mathcal{C} \to \mathcal{A}$ *be a direct system in* $\mathcal{A}$*. The* **colimit** *(also called the* **injective** *or* **direct limit***) of* $\mathcal{F}$ *is an object* $\mathrm{colim}\, \mathcal{F} \in \mathrm{obj}(\mathcal{A})$ *such that for all* $C \in \mathcal{C}$*, there are morphisms* $\psi_C : \mathcal{F}(C) \to \mathrm{colim}\, \mathcal{F}$ *making the diagrams*

$$\begin{array}{ccc} \mathcal{F}(C) & \longrightarrow & \mathcal{F}(C') \\ & {\scriptstyle \psi_C} \searrow \quad \swarrow {\scriptstyle \psi_{C'}} & \\ & \operatorname{colim} \mathcal{F} & \end{array}$$

commute whenever $\operatorname{Hom}_{\mathcal{A}}(\mathcal{F}(C), \mathcal{F}(C')) \neq \varnothing$, *and such that* $\operatorname{colim} \mathcal{F}$ *is universal among all such objects in* $\mathcal{A}$.

Example A.3.2. Let $\mathcal{C}$ be the same category as in Example A.3.1. Then a direct system $\mathcal{F} : \mathcal{C} \to \mathcal{A}$ is once again defined by specifying $\mathcal{F}(1) = a_1$ and $\mathcal{F}(2) = a_2$, but $\operatorname{colim} \mathcal{F}$ is the coproduct of these elements, $a_1 \sqcup a_2$.

There are plenty of other important examples of limits and colimits in category theory.

Definition. *Let* $\mathcal{C}$ *be the diagram category*

$$\begin{array}{ccc} & & \bullet \\ \bullet & \begin{array}{c} \nearrow \\ \searrow \end{array} & \\ & & \bullet \end{array}$$

The colimit of a direct system $\mathcal{F} : \mathcal{C} \to \mathcal{A}$ *is called the* **pushout** *of* $\mathcal{F}$ *and the limit of an inverse system* $\mathcal{F} : \mathcal{C} \to \mathcal{A}$ *is called the* **pullback** *of* $\mathcal{F}$.

Explicitly, if a direct system $\mathcal{F}$ has image $A \begin{array}{c} \nearrow B \\ \searrow C \end{array}$ then we write the pushout $Q = \operatorname{colim} \mathcal{F}$ as a square diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ & & \lrcorner \\ C & \longrightarrow & Q \end{array}$$

Similarly, if $\mathcal{F}$ is an inverse system with image $\begin{array}{c} B \searrow \\ C \nearrow \end{array} A$ then we write the pullback $P = \lim \mathcal{F}$ as a square diagram

$$\begin{array}{ccc} P & \longrightarrow & B \\ \downarrow & \ulcorner & \downarrow \\ C & \longrightarrow & A \end{array}$$

Definition. *Suppose $\mathcal{A}$ is a category with an object $0 \in \operatorname{obj}(\mathcal{A})$ that is both initial and terminal. For any morphism $f : A \to B$, the pushout of the diagram* $\begin{matrix} & \overset{f}{\nearrow} & B \\ A & & \\ & \searrow & 0 \end{matrix}$ *is called the* **cokernel** *of f, written $\operatorname{coker} f$. Similarly, for such a morphism f, the pullback of the diagram* $\begin{matrix} 0 & \searrow & \\ & & B \\ A & \underset{f}{\nearrow} & \end{matrix}$ *is called the* **kernel** *of f, written $\ker f$.*

Example A.3.3. The category `AbGps` of abelian groups has the trivial group 0 as a zero object, i.e. an object that is both initial and terminal. Then for a homomorphism of abelian groups $f : A \to B$, the kernel and cokernel of f coincide exactly with these notions from abstract algebra:

$$\begin{aligned} \ker f &= \{a \in A \mid f(a) = 0\} \\ \text{and} \quad \operatorname{coker} f &= \{[b] \mid b \in B \text{ and } [b] = [b'] \text{ if } b' = b + f(a)\}. \end{aligned}$$

This also holds in $\mathtt{Mod}_R$ for any ring R, and indeed in any *abelian category* as we shall see in the next section.

A.4 Abelian Categories

Abelian categories are the preferred setting for working with derived functors, which are the main tools used in homological algebra.

Definition. *$\mathcal{C}$ is an* **additive category** *if*

- $\operatorname{Hom}_{\mathcal{C}}(A, B)$ *is an abelian group for all $A, B \in \operatorname{obj}(\mathcal{C})$. Note that we normally only require* Hom *to be a set.*
- *For all $a \in A, b \in B, f \in \operatorname{Hom}(A, B)$ and $g \in \operatorname{Hom}(B, A)$,*

$$b(f + g) = bf + bf \qquad \textit{and} \qquad (f + g)a = fa + ga.$$

- *There exists a 0 object, which is both initial and terminal, meaning for all $A \in \operatorname{obj}(\mathcal{C})$, there exist maps $0 \to A$ and $A \to 0$.*
- *$\mathcal{C}$ has finite products ($\prod$) and coproducts ($\bigoplus$) and they agree on finite-index sets.*

The last condition, that $\prod$ and $\bigoplus$ agree on finite sets, can be proven using the other axioms. We can also define **additive functors** between additive categories, which are functors that preserve the additive structure of the category.

To fully understand abelian categories, it requires us to redefine our concept of kernels and cokernels. These are best understood in terms of monics and epics.

Definition. *A* **monomorphism***, or* **monic***, in a category $\mathcal{C}$ is a morphism $u : B \to C$ so that if there are maps $f, g : A \to B$ such that $uf = ug$, then $f = g$.*

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{u} C$$

Definition. *An* **epimorphism**, *or* **epic**, *is the dual notion, that is π is an epic if for every $f, g : B \to C$, $f\pi = g\pi$ implies $f = g$.*

$$A \xrightarrow{\pi} B \underset{g}{\overset{f}{\rightrightarrows}} C$$

In general, if u is one-to-one then u is monic, and if π is onto then π is epic, but the converse does *not* hold in a general category. We use the converse statements to define an abelian category.

Definition. *An additive category $\mathcal{A}$ is* **abelian** *if*

(1) Every morphism has a kernel and a cokernel.

(2) If $f : M \to N$ is monic, then M is the kernel of the diagram $\begin{matrix} 0 \searrow & \\ & \operatorname{coker} f \\ N \nearrow & \end{matrix}$

(3) If f is epic, then N is the cokernel of the diagram $\begin{matrix} & \nearrow M \\ \ker f & \\ & \searrow 0 \end{matrix}$

Note that condition (2) implies that if f is monic, then there exists a short exact sequence

$$0 \to M \xrightarrow{f} N \to \operatorname{coker} f \to 0.$$

Likewise, condition (3) implies that if f is epic, there exists a short exact sequence

$$0 \to \ker f \to M \xrightarrow{f} N \to 0.$$

Example A.4.1. For any ring R, the category $\texttt{Mod}_R$ is an abelian category. If R is (left) noetherian, then the subcategory $\texttt{mod}_R$ of finitely generated (left) R-modules is also an abelian category.

Theorem A.4.2. *Any abelian category is naturally isomorphic to a full subcategory of $\texttt{Mod}_R$ for some ring R.*

A.5 Grothendieck Topologies and Sites

To every topological space X, we can associate a category $\texttt{Top}(X)$ consisting of the open subsets $U \subseteq X$ with morphisms given by inclusions of open sets $U \hookrightarrow V$. A *presheaf* on X is a functor $F : \texttt{Top}(X)^{op} \to \texttt{Set}$, i.e. a contravariant functor on the category $\texttt{Top}(X)$. The conditions for F to be a *sheaf* on X can be summarized by saying that for every open set $U \in \texttt{Top}(X)$ and every open covering $U = \bigcup U_i$, the set $F(U)$ is an equalizer in the following diagram:

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

This generalizes as follows.

Definition. *A* **Grothendieck topology** *on a category* $\mathcal{C}$ *is a set of collections of morphisms* $\mathrm{Cov}(X) = \{\{X_i \to X\}_i\}$ *for every objects* $X \in \mathcal{C}$*, called* **coverings***, satisfying:*

(i) Every isomorphism $X' \to X$ *defines a covering* $\{X' \to X\}$ *in* $\mathrm{Cov}(X)$*.*

(ii) For any covering $\{X_i \to X\}$ *of* X *and any morphism* $Y \to X$ *in* $\mathcal{C}$*, the fibre products* $X_i \times_X Y$ *exist and the induced maps* $\{X_i \times_X Y \to Y\}$ *are a covering of* Y*.*

(iii) If $\{X_i \to X\}_i$ *is a covering of* X *and* $\{Y_{ij} \to X_i\}_j$ *is a covering of* X_i *for each* i*, then the compositions* $\{Y_{ij} \to X_i \to X\}_{i,j}$ *are a covering of* X*.*

A category equipped with a Grothendieck topology is called a **site***.*

Example A.5.1. For a topological space X, the category $\texttt{Top}(X)$ is a site with coverings

$$\mathrm{Cov}(U) = \left\{ \{U_i \hookrightarrow U\} : U_i \subseteq U \text{ are open and } U = \bigcup_i U_i \right\}.$$

When X is a scheme with the Zariski topology, $\texttt{Top}(X)$ is called the (*small*) *Zariski site* on X.

Example A.5.2. The category $\texttt{Top}$ of all topological spaces with continuous maps between them is a site, called the *big topological site*, whose coverings are defined by

$$\mathrm{Cov}(X) = \left\{ \{f_i : X_i \hookrightarrow X\} : f_i \text{ is an open embedding and } X = \bigcup_i X_i \right\}.$$

Example A.5.3. Similarly, for a scheme X, let $\texttt{Sch}_X$ be the category of X-schemes (the category $\texttt{Sch}$ of all schemes can be viewed in this framework by setting $X = \operatorname{Spec} \mathbb{Z}$ since this is a terminal object in $\texttt{Sch}$). Then $\texttt{Sch}_X$ is a site, called the *big Zariski site* on X, with coverings

$$\mathrm{Cov}(Y) = \left\{ \{\varphi_i : Y_i \to Y\} : \varphi_i \text{ is an open embedding and } Y = \bigcup_i Y_i \right\}.$$

Example A.5.4. Let $\mathcal{C}$ be a site and $X \in \mathcal{C}$ be an object. Define the *localized site* (or the *slice category*) $\mathcal{C}/X$ to be the category with objects $Y \to X \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$, morphisms $Y \to Z$ in $\mathcal{C}$ such that

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

commutes. Then $\mathcal{C}/X$ can be equipped with a Grothendieck topology by defining

$$\mathrm{Cov}(Y \to X) = \{\{Y_i \to Y\} : Y_i \to Y \in \mathrm{Hom}_X(Y_i, Y), \{Y_i \to Y\} \in \mathrm{Cov}_{\mathcal{C}}(Y)\}.$$

Example A.5.5. Let X be a scheme and define the (*small*) *étale site* on X to be the category $\acute{\mathrm{E}}\mathrm{t}(X)$ of X-schemes with étale morphisms $Y \to X$ and covers $\{Y_i \to Y\} \in \mathrm{Cov}(Y)$ such that $\coprod Y_i \to Y$ is surjective.

Example A.5.6. In contrast, we can equip the slice category $\mathtt{Sch}/X$ with a Grothendieck topology by declaring $\{Y_i \to Y\}$ to be a covering of $Y \to X$ if each $Y_i \to Y$ is étale and $\coprod Y_i \to Y$ is surjective. The resulting site is referred to as the *big étale site* on X.

Example A.5.7. Similar constructions can be made by replacing "étale" with other properties, such as:

- The *fppf site* is the category $\mathtt{Sch}/X$ with coverings $\{Y_i \to Y\} \in \mathrm{Cov}(Y)$ such that $Y_i \to Y$ are flat and locally of finite presentation and $\coprod Y_i \to Y$ is surjective.
- The *lisse-étale site* $\mathrm{Lis}\acute{\mathrm{E}}\mathrm{t}(X)$ is the category of X-schemes with smooth morphisms between them, whose coverings are $\{Y_i \to Y\} \in \mathrm{Cov}(Y)$ such that the $Y_i \to Y$ are *étale* and $\coprod Y_i \to Y$ is surjective.
- The *smooth site* $\mathtt{Sm}(X)$ is the category of X-schemes with smooth morphisms between them and surjective families of *smooth* coverings.
- Most generally, the *flat site* is $\mathtt{Sch}/X$ with surjective families of flat morphisms of finite type as coverings.

Definition. *A* **continuous map between sites** $f : \mathcal{C}_1 \to \mathcal{C}_2$ *is a functor* $F : \mathcal{C}_2 \to \mathcal{C}_1$ *that preserves fibre products and takes coverings in* $\mathcal{C}_2$ *to coverings in* $\mathcal{C}_1$.

Remark. Notice that a continuous map between sites is a functor *in the opposite direction.* This is in analogy with the topological notion: a continuous map $f : X \to Y$ between topological spaces induces a functor $F : \mathtt{Top}(Y) \to \mathtt{Top}(X)$ given by $V \mapsto f^{-1}(V)$.

Example A.5.8. When X is a scheme, there are continuous maps between the various sites we have defined on $\mathtt{Sch}/X$. We collect some of these sites in the following table, along with their relevant features. (The arrows between sites represent continuous maps between sites, so the functors on the underlying categories go in the opposite direction. Note that when we define sheaves in the next section, sheaves will pull back in the *same direction* as these arrows.)

	X_{flat}	$\to$	X_{fppf}	$\to$	X_{smooth}	$\to$	$X_{\text{ét}}$	$\to$	X_{Nis}	$\to$	X_{Zar}
name	flat		fppf		smooth		étale		Nisnevich		Zariski
maps	all		flat, locally f.p.		smooth		étale		étale, with residue field isomorphisms		all

Example A.5.9. Let G be a profinite group and let $\mathcal{C}_G$ be the category of all finite, discrete G-sets. Then the collections of G-homomorphisms $\{X_i \to X\}$ such that $\coprod_i X_i \to X$ is surjective form a Grothendieck topology on $\mathcal{C}_G$. When $G = \mathrm{Gal}(\bar{k}/k)$ for some field k, the category $\mathcal{C}_G$ is equivalent to $X_{\text{ét}}$ for $X = \operatorname{Spec} k$.

The notions of presheaf and sheaf generalize quite naturally to an arbitrary category with a Grothendieck topology. In fact, one may view the axioms of a Grothendieck topology as precisely those necessary to define a sheaf theory on a category.

Definition. *A* **presheaf** *on a category $\mathcal{C}$ is a functor $F : \mathcal{C}^{op} \to \mathtt{Set}$, that is, a contravariant functor from $\mathcal{C}$ to the category of sets. The category of presheaves on $\mathcal{C}$ (with natural transformations between them) will be denoted $\mathtt{PreSh}_{\mathcal{C}}$.*

Definition. *We say F is* **separated** *if for every collection of maps $\{X_i \to X\}$, the map $F(X) \to \coprod_i F(X_i)$ is injective.*

Definition. *Let $\mathcal{C}$ be a site. A* **sheaf** *on $\mathcal{C}$ is a presheaf $F : \mathcal{C}^{op} \to \mathtt{Set}$ such that for every object $X \in \mathcal{C}$ and every covering $\{X_i \to X\} \in \mathrm{Cov}(X)$, the sequence of based sets*

$$F(X) \longrightarrow \prod_i F(X_i) \rightrightarrows \prod_{i,j} F(X_i \times_X X_j)$$

is exact, or equivalently, $F(X)$ is an equalizer in the diagram. The category of sheaves on $\mathcal{C}$ will be denoted $\mathtt{Sh}_{\mathcal{C}}$.

As in topology, we can consider sheaves on $\mathcal{C}$ with values in set categories with further structure, e.g. $\mathtt{Group}$, $\mathtt{Ring}$, $\mathrm{R}{-}\mathtt{Mod}$, $\mathtt{Alg}_k$.

Theorem A.5.10 (Sheafification). *The forgetful functor $\mathtt{Sh}_{\mathcal{C}} \to \mathtt{PreSh}_{\mathcal{C}}$ has a left adjoint $F \mapsto F^a$.*

Proof. First consider the forgetful functor $\mathtt{Sep}_{\mathcal{C}} \to \mathtt{PreSh}_{\mathcal{C}}$ defined on the subcategory of separated presheaves on $\mathcal{C}$. For a presheaf F on $\mathcal{C}$, let F^{sep} be the presheaf

$$X \longmapsto F^{sep}(X) := F(X)/\sim$$

where, for $a, b \in F(X)$, $a \sim b$ if there is a covering $\{X_i \to X\}$ of X such that a and b have the same image under the map

$$F(X) \to \coprod_i F(X_i).$$

By construction, F^{sep} is a separated presheaf on $\mathcal{C}$ and for any other separated presheaf F', any morphism of presheaves $F \to F'$ factors through F^{sep} uniquely. Hence $F \mapsto F^{sep}$ is left adjoint to the forgetful functor $\mathtt{Sep}_{\mathcal{C}} \to \mathtt{PreSh}_{\mathcal{C}}$ so it remains to construct a sheafification of every separated presheaf on $\mathcal{C}$.

For a separated presheaf F, define F^a to be the presheaf

$$X \longmapsto F^a(X) := (\{X_i \to X\}, \{\alpha_i\})/\sim$$

where $\{X_i \to X\} \in \mathrm{Cov}_{\mathcal{C}}(X)$, $\{\alpha_i\}$ is a collection of elements in the equalizer

$$\mathrm{Eq}\left(\prod_i F(X_i) \rightrightarrows \prod_{i,j} F(X_i \times_X X_j) \right),$$

and $(\{X_i \to X\}, \{\alpha_i\}) \sim (\{Y_j \to Y\}, \{\beta_j\})$ if α_i and β_j have the same image in $F(X_i \times_X Y_j)$ for all i, j. Then as above, F^a is a sheaf which is universal with respect to all morphisms of sheaves $F \to F'$. Thus $F \mapsto F^a$ defines a left adjoint to the forgetful functor $\mathtt{Sh}_{\mathcal{C}} \to \mathtt{Sep}_{\mathcal{C}}$ and composition with the first construction proves the theorem. □