Arithmetic Fundamental Group

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0 Introduction

The following notes are taken from a reading course on étale fundamental groups led by Dr. Lloyd West at the University of Virginia in Spring 2017. The contents were presented by students throughout the course and mostly follow Szamuely’s *Galois Groups and Fundamental Groups*. Main topics include:

- A review of covering space theory in topology (universal covers, monodromy, locally constant sheaves)
- Covers and ramified covers of normal curves
- The algebraic fundamental group for curves
- An introduction to schemes
- Finite étale covers of schemes and the étale fundamental group
- Grothendieck’s main theorems for the fundamental group of a scheme
- Applications.

0.1 Topology Review

There are two key concepts in algebraic topology that, for various reasons, one might want to consider in an algebraic setting. These are covering spaces and fundamental groups, and they are intimately connected. The more familiar concept might be that of the fundamental group, which at the beginning is usually defined in terms of homotopy classes of based loops in a given topological space. To define an algebraic analogue, we will need an alternative perspective on the fundamental group.

Let $X$ be a connected, locally simply connected topological space.

**Definition.** A **cover** of $X$ is a space $Y$ and a map $p : Y \to X$ that is a local homeomorphism. That is, for every $x \in X$ there is a neighborhood $U \subseteq X$ of $x$ such that $p^{-1}(U)$ is a disjoint union of open sets in $Y$ and $p$ restricts to a homeomorphism on each open set.

One consequence of this definition is that for all $x, y \in X$, $p^{-1}(x)$ is a discrete space and $p^{-1}(x) \cong p^{-1}(y)$. A primary goal in topology is to study and classify all such covers $Y \xrightarrow{p} X$.

**Definition.** A **morphism of covers** between $Y \xrightarrow{p} X$ and $Y' \xrightarrow{p'} X$ is a map $f : Y \to Y'$ making the following diagram commute:

$$
\begin{array}{ccc}
Y_1 & \xrightarrow{f} & Y_2 \\
p_1 \downarrow & & \downarrow p_2 \\
X & & \\
\end{array}
$$
This defines a category \( \text{Cov}_X \) of covers over \( X \). In this category, we will abbreviate \( \text{Hom}_{\text{Cov}_X}(Y, Z) \) by \( \text{Hom}_X(Y, Z) \).

**Example 0.1.1.** The unit interval \([0,1] \subseteq \mathbb{R}\) has no nontrivial covers. However, \( S^1 \subseteq \mathbb{C} \) does: for each \( n \in \mathbb{Z} \), the map

\[
p_n : S^1 \to S^1, z \mapsto z^n
\]

is a cover. Further, there is a special cover

\[
\pi : \mathbb{R} \to S^1, t \mapsto e^{2\pi it}
\]

such that for every \( n \in \mathbb{Z} \), the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{R} & \to & S^1 \\
\pi & \downarrow & p_n \\
S^1 & \nearrow & \pi^2
\end{array}
\]

All covers of the circle arise in this way.

The special property of \( \mathbb{R} \to S^1 \) leads to the notion of a universal cover.

**Definition.** A covering space \( \pi : \tilde{X} \to X \) is a **universal cover** for \( X \) if for every other cover \( p : Y \to X \), there is a unique map \( f : \tilde{X} \to Y \) making the diagram commute:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{f} & Y \\
\pi \downarrow & & \downarrow p \\
X & \xrightarrow{p} & X
\end{array}
\]

It is equivalent to say that a universal cover is any simply connected cover of \( X \), and one shows easily that universal covers are unique up to equivalence of covers. An important result is that a universal cover exists, under certain mild conditions on \( X \).

In topology, the topological fundamental group is defined using homotopy:

\[
\pi_1^{\text{top}}(X, x) := \left\{ \text{homotopy classes of loops in } X \text{ based at } x \right\}.
\]

The universal cover has important connections to this fundamental group. In particular, consider an automorphism \( \alpha \in \text{Aut}_X(\tilde{X}) = \text{Hom}_X(\tilde{X}, \tilde{X}) \). Fix \( x \in X \) and a lift \( \tilde{x} \in \tilde{X} \) of \( x \). Then \( \pi\alpha(\tilde{x}) = x \). Moreover, since \( \tilde{X} \) is simply connected, any path \( \tilde{x} \to \alpha \tilde{x} \) is unique up to homotopy. This determines a map \( \text{Aut}_X(\tilde{X}) \to \pi_1^{\text{top}}(X, x) \).

**Theorem 0.1.2.** For any \( x \in X \), \( \text{Aut}_X(\tilde{X}) \to \pi_1^{\text{top}}(X, x) \) is an isomorphism.
Unfortunately, such a space \( \tilde{X} \) does not exist in the algebraic world. Thus we describe a slightly different interpretation of the fundamental group.

**Definition.** The fibre functor over \( x \in X \) is the assignment

\[
\text{Fib}_x : \text{Cov}_X \longrightarrow \text{Sets} \\
(Y \xrightarrow{p} X) \longmapsto p^{-1}(x).
\]

By the universal property of a universal cover \( \tilde{X} \in \text{Cov}_X \), to give a morphism of covers \( f : \tilde{X} \to Y \) is the same as to choose a point \( y = f(\tilde{x}) \in p^{-1}(x) \). In other words, \( \text{Fib}_x \) is a representable functor, i.e. for \( x \in X \), there is a natural isomorphism

\[
\text{Fib}_x(-) \cong \text{Hom}_X(\tilde{X}, -),
\]

where the Hom set consists of morphisms based at \( \tilde{x} \). This fibre functor is constructible in algebraic categories, though it fails to be representable.

Going further, there is a natural left action of \( \text{Aut}_X(\tilde{X}) \) on \( \tilde{X} \); however, it will be more convenient to view this as a right action of \( \text{Aut}_X(\tilde{X})^{\text{op}} \) on \( \tilde{X} \). This induces a left action of \( \text{Aut}_X(\tilde{X}) = \pi_1^{\text{top}}(X, x) \) on \( \text{Hom}_X(\tilde{X}, Y) \):

\[
\alpha \cdot f = f \circ \alpha \quad \text{for any } \alpha \in \text{Aut}_X(\tilde{X}), f : \tilde{X} \to Y.
\]

This action is called the monodromy action. Often, one views this as an action of \( \pi_1^{\text{top}}(X, x) \) on the fibre \( \text{Fib}_x(Y) \) given by lifting paths. In any case, we get a map

\[
\pi_1^{\text{top}}(X, x) \longrightarrow \text{Aut}(\text{Fib}_x),
\]

where \( \text{Aut}(\text{Fib}_x) \) is the automorphism group of the fibre functor in the following sense. For any functor \( F : \mathcal{C} \to \mathcal{D} \), an automorphism of \( F \) is a natural transformation of \( F \) that has a two-sided inverse. The set \( \text{Aut}(F) \) of all automorphisms of \( F \) is then a group under composition. Moreover, \( \text{Aut}(F) \) has a natural action on \( F(C) \) for any object \( C \in \mathcal{C} \).

**Theorem 0.1.3.** For all \( x \in X \), \( \pi_1^{\text{top}}(X, x) \to \text{Aut}(\text{Fib}_x) \) is an isomorphism.

**Theorem 0.1.4.** Let \( X \) be a connected, locally simply connected space and fix \( x \in X \). Then the fibre functor \( \text{Fib}_x \) defines an equivalence of categories

\[
\text{Cov}_X \xrightarrow{\sim} \{ \text{left } \pi_1(X, x)\text{-sets} \}
\]

with connected covers corresponding to transitive \( \pi_1(X, x)\text{-sets} \) and Galois covers to coset spaces of \( \tilde{X}_x \) by normal subgroups.

**Proof.** For a transitive \( \pi_1(X, x)\)-set \( S \), define \( Y_S = \tilde{X}/H \) where \( H = \text{Stab}_{\pi_1(X, x)}(s) \) for any point \( s \in S \). This defines a Galois cover \( Y_S \to X \) and one can extend this to arbitrary \( \pi_1(X, x)\)-sets orbitwise for the full correspondence. \( \square \)
The picture gets more interesting if we restrict ourselves to finite covers. Given such a cover \( p : Y \to X \), there is an exact sequence of groups

\[
1 \to N \to \pi_1^{\text{top}}(X, x) \to \text{Aut}_X(p^{-1}(x)) \to 1
\]

where \( N \) is some finite index kernel. This shows that the monodromy action factors through a finite quotient. As a result, this action can be defined on the level of a profinite group, namely the profinite completion of \( \pi_1^{\text{top}}(X, x) \):

\[
\hat{\pi}_1^{\text{top}}(X, x) := \varprojlim \pi_1^{\text{top}}(X, x)/N,
\]

where the inverse limit is over all finite index subgroups \( N \leq \pi_1^{\text{top}}(X, x) \).

**Corollary 0.1.5.** The fibre functor \( \text{Fib}_x \) defines an equivalence of categories

\[
\{\text{finite covers of } X\} \sim \{\text{finite, continuous } \hat{\pi}_1^{\text{top}}(X, x)-\text{sets}\}.
\]

Moreover, the correspondence restricts to

\[
\{\text{connected covers}\} \sim \{\text{finite } \hat{\pi}_1^{\text{top}}(X, x)-\text{sets with transitive action}\}
\]

and

\[
\{\text{Galois covers}\} \sim \{\pi_1^{\text{top}}(X, x)/N \mid N \text{ an open normal subgroup}\}.
\]

### 0.2 Finite Étale Algebras

Fix a field \( k \) and an algebraic closure \( \bar{k} \), which comes equipped with a separable closure \( k_s \subseteq \bar{k} \). Set \( G_k = \text{Gal}(k_s/s) \) and let \( L/k \) be any finite, separable extension.

**Lemma 0.2.1.** \( \text{Hom}_k(L, k_s) \) is a finite, continuous, transitive \( G_k \)-set.

**Proof.** By Galois theory, \( \# \text{Hom}_k(L, k_s) = \left[ L : k \right] \), so this is finite when the extension is assumed to be finite. The \( G_k \)-action is defined by \( \sigma \cdot f = \sigma \circ f \) for \( \sigma \in G_k \) and \( f \in \text{Hom}_k(L, k_s) \); it is routine to verify that this is indeed a group action. Now to show the action is continuous, since \( \text{Hom}_k(L, k_s) \) is discrete, this is equivalent to showing the stabilizer \( \text{Stab}_{G_k}(f) \) is open for each \( f \in \text{Hom}_k(L, k_s) \). Notice that

\[
\text{Stab}_{G_k}(f) = \{\sigma \in G_k \mid \sigma \circ f = f\} = \{\sigma \in G_k \mid \sigma \text{ fixes } f(L)\}
\]

and this is open by Galois theory / the topology of the profinite group \( G_k \). Finally, since \( L/k \) is separable, we may pick a minimal polynomial \( h(t) \) for a primitive element of \( L/k \). Then \( G_k \) permutes the roots of \( h(t) \) transitively, so it follows that \( G_k \) acts transitively on \( \text{Hom}_k(L, k_s) \).

**Corollary 0.2.2.** There exists an open subgroup \( H \leq G_k \) such that \( \text{Hom}_k(L, k_s) \cong G_k/H \) as \( G_k \)-sets. Further, if \( L/k \) is Galois, \( H \) may be chosen to be an open normal subgroup.

**Proof.** We may pick \( H = \text{Stab}_{G_k}(h) \), the stabilizer of the minimal polynomial of a primitive element of \( L/k \). The Galois case follows from the fundamental theorem of Galois theory.
Theorem 0.2.3. The assignment \( L/k \mapsto \text{Hom}_k(L,k_s) \) is a contravariant functor
\[
\{\text{finite separable extensions of } k\} \longrightarrow \{\text{finite, continuous, transitive } G_k\text{-sets}\}
\]
which is an anti-equivalence of categories. Moreover, Galois extensions \( L/k \) correspond to finite quotients of \( G_k \).

Proof. For any finite, continuous, transitive \( G_k\)-set \( S \), define a finite separable extension \( L_S/k \) by taking the subfield of \( k_s/k \) fixed by the stabilizer \( \text{Stab}_{G_k}(s) \) of any point \( s \in S \). One now checks that this is an inverse functor to the \( \text{Hom}_k(-,k_s) \) functor. \( \square \)

To study all finite continuous \( G_k\)-sets, we replace separable field extensions with a slightly more general object.

Definition. A \( k\)-algebra \( A \) is a \textbf{finite étale algebra} if \( A \cong L_1 \times \cdots \times L_r \) for finite separable extensions \( L_i/k \).

Corollary 0.2.4 (Grothendieck). The assignment \( A/k \mapsto \text{Hom}_k(A,k_s) \) is an anti-equivalence of categories
\[
\{\text{finite étale } k\text{-algebras}\} \cong \{\text{finite continuous } G_k\text{-sets}\}
\]
which reduces to the above case when \( A = L \) is a finite separable extension of \( k \).

Topologically, we may view \( k \) as a point space covering the ‘smaller’ point space \( k_s \), and any intermediate extension \( L/k \) as an intermediate cover. In this setting, \( G_k \) plays the role of the deck transformations of the ‘universal cover’ \( k \rightarrow k_s \) and \( \text{Hom}_k(L,k_s) \) plays the role of the fibre of a cover. Also, under this analogy, a finite separable extension \( L \) represents a connected cover while a finite étale algebra \( A \) may be viewed as a disconnected cover. Finally, the choice of a algebraic (and separable) closure of \( k \) is analagous to the choice of a basepoint of a topological space, which also determines a universal cover. We will see that this subtlety conceals a wealth of information about the algebraic fundamental group.

0.3 Locally Constant Sheaves

In this setting we unite the topological and field-theoretic approaches to the fundamental group introduced in Sections 0.1 and 0.2. Let \( X \) be a space.

Definition. A \textbf{presheaf} on \( X \) with values in a category of sets \( C \) (e.g. \textbf{Sets}, \textbf{Groups}, \textbf{AbGps}, \textbf{Rings}, etc.) is a functor
\[
\mathcal{F} : \text{Top}_X \longrightarrow C
\]
\[
U \longmapsto \mathcal{F}(U)
\]
\[
(V \rightarrow U) \mapsto \rho_{UV} \in \text{Hom}_C(\mathcal{F}(U),\mathcal{F}(V)),
\]
such that for each object \( U \), \( \rho_{UU} = \text{id}_U \), and for any inclusions \( W \hookrightarrow V \hookrightarrow U \), \( \rho_{UV} = \rho_{VW} \circ \rho_{UV} \).
For an element \( s \in \mathcal{F}(U) \) and an inclusion \( V \hookrightarrow U \), we will write \( s|_V = \rho_{UV}(s) \). Suggestively, these elements are called sections over \( U \) and the \( \rho_{UV} \) restriction maps.

**Definition.** Two presheaves \( \mathcal{F} \) and \( \mathcal{G} \) on \( X \) are said to be **isomorphic** if there exists a natural isomorphism \( \mathcal{F} \rightarrow \mathcal{G} \).

**Example 0.3.1.** The classic example of a presheaf on a space \( X \) is the assignment to each open \( U \subseteq X \) of the space of continuous (or smooth or holomorphic, when appropriate) functions on \( U \), with \( \rho_{UV} \) given by restriction of functions. This example in fact enjoys several important properties, e.g. a function is 0 on an open set if and only if it is 0 on all open covers of the set. These properties are generalized below.

**Definition.** A presheaf \( \mathcal{F} \) on \( X \) is a **sheaf** if:

1. For every covering \( \{U_i\} \) of an open set \( U \subseteq X \) and sections \( s, t \in \mathcal{F}(U) \) such that \( s|_{U_i} = t|_{U_i} \) for all \( i \), we must have \( s = t \).

2. For every covering \( \{U_i\} \) of \( U \) and sections \( \{s_i \in \mathcal{F}(U_i)\} \), if \( s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \) whenever \( U_i \cap U_j \neq \emptyset \), then there exists a section \( s \in \mathcal{F}(U) \) such that \( s|_{U_i} = s_i \), for each \( i \).

**Remark.** Let \( \mathcal{F} \) be a sheaf on \( X \). For any inclusion of an open set \( U \hookrightarrow X \), there is an obvious restriction of \( \mathcal{F} \) to a sheaf \( \mathcal{F}|_U \) defined on \( U \).

**Definition.** Let \( S \) be a set. Then the **constant sheaf** on \( X \) with values in \( S \) is the sheaf \( U \mapsto \mathcal{F}_S(U) \), where \( \mathcal{F}_S(U) = C(U,S) \) is the set of continuous functions \( U \rightarrow S \).

Notice that if \( S \) is a discrete set and \( U \) is a connected, open set, then \( \mathcal{F}_S(U) = S \), hence the name “constant sheaf”.

**Definition.** A sheaf \( \mathcal{F} \) on \( X \) is **locally constant** if for each \( x \in X \), there is a neighborhood \( U \) of \( x \) such that \( \mathcal{F}|_U \) is isomorphic to a constant sheaf.

We now rephrase the theory of covers from Section 0.1 in terms of locally constant sheaves. For a space \( X \), let \( \text{LCSh}_X \) be the category of locally constant sheaves on \( X \) (as a subcategory of \( \text{Sh}_X \)).

**Definition.** Let \( p : Y \rightarrow X \) be a cover and \( U \subseteq X \) an open set. Then a **section** of \( p \) over \( U \) is a continuous map \( s : U \rightarrow Y \) satisfying \( p \circ s = id_U \). The set of all sections over \( U \) is denoted \( \Gamma(U,p) \).

**Proposition 0.3.2.** For a cover \( p : Y \rightarrow X \), the assignment \( U \mapsto \mathcal{F}_p(U) := \Gamma(U,p) \) is a locally constant sheaf on \( X \).

**Proof.** The sheaf axioms are easily verified using properties of continuous functions. To prove the locally constant condition, fix \( x \in X \) and choose a connected, locally trivial neighborhood \( U \subseteq X \) of \( x \) such that \( p^{-1}(U) \cong U \times p^{-1}(x) \). For a section \( s : U \rightarrow Y \), \( p \) maps \( s(U) \) homeomorphically onto \( U \), so since \( U \) is connected, \( s(U) \) must be a connected component of \( p^{-1}(U) \). In this manner \( \Gamma(U,p) \cong p^{-1}(x) \) and therefore \( \mathcal{F}_p|_U \) is isomorphic to the constant sheaf \( \mathcal{F}_{p^{-1}(x)} \). Hence \( \mathcal{F}_p \) is locally constant. \( \square \)
We will show that the section functor establishes an equivalence between covers of $X$ and locally constant sheaves on $X$. To do so requires the construction of a covering space from the data of a locally constant sheaf $\mathcal{F}$. Such a cover is called the étale space of $\mathcal{F}$. To define this, we need:

**Definition.** Let $\mathcal{F}$ be a presheaf on $X$ and fix $x \in X$. The stalk of $\mathcal{F}$ at $x$ is the direct limit

$$\mathcal{F}_x := \lim_{\longrightarrow} \mathcal{F}(U)$$

over all open neighborhoods $U$ of $x$, directed by inclusion.

**Lemma 0.3.3.** The assignment $\mathcal{F} \mapsto \mathcal{F}_x$ is a functor for all $x \in X$.

**Theorem 0.3.4.** The functor $(Y \xrightarrow{p} X) \mapsto \mathcal{F}_p$ induces an equivalence of categories

$$\text{Cov}_X \xrightarrow{\sim} \text{LCSh}_X.$$

**Proof.** Given a presheaf $\mathcal{F}$ on $X$, we construct its étale space $X_{\mathcal{F}}$ by $X_{\mathcal{F}} = \bigsqcup_{x \in X} \mathcal{F}_x$, the disjoint union of the stalks over each $x$ in the base. The maps $\mathcal{F}_x \to \{x\}$ induce a map $p : X_{\mathcal{F}} \to X$. Further, we endow $X_{\mathcal{F}}$ with the topology induced by the maps $p$ and $\{i_s\}_{s \in \mathcal{F}(U)}$ for all open $U \subseteq X$, where $i_s : U \to X_{\mathcal{F}}$ is the map sending $x$ to the unique element of $s(U) \cap \mathcal{F}_x$.

Now suppose $\mathcal{F}$ is a locally constant sheaf; we claim $p : X_{\mathcal{F}} \to X$ is a topological cover. If $U$ is a locally constant neighborhood, then $\mathcal{F}_x = S$ for all $x \in U$ and some fixed discrete set $S$. Thus $p^{-1}(U) \cong U \times S$ so this shows $X_{\mathcal{F}}$ is in fact a cover of $X$. Now consider the assignments

$$\text{Cov}_X \xrightarrow{\sim} \text{LCSh}_X \quad \text{and} \quad \text{LCSh}_X \xrightarrow{\sim} \text{Cov}_X$$

$$(Y \xrightarrow{p} X) \mapsto \mathcal{F}_p \quad \text{and} \quad \mathcal{F} \mapsto (X_{\mathcal{F}} \xrightarrow{p} X).$$

We claim that these functors are naturally inverse isomorphisms. It is enough to prove that the natural transformations

$$A : \mathcal{F} \to \mathcal{F}_{X_{\mathcal{F}}} \quad \text{and} \quad B : Y \to X_{\mathcal{F}}$$

$$s \in \mathcal{F}(U) \longmapsto i_s \quad \text{and} \quad y \in p^{-1}(x) \longmapsto y \in (\mathcal{F}_p)_x$$

are isomorphisms. Choose a covering $\{U_i\}$ such that $\mathcal{F}|_{U_i}$ is isomorphic to a constant sheaf on some $F_i$. Then $(U_i)_{\mathcal{F}|_{U_i}} \cong U_i \times F_i$ and $\Gamma(U_i, U_i \times F_i \to U_i) = \mathcal{F}_{F_i}$ for each $i$. Thus $A$ and $B$ are isomorphisms. Checking naturality is straightforward, so we have the desired equivalence of categories.

**Corollary 0.3.5.** Let $X$ be a connected, locally simply connected space and fix $x \in X$. Then the functor $\mathcal{F} \mapsto \mathcal{F}_x$ induces an equivalence of categories

$$\text{LCSh}_X \xrightarrow{\sim} \{\text{left } \pi_1(X,x)\text{-sets}\}.$$
0.4 Étale Morphisms

Let us now shift focus to the algebraic setting, meaning a category of algebro-geometric objects such as nonsingular varieties over a field or schemes. The Zariski topology makes it tricky to define a cover in terms of a local homeomorphism – in fact, by Zariski’s theorem, such a map is only ever a trivial inclusion of an open set, so nothing terribly interesting. However, in the geometric setting, we have the notion of the differential on tangent spaces.

Definition. For a morphism \( p : Y \to X \) of nonsingular varieties \( X \) and \( Y \) over an algebraically closed field \( k \), we say \( p \) is \( \text{étale} \) at \( y \in Y \) if the differential \( dp_y : T_y Y \to T_{p(y)} X \) is an isomorphism. The map is said to be \( \text{étale} \) if it is \( \text{étale} \) at every \( y \in Y \).

If \( k \) is not algebraically closed, consider the base change \( \bar{p} : Y(\bar{k}) \to X(\bar{k}) \). Then we say \( p \) is \( \text{étale} \) at \( y \in Y \) if for all geometric points \( \bar{y} \to y \), \( \bar{p} \) is \( \text{étale} \) at \( \bar{y} \). As above, \( p \) is \( \text{étale} \) if it is \( \text{étale} \) at every \( y \in Y \).

A consequence of this definition is that the fibres of an \( \text{étale} \) morphism \( p : Y \to X \) are finite and their cardinality is constant (at least, on connected components). The definition of an \( \text{étale} \) morphism for schemes requires a little more care.

Definition. Suppose \( f : A \to B \) is a morphism of local rings. We say \( f \) is \( \text{flat} \) if the functor \( M \to M \otimes_A B \) is exact.

Definition. A morphism \( f : A \to B \) of local rings is \( \text{unramified} \) if \( B/f(m_A)B \) is a finite, separable field extension of \( A/m_A \).

We will typically encounter the case when \( f(m_A)B = m_B \), so that unramified is the same as \( B/m_B \) being a finite, separable extension of \( A/m_A \).

Definition. A morphism of schemes \( p : X \to Y \) is \( \text{étale} \) at \( y \in Y \) if the induced morphism on local rings \( p^\#: \mathcal{O}_{X,p(y)} \to \mathcal{O}_{Y,y} \) is a \( \text{étale} \) morphism of local rings, that is, \( p^\# \) is flat and unramified. We say \( p \) is \( \text{étale} \) if it is \( \text{étale} \) at every \( y \in Y \).

Let \( \text{Fét}_X \) be the category consisting of finite \( \text{étale} \) covers of \( X \), together with morphisms of covers defined as in the topological case (i.e. commuting with covering maps over \( X \)). For a geometric point \( \bar{x} : \text{Spec} \Omega \to X \) in a scheme \( X \), define the fibre functor

\[
\text{Fib}_{\bar{x}} : \text{Fét}_X \to \text{Sets}
\]

\[
(Y \xrightarrow{\bar{p}} X) \mapsto \text{Spec} \Omega \times_X Y.
\]

Definition. The \( \text{étale fundamental group} \) of a scheme \( X \) at a geometric point \( \bar{x} : \text{Spec} \Omega \to X \) is the automorphism group of the fibre functor over \( \bar{x} \),

\[
\pi_1^{\text{ét}}(X, \bar{x}) = \text{Aut(\text{Fib}_{\bar{x}})}.
\]

As in the topological case, we will prove:

Theorem 0.4.1 (Grothendieck). For \( X \) a connected scheme and \( \bar{x} : \text{Spec} \Omega \to X \) a geometric point,
(1) \( \pi_1^{\text{ét}}(X, \bar{x}) \) is a profinite group which acts continuously on each fibre \( \text{Fib}_X(Y) \) for \( Y \to X \) a cover.

(2) \( \text{Fib}_X : \text{Fét}_X \to \{ \text{continuous, finite } \pi_1^{\text{ét}}(X, \bar{x})\text{-sets} \} \) is an equivalence of categories.

Further, we will see that \( \pi_1^{\text{ét}}(X, \bar{x}) = \lim \leftarrow \text{Aut}_X(\text{Fib}_X(Y)) \), where the inverse limit may be taken over all Galois covers \( p : Y \to X \).

**Example 0.4.2.** Let \( k \) be a field and consider the scheme \( X = \text{Spec } k \) which is a point. A geometric point \( \bar{x} : \text{Spec } \Omega \to X \) is a choice of algebraic closure \( \Omega \supseteq k \); such a choice also defines a separable closure \( k_{\text{sep}} \supseteq k \). In this case, the correspondence of Theorem 0.4.1 is:

\[
\text{Fét}_X \leftrightarrow \{ \text{Spec } A | A \text{ is a finite étale } k\text{-algebra} \}.
\]

Here, the functor \( \text{Fib}_X \) is representable by \( \text{Hom}_k(A, \Omega) \cong \text{Hom}_k(A, k_{\text{sep}}) \). Moreover, we have a Galois action of \( G_k := \text{Gal}(k_{\text{sep}}/k) \) on \( \text{Hom}_k(A, k_{\text{sep}}) \) given by \( \sigma \cdot f = \sigma \circ f \) for any \( f : A \to k_{\text{sep}} \) and \( \sigma \in G_k \). In particular, \( \pi_1(X, \bar{x}) = G_k \) and this action is the monodromy action as described in Section 0.1.

**Theorem 0.4.3.** For a field \( k \), \( \text{Hom}_k(\quad, k_{\text{sep}}) \) induces an equivalence of categories

\[
\{ \text{finite étale } k\text{-algebras} \} \cong \{ \text{finite continuous } G_k\text{-sets} \}.
\]

**Example 0.4.4.** Let \( (A, m, k) \) be a complete DVR and set \( X = \text{Spec } A \). There is an equivalence between étale covers

\[
\text{Fét}_A \leftrightarrow \text{Fét}_k.
\]

(One direction is obvious; the other is an application of Hensel’s Lemma.) Therefore \( \pi_1^{\text{ét}}(\text{Spec } A, \bar{x}) \cong \pi_1^{\text{ét}}(\text{Spec } k, \bar{x}) \) for any geometric point \( \bar{x} \) of \( A \).

**Theorem 0.4.5.** Suppose \( X \) is connected and normal, \( K = k(X) \) is the function field of \( X \) and \( L/K \) is a finite, separable field extension. Let \( Y \) be the normalization of \( X \) in \( L \). Then \( p : Y \to X \) is a ramified cover and for some set \( S \subset Y \), \( p : Y \setminus S \to X \) is étale. Further, every étale cover of \( X \) arises in this way.

**Corollary 0.4.6.** If \( X \) is a connected, normal scheme with function field \( K \) and \( K_{\text{ur}} \) is the compositum of all extensions of \( K \) in which normalization of \( X \) is unramified, then

\[
\pi_1^{\text{ét}}(X) \cong \text{Gal}(K_{\text{ur}}/K).
\]

**Example 0.4.7.** A famous result in number theory says that \( \mathbb{Q} \) has no nontrivial unramified extensions. Therefore \( \pi_1^{\text{ét}}(\text{Spec } \mathbb{Z}) = 1 \). However, ‘removing some points’ from \( \text{Spec } \mathbb{Z} \), e.g. localizing, yields nontrivial fundamental groups, such as \( \pi_1^{\text{ét}}(\text{Spec } \mathbb{Z}[\frac{1}{n}]) \).

**Example 0.4.8.** We will show that \( \pi_1^{\text{ét}}(\mathbb{A}_C) = 1 \) (compare to the topological case!), but for \( \text{char } k > 0 \), we may have \( \pi_1^{\text{ét}}(\mathbb{A}_k) \neq 1 \).

There is a classic and important connection between curves and Riemann surfaces which we will explore further in Chapter 2. The arithmetic version of this correspondence is borne out by the following theorem.
Theorem 0.4.9. Let $X$ be a normal curve of genus $g$ over $k$. If $\text{char } k = p > 0$, then

$$
\pi_1^{\text{ét}}(X)^{(p')} \cong \left\langle x_1, y_1, \ldots, x_g, y_g : \prod_{i=1}^{g} [x_i, y_i] = 1 \right\rangle [p]
$$

where $(\cdot)^{(p')}$ denotes the prime-to-$p$ part of a profinite group. Further, if $k = \mathbb{C}$, then $\pi_1^{\text{ét}}(X) \cong \widehat{\pi_1^{\text{top}}(X)}$. 
1 Fundamental Groups of Algebraic Curves

1.1 Curves Over Algebraically Closed Fields

In the first two sections of Chapter 1, we will review the basics of algebraic curves. Fix an algebraically closed field $k$. Denote by $\mathbb{A}^n_k$ the affine $n$-space over $k$, which as a set is equal to $k^n$. Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over $k$. An ideal $I \subseteq R$ defines an algebraic set $X = V(I)$, whose coordinate ring is $\mathcal{O}(X) = R/I$.

Lemma 1.1.1. $\mathcal{O}(X)$ is a finitely generated, reduced $k$-algebra.

Definition. If $I$ is a prime ideal of $R = k[x_1, \ldots, x_n]$, then we call $X = V(I)$ an integral affine variety.

Definition. Let $Y = V(J) \subseteq \mathbb{A}^n_k$ be an integral affine variety. A morphism (or regular map) $\varphi : Y \to \mathbb{A}^m_k$ is a choice of functions $(f_1, \ldots, f_m) \in \mathcal{O}(Y)^m$. In general, for another integral affine variety $X \subseteq \mathbb{A}^m_k$, we say $\varphi : Y \to X$ is a morphism if $\varphi = (f_1, \ldots, f_m)$ such that $(f_1(P), \ldots, f_m(P)) \in X$ for all $P \in Y$.

Every morphism $\varphi : Y \to X$ induces a $k$-algebra homomorphism $\varphi^* : \mathcal{O}(X) \to \mathcal{O}(Y)$ given by $f \mapsto f \circ \varphi$. When we equip $X$ and $Y$ with the Zariski topology, a morphism $\varphi : Y \to X$ is continuous with respect to these topologies.

Proposition 1.1.2. The assignments $X \mapsto \mathcal{O}(X)$ and $\varphi \mapsto \varphi^*$ define an anti-equivalence of categories

$$\{\text{affine varieties over } k\} \longrightarrow \{\text{finitely generated, reduced } k\text{-algebras}\}$$

$$X \longmapsto \mathcal{O}(X)$$

$$V(I) \longleftarrow A = k[x_1, \ldots, x_n]/I.$$

Let $X$ be an integral affine variety. Then $\mathcal{O}(X)$ is a domain, so we can define the following:

- The fraction field of $X$, $k(X) := \text{Frac} \mathcal{O}(X)$.

- For any point $P \in X$, the local ring at $P$, $\mathcal{O}_{X,P} := \mathcal{O}(X)_{m_P}$, where $m_P = (x_1 - \alpha_1, \ldots, x_n - \alpha_n)$ is the maximal ideal of $\mathcal{O}(X)$ corresponding to $P = (\alpha_1, \ldots, \alpha_n)$.

- For any open subset $U \subseteq X$, the ring $\mathcal{O}_X(U) := \bigcap_{P \in U} \mathcal{O}_{X,P}$.

Lemma 1.1.3. For any integral affine variety $X$, $\mathcal{O}(X) = \bigcap_{P \in X} \mathcal{O}_{X,P}$.

Proof. By commutative algebra, any domain can be written as the intersection of its localizations at maximal ideals. Thus for $\mathcal{O}(X)$, we have

$$\mathcal{O}(X) = \bigcap_{\text{max. ideals } m \subseteq \mathcal{O}(X)} \mathcal{O}(X)_m = \bigcap_{P \in X} \mathcal{O}_{X,P}.$$
**Definition.** The dimension of an integral affine variety $X$ is $\dim X := \text{tr deg}_k k(X)$. We say $X$ is an integral affine curve if $\dim X = 1$.

By dimension theory, $\dim X = \dim \mathcal{O}(X)$, the Krull dimension of the coordinate ring of $X$.

**Definition.** An integral affine variety $X$ is normal at a point $P \in X$ if the local ring $\mathcal{O}_{X,P}$ is integrally closed in $k(X)$. We say $X$ is a normal variety if it is normal at every point $P \in X$.

**Proposition 1.1.4.** $X$ is normal if and only if $\mathcal{O}(X)$ is an integrally closed domain.

**Proof.** ($\iff$) follows from the fact that if a domain is integrally closed, then so is every localization.

($\Rightarrow$) By Lemma 1.1.3, $\mathcal{O}(X) = \bigcap_{P \in X} \mathcal{O}_{X,P}$ so if each local ring is integrally closed, then so is their intersection. \qed

**Remark.** When $\dim X = 1$, $X$ is normal at $P$ if and only if $\mathcal{O}_{X,P}$ is a DVR.

**Definition.** An integral affine variety $X$ is nonsingular (or smooth) at $P \in X$ if $\mathcal{O}_{X,P}$ is a regular local ring. Equivalently (though this requires proof), $X$ is nonsingular at $P$ if the dimension of the tangent space to $X$ at $P$ is equal to $\dim X$.

**Proposition 1.1.5.** If $X$ is an integral affine curve, then $X$ is nonsingular at $P \in X$ if and only if $X$ is normal at $P$.

**Example 1.1.6.** Let $X \subset \mathbb{A}^2_k$ be an integral affine plane curve. Then $X$ is smooth at $P \in X$ as long as $d_x f(P) \neq 0$ or $d_y f(P) \neq 0$.

**Proposition 1.1.7.** Let $X$ be an integral affine curve and $P \in X$ a normal point. Then there exists an affine open neighborhood $U$ of $P$ in $X$ which is isomorphic to an open neighborhood of some smooth point in a plane curve.

### 1.2 Curves Over Arbitrary Fields

When $k$ is not algebraically closed, there is no guarantee of a correspondence between ideals of $k[x_1, \ldots, x_n]$ and points in affine space. For example, there are no $\mathbb{R}$-points $(x, y) \in \mathbb{A}^2_{\mathbb{R}}$ satisfying $x^2 + y^2 + 1 = 0$, and therefore there are ideals of $\mathbb{R}[x, y]$ which do not correspond to points in affine space. To deal with this complication, we generalize our notion of curves considerably.

**Definition.** A ringed space is a pair $(X, \mathcal{F})$ consisting of a space $X$ and a sheaf of rings $\mathcal{F}$ on $X$.

Let $A$ be a finitely generated $k$-algebra which is an integral domain with field of fractions $K$, and suppose $\text{tr deg}_k K = 1$. Consider the space $X = \text{Spec } A = \{ \mathfrak{p} \subset A \mid \mathfrak{p} \text{ is a prime ideal} \}$. Taking closed sets to be of the form $V(I) = \{ \mathfrak{p} \in \text{Spec } A \mid I \supseteq \mathfrak{p} \}$ for ideals $I \subset A$ defines a Zariski topology on $X$. The zero ideal $0$ defines a generic point.
of $X$: for any nonempty open subset $U \subseteq X$, $0 \in U$. It follows that $A_0 = K$. On the other hand, any nonzero prime $p$ is maximal (since $\dim A = \text{tr. deg}_k K = 1$), so $V(p) = \{p\}$. Thus prime ideals correspond to closed points in $X$. In particular, all proper closed subsets of $X$ are finite.

**Definition.** For $P \in X$, let $\mathcal{O}_{X,P} := A_P$ be the local ring at $P$. For any open $U \subseteq X$, set

$$\mathcal{O}_X(U) := \bigcap_{P \in U} \mathcal{O}_{X,P}.$$ 

**Proposition 1.2.1.** $U \mapsto \mathcal{O}_X(U)$ defines a sheaf of functions on $X$.

**Definition.** We call the ringed space $(X, \mathcal{O}_X)$ an integral affine curve.

**Definition.** A morphism of sheaves is a natural transformation $\mathcal{F} \to \mathcal{G}$ of sheaves on a common space $X$, that is, a ring homomorphism $\mathcal{F}(U) \to \mathcal{G}(U)$ for each open $U \subseteq X$ such that for every inclusion of open sets $V \subseteq U$, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\
\downarrow & & \downarrow \\
\mathcal{F}(V) & \longrightarrow & \mathcal{G}(V)
\end{array}$$

**Definition.** For a sheaf $\mathcal{G}$ on $Y$ and a continuous map $f : Y \to X$, the pushforward sheaf is a sheaf on $X$ defined by $U \mapsto f_* \mathcal{G}(U) := \mathcal{G}(f^{-1}(U))$ for all open $U \subseteq X$.

**Lemma 1.2.2.** $f^* \mathcal{G}$ is a sheaf on $X$.

**Definition.** A morphism of ringed spaces $(Y, \mathcal{G}) \to (X, \mathcal{F})$ is a pair $(\varphi, \varphi^\#)$ consisting of a continuous map $\varphi : Y \to X$ and a morphism of sheaves $\varphi^\# : \mathcal{F} \to \varphi_* \mathcal{G}$, where $\varphi_* \mathcal{G}$ is the pushforward.

**Remark.** Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be integral affine curves. We will say $\varphi : X \to Y$ is a morphism of curves if it is a morphism of ringed spaces such that $\varphi^\# : \mathcal{O}_X \to \varphi_* \mathcal{O}_Y$ is a morphism of sheaves of local rings, i.e. one that sends maximal ideals in the local rings $\mathcal{O}_{Y,Q}$ to maximal ideals in the $\mathcal{O}_{X,\varphi(Q)}$.

**Proposition 1.2.3.** There is an anti-isomorphism of categories

$$\begin{array}{ccc}
\{\text{integral affine curves over } k\} & \to & \{\text{finitely generated integral domains}\} \\
(X, \mathcal{O}_X) & \mapsto & \mathcal{O}(X) \\
(\varphi : Y \to X) & \sim & (\varphi^\# : \mathcal{O}(X) \to \mathcal{O}(Y)) \\
\text{Spec } A & \leftrightarrow & A \\
(f^* : \text{Spec } B \to \text{Spec } A) & \leftrightarrow & (f : A \to B).
\end{array}$$
Let $X = \text{Spec}\, A$ be an integral affine curve. Suppose $L/k$ is a field extension such that $A \otimes_k L$ is still a domain. Then $X_L := \text{Spec}(A \otimes_k L)$ is an integral affine curve over $L$, called the base change of $X$ to $L$. The natural map $A \to A \otimes_k L, a \mapsto a \otimes 1$ induces a morphism of ringed spaces $X_L \to X$.

**Definition.** If $A \otimes_k \bar{k}$ is an integral domain, we say $X = \text{Spec}\, A$ is geometrically integral. In this case, $X_L$ is integral for all intermediate extensions $\bar{k} \supset L \supset k$. For any of these $L$, there is a functor $F_L : \{\text{geometrically integral affine curves over } k\} \to \{\text{integral affine curves over } k\}$.

**Example 1.2.4.** Let $X = \text{Spec}(\mathbb{R}[x, y]/(x^2 + y^2 + 1))$. Then closed points of $X$ are in correspondence with maximal ideals of the ring $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$. These in turn correspond with Galois orbits of $X_\mathbb{C}$. Explicitly, each closed point $(x, y - i)$ and $(x, y + i)$ via $X_\mathbb{C} \to X$.

**Definition.** A curve $X$ is normal at $P \in X$ if $O(X)_P$ is integrally closed. As before, $X$ is normal if it is normal at every $P \in X$.

**Definition.** A morphism $\varphi : Y \to X$ of integral affine curves is finite if $O(Y)$ is a finitely generated $O(X)$-module via $\varphi^* : O(X) \to O(Y)$.

**Lemma 1.2.5.** Let $\varphi : Y \to X$ be a finite morphism. Then

1. $O(Y)$ is integral over $O(X)$.
2. $\varphi$ is surjective.
3. $\varphi^*$ is injective.
4. $k(Y)/k(X)$ is a finite extension.
5. $\varphi$ has finite fibres.

**Theorem 1.2.6.** Let $X$ be an integral normal affine curve. Then the assignments $Y \mapsto k(Y)$ and $\varphi \mapsto \varphi^*$ induce an anti-equivalence of categories

$$\{\text{normal affine curves over } k \text{ with finite maps } Y \to X\} \to \{\text{embeddings of fields } k(X) \hookrightarrow L\}.$$ 

**Proof.** Given an embedding $i : k(X) \hookrightarrow L$, let $B$ be the integral closure of $O(X)$ in $L$. Then by commutative algebra, $B$ is a finitely generated $k$-algebra which is integrally closed, and since $L/k(X)$ is a finite extension, $\text{tr.deg}_k L = 1$. Hence by Proposition 1.2.3, $Y = \text{Spec}\, B$ is a normal affine curve with a morphism $\varphi = i^* : Y \to X$. One now checks that these assignments are natural inverses. ∎

**Definition.** For an integral affine curve $X$ and a field extension $L/k(X)$, the curve $Y$ constructed above is called the normalization of $X$ in $L$.

**Remark.** If $X$ is geometrically integral, then $X_L$ is equal to the normalization of $X$ in $L$ for any $L \supset k(X)$. 

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1.3 Proper Normal Curves

To fill out the analogy with the topological case (Section 0.1), we want to consider some version of compact curves. Let $X$ be an affine curve with function field $K = k(X)$.

**Lemma 1.3.1.** The set $\{O_{X, P} \mid P \neq 0\}$ is equal to the set of DVRs of $K$ containing $O(X)$.

**Proof.** On one hand, $(\subseteq)$ is obvious. For the other containment, let $R$ be such a DVR and call its maximal ideal $m$. Then $p := m \cap O(X)$ is a nonzero maximal ideal in $O(X)$. This means $O(X)_p \subseteq R$ but each of these is a DVR with the same function field, so we must have $O_{X, p} = R$. \qed

**Example 1.3.2.** Let $X = \mathbb{A}^1_k$ so that $O(\mathbb{A}^1_k) = k[t]$, the polynomial ring in a single indeterminate. If $Y$ is also $\mathbb{A}^1_k$, we may take $O(Y) = k[t^{-1}]$. Let $R$ be a DVR with $\text{Frac}(R) = k(X) \cong k(t) \cong k(t^{-1})$. Then either $R \supset O(X)$ or $R \supset O(Y)$. In fact, the only DVR not containing $O(X)$ is $R = k[t^{-1}](t^{-1})$, while for $O(Y)$ the exception is $R' = k[t](t)$. We want some way of “gluing” these two DVRs together so that all DVRs of the function field over $k$ are considered.

Now assume $X$ is a normal affine curve. By Noether normalization, there exists a function $f \in O(X)$ such that $O(X)$ is a finitely generated $k[f]$-module. Define $X^-$ to be the normal curve associated to the integral closure of $k \left[ \frac{1}{f} \right]$ in $K$. Then every DVR $R$ with fraction field $K$ is a local ring for $X^+ = X$ or for $X^-$, corresponding to whether $f \in R$ or $\frac{1}{f} \in R$.

More abstractly, for a finitely generated field extension $K/k$ of transcendence degree 1, let $X_K$ be the set of DVRs of $K/k$. Define a topology on $X_K$ by declaring the complements of finite subsets to be open. Define a sheaf of rings on $X_K$ by $U \mapsto O_{X_K}(U) := \bigcap_{P \in U} R$.

**Theorem 1.3.3.** $(X_K, O_{X_K})$ is a ringed space which is “locally affine” in the sense that there is a decomposition $X_K = X_K^+ \cup X_K^-$ where $X_K^+$ and $X_K^-$ are affine curves.

**Definition.** The ringed space $(X_K, O_{X_K})$ is called an integral proper normal curve.

**Definition.** A morphism of (integral) proper normal curves $\varphi : Y_L \to X_K$ is a morphism of the underlying ringed spaces in which the morphism of sheaves $\varphi^\# : O_{X_K} \to O_{Y_L}$ is a morphism of local rings, i.e. sends maximal ideals to maximal ideals.

**Proposition 1.3.4.** The functor that sends a morphism of proper normal curves $Y_L \to X_K$ to the induced injection of fields $\varphi^* : K \hookrightarrow L$ is an anti-equivalence of categories

\[
\left\{ \text{proper normal curves with finite surjections } Y_L \to X_K \right\} \xrightarrow{\sim} \left\{ \text{finite field extensions } K \hookrightarrow L \text{ over } k \right\}.
\]

We will now drop the subscript on $X_K$ denoting the field unless it becomes convenient to have it. Take a morphism of proper normal curves $\varphi : Y \to X$ with affine covers $X = X^+ \cup X^-$ and $Y = Y^+ \cup Y^-$. Then there are induced morphisms of affine curves $\varphi^+ : Y^+ \to X^+$ and $\varphi^- : Y^- \to X^-$.\)

**Definition.** We say an open set $U \subseteq X$ in a proper normal curve is affine if $O_X(U)$ is a finitely generated $k$-algebra.
Lemma 1.3.5. Let $X$ be a proper normal curve. Then a subset $U \subseteq X$ is open if and only if $U \cong \text{Spec} \ A$ for some finitely generated $k$-algebra $A$.

Proposition 1.3.6. There is an equivalence of categories

$$\{\text{normal affine curves over } k\} \xrightarrow{\sim} \left\{\text{affine open subsets of proper normal curves over } k\right\}.$$ 

Proof. (Sketch) Given a normal affine curve $X$, set $X^+ = X$ and embed $X$ into $X' = X^+ \cup X^-$ as shown above. This yields a proper normal curve, and to show the bijection is an equivalence of categories, one need only check that a morphism of affine curves $\varphi : Y \to X$ extends uniquely to $\tilde{\varphi} : Y' \to X'$.

Definition. We say a morphism of proper normal curves $\varphi : Y \to X$ is finite if for any affine $U \subseteq X$,

1. $\varphi^{-1}(U)$ is affine in $Y$.
2. $\varphi_* \mathcal{O}(U)$ is a finitely generated $\mathcal{O}(U)$-module.

Notice that for any affine set $U \subseteq X$ and finite morphism $\varphi : Y \to X$, the restriction $\varphi|_{\varphi^{-1}(U)} : \varphi^{-1}(U) \to U$ is a finite morphism of affine curves in the sense of Section 1.2. Thus Lemma 1.2.5 yields:

Corollary 1.3.7. Any finite morphism is surjective.

Conversely, we have:

Proposition 1.3.8. Let $\varphi : Y_L \to X_K$ be a surjective morphism of proper normal curves. Then $\varphi$ is finite.

Proof. Let $U \subseteq X$ be affine. Then $\varphi^{-1}(U) = \{R \mid R$ is a DVR of $L$ containing $\mathcal{O}(U)\}$. For any $R \in \varphi^{-1}(U)$, the integral closure $B$ of $\mathcal{O}(U)$ is contained in $R$ since $R$ is itself integrally closed. But by Theorem 1.2.6, this $B$ is precisely the integral closure of the normalization $V$ of $U$ in $L$. Hence $V = \varphi^{-1}(U)$ so (1) is satisfied. It also follows from the identification of $V$ as the normalization of $U$ in $L$ that $\varphi_* \mathcal{O}(U) = \mathcal{O}(\varphi^{-1}(U)) = \mathcal{O}(V)$ is finitely generated over $\mathcal{O}(U)$.

Corollary 1.3.9. For any proper normal curve $X$, the assignment $Y \mapsto k(Y)$ induces an anti-equivalence of categories

$$\left\{\text{proper normal curves with finite morphisms } Y \to X\right\} \xrightarrow{\sim} \left\{\text{finitely generated field extensions } k(X) \hookrightarrow L \text{ with } \text{tr.deg}_k L = 1\right\}.$$  

1.4 Finite Branched Covers

In this section we describe the analogue of branched covers in the algebraic setting. We begin with the case of affine curves and later generalize to proper normal curves. Fix a morphism $\varphi : Y \to X$ of integral affine curves.
Definition. We say $\varphi : Y \to X$ is separable if $\varphi^* : k(X) \hookrightarrow k(Y)$ is a separable field extension. If $\varphi$ is finite and separable, then we say $\varphi$ is étale at a closed point $P \in X$ if $O(Y)/PO(Y)$ is a finite étale algebra over the residue field $\kappa(P) = O(X)/P$. For an open set $U \subseteq X$, $\varphi$ is étale over $U$ if it is étale at every point $P \in U$.

Further suppose $X$ and $Y$ are normal affine curves. Then $O(X)$ and $O(Y)$ are Dedekind domains, so $PO(Y)$ can be written

$$PO(Y) = \prod_{i=1}^{r} Q_{i}^{e_{i}}$$

for distinct closed points $Q_{i} \in Y$ and integers $e_{i}$. By the Chinese remainder theorem,

$$O(Y)/PO(Y) \cong \prod_{i=1}^{r} (O(Y)/Q_{i}^{e_{i}}).$$

Definition. The integer $e_{i}$ is called the ramification index of $\varphi$ at $Q_{i}$ (over $P$). If $e_{i} > 1$ for any $Q_{i}$, we say $\varphi$ is ramified at $P$ (or $P$ is a branch point of $\varphi$).

Proposition 1.4.1. For a morphism of normal affine curves $\varphi : Y \to X$ and a point $P \in X$ such that $PO(Y) = \prod_{i=1}^{r} Q_{i}^{e_{i}}$, the following are equivalent:

1. $\varphi$ is étale at $P$.

2. $e_{i} = 1$ for each $1 \leq i \leq r$ and each residue field $\kappa(Q_{i}) = O(Y)/Q_{i}$ is separable over $\kappa(P)$.

3. For each $Q_{i}$ over $P$, $\kappa(Q_{i})/\kappa(P)$ is separable and $m_{P}O_{Y,Q_{i}} = m_{Q_{i}}$, where $m_{P} \subset O_{X,P}$ and $m_{Q_{i}} \subset O_{Y,Q_{i}}$ are the maximal ideals in the given local rings.

Note that $O(Y)/PO(Y) \cong (\varphi_{*}O_{Y,P}) \otimes_{O_{X,P}} \kappa(P)$. We can view the set $\{Q_{i}\}$ as the geometric fibre $\varphi^{-1}(P)$, and under this identification $O(Y)/PO(Y)$ acts as the space of regular functions on $\varphi^{-1}(P)$.

Example 1.4.2. Let $k = \mathbb{C}$ and $X = \mathbb{A}_{\mathbb{C}}^{1}$. Then the space of regular functions at the points 0 and 2 is given by

$$\mathbb{C}[t]/(t(t - 2)) \cong \mathbb{C} \times \mathbb{C}.$$  

Example 1.4.3. Let $k = \mathbb{C}$ and define the map

$$\varphi_{n} : \mathbb{A}_{\mathbb{C}}^{1} \to \mathbb{A}_{\mathbb{C}}^{1}, \quad x \mapsto x^{n}.$$  

This induces the ring extension $\mathbb{C}[t^{n}] \hookrightarrow \mathbb{C}[t]$. Here, the maximal ideals are $(t^{n} - a)$ for $a \in \mathbb{C}$, and we have the following ramification behavior:

$$\mathbb{C}[t]/(t^{n} - a) \cong \begin{cases} \prod_{i=0}^{n-1} (\mathbb{C}[t]/(t - \zeta_{i}^{n} \sqrt[n]{a})) \cong \mathbb{C}^{n}, & a \neq 0 \\ \mathbb{C}[t]/(t^{n}), & a = 0. \end{cases}$$

Thus $\varphi_{n}$ is étale at each $a \neq 0$ but not at $a = 0$.  

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Lemma 1.4.4. If $Z \xrightarrow{\varphi} Y \xrightarrow{\psi} X$ are finite separable morphisms of affine curves and $P \in X$ is a closed point, then

1. If $\varphi$ is étale at $P$ and $\psi$ is étale over $\varphi^{-1}(P)$, then $\psi \circ \varphi$ is étale at $P$.

2. If $X, Y$ and $Z$ are normal affine curves, then the converse holds.

Proof. Let $X = \text{Spec } A, Y = \text{Spec } B$ and $Z = \text{Spec } C$ for $k$-algebras $A, B$ and $C$. Then we have ring extensions $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ as $\kappa(P)$-algebras. By the above comments,

$$
\frac{C}{PC} \cong C \otimes_A \kappa(P) \\
\cong C \otimes_B \left( B \otimes_A \kappa(P) \right) \\
\cong C \otimes_B \left( \prod_{Q \rightarrow P} \kappa(Q) \right) \\
\cong \prod_{R \rightarrow P} \kappa(R).
$$

Thus the composition is étale at $P$ if each morphism is. Now suppose the curves are all normal. Then the ramification indices satisfy $\sum_{i=1}^{r} e_i f_i = [\text{Frac}(B) : \text{Frac}(A)]$, where $f_i = [\kappa(Q_i) : \kappa(P)]$ and likewise for the ring extensions $C/B$ and $C/A$. It follows easily that if one of $\varphi, \psi$ is ramified, then so is the composition. \hfill \Box

Proposition 1.4.5. Let $\varphi : Y \rightarrow X$ be a finite separable morphism of affine curves. Then there is a nonempty open set $U \subseteq X$ such that $\varphi$ is étale over $U$. In particular, $\varphi$ has finitely many branch points.

Proof. Let $X = \text{Spec } A$ and $Y = \text{Spec } B$. By hypothesis, $B$ is a finitely generated $A$-module, so write $B = A[f_1, \ldots, f_r]$ for integral elements $f_1, \ldots, f_r$. Then there is a tower of ring extensions

$$
A \subseteq A[f_1] \subseteq A[f_1, f_2] \subseteq \cdots \subseteq A[f_1, \ldots, f_{r-1}] \subseteq B.
$$

This determines a sequence of morphisms

$$
Y \rightarrow X_{r-1} \rightarrow \cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X
$$

whose composition is $\varphi$, where $X_i = \text{Spec } A[f_1, \ldots, f_i]$ for $1 \leq i \leq r - 1$. By induction and using Lemma 1.4.4, we may reduce to the case $Y = X_1 \rightarrow X$, i.e. $B = A[f]$ for some integral $f = f_1 \in B$. Write $B = A[t]/(F(t))$ where $F \in A[t]$ is the minimal polynomial over $A$ of $f$. Since $\varphi$ is separable, $k(Y)/k(X)$ is a separable extension of fields and thus $(F', F) = 1$ in $k(X)[t]$. Thus there exist $G_1, G_2 \in k(X)[t]$ such that $G_1 F' + G_2 F = 1$. Cancelling denominators, there exists some $g \in A$ such that $H_1 = G_1 g$ and $H_2 = G_2 g$ in $A[t]$, and so we have

$$
H_1 F' + H_2 F = g \in A[t].
$$

Let $U = D(g) = \{ P \in X \mid g(P) \neq 0 \}$ which is a nonempty open subset of $X$. We claim $\varphi$ is étale over $U$. For any $P \in U$, take the reduction of the above equation in $\kappa(P)[t]$ to see that

$$
\overline{H_1} F' + \overline{H_2} F = \overline{g} \neq 0 \quad \text{in } \kappa(P)[t].
$$
Thus for all $\alpha \in A$, $\overline{F}(\alpha) = 0$ implies $\overline{F}'(\alpha) \neq 0$, or in other words, $\overline{F}$ has no multiple roots. Therefore we can write

$$B/PB \cong \kappa(P)[t]/(\overline{F}) \cong \prod_{i=1}^{s} \kappa(P)[t]/(\overline{F}_i)$$

where $\overline{F}_1, \ldots, \overline{F}_s$ are the irreducible factors of $\overline{F}$ in $\kappa(P)[t]$. The work above shows that each $\kappa(P)[t]/(\overline{F}_i)$ is separable, so $B/PB$ is a finite étale $\kappa(P)$-algebra. The result follows.  

In light of Proposition 1.4.5, the following definitions make sense:

**Definition.** We call a morphism $\varphi : Y \to X$ of integral affine curves a finite branched cover if it is finite and separable. Further, we call $\varphi$ a (finite) Galois branched cover if the field extension $k(X) \hookrightarrow k(Y)$ is Galois.

If, in addition to $\varphi$ being a finite Galois cover, $X$ and $Y$ are normal curves, then $\text{Gal}(k(Y)/k(X))$ acts transitively on the fibres $\varphi^{-1}(P)$ of $\varphi$, just as in the case of a Galois topological cover.

**Proposition 1.4.6.** If $\varphi : Y \to X$ is a Galois branched cover over a perfect field $k$ and $X$ and $Y$ are normal affine curves, then $\varphi$ is étale at $P \in X$ if and only if $I_{Q_i} = \{1\}$ for all $Q_i \in \varphi^{-1}(P)$, where $I_{Q_i}$ is the inertia subgroup of $\text{Gal}(k(Y)/k(X))$.

**Proof.** Recall the definitions of the decomposition and inertia groups for a point $Q_i \in \varphi^{-1}(P)$:

$$D_{Q_i} = \{ \sigma \in \text{Gal}(k(Y)/k(X)) \mid \sigma(Q_i) = Q_i \}$$

$$I_{Q_i} = \{ \sigma \in \text{Gal}(k(Y)/k(X)) \mid \sigma \text{ acts trivially on } \kappa(Q_i) \}.$$ 

Since $k$ is perfect, we know from algebraic number theory that $e_i = [\kappa(Q_i) : \kappa(Q_i)^{I_{Q_i}}]$, where $\kappa(Q_i)^{I_{Q_i}}$ is the subfield of $\kappa(Q_i)$ fixed by the inertia group, but this index is precisely the order of $I_{Q_i}$. Hence $e_i = 1$ if and only $I_{Q_i} = \{1\}$, so the result follows by Proposition 1.4.1.  

For a normal affine curve $X$ and a tower of fields (finite over $k(X)$)

$$k(X) \subset K_1 \subset K_2 \subset \cdots$$

set $L = \bigcup_{j=1}^{\infty} K_j$ and let $A_j$ be the integral closure of $\mathcal{O}(X)$ in $K_j$ for each $j \geq 1$. From this we get a sequence of morphisms of curves

$$\cdots \to X_2 \to X_1 \to X.$$ 

Fix $P \in X$. Choose maximal ideals $\{P_j\}_{j \geq 1}$ with $P_j \in A_j$ such that $P_j \cap \mathcal{O}(X) = P$ and $P_{j+1} \cap A_j = P_j$ for all $j \geq 1$. If $I_j$ is the inertia subgroup of $P_j$ in $G_j := \text{Gal}(K_i/k(X))$, then the $I_j$ form an inverse system with the natural maps coming from each quotient of Galois groups. Define the **inertia subgroup** of $P$ to be the inverse limit

$$I_P := \varprojlim I_j.$$ 

Since each $I_j$ is a normal subgroup of $G_j$, it follows that $I_P$ is a closed subgroup of the profinite group $G := \text{Gal}(L/k(X))$. Note that $I_P$ depends on the choices of $P_1, P_2, \ldots$ but a different choice of points lying over $P$ yields a conjugate subgroup.
Corollary 1.4.7. Let \( \varphi : Y \to X \) be a Galois branched cover of normal affine curves over a perfect field and assume \( k(Y) \subseteq L \) where \( L \) is as above. Then \( \varphi \) is étale at \( P \in X \) if and only if the image of \( I_P \) in \( \text{Gal}(k(Y)/k(X)) = G/\text{Gal}(L/k(Y)) \) is trivial.

Proof. Pick \( \ell \geq 1 \) so that \( k(Y) \subseteq K_\ell \). Let \( Q = P_\ell \cap \mathcal{O}(Y) \). Then the image of \( I_P \) in \( \text{Gal}(k(Y)/k(X)) \) is precisely the inertia subgroup \( I_Q \leq \text{Gal}(k(Y)/k(X)) \). Now apply Proposition 1.4.6.

We now extend the notion of finite branched covers to proper normal curves. Suppose \( \varphi : Y \to X \) is a finite separable morphism of proper normal curves (as defined in Section 1.3).

Definition. We say \( \varphi : Y \to X \) is étale at \( P \in X \) if there exists an affine open neighborhood \( U \subseteq X \) of \( P \) such that \( \varphi|_{\varphi^{-1}(U)} : \varphi^{-1}(U) \to U \) is étale at \( P \) as a morphism of affine curves. As in the affine case, call \( \varphi \) étale over a subset \( S \subseteq X \) if it is étale at every point \( P \in S \), and simply étale if it is étale at every \( P \in X \).

Note that by Lemma 1.4.1, this definition does not depend on the neighborhood \( U \) of \( P \) chosen. The last few results for affine curves generalize to proper normal curves as an immediate consequence of the definition:

Theorem 1.4.8. Let \( \varphi : Y \to X \) be a morphism of proper normal curves. Then

1. There exists a nonempty open set \( U \subseteq X \) such that \( \varphi \) is étale over \( U \).
2. If \( \varphi \) is a Galois branched cover and \( k \) is perfect, then \( \varphi \) is étale at \( P \in X \) if and only if the inertia group is trivial for each point \( Q_i \) over \( P \).
3. If \( \varphi \) is a Galois branched cover, \( k \) is perfect, \( L = \bigcup_{j=1}^{\infty} K_j \) and \( k(Y) \subseteq L \), then \( \varphi \) is étale at \( P \in X \) if and only if \( I_P \) maps trivially into \( \text{Gal}(k(Y)/k(X)) \).

Example 1.4.9. For a proper normal curve \( X \) over \( \mathbb{C} \), \( X(\mathbb{C}) \) has the structure of a compact Riemann surface such that the cover \( X(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) \) corresponding to \( X \to \mathbb{P}^1 \) is proper and holomorphic.

Proposition 1.4.10. For a proper normal curve \( X \) over \( \mathbb{C} \), the following categories are anti-equivalent:

\[
\begin{align*}
\left\{ \text{ proper normal curves } Y \text{ with finite morphism } Y \to X \right\} & \leftrightarrow \left\{ \text{ finite extensions } k(X) \hookrightarrow L \right\} & \leftrightarrow \left\{ \text{ compact connected Riemann surfaces } Y \text{ with proper holomorphic maps } Y \to X(\mathbb{C}) \right\}.
\end{align*}
\]

Example 1.4.11. Consider the squaring map \( \rho_2 : \mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^1_{\mathbb{R}}, \ t \mapsto t^2 \) corresponding to the ring extension \( \mathbb{R}[t^2] \hookrightarrow \mathbb{R}[t] \). The maximal ideals of \( \mathbb{R}[t] \) are of the form

1. \( (t - a) \) for \( a \in \mathbb{R} \)
2. \( (t^2 + bt + c) \) for \( b^2 - 4c < 0 \).
For (1), we have the following ramification behavior:

\[
\mathbb{R}[t]/(t^2 - a) \cong \begin{cases} 
\mathbb{R} \times \mathbb{R}, & a > 0 \\
\mathbb{R}[t]/(t^2), & a = 0 \\
\mathbb{C}, & a < 0
\end{cases}
\]

while for (2), we get

\[
\mathbb{R}[t]/(t^4 + bt^2 + c) \cong \mathbb{C} \times \mathbb{C}.
\]

Note that \(\kappa(t - a) = \mathbb{R}\), while \(\kappa(t^2 + bt + c) = \mathbb{C}\) so we see that \(\rho_2\) is étale off of \(\{0, \infty\}\). (To see that \(\rho_2\) is not étale at \(\infty\), repeat the above argument replacing \(t\) with \(1/t\).)

**Example 1.4.12.** Let \(E\) be an elliptic curve over an algebraically closed field \(k\) of characteristic \(\text{char } k = p\). Take \(m \in \mathbb{Z}\) such that \(p \nmid m\) and consider the multiplication-by-\(m\) isogeny:

\([m] : E \to E, \quad P \mapsto mP.\)

We know that \([m]\) is separable when \(p \nmid m\), and moreover that \(E[m] = (\mathbb{Z}/m\mathbb{Z})^2\) and thus \(\#E[m] = m^2\). For each \(P \in E\), we have

\[\mathcal{O}(E)/[m]P \cong \prod_{i=1}^{m^2} \kappa(P) \cong \prod_{i=1}^{m^2} k\]

and hence \([m]\) is étale. In general, any degree-\(m\) isogeny \(\varphi : E_1 \to E_2\) of elliptic curves over an algebraically closed field of characteristic not dividing \(m\) is separable and therefore étale by the same argument.

**Example 1.4.13.** Let \(k\) be any field of characteristic \(p > 0\) and consider the field

\[L = k(x)[y]/(y^p - (f(x)g(x))^{p-1}y - f(x)).\]

Note that \(y^p - (f(x)g(x))^{p-1}y - f(x)\) is irreducible, e.g. by Eisenstein’s criterion, so \(L\) is indeed a field. Let \(B\) be the integral closure of \(k[x]\) in \(L\) and let \(X\) be the proper normal curve corresponding (via Proposition 1.3.4) to \(L\). Then there is a morphism of curves \(\varphi : X \to \mathbb{P}_k^1\) coming from \(k(x) \hookrightarrow L\). For \(a \in k\), consider the maximal ideal \((x - a) \in B\). We consider three cases:

1. If \(f(a) = 0\), then \((x - a) = (y^p)\) in \(B\), so we have
   \[B/(x - a)B \cong B/(y^p)B.\]
   Thus \(\varphi\) is not étale at \((x - a)\).

2. If \(f(a) \neq 0\) but \(g(a) = 0\), \((x - a) = (y^p - f(a))\) so
   \[B/(x - a)B \cong \begin{cases} 
B/(y - f(a)^{1/p})^p, & f(a)^{1/p} \in k \\
B/(y^p - f(a)), & \text{otherwise.}
\end{cases}\]
   In the first case, \(B/(y - f(a)^{1/p})^p\) is not a field extension of \(\kappa(x - a) = k\), while in the second, \(B/(y^p - f(a))\) is an inseparable field extension of \(k\). Thus \(\varphi\) is not étale at any of these \((x - a)\).
(3) Finally, if \( f(a) \neq 0 \) and \( g(a) \neq 0 \), then \( (x-a) = (y^p - (f(a)g(a))^{p-1}y - f(a)) \). If there is a \( c \in k \) such that \( f(a) = c^p - c \), then
\[
(x-a) = \prod_{i=0}^{p-1} (x - (c + if(a)g(a))) \implies B/(x-a)B \cong \prod_{i=0}^{p-1} k.
\]

Thus \( \varphi \) is étale at \( (x-a) \) in this case. On the other hand, if there does not exist such a \( c \), then \( (x-a) \) remains a prime ideal in \( B \) and thus \( B/(x-a) \) is a finite separable extension of \( k \).

Hence \( \varphi \) is étale at the point \( (x-a) \) if and only if \( f(a) \neq 0 \) and \( g(a) \neq 0 \).

### 1.5 The Fundamental Group of Curves

Let \( k \) be a perfect field, \( X \) an integral proper normal curve over \( k \), \( K = k(X) \) its function field and fix a separable closure \( K_s \) of \( K \). For a nonempty open subset \( U \subseteq X \), define \( K_U \) to be the compositum of all finite subextensions \( K_s \supseteq L \supseteq K \) corresponding to finite morphisms of proper normal curves \( Y \to X \) which are étale over \( U \).

**Proposition 1.5.1.** For any nonempty open subset \( U \subseteq X \), \( K_U/K \) is a Galois extension and each finite subextension corresponds to a cover of \( X \) which is étale over \( U \).

**Proof.** To show \( K_U/K \) is a Galois extension, it’s enough to see that \( K_U \) is stable under the \( \text{Gal}(K_s/K) \)-action, but this follows from the definition of the field \( K_U \). To prove the second statement, it’s enough to show for a finite subextension \( L/K \) of \( K_U/K \) that:

(i) If \( L \) comes from a cover of \( X \) which is étale over \( U \), then so does any subfield \( L \supseteq L' \supseteq K \).

(ii) If \( M/K \) is any other finite subextension of \( K_U/K \) coming from a cover which is étale over \( U \), then \( L/M \) also comes from a cover étale over \( U \).

Then the result will follow since \( K_U \) is the compositum of a tower of finite extensions \( L_1 \subseteq L_1L_2 \subseteq L_1L_2L_3 \subseteq \cdots \)

For (i), let \( L/K \) correspond to a cover \( Y \to X \) which is étale over \( U \). Then for any \( L \supseteq L' \supseteq K \), we get a composition of covers \( Y \to Y' \to X \). Applying Lemma 1.4.4, we get that \( \varphi' \) is étale over \( U \). For (ii), fix \( P \in U \) and consider the inertia group \( I_P \subseteq \text{Gal}(K_s/K) \) as defined in Section 1.4. Then by (3) of Theorem 1.4.8, \( I_P \) is trivial in both \( \text{Gal}(L/K) \) and \( \text{Gal}(M/K) \). Since \( \text{Gal}(L/K) = \text{Gal}(K_s/K)/\text{Gal}(K_s/L) \), this means \( I_P \subseteq \text{Gal}(K_s/L) \); likewise \( I_P \subseteq \text{Gal}(K_s/M) \). Hence \( I_P \subseteq \text{Gal}(K_s/LM) \), which means \( I_P \) is trivial in \( \text{Gal}(LM/K) \), so by Theorem 1.4.8 again, the curve corresponding to \( LM/K \) is étale at \( P \). \( \square \)

**Definition.** For a proper normal curve \( X \) over a perfect field \( k \) and a nonempty open subset \( U \subseteq X \), we define the **algebraic fundamental group** over \( U \) to be
\[
\pi_1(U) := \text{Gal}(K_U/K).
\]
Note that \( \pi_1(U) \) is a profinite group which depends on the choice of separable closure \( K_s \). Here, this choice of \( K_s \) acts as the “basepoint” in analogy with the topological case (see Section 0.1).

**Definition.** Extending the notion of curves from the last section, we define a proper normal curve to be a finite disjoint union of integral proper normal curves.

The ring of rational functions on a proper normal curve \( X = \bigsqcup X_i \) is defined to be the direct sum of the function fields of each component, \( \mathcal{O}(X) = \bigoplus k(X_i) \); this is naturally a finite dimensional algebra over any of the components \( k(X_i) \). In general, we say a morphism \( Y \to X \), with \( X \) integral, is separable if \( k(Y) \) is étale over \( k(X) \).

**Theorem 1.5.2.** Suppose \( X \) is an integral proper normal curve over a perfect field \( k \) and \( U \subseteq X \) is a nonempty open subset. Then there is an equivalence of categories

\[
\left\{ \text{covers of proper normal curves } Y \xrightarrow{\varphi} X \right. \left. \begin{array}{l}
\text{where } \varphi \text{ is finite, separable, étale over } U\end{array} \right\} \sim \left\{ \text{finite continuous left } \pi_1(U)\text{-sets} \right\}.
\]

**Proof.** Propositions 1.3.4 and 1.5.1 extend to direct sums of fields, so for a cover \( Y \to X \) which is a proper normal curve \( Y = \bigsqcup Y_i \), let \( A = \bigoplus L_i \) be the finite étale \( K \)-algebra corresponding to these \( Y_i \to X \). Finally, by Corollary 0.2.4, \( \text{Hom}(A, K_s) \) is a finite continuous \( \text{Gal}(K_s/K) \)-set.

Now, for an integral normal affine curve \( U \) over \( k \), we know \( U \) is an affine open subset of an integral proper normal curve \( X = U \cup U^- \). Thus \( \pi_1(U) \) is defined.

**Corollary 1.5.3.** For any integral normal affine curve \( U \) over a perfect field \( k \), there is an equivalence of categories

\[
\left\{ \text{normal affine curves with finite étale covers } V \to U \right\} \sim \left\{ \text{finite continuous left } \pi_1(U)\text{-sets} \right\}.
\]

**Proof.** Any finite étale cover of normal affine curves \( V \to U \) extends by Proposition 1.3.6 to a cover of proper normal curves \( Y^L \to X^K \) which is in turn finite by Proposition 1.3.8. Finally, Theorem 1.5.2 gives the desired correspondence.

We now discuss how to capture the notion of a “universal cover” as we have in the topological case (Section 0.1). Let \( U \subseteq X \) be a nonempty open subset of an integral proper normal curve and let \( A = \mathcal{O}(U) \) be its ring of rational functions. Take \( \tilde{A} \) be the integral closure of \( A \) in \( K_U \). Then for a finite étale cover \( V \to U \) with fraction field \( K(V) \), we have \( \mathcal{O}(V) = \tilde{A} \cap K(V) \).
1.5 The Fundamental Group of Curves

1 Fundamental Groups of Algebraic Curves

Note that for any maximal ideal \( m \subseteq \tilde{A} \), \( m \cap \mathcal{O}(V) \) is a closed point in \( V \). Although \( \tilde{A} \) is not itself a finitely generated ring over \( A \), it is a compositum of such rings, i.e. the \( \mathcal{O}(V) \) are finite over \( A \). Define \( \tilde{U} = \text{MaxSpec} \tilde{A} \) and equip this with the inverse limit topology with respect to the inverse system \( V \to V' \to U \). We define a sheaf of rings \( \mathcal{O}_{\tilde{U}} \) on \( \tilde{U} \) stalkwise by setting \( \mathcal{O}_{\tilde{U}, \tilde{Q}} = \tilde{A}_{\tilde{Q}} \) for any \( \tilde{Q} \in \tilde{U} \). Then for any open set \( \tilde{V} \subseteq \tilde{U} \), we put

\[
\mathcal{O}_{\tilde{U}}(\tilde{V}) = \bigcap_{\tilde{Q} \in \tilde{U}} \mathcal{O}_{\tilde{U}, \tilde{Q}}.
\]

Definition. \( \tilde{U} \) is called the pro-étale cover of \( U \).

Lemma 1.5.4. \( (\tilde{U}, \mathcal{O}_{\tilde{U}}) \) is a locally ringed space.

Remark. Note that \( \tilde{U} \) is not an object in the category of proper normal curves (or even in the category of schemes, as we shall see!) However, constructing a “universal cover” is useful to illustrate the analogy with the topological case. For example, we have the following theorem which will be proven in Section 2.3.

Theorem 1.5.5. Let \( X \) be an integral proper normal curve over \( \mathbb{C} \) and \( U \subseteq X \) a nonempty open subset. Then \( \pi_1(U) \) is isomorphic to \( \pi_1^{top}(U(\mathbb{C})) \), the profinite completion of the topological fundamental group of the Riemann surface \( U(\mathbb{C}) \). In particular, \( \pi_1(U) \) is the profinite completion of

\[
\langle a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_n | [a_1, b_1] \cdots [a_g, b_g] \gamma_1 \cdots \gamma_n = 1 \rangle,
\]

where \( n \) is the number of points in \( X \setminus U \) and \( g \) is the genus of the Riemann surface \( U(\mathbb{C}) \).

Let \( X/\mathbb{C} \) and \( U \) be as above. If \( G \) is any finite quotient of \( \pi_1(U) \), then \( G \) corresponds to a finite Galois branched cover \( Y \to X \). Here, the image of each generator \( \gamma_i \in \pi_1(U) \) in \( G \) is a cyclic generator of the stabilizer of a point \( Q_i \to P_i \), where \( P_i \in X \setminus U \) is the point corresponding to \( \gamma_i \).

Let \( X/\mathbb{C} \) be an integral proper normal curve over an arbitrary (perfect) field \( k \). If \( L/k \) is a finite extension, then \( k(X) \otimes_k L \) is a finite dimensional algebra over \( L(t) \) which is not, in general, a field. Assume \( k(X) \otimes_k L \) is a finite direct product of fields \( L_1, \ldots, L_n \), each of which is a finitely generated extension of \( L(t) \) of transcendence degree 1 over \( L \). Then each \( L_i \) corresponds to an integral proper normal curve \( X_i/L \) by Proposition 1.3.4. Define the base change \( X_L = \prod X_i \). This comes equipped with a natural morphism of proper normal curves \( X_L \to X \). We will prove:
Theorem 1.5.6. If \( k \) is an algebraically closed field of characteristic 0, \( L/k \) is a finite extension, \( X \) is an integral proper normal curve over \( k \) and \( U \subseteq X \) is open, then the base change functor \( X \mapsto X_L \) induces an equivalence of categories \[
\left\{ \text{finite branched Galois covers } Y \to X \right\} \xrightarrow{\sim} \left\{ \text{finite branched Galois covers } Y_L \to X_L \right\}.
\]

Corollary 1.5.7. For \( U \subseteq X \) as above, there is an isomorphism \( \pi_1(U_L) \cong \pi_1(U) \) for all finite extensions \( L/k \).

Theorem 1.5.5 generalizes to curves over an algebraically closed field \( k \) of characteristic 0 in the following sense.

Corollary 1.5.8. If \( k \) is algebraically closed of characteristic 0 and \( U \subseteq X \) is a nonempty open subset of an integral proper normal curve \( X \) over \( k \), then \( \pi_1(U) \) is isomorphic to the profinite completion of a group with presentation
\[
\langle a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_n \mid [a_1, b_1] \cdots [a_g, b_g] \gamma_1 \cdots \gamma_n = 1 \rangle,
\]
where \( g = g(X) \) is the genus of \( X \) and \( n \) is the number of points in \( X \setminus U \).

Example 1.5.9. If \( k \) is an algebraically closed field of characteristic 0 and \( X \to \mathbb{P}^1_k \) is a finite étale cover, the Riemann-Hurwitz inequality says that
\[
2g(X) - 2 = n(0 - 2) + 0
\]
but this is only possible if \( g(X) = 0 \) and \( n = 1 \). Thus \( X \to \mathbb{P}^1_k \) is a birational isomorphism, but since \( X \) is complete, it must be a regular isomorphism. Hence there are no nontrivial extensions \( L \) of \( k(\mathbb{P}^1_k) = k \) which proves \( \pi_1(\mathbb{P}^1_k) = 1 \).

Example 1.5.10. A similar argument shows \( \pi_1(\mathbb{A}^1_k) = 1 \) when \( k \) is algebraically closed and \( \text{char } k = 0 \). Both this and the previous result also follow easily from Theorem 1.5.5 and Corollary 1.5.8.

Example 1.5.11. Theorem 1.5.5 and Corollary 1.5.8 also imply that \( \pi_1(\mathbb{P}^1_k \setminus \{0, \infty\}) \cong \hat{\mathbb{Z}} \), the profinite completion of the integers. Using Theorem 1.5.2, we get a finite étale cover \( X_n \to \mathbb{P}^1_k \setminus \{0, \infty\} \) for each \( n \geq 1 \) having Galois group \( \mathbb{Z}/n\mathbb{Z} \).

Example 1.5.12. By Theorem 1.5.5 and Corollary 1.5.8, \( \pi_1(\mathbb{P}^1_k \setminus \{0, 1, \infty\}) \) is the free profinite group on two generators.

1.6 The Outer Galois Action

Let \( X \) be an integral proper normal curve over a perfect field \( k \) which is not necessarily algebraically closed. Let \( K = k(X) \). Fixing an algebraic closure \( \bar{k} \), we assume \( X \) is geometrically integral, that is, \( K \otimes_k \bar{k} \) is a field. Then \( K_s \), the separable closure of \( K \) containing \( \bar{k} \), is defined. For any nonempty open \( U \subseteq X \), the base change \( U_k \) is integral and we have \( K_k \subseteq K_U \), where the compositum is taken in \( K_U \). By construction, for any finite extension \( L/k \), the cover \( X_L \to X \) is finite étale with \( \text{Gal}(k(X_L)/K) \cong \text{Gal}(L/k) \). Thus \( \text{Gal}(K_k/K) \cong \text{Gal}(\bar{k}/k) \) so the latter may be thought of as a quotient of \( \pi_1(U) \) for any open \( U \subseteq X \).
1.6 The Outer Galois Action

Proposition 1.6.1. For $X$ a geometrically integral, proper normal curve over a perfect field $k$ and for any nonempty open subset $U \subseteq X$, there is a short exact sequence of profinite groups

$$1 \rightarrow \pi_1(U_k) \rightarrow \pi_1(U) \rightarrow \text{Gal}(\bar{k}/k) \rightarrow 1.$$  

Proof. The surjection $\pi_1(U) \rightarrow \text{Gal}(\bar{k}/k)$ was described above, so it remains to identify $\pi_1(U_k)$ with the kernel of this map. Suppose $G$ is a finite quotient of $\pi_1(U_k)$. Then there is a corresponding finite extension $K_0/K\bar{k}$:

This also determines a finite branched cover $Y_0 \rightarrow X_{\bar{k}}$ which is étale over $U_{\bar{k}} \subseteq X_{\bar{k}}$. Let $f \in K\bar{k}[t]$ be the minimal polynomial of $K_0/K\bar{k}$. Then we can find a finite extension $L/k$ such that $KL/k$ contains all the coefficients of $f$. Let $L_0/KL$ be the finite extension with $\text{Gal}(L_0/KL) = G$. 

\[
\begin{array}{c}
\pi_1(U) \\
\downarrow \quad \downarrow \quad \downarrow \\
K_U \\
\downarrow \quad \downarrow \quad \downarrow \\
K\bar{k} \\
\downarrow \quad \downarrow \quad \downarrow \\
K \\
\end{array}
\]

\[
\begin{array}{c}
\pi_1(U_k) \\
\downarrow \quad \downarrow \quad \downarrow \\
K_{U_k} \\
\downarrow \quad \downarrow \quad \downarrow \\
K_0 \\
\downarrow \quad \downarrow \quad \downarrow \\
K\bar{k} \\
\end{array}
\]

\[
\begin{array}{c}
\pi_1(U) \\
\downarrow \quad \downarrow \quad \downarrow \\
K_0 \\
\downarrow \quad \downarrow \quad \downarrow \\
KL \\
\downarrow \quad \downarrow \quad \downarrow \\
L \\
\downarrow \quad \downarrow \quad \downarrow \\
k \\
\end{array}
\]

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By construction, $L_0\tilde{k} = K_0$ and the tower $L_0 \supseteq KL \supseteq L$ determines a composition of covers $Y \to X_L \to X$ which is étale over $U$, and such that $Y \tilde{k} = Y_0$. This implies $L_0 \subseteq K_U$ and then $K_0 = L_0\tilde{k} \subseteq K_U$. This shows that we can identify any such $G = \text{Gal}(L_0/KL)$ with a finite quotient of $\ker(\pi_1(U) \to \text{Gal}(K\tilde{k}/K))$. Varying the finite quotients $G$ of $\pi_1(U\tilde{k})$, we conclude that $K_{U\tilde{k}} \subseteq K_U$ and hence there is a surjection $\text{Gal}(K_U/K\tilde{k}) \to \text{Gal}(K_{U\tilde{k}}/K\tilde{k}) = \pi_1(U\tilde{k})$ which is bijective on finite quotients. Taking inverse limits – which is a left exact functor – gives $\pi_1(U\tilde{k}) = \ker(\pi_1(U) \to \text{Gal}(K\tilde{k}/K))$ as required. 

**Corollary 1.6.2.** There is a continuous homomorphism $p_U : \text{Gal}(\tilde{k}/k) \to \text{Out}(\pi_1(U\tilde{k}))$ for any nonempty open set $U \subseteq X$.

**Proof.** Any short exact sequence of groups $1 \to N \to G \to P \to 1$ defines a homomorphism $G \to \text{Aut}(N)$ whose restriction to the subgroup $N \leq G$ has image lying in $\text{Inn}(N)$. By definition, $\text{Out}(N) = \text{Aut}(N)/\text{Inn}(N)$ so we get a map $P \to \text{Out}(N)$. Applying this to the short exact sequence from Proposition 1.6.1 gives the result. 

**Definition.** For an open subset $U$ of a geometrically integral proper normal curve $X$ over a perfect field $k$, the subgroup $\pi_1(U\tilde{k}) \leq \pi_1(U)$ is called the geometric fundamental group of $U$. The map $p_U : \text{Gal}(\tilde{k}/k) \to \text{Out}(\pi_1(U\tilde{k}))$ is called the outer Galois action on the geometric fundamental group.

We next study the action of $\pi_1(U)$ on $\tilde{X}$, the pro-étale cover of $X$ defined in Section 1.5. Let $\tilde{Q}$ be a pro-point in $\tilde{X}$ lying over a point $P \in X$. We define $D_{\tilde{Q}}$ to be the stabilizer of $\tilde{Q}$ under the $\pi_1(U)$-action. Then $\kappa(\tilde{Q}) = O_{X,\tilde{Q}}/\tilde{Q}O_{X,\tilde{Q}} = \overline{\kappa(P)}$, the algebraic closure of the residue field $\kappa(P) = O_{X,P}/PO_{X,P}$. We have a natural surjection $D_{\tilde{Q}} \to \text{Gal}(\kappa(\tilde{Q})/\kappa(P))$ with kernel denoted $I_{\tilde{Q}}$, called the inertia group of $\tilde{Q}$. We may alternatively view $\tilde{X}$ as a profinite Galois cover of $X\tilde{k}$ with Galois group $\pi_1(U\tilde{k})$, corresponding to the short exact sequence from Proposition 1.6.1:

\[
\begin{array}{c}
\tilde{X} \\
\pi_1(U) \\
X \end{array} \quad \begin{array}{c}
\pi_1(U\tilde{k}) \\
X_k \\
\text{Gal}(\tilde{k}/k) \end{array}
\]

**Definition.** Let $P \in X$ be a closed point such that $\kappa(P) \cong k$. Then we say $P$ is $k$-rational.

**Lemma 1.6.3.** Let $P \in X$ be a $k$-rational (closed) point. Then the stabilizer of any pro-point $\tilde{Q} \in \tilde{X}$ lying over $P$ in $\pi_1(U\tilde{k})$ is equal to the inertia group $I_{\tilde{Q}}$.

**Proof.** We have a surjection $D_{\tilde{Q}} \to \text{Gal}(\kappa(\tilde{Q})/\kappa(P)) = \text{Gal}(\tilde{k}/k)$ by assumption, but this is just the restriction of the map $\pi_1(U) \to \text{Gal}(\tilde{k}/k)$ to the action on $\tilde{Q}$. By definition the kernel of the latter map is $I_{\tilde{Q}}$. 

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Corollary 1.6.4. If \( U \) contains a \( k \)-rational point of \( X \) then the short exact sequence

\[
1 \to \pi_1(U_k) \to \pi_1(U) \to \text{Gal}(\bar{k}/k) \to 1
\]

is split.

Proof. If \( \bar{Q} \) is a pro-point over some \( k \)-rational point \( P \in U \), then the stabilizer of \( \bar{Q} \) in \( \pi_1(U_k) \) is trivial. Therefore Lemma 1.6.3 says that \( I_{\bar{Q}} = 1 \), but by definition of the inertia group this implies \( D_{\bar{Q}} \cong \text{Gal}(\bar{k}/k) \). Thus there is a map \( \text{Gal}(\bar{k}/k) \to \pi_1(U) \) mapping isomorphically onto \( D_{\bar{Q}} \subseteq \pi_1(U) \) which defines the required splitting. \( \square \)

Example 1.6.5. Let \( X = \mathbb{P}^1_k \) and consider the open set \( U = \mathbb{P}^1_k \setminus \{0, \infty\} \). By Example 1.5.11, \( \pi_1(U) \cong \hat{\mathbb{Z}} \), a free abelian profinite group. Hence \( \text{Gal}(\bar{k}/k) \) acts on \( \pi_1(U_k) \) directly, not just by outer automorphisms. Let \( n \geq 1 \). Then the quotient \( \pi_1(U_k)/n\pi_1(U_k) \cong \mathbb{Z}/n\mathbb{Z} \) corresponds to \( \text{Gal}(L_n/\bar{k}(t)) \) where \( L_n/\bar{k}(t) \) is the field extension defined by \( x^n - t \). Consider the short exact sequence of profinite groups

\[
1 \to \text{Gal}(L_n/\bar{k}(t)) \to \text{Gal}(L_n/\bar{k}(t)) \to \text{Gal}(\bar{k}/k) \to 1.
\]

Identifying \( \pi_1(U_k)/n\pi_1(U_k) = \mathbb{Z}/n\mathbb{Z} \), we see that this sequence defines the \( \text{Gal}(\bar{k}/k) \)-action on \( \pi_1(U_k)/n\pi_1(U_k) \). Explicitly, \( \text{Gal}(L_n/\bar{k}(t)) \) is generated by \( \sqrt[n]{t} \mapsto \zeta_n \sqrt[n]{t} \), where \( \zeta_n \in \bar{k} \) is a primitive \( n \)th root of unity. Then for \( \sigma \in \text{Gal}(\bar{k}/k) \), the Galois action is given by

\[
\sigma \cdot (\sqrt[n]{t} \mapsto \zeta_n \sqrt[n]{t}) = \sqrt[n]{t} \mapsto \sigma(\zeta_n) \sqrt[n]{t}.
\]

Fixing a compatible system of primitive roots of unity \((\zeta_n)_{n \in \mathbb{N}}\), we get isomorphisms

\[
\hat{\mathbb{Z}} \cong \text{Gal} \left( \lim_{\leftarrow} L_n/\bar{k}(t) \right) = \text{Gal}(\bar{k}(t)^\text{cyc}/\bar{k}(t)) \cong \pi_1(\mathbb{P}^1_k \setminus \{0, \infty\}).
\]

This defines a representation \( \text{Gal}(\bar{k}/k) \to \text{Aut}(\hat{\mathbb{Z}}) = \hat{\mathbb{Z}}^\times \), called the cyclotomic character of \( \text{Gal}(\bar{k}/k) \). Notice that this descends to a character \( \text{Gal}(\bar{k}/k) \to (\mathbb{Z}/n\mathbb{Z})^\times \) for each \( n \geq 1 \).

One of the most important objects in arithmetic geometry is \( \pi_1(\mathbb{P}^1_\mathbb{Q} \setminus \{0, 1, \infty\}) \). If \( U = \mathbb{P}^1_\mathbb{Q} \setminus \{0, 1, \infty\} \), there is an outer Galois action \( \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Out}(\pi_1(U_\mathbb{Q})) \). We will prove that this action is faithful, but first we need:

Theorem 1.6.6 (Belyi). Suppose \( X \) is an integral proper normal curve defined over an algebraic closed field \( k \) of characteristic 0. Then there exists a morphism \( X \to \mathbb{P}^1_k \) which is étale over \( \mathbb{P}^1_k \setminus \{0, 1, \infty\} \) if and only if \( X \) is defined over \( \mathbb{Q} \).

Proof. (\( \Rightarrow \)) follows from Theorem 1.5.6.

(\( \Leftarrow \)) If \( X \) is defined over \( \overline{\mathbb{Q}} \), there exists a map \( \pi : X \to \mathbb{P}^1_{\overline{\mathbb{Q}}} \), which is étale over \( \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus S \) for some finite set \( S \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}} \). We first show \( S \) consists of \( \mathbb{Q} \)-rational points. Since \( S \) is finite, we can find a point \( P \in S \) with \( \kappa(P) : \overline{\mathbb{Q}} = n \) maximal among the points in \( S \). Let \( f \in \overline{\mathbb{Q}}[t] \) be the minimal polynomial of \( \kappa(P)/\overline{\mathbb{Q}} \). Define \( \varphi_f : \mathbb{P}^1_{\overline{\mathbb{Q}}} \to \mathbb{P}^1_{\overline{\mathbb{Q}}} \) by \( x \mapsto F(x) \), where \( F \) is the homogenization of \( f \). Then \( \varphi_f \) is defined over \( \mathbb{Q} \) and is étale over \( \mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus S_f \), where
Belyi’s theorem says that any such cover defined over $U$ must have $\pi$-rational. The composition $\varphi_j \circ \pi : X \to \mathbb{P}^1_\mathbb{Q}$ is now étale away from $S' := \varphi_j(S) \cup \{\infty\} \cup \varphi_f(S_f)$. But now $\infty$ has degree 1, points in $\varphi_j(S)$ have degree at most $n$ and points in $\varphi_f(S_f)$ have degree at most $n-1$, so because $\varphi_j(P) = 0$, there are strictly less points of degree $n$ in $S'$ than in $S$. Repeat this procedure until $n = 1$, at which time all points in $S'$ will be $\mathbb{Q}$-rational.

Next, suppose $S$ has more than three points. We may assume $0, 1, \infty \in S$ and take $\alpha \in S \setminus \{0, 1, \infty\}$. Define the Belyi function

$$\varphi : \mathbb{P}^1_\mathbb{Q} \to \mathbb{P}^1_\mathbb{Q}$$

$$x \mapsto x^A(x-1)^B$$

where $A, B \in \mathbb{Z} \setminus \{0\}$ are such that $\alpha = \frac{A}{A+B}$. As above, $\varphi$ is étale away from $S' = \varphi(S) \cup \{\infty\} \cup \varphi(S_{\alpha})$ where $S_{\alpha} = \{R \in \mathbb{A}^1_\mathbb{Q} \mid \varphi'(R) = 0\}$. Note that $\varphi'(x) = Ax^{A-1}(x-1)^B + Bx^A(x-1)^{B-1}$ so $\varphi'(R) = 0$ precisely when $R \in \{0, 1, \infty, \alpha\}$. This shows that $\varphi(S) = S'$. Composing with $\pi$, we get a map $\varphi \circ \pi : X \to \mathbb{P}^1_\mathbb{Q}$ which is étale outside $\varphi(S)$ and maps $\{0, 1, \infty\}$ to $\{0, \infty\}$. Hence $|\varphi(S)| < |S|$, so repeating this process reduces to the case when $S'$ has (at most) three elements.

**Theorem 1.6.7.** The outer Galois representation $\rho : \text{Gal}(\mathbb{Q}/\mathbb{Q}) \to \text{Out}(\pi_1(\mathbb{P}^1_\mathbb{Q} \setminus \{0, 1, \infty\}))$ is faithful.

**Proof.** Let $L = \overline{\mathbb{Q}}^{\ker \rho}$ be the subfield fixed by the kernel of the outer Galois representation. Then for $U = \mathbb{P}^1_\mathbb{Q} \setminus \{0, 1, \infty\}$, Proposition 1.6.1 gives us a short exact sequence of profinite groups

$$1 \to \pi_1(U_\overline{\mathbb{Q}}) \to \pi_1(U_L) \to \text{Gal}(\overline{\mathbb{Q}}/L) \to 1.$$  

By construction, $\rho_{U_L} : \text{Gal}(\overline{\mathbb{Q}}/L) \to \text{Out}(\pi_1(U_\overline{\mathbb{Q}}))$ has trivial kernel, which means any automorphism of $\pi_1(U_\overline{\mathbb{Q}})$ in $\pi_1(U_L)$ is given by conjugation by some $y \in \pi_1(U_\overline{\mathbb{Q}})$. That is, for any $x \in \pi_1(U_L)$, there exists a $y \in \pi_1(U_\overline{\mathbb{Q}})$ such that for all $a \in \pi_1(U_\overline{\mathbb{Q}})$, we have $yax = yay$. This is equivalent to $y^{-1}xay^{-1} = a$, i.e. $y^{-1}x$ belongs to $C$, the centralizer of $\pi_1(U_\overline{\mathbb{Q}})$ in $\pi_1(U_L)$. Hence $\pi_1(U_L)$ is generated by $\pi_1(U_\overline{\mathbb{Q}})$ and $C$. However, Corollary 1.5.8 says that $\pi_1(U_\overline{\mathbb{Q}})$ is a free profinite group on 2 generators, and such a group has trivial centralizer, so we must have $\pi_1(U_\overline{\mathbb{Q}}) \cap C = \{1\}$. Thus $\pi_1(U_L) \cong \pi_1(U_\overline{\mathbb{Q}}) \times C$ as a direct product, or in other words, $\pi_1(U_L) \to \pi_1(U_\overline{\mathbb{Q}})$ is a continuous retraction of profinite groups. This says that any continuous left $\pi_1(U_\overline{\mathbb{Q}})$-set comes from a continuous left $\pi_1(U_L)$-set, but by Theorem 1.5.6, any finite cover of $U_\overline{\mathbb{Q}}$ is obtained by base change from some finite cover of $U_L$. Finally, Belyi’s theorem says that any such cover defined over $\overline{\mathbb{Q}}$ can be defined over $L$, but we will demonstrate that this is impossible unless $L = \overline{\mathbb{Q}}$, in which case ker $\rho$ is trivial.

If $L \neq \overline{\mathbb{Q}}$, take $x \in \overline{\mathbb{Q}} \setminus L$. Let $E'$ be an elliptic curve over $\overline{\mathbb{Q}}$ with $j$-invariant $j(E') = x$. If $E'$ can be obtained by base change from some genus 1 curve $X$ defined over $L$, as Belyi’s theorem indicates, then $X$ has a Jacobian $E/L$ which is an elliptic curve. There exists an embedding $\phi : E \to \mathbb{P}^2_L$ sending the distinguished $L$-point of $E$ to $[0, 1, 0] \in \mathbb{P}^2_L$. Moreover, it is known that $j(E) \in L$ and this $j$-invariant is preserved by any $\overline{\mathbb{Q}}$-isomorphism. Since $E'$ is the base change of $X$, we have $E' \cong X_{\overline{\mathbb{Q}}} \to E$, but this implies $j(E) = j(E') = x \notin L$, a contradiction. Hence $L = \overline{\mathbb{Q}}$, so the outer Galois representation of $U$ is faithful.
Example 1.6.8. One can similarly prove that for any elliptic curve $E$ over $\mathbb{Q}$ (or any number field $K$), the outer Galois representation $\rho : \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \pi_1(E \setminus \{0\})$ is faithful.

1.7 The Inverse Galois Problem

For a field $k$, let $G_k = \operatorname{Gal}(\overline{k}/k)$ be the absolute Galois group. We showed in the previous section (Theorem 1.6.7) that there is a faithful representation $G_{\mathbb{Q}} \hookrightarrow \operatorname{Out}(\pi_1(P^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}))$, but this group of automorphisms is inordinately complicated so we don’t have much understand of $G_{\mathbb{Q}}$. A related question we might answer is: what are the finite quotients of $G_{\mathbb{Q}}$.

**Question** (Inverse Galois Problem). Which groups arise as Galois groups $\operatorname{Gal}(L/\mathbb{Q})$ for some finite Galois extension $L/\mathbb{Q}$?

**Definition.** If $L/\mathbb{Q}$ has Galois group $G$, we say $L$ is a $G$-extension of $\mathbb{Q}$.

**Example 1.7.1.** Let $C_n = \mathbb{Z}/n\mathbb{Z}$ be a cyclic group of order $n$. We know that the cyclotomic extension $\mathbb{Q}(\zeta_p)$ has Galois group $\operatorname{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong \mathbb{Z}/(p-1)\mathbb{Z}$, so by group theory, if $n \mid p - 1$, this Galois group has a subgroup $H$ of order $(p-1)/n$. Hence the subextension $\mathbb{Q}(\zeta_p)^H/\mathbb{Q}$ has group $\operatorname{Gal}(\mathbb{Q}(\zeta_p)^H/\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$. Thus the inverse Galois problem for cyclic groups reduces to finding a prime $p$ such that $p \equiv 1 \pmod{n}$, but by Dirichlet’s theorem, there are infinitely many such primes $p$. Thus all cyclic groups arise as Galois groups over $\mathbb{Q}$. Using the theory of cyclotomic extensions, this can be generalized to give a positive solution to the inverse Galois problem for all finite abelian groups.

Combined with the Kronecker-Weber theorem (every finite abelian extension of $\mathbb{Q}$ lies in some cyclotomic extension $\mathbb{Q}(\zeta_m)/\mathbb{Q}$), the conclusion of Example 1.7.1 gives a complete description of abelian extensions of $\mathbb{Q}$, namely that the maximal abelian extension of the rationals $\mathbb{Q}^{ab}$ is equal to the compositum $\mathbb{Q}^{\text{cyc}} := \bigcup_{m \geq 1} \mathbb{Q}(\zeta_m)$. The study of $\operatorname{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ and its subgroups is the content of global class field theory.

If a group $G$ arises as a Galois group over $\mathbb{Q}$, we may then ask for all $G$-extensions of $\mathbb{Q}$. This may be a much harder question to answer.

**Example 1.7.2.** One can show that the splitting field of $f(x) = x^3 + x^2 - 2x - 1$ defines a $C_3$-extension of $\mathbb{Q}$. How do we determine all $C_3$-extensions? Well notice that $C_3$ embeds as a subgroup of $\operatorname{PGL}_2(\mathbb{Q}) = \operatorname{Aut}(\mathbb{P}^1_{\mathbb{Q}})$ via

$$C_3 = \langle \sigma \rangle \longrightarrow \operatorname{Aut}(\mathbb{P}^1)$$

$$\sigma \longmapsto \left( x \longmapsto \frac{1}{1-x} \right).$$

Then $\mathbb{P}^1/C_3$ is birationally equivalent to $\mathbb{P}^1$ and the rational expression $T = x + \sigma x + \sigma^2 x = \frac{x^3 - 3x - 1}{x^2 - x}$ is an invariant under the $C_3$-action. Then the fixed subfield $\mathbb{Q}(x)^{C_3} = \mathbb{Q}(T)$ corresponds to a cover

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^1/C_3 \cong \mathbb{P}^1$$

$$x \longmapsto T = \frac{x^3 - 3x - 1}{x^2 - x}.$$
By construction, \( \text{Gal}(\mathbb{Q}(x)/\mathbb{Q}(T)) \cong C_3 \) and this extension has minimal polynomial
\[
f(x, T) = x^3 - Tx^2 + (T - 3)x + 1.
\]
Notice that the discriminant of \( f \) is \((T^2 - 3T - 9)^2\), so for any \( t \in \mathbb{Q} \) such that \( t^2 - 3t - 9 \neq 0 \), \( f(x, t) \) becomes an irreducible polynomial over \( \mathbb{Q} \) with Galois group \( \text{Gal}(f(x, t)) \cong C_3 \). In fact, one can show that \( f(x, T) \) parametrizes all \( C_3 \)-extensions of \( \mathbb{Q} \).

**Definition.** We say a polynomial \( f(x, T) \), with \( T = (T_1, \ldots, T_n) \) an \( n \)-tuple of indeterminates, is a **generic polynomial** for a group \( G \) over a field \( K \) if for all extensions \( F/K \),
\[
(1) \text{ The splitting field } F_f/F(T) \text{ has group } \text{Gal}(F_f/F(T)) \cong G.
\]
\[
(2) \text{ Every } G \text{-extension } L/F \text{ is obtained as the splitting field of } f(x, t) \text{ for some } t = (t_1, \ldots, t_n) \in F^n.
\]
A polynomial \( f(x, T) \) that satisfies these conditions for some fixed \( F \) is called a **versal family** for \( G \)-extensions of \( F \). Thus a generic polynomial for \( K \) is one that is a versal family for all extensions of \( K \).

**Example 1.7.3.** We saw in Example 1.7.2 that \( f(x, T) = x^3 - Tx^2 + (T - 3)x + 1 \) is a versal family for \( C_3 \)-extensions of \( \mathbb{Q} \). One can show (see Theorem 2.2.1 of Jensen, Ledet and Yui’s *Generic Polynomials: Constructive Aspects of the Inverse Galois Problem*) that in fact this \( f(x, T) \) is a generic polynomial for \( C_3 \)-extensions of \( \mathbb{Q} \).

**Example 1.7.4.** For \( C_8 \)-extensions of \( \mathbb{Q} \), there is a versal family for \( \mathbb{Q} \), but there is no generic polynomial for all extensions of \( \mathbb{Q} \).

**Remark.** Why could we find a one-parameter generic polynomial for \( C_3 \)-extensions? It is a fact that if there exists a one-parameter generic polynomial, i.e. \( f(x, T_1) = f(x, T) \), for \( G \)-extensions of \( K \), then \( G \) embeds as a subgroup of \( \text{PGL}_2(K) \).

**Example 1.7.5.** By the Remark, the fact that \( C_4 \) does not embed as a subgroup of \( \text{PGL}_2(\mathbb{Q}) \) implies that there does not exist a 1-parameter generic polynomial for \( C_4 \)-extensions of \( \mathbb{Q} \). However, there does exist an embedding \( C_4 \hookrightarrow \text{PGL}_2(\mathbb{Q}(i)) \), so there may be a 1-parameter generic polynomial over \( \mathbb{Q}(i) \).

Let \( G \) be a finite group. By Cayley’s theorem, \( G \) embeds as a subgroup of some \( S_n \), inducing a \( G \)-action on the set \( \{x_1, \ldots, x_n\} \). Define \( E = \mathbb{Q}(x_1, \ldots, x_n)^G \) and let \( \pi: \mathbb{A}^n_{\mathbb{Q}} \to \mathbb{A}^n_{\mathbb{Q}}/G \) be the corresponding covering map.

**Question** (Noether’s Problem). For which groups \( G \) is the field \( E = \mathbb{Q}(x_1, \ldots, x_n)^G \) transcendental over \( \mathbb{Q} \)?

The inverse Galois problem is known to be a consequence of the so-called **regular** inverse Galois problem:

**Question** (Regular Inverse Galois Problem). Which groups \( G \) arise as Galois groups of finite Galois extensions \( K/\mathbb{Q}(t) \) which are regular over \( \mathbb{Q} \)?
(A regular extension $K/Q(t)$ is one which does not contain a subextension of the form $L(t)$ for $L/Q$ a nontrivial extension.) To see this implication, we need Hilbert’s irreducibility theorem:

**Theorem 1.7.6** (Hilbert Irreducibility). Let $G$ be a finite group acting on $\{x_1, \ldots, x_n\}$. If $A^n_Q/G$ is rational over $Q$, then there exist infinitely many points $P \in A^n_Q/G$ with $\text{Gal}(Q(Q)/Q) \cong G$ for each $Q \in \pi^{-1}(P)$.

**Example 1.7.7.** Let $G = S_n$ be the symmetric group on $n$ symbols. Let $F = Q(x_1, \ldots, x_n)$. Then $E = F^{S_n} = Q(\sigma_1, \ldots, \sigma_n)$ where $\sigma_i$ is the $i$th elementary symmetric polynomial and the extension $F/E$ has minimal polynomial

$$f(x, \sigma) = x^n - \sigma_1 x^{n-1} + \ldots + (-1)^n \sigma_n$$

for $\sigma = (\sigma_1, \ldots, \sigma_n)$. This defines an $S_n$-cover $A^n_Q \to A^n_Q/S_n$. By Hilbert’s irreducibility theorem, $f(x, s) = x^n - s_1 x^{n-1} + \ldots + (-1)^n s_n$ is irreducible for infinitely many tuples $s = (s_1, \ldots, s_n) \in Q^n$. It is a fact that

$$\# \{ s = (s_1, \ldots, s_n) \in Q^n \mid 1 \leq s_i \leq N, f(x, s) \text{ is irreducible} \} = O(N^{n-1/2} \log N).$$

Using this, one can prove that this polynomial $f(x, \sigma)$ is a generic polynomial for $S_n$-extensions of $Q$.

Similar proofs using Hilbert’s irreducibility theorem can be used to verify generic polynomials for other classes of $G$-extensions. A question related to Noether’s Problem and the regular IGP is:

**Question.** For which groups $G$ and fields $K$ is $A^n_K/G$ rational?

The answer to this was demonstrated to be no the general case: Lenstra provided counterexamples for $K = Q$ and $G = C_5$, and Saltman even showed a counterexample for $K = C$. The question is even unknown for $G = A_n, n > 5$.

Here is a strategy to solving the regular Inverse Galois Problem for a given group $G$.

1. If $G$ has a generating set consisting of $n - 1$ elements, then $G$ arises as a quotient of the topological fundamental group $\pi_1(\mathbb{P}^1_C \setminus \{ P_1, \ldots, P_n \})$, since this is the free group on $n - 1$ elements. Set $\pi^{top}(n) = \pi_1(\mathbb{P}^1_C \setminus \{ P_1, \ldots, P_n \})$. In particular, suppose we can find $g_1, \ldots, g_n \in G$ such that $G = \langle g_1, \ldots, g_n \rangle$, $g_1 \cdots g_n = 1$ and create a map $\varphi : \pi^{top}(n) \to G$ with $\varphi(\gamma_i) = g_i$ for $1 \leq i \leq n$. We call $(g_1, \ldots, g_n) \in G$ a generating $n$-tuple for $G$.

2. Set $\pi(n) = \pi_1(\mathbb{P}^1_Q \setminus \{ P_1, \ldots, P_n \})$ so that by Corollary 1.5.8, $\pi(n) = \hat{\pi}^{top}(n)$. We may even pick $P_1, \ldots, P_n \in \mathbb{P}^1_Q(Q)$.

3. Also set $\Pi(n) = \pi_1(\mathbb{P}^1_Q \setminus \{ P_1, \ldots, P_n \})$ so that Proposition 1.6.1 gives us a short exact sequence of profinite groups

$$1 \to \pi(n) \to \Pi(n) \to G^\varphi_Q \to 1.$$
In particular, can we extend a continuous, surjective homomorphism \( \varphi : \pi(n) \to G \) to a continuous, surjective \( \hat{\varphi} : \Pi(n) \to G \)?

In general, consider a lifting problem of the form

\[
\begin{array}{ccc}
N & \longrightarrow & \Gamma \\
\downarrow & & \downarrow \\
G & & \\
\end{array}
\]

where \( N \) is a normal subgroup of a profinite group \( \Gamma \) and \( G \) is any finite group. The set \( \text{Hom}_{cts}(N, G) \) comes equipped with two actions:

\[
G \times \text{Hom}_{cts}(N, G) \longrightarrow \text{Hom}_{cts}(N, G)
\]

\[
(g, \varphi) \mapsto g\varphi(-)g^{-1}
\]

and

\[
\text{Hom}_{cts}(N, G) \times \Gamma \longrightarrow \text{Hom}_{cts}(N, G)
\]

\[
(\varphi, \sigma) \mapsto (\varphi_{\sigma} : n \mapsto \varphi(\sigma n \sigma^{-1}))
\]

which are compatible in the sense that \( g\varphi_{\sigma} = (g\varphi)_{\sigma} \).

**Lemma 1.7.8.** Let \( S \subseteq \text{Hom}_{cts}(N, G) \) be a set of maps which are stable under the \( G \)- and \( \Gamma \)-actions and such that \( G \) acts freely and transitively on \( S \). Then any \( \varphi \in S \) extends to a continuous homomorphism \( \hat{\varphi} : \Gamma \to G \).

**Proof.** For \( \varphi \in S \) and \( \sigma \in \Gamma \), \( \varphi_{\sigma} \in S \) by stability, but \( \varphi_{\sigma}(n) = \varphi(\sigma n \sigma^{-1}) = g_{\sigma}\varphi(n)g_{\sigma}^{-1} \) for some \( g_{\sigma} \in G \), since \( G \) acts transitively. Moreover, freeness of the \( G \)-action implies this \( g_{\sigma} \) is unique. Define \( \hat{\varphi}(\sigma) = g_{\sigma} \). We claim that this is the desired extension. First, for \( \sigma, \tau \in \Gamma \),

\[
\varphi_{\sigma \tau}(n) = \varphi(\sigma \tau n \tau^{-1} \sigma^{-1}) = \varphi_{\sigma}(\tau n \tau^{-1}) = \varphi_{\sigma} \varphi_{\tau}(n)
\]

so \( \hat{\varphi} \) is a group homomorphism. Next, for \( \sigma \in N \subseteq \Gamma \),

\[
\varphi_{\sigma}(n) = \varphi(\sigma n \sigma^{-1}) = \varphi(\sigma)\varphi(n)\varphi(\sigma)^{-1}
\]

since \( \varphi \) is a homomorphism on \( N \), so by uniqueness we have \( g_{\sigma} = \varphi(\sigma) \). Thus \( \hat{\varphi} \) extends \( \varphi \) as claimed. \( \square \)
Now to access Lemma 1.7.8, we want to construct such a set \( S \subseteq \text{Hom}_{cts}(\pi(n), G) \). First, we need \( S \) to be stable under the action of \( \pi(n) \). Note that for \( \varphi \in S \), having \( \varphi_\sigma \in S \) for all \( \sigma \in \pi(n) \) is equivalent to having \( \varphi_\sigma(n) = \varphi(\sigma)\varphi(n)\varphi^{-1} = (\varphi(\sigma) \cdot \varphi)(n) \), under the left action of \( G \) on \( \varphi \). Let \((g_1, \ldots, g_n)\) be a generating \( n \)-tuple for \( G \). Then we must have \( g_i = \varphi(\gamma_i) \) stable under the conjugacy action of \( G \). In particular, we will specify conjugacy classes \( C_1, \ldots, C_n \) in \( G \) and consider sets

\[
S = \{ \varphi \in \text{Hom}_{cts}(\pi(n), G) \mid \varphi(\gamma_i) \in C_i, (\varphi(\gamma_1), \ldots, \varphi(\gamma_n)) \text{ is a } n \text{-tuple} \}.
\]

By construction, this \( S \) is stable under the \( \pi(n) \)- and \( G \)-actions. Now we impose further conditions on the \( C_i \) to force

(i) transitivity;
(ii) free action;
(iii) \( \Pi(n) \)-stability.

**Definition.** Let \( G \) be a finite group. We say a set of conjugacy classes \( C_1, \ldots, C_n \) in \( G \) is a rigid system if there exists a generating \( n \)-tuple \((g_1, \ldots, g_n)\) in \( G^n \) for \( G \) with \( g_i \in C_i \) and \( G \) acts transitively on the set of all such tuples.

To ensure (i), we pick a rigid system of conjugacy classes in \( G \) represented by the \( \varphi(\gamma_i) \). To guarantee (ii), it is enough to assume that \( Z(G) = 1 \) (\( G \) has trivial center). Next, set

\[
\Sigma = \{ (g_1, \ldots, g_n) \in G^n \mid g_1 \cdots g_n = 1 \text{ and } g_i \in C_i \}
\]

\[
\Sigma = \{ (g_1, \ldots, g_n) \in \Sigma \mid \langle g_1, \ldots, g_n \rangle = G \}.
\]

**Definition.** If \( \Sigma = \overline{\Sigma} \), we say the \( C_i \) are strictly rigid.

\( G \) acts on \( \Sigma \) and \( \overline{\Sigma} \), so supposing \( Z(G) = 1 \), it is clear that rigidity is equivalent to \( |G| = |\Sigma| \). One can even write down formulas relating \( |G| \) and \( |\Sigma| \) (see Serre).

**Example 1.7.9.** Consider \( G = S_n \) for \( n \geq 3 \). For \( 1 \leq k \leq n \), let \( kA \) be the conjugacy class of \( k \)-cycles in \( S_n \). Also, let \( C^{(k)} \) be the conjugacy class of \((1 \ 2 \ \cdots \ k)(k+1 \ \cdots \ n)\). In particular, \( C^{(1)} = (n-1)A \). We claim that the system of conjugacy classes \( \{nA, 2A, C^{(1)}\} \) is strictly rigid. By definition,

\[
\overline{\Sigma} = \{ (x, y, z) \in S_n^3 \mid xyz = 1, x \in nA, y \in 2A, z \in C^{(1)} \}.
\]

Thus determining the size of \( \overline{\Sigma} \) comes down to determining when \( xy \) is an \((n-1)\)-cycle. In general, one can see through elementary calculations that \( xy \in C^{(n-k)} \) if \( y = (y_1 \ y_2) \) with \( y_1 \) and \( y_2 \) exactly \( k \) numbers apart (e.g. if \( y = (2 \ 4) \) then \( k = 2 \)). One then shows that the conjugacy action of \( S_n \) is transitive on these \( 3 \)-tuples, so we get \( |\overline{\Sigma}| = |G| \). But now it is obvious that this system of conjugacy classes is strictly rigid, since by group theory \( \langle nA, 2A \rangle = G \). Hence \( \{nA, 2A, C^{(1)}\} \) is a good choice of conjugacy classes. Even \( \{nA, 2A, C^{(k)}\} \) will work in most situations.

To ensure that the chosen \( S \) is stable under the action of \( \Pi(n) \), we make a final definition.
Definition. A conjugacy class $C$ in $G$ is said to be rational if for every $g \in C$, $g^m \in C$ for every $m \in \mathbb{Z}$ coprime to $|G|$.

Lemma 1.7.10. If $C_1, \ldots, C_n$ are rational conjugacy classes of $G$ and $\varphi : \pi(n) \rightarrow G$ is a continuous homomorphism with $\varphi(\gamma_i) \in C_i$ for each $1 \leq i \leq n$, then for all $\sigma \in \Pi(n)$, we have $\varphi(\gamma_i) \in C_i$.

Proof. Topologically, each $\gamma_i$ generates an inertia group $I_{\bar{Q}_i}$ in $\pi(n)$ for some pro-point $\bar{Q}_i \rightarrow P_i$. Also, for each $\sigma \in \Pi(n)$, $\sigma(\bar{Q}_i)$ is another pro-point over $P_i$ and $\sigma\gamma_i\sigma^{-1}$ generates $I_{\sigma(\bar{Q}_i)}$. But $P_i \in \mathbb{P}^1_{\bar{Q}(Q)}$ so $\bar{Q}_i$ and $\sigma(\bar{Q}_i)$ are both pro-points above the same point $\bar{P}_i \in \mathbb{P}^1_{\bar{Q}}$ which lies over $P_i$. Thus $I_{\bar{Q}_i}$ and $I_{\sigma(\bar{Q}_i)}$ are both stabilizers in $\pi_1(\mathbb{P}^1_{\bar{Q}})$ of points above some $\bar{P}_i$. This implies these groups are conjugate in $\pi(n)$. Write $I_{\sigma(\bar{Q}_i)} = \alpha I_{\bar{Q}_i} \alpha^{-1}$ for $\alpha \in \pi(n)$. Then $\varphi(I_{\sigma(\bar{Q}_i)}) = \varphi(\alpha)\varphi(I_{\bar{Q}_i})\varphi(\alpha)^{-1}$ so $\varphi(I_{\bar{Q}_i})$ and $\varphi(I_{\sigma(\bar{Q}_i)})$ are conjugate in $G$ and cyclically generated by $\varphi(\gamma_i)$ and $\varphi(\sigma\gamma_i\sigma^{-1})$, respectively. So for some $g \in G$ and $m$ prime to $|G|$, we have $\varphi(\sigma\gamma_i\sigma^{-1}) = g \varphi(\gamma_i)^mg^{-1}$. Finally, since $C_i$ is rational, this means $\varphi(\gamma_i)^m \in C_i$ and so $\varphi(\sigma\gamma_i\sigma^{-1}) \in C_i$. Hence $\varphi$ is $\Pi(n)$-stable.$\Box$

Theorem 1.7.11. For a finite group $G$ with $Z(G) = 1$, suppose there exists a rigid system of rational conjugacy classes $C_1, \ldots, C_n$ in $G$. Then for any $P_1, \ldots, P_n \in \mathbb{P}^1_{\bar{Q}(Q)}$, there is a continuous, surjective homomorphism

$$\varphi : \pi_1(\mathbb{P}^1_{\bar{Q}} \setminus \{P_1, \ldots, P_n\}) \longrightarrow G$$

with $\varphi(\gamma_i) \in C_i$ for $1 \leq i \leq n$. In particular, $G = \text{Gal}(F/\mathbb{Q}(T))$ for some regular Galois extension $F/\mathbb{Q}(T)$.

Example 1.7.12. For $S_n$, we saw that $\{nA, 2A, C^{(1)}\}$ is a strictly rigid system. Also, any conjugacy class in $S_n$ is rational since $g \mapsto g^m$ preserves cycle type. In general, one can prove this holds for $\{nA, 2A, C^{(k)}\}$, $1 \leq k \leq n$. This choice of rigid system corresponds to the cover

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

$$x \mapsto x^k(x-1)^{n-k}$$

with $t = 0$ corresponding to $C^{(k)}$, $t = \infty$ corresponding to $nA$ and $t = k^n(k-n)^{n-k}n^{-n}$ corresponding to $2A$. Taking a splitting field of $x^k(x-1)^{n-k}-T$ determines an $S_n$-extension of $\mathbb{Q}(T)$, so the regular Inverse Galois Problem is solvable for $G = S_n$.  

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2 Riemann’s Existence Theorem

2.1 Riemann Surfaces

Riemann surfaces are a mix of the topology of covering spaces and the complex analysis of analytic continuation. The main problem one encounters in the latter setting is that a holomorphic function does not always admit a uniquely defined analytic continuation. The normal strategy then is to employ ‘branch cuts’, but this tactic seems ad hoc and not suited to generalization. Riemann’s idea was to replace the branches of a function with a covering space on which the analytic continuation is an actual function.

**Definition.** Let $X$ be a surface, i.e. a two-dimensional manifold. A **complex atlas** on $X$ is a choice of open covering $\{U_i\}$ of $X$ together with homeomorphisms $\varphi_i : U_i \rightarrow \varphi_i(U_i) \subseteq \mathbb{C}$ such that for each pair of overlapping charts $U_i, U_j$, the transition map

$$\varphi_{ij} := \varphi_j \circ \varphi^{-1}_i : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$

and its inverse are holomorphic. A **complex structure** on $X$ is the choice of a complex atlas, up to holomorphic equivalence of charts, defined by a similar condition to the above. A connected surface which admits a complex structure is called a **Riemann surface**.

**Example 2.1.1.** The complex plane $\mathbb{C}$ is a trivial Riemann surface. Any connected open subset $U$ in $\mathbb{C}$ is also a Riemann surface via the given embedding $U \hookrightarrow \mathbb{C}$.

**Example 2.1.2.** The complex projective line $\mathbb{P}^1 = \mathbb{P}^1_\mathbb{C} = \mathbb{C} \cup \{\infty\}$ admits a complex structure defined by the open sets $U_0 = \mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}$ and $U_1 = \mathbb{P}^1 \setminus \{0\} = \mathbb{C}^\times \cup \{\infty\}$, together with charts

$$\varphi_0 : U_0 \rightarrow \mathbb{C}, z \mapsto z \quad \text{and} \quad \varphi_1 : U_1 \rightarrow \mathbb{C}, z \mapsto \frac{1}{z},$$

where $\frac{1}{\infty} = 0$ by convention. Note that $\varphi_1 \circ \varphi_0^{-1}$ is the function $z \mapsto \frac{1}{z}$ on $\mathbb{C}^\times$ which is holomorphic.

**Example 2.1.3.** Let $\Lambda \subseteq \mathbb{C}$ be a lattice with basis $[\omega_1, \omega_2]$. 

\[\begin{array}{c}
\omega_1 \\
\omega_2
\end{array}\]

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Then the quotient $\mathbb{C}/\Lambda$ admits a complex structure as follows. Let $\pi : \mathbb{C} \rightarrow \mathbb{C}/\Lambda$ be the quotient map and suppose $\Pi \subseteq \mathbb{C}$ is a fundamental domain for $\Lambda$, meaning no two points in $\Pi$ are equivalent mod $\Lambda$. Set $U = \pi(\Pi) \subseteq \mathbb{C}/\Lambda$. Then $\pi|_{\Pi} : \Pi \rightarrow U$ is a homeomorphism, so let $\varphi : U \rightarrow \Pi$ be its inverse. Letting $\{U_i\}$ be the collection of all images under $\pi$ of fundamental domains for $\Lambda$, we get a complex atlas on $\mathbb{C}/\Lambda$ (one can easily check that the transition functions between the $U_i$ are locally constant, hence holomorphic). Topologically, $\mathbb{C}/\Lambda$ is homeomorphic to a torus.

**Definition.** A function $f : U \rightarrow \mathbb{C}$ on an open subset $U$ of a Riemann surface $X$ is holomorphic if for every complex chart $\varphi : V \rightarrow \varphi(V) \subseteq \mathbb{C}$, the function $f \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow U \cap V \rightarrow \mathbb{C}$ is holomorphic.

Let $\mathcal{O}(U)$ denote the set of all holomorphic functions $U \rightarrow \mathbb{C}$.

**Lemma 2.1.4.** For any open set $U$ of a Riemann surface $X$, $\mathcal{O}(U)$ is a commutative $\mathbb{C}$-algebra.

**Proposition 2.1.5** (Holomorphic Continuation). For any open set $U \subseteq X$ of a Riemann surface and any $x \in U$, if $f \in \mathcal{O}(U \setminus \{x\})$ is bounded in a neighborhood of $x$, then $f$ extends uniquely to some $\tilde{f} \in \mathcal{O}(U)$.

More generally, we can define holomorphic maps between two Riemann surfaces.

**Definition.** A continuous map $f : X \rightarrow Y$ between Riemann surfaces is called holomorphic if for every pair of charts $\varphi : U \rightarrow \varphi(U) \subseteq \mathbb{C}$ on $X$ and $\psi : V \rightarrow \psi(V) \subseteq \mathbb{C}$ on $Y$ such that $f(U) \subseteq V$, the map $\psi \circ f \circ \varphi^{-1} : \varphi(U) \rightarrow U \rightarrow V \rightarrow \psi(V)$ is holomorphic. We say $f$ is biholomorphic if it is a bijection and its inverse $f^{-1}$ is also holomorphic. In this case $X$ and $Y$ are said to be isomorphic as Riemann surfaces.

**Lemma 2.1.6.** If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are holomorphic maps between Riemann surfaces, then $g \circ f : X \rightarrow Z$ is also holomorphic.

**Proposition 2.1.7.** Let $f : X \rightarrow Y$ be a holomorphic map. Then for all open $U \subseteq X$, there is an induced $\mathbb{C}$-algebra homomorphism $f^* : \mathcal{O}(U) \rightarrow \mathcal{O}(f^{-1}(U))$

$\psi \mapsto f^* \psi := \psi \circ f$.

**Proof.** The fact that $f^* \psi$ is an element of $\mathcal{O}(f^{-1}(U))$ follows from the above definitions of $\mathcal{O}$ and a holomorphic map between Riemann surfaces. The ring axioms are also easy to verify.

**Theorem 2.1.8.** Suppose $f, g : X \rightarrow Y$ are holomorphic maps between Riemann surfaces such that there exist a set $A \subseteq X$ containing a limit point $a \in A$ and $f|_A = g|_A$. Then $f = g$. 


Proof. Let $U \subseteq X$ be the set of all $x \in X$ with an open neighborhood $W$ on which $f|_W = g|_W$. Then $U$ is open and $a \in U$; we will show it is also closed. If $x \in \partial U$, we have $f(x) = g(x)$ since $f$ and $g$ are continuous. Choose a neighborhood $W \subseteq X$ of $x$ and charts $\varphi : W \to \varphi(W) \subseteq \mathbb{C}$ and $\psi : W' \to \psi(W') \subseteq \mathbb{C}$ in $Y$ with $f(W) \subseteq W'$ and $g(W) \subseteq W'$. Consider

$$F = \psi \circ f \circ \varphi^{-1} : \varphi(W) \to \psi(W') \quad \text{and} \quad G = \psi \circ g \circ \varphi^{-1} : \varphi(W) \to \psi(W').$$

Then $F$ and $G$ are holomorphic and $W \cap U \neq \emptyset$, so we must have $F = G$. Therefore $f|_W = g|_W$, so $x \in U$ after all. This implies $U = X$. \qed

Definition. A meromorphic function on an open set $U \subseteq X$ consists of an open subset $V \subseteq U$ and a holomorphic function $f : V \to \mathbb{C}$ such that $U \setminus V$ contains only isolated points, called the poles of $f$, and $\lim_{x \to p}|f(x)| = \infty$ for every pole $p \in U \setminus V$.

Denote the set of meromorphic functions on $U$ by $\mathcal{M}(U)$. Then $\mathcal{M}(U)$ is a $\mathbb{C}$-algebra, where $f + g$ and $fg$ are defined by meromorphic continuation.

Example 2.1.9. Any polynomial $f(z) = c_0 + c_1z + \ldots + c_nz^n$ is a holomorphic function $\mathbb{C} \to \mathbb{C}$. Viewing $\mathbb{C} \subseteq \mathbb{P}^1$, $f$ is a meromorphic function on $\mathbb{P}^1$ with only a pole at $\infty$ of order $n$ (assuming $c_n \neq 0$).

Example 2.1.10. Any meromorphic function $f \in \mathcal{M}(X)$ may be represented by a Laurent series expansion about any of its poles $p$ by choosing a complex chart $U \to \mathbb{C}$ containing $p$, lifting $z$ to a parameter $t$ on $U$ and writing

$$f(t) = \sum_{n=-N}^{\infty} c_nt^n \text{for some } c_n \in \mathbb{C}.$$

Theorem 2.1.11. Suppose $X$ is a Riemann surface. Then the set of meromorphic functions $\mathcal{M}(X)$ is in bijection with the set of holomorphic maps $X \to \mathbb{P}^1$.

Proof. If $f \in \mathcal{M}(X)$ is a meromorphic function, then setting $f(p) = \infty$ for every pole $p$ of $f$ defines a holomorphic map $f : X \to \mathbb{P}^1$. Indeed, it is clear that $f$ is continuous. Let $P$ be the set of its poles. If $\varphi : U \to \varphi(U) \subseteq \mathbb{C}$ is a chart on $X$ and $\psi : V \to \psi(V) \subseteq \mathbb{C}$ is a chart on $\mathbb{P}^1$ with $f(U) \subseteq V$, then since $f$ is holomorphic on $X \setminus P$, $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is holomorphic on $\varphi(U) \setminus \varphi(P)$. By Proposition 2.1.5, $\psi \circ f \circ \varphi^{-1}$ is actually holomorphic on $\varphi(U)$, so $f$ is a holomorphic map of Riemann surfaces.

Conversely, if $g : X \to \mathbb{P}^1$ is holomorphic, then by Theorem 2.1.8, either $g(X) = \{\infty\}$ or $g^{-1}(\infty)$ is a set of isolated points in $X$. It is then easy to see that $g : X \setminus g^{-1}(\infty) \to \mathbb{C}$ is meromorphic. \qed

Corollary 2.1.12 (Meromorphic Continuation). For any open set $U \subseteq X$ and any $x \in U$, if $f \in \mathcal{M}(U \setminus \{x\})$ is bounded in a neighborhood of $x$, then $f$ extends uniquely to some $f \in \mathcal{M}(U)$.

Proof. Apply Proposition 2.1.5 and Theorem 2.1.11. \qed

Corollary 2.1.13. Any nonzero function in $\mathcal{M}(X)$ has only isolated zeroes. In particular, $\mathcal{M}(X)$ is a field.
Theorem 2.1.14. Let \( f : X \to Y \) be a nonconstant holomorphic map between Riemann surfaces. Then for every \( x \in X \) with \( y = f(x) \in Y \), there exists \( k \in \mathbb{N} \) and complex charts \( \varphi : U \to \varphi(U) \subseteq \mathbb{C} \) of \( X \) and \( \psi : V \to \psi(V) \subseteq \mathbb{C} \) of \( Y \) with \( f(U) \subseteq V \) such that

1. \( x \in U \) with \( \varphi(x) = 0 \) and \( y \in V \) with \( \psi(y) = 0 \).
2. \( F = \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V) \) is given by \( F(z) = z^k \) for all \( z \in \varphi(U) \).

Proof. It is easy to arrange (1) by replacing \((U, \varphi)\) with another chart obtained by composing \( \varphi \) with an automorphism of \( \mathbb{C} \) taking \( \varphi(x) \mapsto 0 \). So without loss of generality assume (1) is satisfied. By Theorem 2.1.8, \( F = \psi \circ f \circ \varphi^{-1} \) is nonconstant. Thus since \( f(0) = 0 \), we may write \( F(z) = z^k g(z) \) for some \( k \geq 1 \) and some \( g \in \mathcal{O}(\varphi(U)) \) with \( g(0) \neq 0 \). Then \( g(z) = h(z)^k \) for some holomorphic function \( h \) on \( \varphi(U) \), and \( H(z) = z h(z) \) defines a biholomorphic map \( \alpha \) of some open neighborhood \( W \subseteq \varphi(U) \) of 0 onto another open neighborhood of 0. Finally, replace \((U, \varphi)\) by \((\varphi^{-1}(W), \alpha \circ \varphi)\). By construction, \( F = \psi \circ f \varphi^{-1} \) is now of the form \( F(z) = z^k \).

Definition. The integer \( k \) for which \( F \) can be written \( F(z) = z^k \) about \( x \in X \) is called the multiplicity of \( f \) at \( x \).

Corollary 2.1.15. If \( f : X \to Y \) is a nonconstant holomorphic map between Riemann surfaces, then \( f \) takes open sets to open sets.

Corollary 2.1.16. If \( f : X \to Y \) is an injective holomorphic map, then \( f \) is biholomorphic \( X \to f(X) \).

Proof. If \( f \) is injective, then locally \( F(z) = z^k \) with \( k = 1 \). Hence \( f^{-1} \) is holomorphic.

Theorem 2.1.17. If \( f : X \to Y \) is a nonconstant holomorphic map and \( X \) is compact, then \( Y \) is also compact and \( f \) is surjective.

Proof. By Corollary 2.1.15, \( f(X) \) is open but since \( X \) is compact, \( f(X) \) is also compact and in particular closed. Therefore \( f(X) = Y \).

Corollary 2.1.18. Every holomorphic function on a compact Riemann surface is constant.

Proof. \( \mathbb{C} \) is not compact, so Theorem 2.1.17 implies that every holomorphic function from a compact space into \( \mathbb{C} \) must be constant.

Corollary 2.1.19. Every meromorphic function on \( \mathbb{P}^1 \) is rational.

Proof. First, note that the only way for such an \( f \in \mathcal{M}(\mathbb{P}^1) \) to have infinitely many poles is if it had a limit point, but then Theorem 2.1.8 would imply \( f \equiv \infty \). Thus \( f \) has finitely many poles, say \( a_1, \ldots, a_n \in \mathbb{P}^1 \); we may assume \( \infty \) is not one of the poles, or else consider the function \( \frac{1}{f} \) instead. For \( 1 \leq i \leq n \), expand \( f \) as a Laurent series about \( a_i \):

\[
f_i(z) = \sum_{j=1}^{m_i} c_{ij}(z - a_i)^{-j} \quad \text{for } c_{ij} \in \mathbb{C}.
\]

Then \( g = f - (f_1 + \ldots + f_n) \) is holomorphic on \( \mathbb{P}^1 \) and thus constant by Corollary 2.1.18 since \( \mathbb{P}^1 \) is compact. This shows \( f \) is rational.
Corollary 2.1.20 (Liouville’s Theorem). Every bounded holomorphic function on \( \mathbb{C} \) is constant.

Proof. By Proposition 2.1.5, \( f \) has a holomorphic continuation to \( \tilde{f} : \mathbb{P}^1 \to \mathbb{C} \), but by Corollary 2.1.18, \( \tilde{f} \) must be constant. \( \square \)

The idea in the rest of the section is to relate holomorphic maps between Riemann surfaces to covering space theory.

Theorem 2.1.21. If \( p : Y \to X \) is a nonconstant holomorphic map between Riemann surfaces then \( p \) is open and has discrete fibres.

Proof. By Corollary 2.1.15, \( p \) is open and Theorem 2.1.8 implies each fibre is discrete. \( \square \)

Let \( p : Y \to X \) be a cover of Riemann surfaces. Traditionally, holomorphic functions \( f : Y \to \mathbb{C} \) are treated as multi-valued functions on \( X \) by setting \( f(x) = \{ f(y_1), \ldots, f(y_n) \} \) where \( p^{-1}(x) = \{ y_1, \ldots, y_n \} \).

Example 2.1.22. Let \( \exp : \mathbb{C} \to \mathbb{C}^\times \) be the exponential map \( z \mapsto e^z \) and \( f = \text{id} : \mathbb{C} \to \mathbb{C} \) the identity map. Then the resulting multi-valued function \( \mathbb{C}^\times \to \mathbb{C} \) is the complex logarithm, which is only defined as a function after making a particular choice of branch of the function. We can describe this idea more cleanly with Riemann surfaces and branched covers.

Definition. Suppose \( p : Y \to X \) is a nonconstant holomorphic map. A ramification point of \( p \) is a point \( y \in Y \) such that for every neighborhood \( V \subseteq Y \) of \( y \), \( p|_V : V \to p(V) \) is not injective. The image \( x = p(y) \) is called a branch point of \( p \). If \( p \) has no ramification points (and hence no branch points), then we call \( p \) an unramified map.

Theorem 2.1.23. A nonconstant holomorphic map \( p : Y \to X \) is unramified if and only if it is a local homeomorphism.

Proof. Suppose \( p \) is unramified. Then for any \( y \in Y \), there exists a neighborhood \( V \subseteq Y \) of \( y \) such that \( p|_V : V \to p(V) \) is injective and open. Therefore \( p|_V \) is a homeomorphism onto \( p(V) \). The converse follows from basically the same argument. \( \square \)

Example 2.1.24. For each \( n \geq 2 \), the map \( p_n : \mathbb{C} \to \mathbb{C} \) defined by \( p_n(z) = z^n \) is ramified at \( 0 \in \mathbb{C} \) and unramified everywhere else. Therefore \( p_n : \mathbb{C}^\times \to \mathbb{C} \) is an unramified cover. Moreover, Theorem 2.1.14 says that every ramified cover of Riemann surfaces \( Y \to X \) is locally of the form \( \mathbb{C} \to \mathbb{C}, z \mapsto z^n \).

Example 2.1.25. The exponential map \( \exp : \mathbb{C} \to \mathbb{C}^\times, z \mapsto e^z \) is an unramified cover. In fact, as in the topological case, \( \exp \) gives a universal cover of \( \mathbb{C} \) via the inverse system of the covers \( p_n \).

Example 2.1.26. The quotient map \( \pi : \mathbb{C} \to \mathbb{C}/\Lambda \) from Example 2.1.3 is an unramified cover of Riemann surfaces.

Theorem 2.1.27. Suppose \( p : Y \to X \) is a local homeomorphism of Hausdorff topological spaces and \( X \) is a Riemann surface. Then \( Y \) admits a unique complex structure making \( p \) a holomorphic map.
Proof. Let \( \varphi : V \to \mathbb{C} \) be a chart of \( X \). Then there exists an open subset \( U \subseteq V \) over which \( p|_U : p^{-1}(U) \to U \) is a homeomorphism. Set \( \tilde{U} = p^{-1}(U) \) and \( \tilde{\varphi} = \varphi \circ p|_U : \tilde{U} \to \mathbb{C} \). Then \( \tilde{\varphi} \) is a complex chart on \( Y \) and the collection \( \{ \tilde{U}, \tilde{\varphi} \} \) obtained in this way forms a complex atlas on \( Y \). Since \( p : Y \to X \) is locally biholomorphic by construction, it is a holomorphic map between Riemann surfaces. Uniqueness is easy to check. \( \square \)

Example 2.1.28. Now that we can view nonconstant holomorphic maps as local homeomorphisms, and in most cases covering spaces, we can rephrase the language of branch cuts as a lifting problem. For example, let \( \exp : \mathbb{C} \to \mathbb{C}^\times \) be the exponential map and suppose \( f : X \to \mathbb{C}^\times \) is a holomorphic map of Riemann surfaces, with \( X \) simply connected. Then by covering space theory, for each fixed \( x_0 \in X \) and \( z_0 \in \mathbb{C} \) such that \( f(x_0) = e^{z_0} \), there exists a unique lift \( F : X \to \mathbb{C} \) making the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & \mathbb{C}^\times \\
\downarrow & & \downarrow \exp \\
\downarrow & & \\
\end{array}
\]

commute. Theorem 2.1.27 can be used to show that any such \( F \) is holomorphic. Moreover, any other lift \( G \) of \( f \) differs from \( F \) by \( 2\pi in \) for some \( n \in \mathbb{Z} \). For the special case of a simply connected open set \( X \subseteq \mathbb{C}^\times \), any lift \( F \) is a branch of the complex logarithm on \( X \).

Example 2.1.29. Similarly, one can construct the complex root functions \( z \mapsto z^{1/n}, n \geq 2 \), as lifts along the cover \( p_n : \mathbb{C}^\times \to \mathbb{C} \).

Let \( f : Y \to X \) be a nonconstant holomorphic map that is proper, i.e. the preimage of any compact set in \( X \) is compact in \( Y \). For each \( x \in X \), define the multiplicity of \( f \) at \( x \) to be

\[
\text{ord}_x(f) = \sum_{y \in f^{-1}(x)} v_y(f)
\]

where \( v_y(f) \) is the multiplicity of \( f \) at \( y \).

Example 2.1.30. If \( f : Y \to X \) is unbranched at \( x \in X \), then \( p^{-1}(x) = \{ y_1, \ldots, y_n \} \) for some \( n \) and \( v_{y_i}(f) = 1 \) for each \( 1 \leq i \leq n \). Thus \( \text{ord}_x(f) = n \).

Theorem 2.1.31. If \( f : Y \to X \) is a proper, nonconstant holomorphic map between Riemann surfaces, then there exists a number \( n \in \mathbb{N} \) such that for every \( x \in X \), \( \text{ord}_x(f) = n \).

Proof. By Theorem 2.1.21, the set \( B \) of ramification points of \( f \) is a closed, discrete subset of \( Y \). Let \( A = f(B) \subseteq X \). Then since \( f \) is proper, \( A \) is also closed and discrete. The restriction \( f|_{Y \setminus B} : Y \setminus B \to X \setminus A \) is unramified, so it is a finite-sheeted covering space; say \( n \) is the number of sheets of \( f|_{Y \setminus B} \), i.e. the size of any fibre \( f^{-1}(x) \) for an unbranched point \( x \in X \). By the above example, \( f \) has multiplicity \( n \) at every \( y \in Y \setminus B \). Suppose \( a \in A \) and write \( f^{-1}(a) = \{ b_1, \ldots, b_k \} \subseteq B \) and \( m_i = v_{b_i}(f) \). For each \( 1 \leq i \leq k \), we may choose neighborhoods \( V_i \subset Y \) of \( b_i \) and \( U_i \subset X \) of \( a \) such that for all \( x \in U_i \setminus \{ a \} \), \( f^{-1}(x) \cap V_i \) consists of exactly \( m_i \) points. Then there is a neighborhood \( U \subseteq U_1 \cap \cdots \cap U_k \) of \( a \) such that \( f^{-1}(U) \subseteq V_1 \cup \cdots \cup U_k \) and for every \( x \in U \cap (X \setminus A) \), \( f^{-1}(x) \) consists of exactly \( m_1 + \cdots + m_k \) points. However we showed that \( |f^{-1}(x)| = n \), so \( n = m_1 + \cdots + m_k \) as required. \( \square \)
**Corollary 2.1.32.** Let $X$ be a compact Riemann surface and $f : X \to \mathbb{C}$ a nonconstant meromorphic function. Then the number of zeroes of $f$ equals the number of poles of $f$, counted with multiplicity.

**Proof.** View $f$ as a holomorphic function $X \to \mathbb{P}^1$. Since $X$ and $\mathbb{P}^1$ are compact, $f$ is a proper map so $\text{ord}_0(f) = \text{ord}_\infty(f)$. But $\text{ord}_0(f)$ is precisely the number of zeroes of $f$, while $\text{ord}_\infty(f)$ is the number of poles. \qed

**Corollary 2.1.33.** Any complex polynomial $f(z) \in \mathbb{C}[z]$ of degree $n$ has exactly $n$ zeroes, counted with multiplicity.

**Proof.** We may view $f$ as a holomorphic map $\mathbb{P}^1 \to \mathbb{P}^1$. Then it is easy to see $\text{ord}_\infty(f) = n$, so once again $\text{ord}_0(f) = n$. \qed

### 2.2 The Existence Theorem over $\mathbb{P}^1_{\mathbb{C}}$

Consider the complex projective line $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, also known as the Riemann sphere, and take a finite set of points $P \subset \mathbb{P}^1_{\mathbb{C}}$. Then $\mathbb{P}^1_{\mathbb{C}} \setminus P$ is both a topological space and an algebraic curve. In the topological case, the first invariant that describes the projective line is $\Gamma := \pi_1^{\text{top}}(\mathbb{P}^1 \setminus P)$, while in the algebraic case we now have an algebraic invariant $\pi_1(\mathbb{P}^1 \setminus P) = \pi_1^{\text{ét}}(\mathbb{P}^1 \setminus P)$. We will prove the following case of Theorem 1.5.5:

**Theorem 2.2.1** (Riemann Existence). For any finite set $P \subset \mathbb{P}^1$, there is an isomorphism $\pi_1^{\text{ét}}(\mathbb{P}^1 \setminus P) \cong \hat{\Gamma}$, where $\hat{\Gamma}$ is the profinite completion of the topological fundamental group $\Gamma = \pi_1^{\text{top}}(\mathbb{P}^1 \setminus P)$.

Set $K = \mathbb{C}(\mathbb{P}^1) = \mathbb{C}(t)$ be the function field of the projective line and let $L(P)/K$ be the maximal Galois extension of $K$ that is unramified outside $P$. Then by the definition of the algebraic fundamental group in Section 1.5, an equivalent statement to Theorem 2.2.1 is that $\text{Gal}(L(P)/K) \cong \hat{\Gamma}$. This statement is quite useful for applications to the Inverse Galois Problem over $K = \mathbb{C}(t)$, as we saw in Section 1.7.

Recall that each point $P_i \in P$ corresponds to a generator in the fundamental group $\Gamma = \langle \gamma_1, \ldots, \gamma_r \mid \gamma_1 \cdots \gamma_r = 1 \rangle$, for a fixed basepoint $x \in \mathbb{P}^1$. 

![Diagram](https://via.placeholder.com/150)
2.2 The Existence Theorem over $\mathbb{P}^1_\mathbb{C}$

Specify a generating $r$-tuple $g_1, \ldots, g_r \in G$ – recall from Section 1.7 that this means $\langle g_1, \ldots, g_r \rangle = G$ and $g_1 \cdots g_r = 1$. Then $G$ is obtained as a quotient of $\Gamma$ by mapping $\gamma_i \mapsto g_i$ for each $1 \leq i \leq r$. We have seen that the inertia groups at the points $P_i \in P$ are cyclicly generated, $I_{P_i} = \langle g_i \rangle$. The following lemma allows us to dispense with worries about “canonical” generators $\gamma_1, \ldots, \gamma_r$ for $\Gamma$.

**Lemma 2.2.2.** If $\Gamma_1$ and $\Gamma_2$ are two finitely generated groups with the same homomorphic images, then their profinite completions $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$ are isomorphic.

On the other hand, with $K = \mathbb{C}(t)$, we have that $\text{Gal}(\overline{K}/K) = \hat{F}_\omega$ is free profinite of (uncountably) infinite rank. By definition $\text{Gal}(\overline{K}/K) = \varprojlim \text{Gal}(L(P)/K)$ where the inverse limit is taken over all finite sets $P \subset \mathbb{P}^1$, ordered by $P \subseteq P'$ which induces $L(P) \subseteq L(P')$ and thus $\text{Gal}(L(P')/K) \to \text{Gal}(L(P)/K)$. In particular, for each containment of finite sets $P \subseteq P'$ we have a diagram

$$
\begin{array}{ccc}
\text{Gal}(L(P')/K) & \longrightarrow & \text{Gal}(L(P)/K) \\
\cong & & \cong \\
\hat{F}_{|P'|-1} & \longrightarrow & \hat{F}_{|P|-1}
\end{array}
$$

The topological side of the proof of Theorem 2.2.1 is essentially contained in the following theorem.

**Theorem 2.2.3.** Let $P \subset \mathbb{P}^1_\mathbb{C}$ be a finite set and $\pi : R \to \mathbb{P}^1_\mathbb{C} \smallsetminus P$ a finite-sheeted Galois covering of topological spaces. Then there exists a smooth projective curve $W$ with a regular map $f : W \to \mathbb{P}^1_\mathbb{C}$ which is unramified outside $P$ and such that the covering

$$W \smallsetminus f^{-1}(P) \to \mathbb{P}^1_\mathbb{C} \smallsetminus P$$

is equivalent to $\pi$. Furthermore, under this identification, the deck transformations of $\pi$ become regular automorphisms of $W \to \mathbb{P}^1_\mathbb{C}$.

The proof requires a sequence of reduction steps, given by the following lemmas.

**Lemma 2.2.4.** For a topological cover $R \to \mathbb{P}^1_\mathbb{C} \smallsetminus P$, $R$ may be given the structure of a compact Riemann surface such that $\pi$ becomes analytic.

**Proof.** Given $R \to \mathbb{P}^1_\mathbb{C} \smallsetminus P$, we first give $R$ the structure of a Riemann surface. This can be done using Theorem 2.1.27, but the following explicit description will be of use in later arguments.
Take $q \in R \setminus \pi^{-1}(P)$ and set $z = \pi(q)$. Since $P$ is finite, we can choose a ball $B(z, \varepsilon)$ which does not contain any point of $P$. Then $\pi^{-1}(B(z, \varepsilon)) = U_1 \amalg \cdots \amalg U_n$ for disjoint open sets $U_i$ for which $\pi|_{U_i} : U_i \to B(z, \varepsilon)$ is a homeomorphism. One of these $U_i$ contains $q$; let this $U_i$ be the coordinate chart defining the surface structure on $R$. For two of these charts, $U = U_i$ and $V = V_j$, the transition maps are just given by lifting the transition maps $B(z, \varepsilon) \to B(z', \varepsilon')$ along the local homeomorphism $\pi$, and these are analytic on the disks $B(z, \varepsilon), B(z', \varepsilon')$ so they are analytic on the $U, V$. This gives $R$ the structure of a Riemann surface, but it may not be compact.

In the next step, we show that there exists a compact Riemann surface $\overline{R}$ with an analytic map $\overline{\pi} : \overline{R} \to \mathbb{P}^1$ and an open embedding $R \hookrightarrow \overline{R}$ such that the diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\pi} & \mathbb{P}^1 \\
\downarrow & & \downarrow \\
\mathbb{P}^1 \setminus P & \hookrightarrow & \mathbb{P}^1 \\
& \xleftarrow{\overline{\pi}} & \\
& \overline{R} & \\
\end{array}
\]

commutes. For any ball $B = B(z, \varepsilon) \subseteq \mathbb{P}^1$, let $B' = B'(z, \varepsilon) = B \setminus \{z\}$ be the disk punctured at its center. Then $\pi^\text{top}(B') \cong \mathbb{Z}$ and in particular $B'$ has (up to isomorphism) only one connected, degree $d$ cover for each $d \geq 1$. Call this cover $\mu_d$. When $B = B(0, 1)$ is the unit ball, this degree $d$ cover is explicitly given by

\[
\mu_d : B' \longrightarrow B' \\
z \longmapsto z^d.
\]

Moreover, $\mu_d$ extends to a branched cover of the entire disk $B$: 

\[
\begin{array}{ccc}
R & \xrightarrow{\pi} & \mathbb{P}^1 \\
\downarrow & & \downarrow \\
\mathbb{P}^1 \setminus P & \hookrightarrow & \mathbb{P}^1 \\
& \xleftarrow{\overline{\pi}} & \\
& \overline{R} & \\
\end{array}
\]
The deck transformations of this cover are also easy to write down: they are generated by \( \tau : z \mapsto \zeta z \) where \( \zeta = \zeta_d \) is a primitive \( d \)th root of unity.

Let \( z_0 \in P \) and take a disk \( D = B(z_0, \varepsilon) \) around \( z_0 \) which does not contain any other points in \( P \). Let \( D' = D \setminus \{ z_0 \} \). Then as above, the preimage of \( D' \) along \( \pi \) can be written as a disjoint union of connected components,

\[
\pi^{-1}(D') = D'_1 \amalg \cdots \amalg D'_m
\]

where \( m \leq n \), such that for each \( 1 \leq i \leq m \), \( \pi|_{D'_i} : D'_i \to D' \) is a covering map. We now construct a Riemann surface \( R_1 \) together with an analytic map \( \pi_1 : R_1 \to \mathbb{P}^1 \setminus (P \setminus \{ z_0 \}) \) and an embedding \( R \hookrightarrow R_1 \) such that \( |R_1 \setminus R| = m \) and the diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\pi} & R_1 \\
\downarrow \pi & & \downarrow \pi_1 \\
\mathbb{P}^1 \setminus P & \xleftarrow{\pi_1} & \mathbb{P}^1 \setminus (P \setminus \{ z_0 \})
\end{array}
\]

commutes. After scaling by some \( \lambda \in \mathbb{C} \), we may replace \( D' \) with \( B' = B(0, 1) \setminus \{ 0 \} \). Then there exists an integer \( d_i \) and a homeomorphism \( \tau_i : D_i \to B' \) making the following diagram commute:

\[
\begin{array}{ccc}
D'_i & \xrightarrow{\tau_i} & B' \\
\downarrow \pi & & \downarrow \mu_{d_i} \\
D' & \xrightarrow{\lambda} & B'
\end{array}
\]

It is an easy exercise to show that \( \tau_i \) is in fact biholomorphic. Notice that while \( \tau_i \) is not unique, it is defined up to a deck transformation of \( \mu_{d_i} \). We now construct \( R_1 \) by taking \( m \) copies of \( B \), say \( B_1, \ldots, B_m \), and the corresponding punctured disks \( B'_1, \ldots, B'_m \) and setting \( T = B_1 \amalg \cdots \amalg B_m \). Obtain \( R_1 \) by gluing \( R \) to \( T \) along the homeomorphisms

\[
D'_1 \amalg \cdots \amalg D'_m \longrightarrow B'_1 \amalg \cdots \amalg B'_m.
\]

Repeating this process, we obtain \( \overline{R} \) after a finite number of steps, i.e. whenever all points of \( \pi^{-1}(P) \) have been “filled in”. 

\[\square\]
Lemma 2.2.5. If $\pi : R \to \mathbb{P}^1 \setminus P$ is a Galois covering, then $\tilde{\pi} : \overline{R} \to \mathbb{P}^1$ is Galois as well.

Proof. Suppose $\sigma$ is a deck transformation of $\pi : R \to \mathbb{P}^1 \setminus P$. We claim $\sigma$ extends uniquely to a biholomorphic automorphism of $\overline{\mathbb{P}}^1$. Let $D', D'_1, \ldots, D'_m$ be as above. On the disjoint union $\pi^{-1}(D') = D'_1 \amalg \cdots \amalg D'_m$, $\sigma$ just acts by permuting the $D'_i$. For one of the $D'_i$, set $\sigma(D'_i) = D'_j$. Then the diagram

\[
\begin{array}{ccc}
D'_i & \xrightarrow{\sigma} & D'_j \\
\downarrow & & \downarrow \\
B'_i & \xrightarrow{\sigma} & B'_j
\end{array}
\]

commutes but by Proposition 2.1.5, since $\sigma$ has bounded image it extends to $B_i \to B_j$ and hence to $\sigma : D_i \to D_j$. Now for any $z_0 \in P$, let $D' = D \setminus \{z_0\}$. Then each $D'_i$ in the disjoint union $\pi^{-1}(D') = D'_1 \amalg \cdots \amalg D'_m$ corresponds to a point $q_i \in R$ such that $\pi(q_i) = z_0$, and which we glue to $R$ to obtain $\overline{R}$. Moreover, for each $q_i, q_j$, we may choose points $x_i \in D'_i$ and $x_j \in D'_j$ such that $x_i \neq q_i$ and $x_j \neq q_j$. Now $\pi$ is Galois, so there exists a deck transformation $\sigma \in \Delta(\pi)$ mapping $\sigma(x_i) = x_j$. Then we must have $\sigma(D'_i) = D'_j$ and hence by the above, the extension $\tilde{\sigma}$ maps $\tilde{\sigma}(q_i) = q_j$. This proves $\tilde{\pi}$ is a Galois covering. \hfill \Box

Suppose $\pi : R \to \mathbb{P}^1 \setminus P$ is a Galois cover and set $G = \Delta(\pi) = \text{Aut}(\pi)$, the group of deck transformations. For any analytic curve $Y$, let $\mathcal{M}(Y)$ be the field of meromorphic functions on $Y$. For $Y = \overline{R}$ as above, we have an action of $G$ on $\mathcal{M}(\overline{R})$ by $(\sigma f)(x) = f(\sigma^{-1}(x))$.

Lemma 2.2.6. For any $f \in \mathcal{M}(\overline{R})$ which is invariant under the $G$-action described above, then $f \in \mathcal{M}(\mathbb{P}^1) \subseteq \mathcal{M}(\overline{R})$. In other words, $f = g \circ \pi$ for some $g \in \mathcal{M}(\mathbb{P}^1)$.

Proof. By the construction in Lemma 2.2.4, there exists a function $g : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f = g \circ \pi$. Thus we need only show that $g$ is meromorphic. Take $p \in \overline{R}$ and consider $q = \pi(p) \in \mathbb{P}^1$. If $q \notin P$, the set of branch points of $\pi$, then locally $f$ is given by $f = g \circ \pi : D(p) \to D(q)$ on local trivializations $D(p) \subseteq \overline{R}$ and $D(q) \subseteq \mathbb{P}^1$ of $p, q$, respectively. But locally, $\pi$ is biholomorphic on these neighborhoods so we can invert $\pi$ and write $g = f \circ \pi^{-1}$ and it follows that $g$ is meromorphic.

If $q \in P$, look at the punctured neighborhoods $\overline{D}(p) = D(p) \setminus \{0\}$, $\overline{D}(q) = D(q) \setminus \{0\}$ and the cover $\pi : \overline{D}(p) \to \overline{D}(q)$. This is not holomorphic, but we know that it’s a degree $m$ map that in fact just corresponds to the multiplication-by-$m$ map on the unit punctured disk $\overline{D}(0, 1)$:

\[
\begin{array}{ccc}
\overline{D}(p) & \xrightarrow{\pi} & \overline{D}(q) \\
\downarrow & & \downarrow \\
\overline{D} & \xrightarrow{[m]} & \overline{D}
\end{array}
\]
Write $f$ as a Laurent series $f(z) = \sum_{j=-N}^{\infty} a_j z^j$ for $a_j \in \mathbb{C}$ and $N \geq 0$. Then we must have $f(\zeta_m z) = f(z)$ for any root of unity $\zeta_m$. Thus $a_j$ is nonzero only when $m \mid j$ and we can rewrite

$$f(z) = \sum_{k=-K}^{\infty} a_{km} z^{km}.$$ 

Let $w = z^m$. Then $w$ is a local parameter on $\overline{D}$ and we have

$$f(z) = \sum_{k=-K}^{\infty} a_{km} w^k = g(w).$$

Thus $g(w)$ is meromorphic as required. \qed

In the next step, we need the so-called analytic version of the existence theorem. A proof will be provided in Section 2.4.

**Theorem 2.2.7** (Riemann Existence – Analytic Version). Let $X$ be a compact Riemann surface. Then for any distinct points $p_1, \ldots, p_n \in X$ and any numbers $c_1, \ldots, c_n \in \mathbb{C}$, possibly not distinct, there exists a meromorphic function $g \in \mathcal{M}(X)$ such that $g(p_i) = c_i$ for each $1 \leq i \leq n$.

Lemma 2.2.6 shows that there exists a finite group $G$ acting on $\mathcal{M}(\overline{\mathbb{R}})$ such that $\mathcal{M}(\overline{\mathbb{R}})^G = \mathcal{M}(\mathbb{P}^1)$. By Serre’s GAGA principle, $\mathcal{M}(\mathbb{P}^1) = \mathbb{C}(t)$, so therefore we have proven that $[\mathcal{M}(\overline{\mathbb{R}}) : \mathbb{C}(t)] < \infty$.

**Theorem 2.2.8.** $\mathcal{M}(\overline{\mathbb{R}})/\mathbb{C}(t)$ is a Galois extension with Galois group $G = \Delta(\pi)$.

**Proof.** The paragraph above implies, by Artin’s Lemma, that $\mathcal{M}(\overline{\mathbb{R}})/\mathbb{C}(t)$ is Galois. Further, $\text{Gal}(\mathcal{M}(\overline{\mathbb{R}})/\mathbb{C}(t)) = G/N$ where $N$ is the kernel of the action of $G$ on $\mathcal{M}(\overline{\mathbb{R}})$. To prove this, pick a point $p \in \mathbb{P}^1$ and write $\pi^{-1}(p) = \{p_1, \ldots, p_n\}$. By the analytic version of Riemann’s existence theorem (2.2.7), there exists a meromorphic $g \in \mathcal{M}(\overline{\mathbb{R}})$ such that the values $g(p_1), \ldots, g(p_n)$ are all distinct. Suppose $\sigma \in N$ is nontrivial. Then $\sigma(p_i) = p_j \neq p_i$ for some $i \neq j$, but $\sigma \in N$ implies $\sigma(g) = g$, or $g \circ \sigma = g$. Thus $g(p_i) = g(\sigma(p_i)) = g(p_j)$, which is impossible. Therefore $N = \{1\}$ so $\text{Gal}(\mathcal{M}(\overline{\mathbb{R}})/\mathbb{C}(t)) = G$ as claimed. \qed

The final step in the proof of Theorem 2.2.3 is contained the following lemma.

**Lemma 2.2.9.** There exists a smooth projective curve $W$ over $\mathbb{C}$ and a biholomorphic isomorphism $\overline{\mathbb{R}} \cong W(\mathbb{C})$.

**Proof.** We know $\mathcal{M}(\overline{\mathbb{R}})$ is a transcendence degree 1 extension of $\mathbb{C}$, so by Corollary 1.3.9 there exists a proper normal curve $W$ with $\mathbb{C}(W) = \mathcal{M}(\overline{\mathbb{R}})$. Abstractly, there is a map $\varphi : \overline{\mathbb{R}} \rightarrow W$ which sends a point $r \in \overline{\mathbb{R}}$ to the associated valuation $v_r$ on the field $\mathcal{M}(\overline{\mathbb{R}})$, which is now identified with a valuation $v$ on $\mathbb{C}(W)$ that is trivial on $\mathbb{C}$, and hence determines a point of $W$.

We may assume $W$ is smooth and projective over $\mathbb{C}$. Let $W \hookrightarrow \mathbb{P}^N$ for large enough $N$. Write $\mathbb{P}^N = \{[x_0, x_1, \ldots, x_N] : x_i \in \mathbb{C}, \text{ some } x_i \neq 0\}$. Then for any $x \in \overline{\mathbb{R}}$, there is an affine patch on which the map $\varphi : \overline{\mathbb{R}} \rightarrow W(\mathbb{C})$ is defined, so on this patch, $\varphi = (f_1, \ldots, f_N)$ where
2.3 The General Case

Let $X$ and $Y$ be Riemann surfaces and let $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ be their corresponding fields of meromorphic functions, as in the previous section. Suppose $\varphi : Y \to X$ is a holomorphic map which is not constant on any connected component. This induces a field homomorphism $\varphi^* : \mathcal{M}(X) \to \mathcal{M}(Y)$ defined by $f \mapsto \varphi^*(f) = f \circ \varphi$.

**Proposition 2.3.1.** If $\varphi : Y \to X$ is a nonconstant holomorphic map of degree $d$ between compact, connected Riemann surfaces, then $\mathcal{M}(Y)/\varphi^*\mathcal{M}(X)$ is a field extension of degree $d$. 

*Proof.* Consider the case $f_i = \frac{x_i}{x_0}$. But by properness, this must in fact be holomorphic. Moreover, Theorem 2.2.7 implies $\varphi$ is also injective. Further, properness also implies $\varphi(\overline{R}) \subseteq W$ is closed. Therefore to prove $\varphi$ is a biholomorphic isomorphism, it suffices to show $W(\mathbb{C})$ is connected.

This is a rather standard property, but we repeat the proof here. Given that $W$ is irreducible, smooth and projective over $\mathbb{C}$, let us assume $W(\mathbb{C}) = M_1 \cup M_2$ for some open sets $M_1, M_2 \subseteq W$. Pick $x_0 \in M_1$. Then by the Riemann-Roch theorem, there exists a rational function $f \in \mathbb{C}(W)$ that has a pole at $x_0$ and is holomorphic everywhere else. View $f : M_2 \to \mathbb{P}^1$; this cannot be constant since $M_2$ is infinite and in particular dense in $W(\mathbb{C})$.

On the other hand, $f(M_2) \subseteq \mathbb{P}^1$ is open and $W(\mathbb{C})$ is compact, so $M_2$ is compact and therefore $f(M_2)$ is also closed. Thus $f(M_2) = \mathbb{P}^1$ so some point $m \in M_2$ has $f(m) = \infty$, contradicting the fact that $f$ was holomorphic on $M_2$. Hence $W(\mathbb{C})$ is connected, so we are finished. 

We derive the following important consequences.

**Corollary 2.2.10.** Let $K = \mathbb{C}(t)$ and $E = \mathbb{C}(W)$. Then the embedding $f^* : K \hookrightarrow E$ satisfies

\[ [E : K] = \deg \pi = |\Delta(\pi)| \]

where $\Delta(\pi)$ is the group of deck transformations of $\pi$.

**Corollary 2.2.11.** $E/K$ is Galois and $|\text{Gal}(E/K)| = |\Delta(\pi)|$.

*Proof.* By Theorem 2.2.3, $\Delta(\pi)$ is a subgroup of $\text{Aut}(E/K)$ but in general we have $|\text{Aut}(E/K)| \leq [E : K]$. Therefore Corollary 2.2.10 implies $\text{Aut}(E/K) = \Delta(\pi)$ and therefore the extension is Galois.

**Corollary 2.2.12.** There is an anti-equivalence of categories

\[
\begin{align*}
\{\text{compact Riemann surfaces}\} & \longrightarrow \{\text{finite étale } \mathbb{C}(t)\text{-algebras}\} \\
Y & \longmapsto \mathcal{M}(Y).
\end{align*}
\]

**Corollary 2.2.13.** Let $K = \mathbb{C}(t)$. Then there is an equivalence of categories

\[
\begin{align*}
\{\text{compact Riemann surfaces}\} & \xrightarrow{\sim} \{\text{finite, continuous left } \text{Gal}(\overline{K}/K)\text{-sets}\}.
\end{align*}
\]

*Proof.* Combine Corollaries 2.2.12 and 0.2.4. 

## 2.3 The General Case

Let $X$ and $Y$ be Riemann surfaces and let $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ be their corresponding fields of meromorphic functions, as in the previous section. Suppose $\varphi : Y \to X$ is a holomorphic map which is not constant on any connected component. This induces a field homomorphism $\varphi^* : \mathcal{M}(X) \to \mathcal{M}(Y)$ defined by $f \mapsto \varphi^*(f) = f \circ \varphi$. 

**Proposition 2.3.1.** If $\varphi : Y \to X$ is a nonconstant holomorphic map of degree $d$ between compact, connected Riemann surfaces, then $\mathcal{M}(Y)/\varphi^*\mathcal{M}(X)$ is a field extension of degree $d$. 

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Proof. Let $f \in \mathcal{M}(Y)$ be a meromorphic function. We first prove $f$ satisfies a polynomial equation of degree $d$ over $\mathcal{M}(X)$. Let $S$ be the set of branch points of the cover. Then for each $x \not\in \varphi(S)$, there is a neighborhood $U \subseteq X$ of $x$ such that $\varphi^{-1}(U) = \bigsqcup_{i=1}^{d} V_i$ for $V_1, \ldots, V_d \subseteq Y$ homeomorphic to $U$ via $\psi_i : U \to V_i$. Set $f_i = f \circ \psi_i$, which is meromorphic on $U$. Define

$$F(t) = \prod_{i=1}^{d} (t - f_i(t)) = t^d + a_{n-1}t^{d-1} + \ldots + a_0$$

for $a_i$ meromorphic on $U$. One shows further that $a_i$ extend meromorphically to all of $X$, so $F(t)$ is defined over $\mathcal{M}(X)$. Lastly, $(\varphi^*F)(f) = 0$ since it restricts to $F(f_i) = 0$ on each $V_i$. This proves the claim.

Now we show that some $f \in \mathcal{M}(Y)$ satisfies an irreducible polynomial of degree $d$ over $\mathcal{M}(X)$. If $x \not\in \varphi(S)$, then $\varphi^{-1}(x) = \{y_1, \ldots, y_d\}$. By the analytic existence theorem (2.2.7), choose $f \in \mathcal{M}(Y)$ which is holomorphic at $y_1, \ldots, y_d$ and has $f(y_i)$ all distinct. Then by the first paragraph, there exist $a_i \in \mathcal{M}(X)$ such that

$$(\varphi^*a_n)f^n + \ldots + \varphi^*a_0 = 0.$$ 

(Here, $n \leq d$.) Suppose one of the $a_i$ has a pole at $x$. Then we may choose a small enough neighborhood about $x$ not containing any branch points such that $f$ is holomorphic and assigns distinct values to all preimages of each point. Replacing $x$ with any of the points of this neighborhood, we may assume each $a_i$ is holomorphic at $x$. Finally, take $g \in \mathcal{M}(Y)$. Then by the primitive element theorem, $\mathcal{M}(X)(f, g) = \mathcal{M}(X)(h)$ for some $h \in \mathcal{M}(Y)$. But then $\mathcal{M}(X)(f) \subseteq \mathcal{M}(X)(h)$ and by the first paragraph, $h$ has degree $\leq d$ over $\mathcal{M}(X)$, so $\mathcal{M}(X)(f) = \mathcal{M}(X)(h)$. In particular, $\mathcal{M}(Y) = \mathcal{M}(X)(f)$ so the proof is complete.

We now generalize the results from Section 2.2 to arbitrary covers of Riemann surfaces.

**Lemma 2.3.2.** Suppose $X$ is a Riemann surface, $S \subset X$ a discrete set of points and $\varphi : Y \to X \setminus S$ a proper holomorphic cover of Riemann surfaces. Then there is a Riemann surface $\overline{Y}$ and proper holomorphic map $\overline{\varphi} : \overline{Y} \to X$ such that $\overline{Y} \setminus \overline{\varphi}^{-1}(S) = Y$.

**Proof.** This is only a slight generalization of the proof of Lemma 2.2.4. For details, see Forster.

**Lemma 2.3.3.** If $\varphi : Y \to X \setminus S$ is a Galois covering, then $\overline{\varphi} : \overline{Y} \to X$ is Galois as well.

**Proof.** A generalization of Lemma 2.2.5.

**Theorem 2.3.4.** Let $X$ be a connected, compact Riemann surface with field of meromorphic functions $\mathcal{M}(X)$. Then for every finite étale $\mathcal{M}(X)$-algebra $A$, there is a compact Riemann surface $Y$ with $\mathcal{M}(Y) \cong A$ and a holomorphic map $Y \to X$.

**Corollary 2.3.5.** For any connected, compact Riemann surface $X$, there is an anti-equivalence of categories

$$\left\{\text{proper holomorphic maps of Riemann surfaces } Y \to X \right\} \cong \{\text{finite étale } \mathcal{M}(X)\text{-algebras}\}.$$
Corollary 2.3.6. Let $X$ be a connected, compact Riemann surface. Then there is an equivalence of categories

$$\left\{ \text{compact Riemann surfaces} \right\} \xrightarrow{\sim} \{ \text{finite, continuous left } \mathcal{M}(X)/\mathcal{M}(X)-\text{sets} \}.$$

We can now give a proof of Theorem 1.5.5.

Proof. Let $K = k(X)$ be the function field of $X$ and $K_U$ the compositum in $K_\bar{s}$ of all finite extensions $L/K$ coming from proper normal curves $Y \to X$ étale over $U$. Then by Theorem 2.3.4, for each of these $L/K$ there is a compact connected Riemann surface $Y_L \to X$ étale over $U$ such that $\mathcal{M}(Y) \cong L$. Let $\widetilde{M}$ be the compositum of these $\mathcal{M}(Y)$. Then $\pi_1(U) = \text{Gal}(K_U/K) \cong \text{Gal}(\widetilde{M}/\mathcal{M}(X))$ so it suffices to show $\text{Gal}(\widetilde{M}/\mathcal{M}(X))$ is the profinite completion of $\hat{G} := \pi_{1,\text{top}}(U(\mathbb{C}))$. But by Corollary 0.1.5, every finite quotient of $\hat{G}$ uniquely determines a finite Galois cover $Y' \to U$; thus a covering $Y \to X$ by Lemma 2.3.2 which is Galois by Lemma 2.3.3; thus a finite Galois field extension $\mathcal{M}(Y)/\mathcal{M}(X)$ by Proposition 2.3.1. It is clear that this correspondence is functorial on both sides, so taking inverse limits, we get the desired identification $\hat{G} = \text{Gal}(\widetilde{M}/\mathcal{M}(X))$. \qed

2.4 The Analytic Existence Theorem

In this section we give a proof of Theorem 2.2.7. First, recall the following results for sheaf cohomology.

Theorem 2.4.1 (Leray). Let $X$ be a space with a sheaf $\mathcal{F}$ and an open cover $U$. If the cover is acyclic, i.e. $\check{H}^1(U, \mathcal{F}) = 0$, then $H^1(X, \mathcal{F}) = \check{H}^1(U, \mathcal{F})$.

For a cover $U = \{U_i\}_{i \in I}$, we can view $U_i$ as a ball $B(a_i, \varepsilon_i)$ – using the biholomorphic map $z$. Writing

$$B(a_i, \varepsilon_i) = \bigcup_{n=1}^{\infty} B \left( a_i, \varepsilon_i - \frac{1}{n} \right),$$

compactness of $X$ implies there is a finite subcover of each $B(a_i, \varepsilon_i)$ and therefore each $U_i$. Taking a finite subcover consisting of such sets, we obtain a cover $W = \{W_j\}_{j=1}^N$ of $X$ such that each $W_j$ is compactly contained in some $U_i$. We say the cover $W$ is compactly contained in $U$.

Proposition 2.4.2. If $W$ is compactly contained in $U$ then the restriction map $\check{H}^1(U, \mathcal{O}) \to \check{H}^1(W, \mathcal{O})$ has finite-dimensional image.

Proof. Consider the sheaf $\mathcal{E}$ of smooth functions on $X$. It is a fact that $H^1(B, \mathcal{E}) = 0$ for any disk $B \subseteq \mathbb{C}$ – in technical terms, $\mathcal{E}$ is a flabby sheaf. Given a cocycle $(f_{ij}) \in Z^1(B, \mathcal{O})$, we can resolve it using smooth functions: $f_{ij} = g_j - g_i$ for $g_i, g_j \in C^0(B, \mathcal{E})$. Now these may not be analytic, but this won’t be a problem. Consider the Cauchy-Riemann operator

$$\frac{\partial}{\partial \bar{z}} : f \mapsto \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).$$
By the Cauchy-Riemann equations, \( \frac{\partial}{\partial \bar{z}} \) is 0 on analytic functions, so in particular
\[
0 = \frac{\partial f_{ij}}{\partial \bar{z}} = \frac{\partial g_j}{\partial \bar{z}} - \frac{\partial g_i}{\partial \bar{z}}.
\]

Now the functions \( \left( \frac{\partial g_j}{\partial \bar{z}} \right) \) agree on any \( U_i \cap U_j \), so we get a lift \( h \in \mathcal{Z}^1(B, \mathcal{E}) \). By Dolbeault’s Lemma from complex analysis, there is some \( g \in \mathcal{E}(B) \) such that \( \frac{\partial g}{\partial \bar{z}} = h \). By the above, \( \frac{\partial g_j - g}{\partial \bar{z}} = 0 \) for all \( j \). Writing \( f_{ij} = (g_j - g) - (g_i - g) \), we now see that \( (f_{ij}) \) is a coboundary. Hence \( \mathcal{H}^1(B, \mathcal{O}) = 0 \).

To extend this to covers, suppose \( W \) is compactly contained in \( U \). Take \( U \in \mathcal{U} \) and \( W \in \mathcal{W} \) with \( W \subseteq U \). Then for any \( f \in \mathcal{O}(U) \), \( f \) may be unbounded but by compactness, \( f|_W \) is bounded. Let \( || \cdot || \) be the \( L^2 \)-norm, so that
\[
||f|| = \int \int_U |f(x + i y)|^2 dx dy.
\]
This may be infinite, but define
\[
L^2(U, \mathcal{O}) = \{ f \in \mathcal{O}(U) : ||f|| < \infty \}.
\]
As spaces, \( L^2(U, \mathcal{O}) \subseteq L^2(U) \), the ordinary Hilbert space of square-integrable complex functions on \( U \). Thus the above says that \( \mathcal{O}(U) \to \mathcal{O}(W) \) has image lying in \( L^2(W, \mathcal{O}) \). We need:

**Lemma 2.4.3.** If \( W' \) is compactly contained in \( U' \subseteq \mathbb{C} \), then for every \( \varepsilon > 0 \) there exists a closed subspace \( A \subseteq L^2(U, \mathcal{O}_C) \) of finite codimension such that for all \( f \in A \),
\[
||f||_{L^2(W')} \leq \varepsilon ||f||_{L^2(U')}.
\]

We next construct a norm on the Čech cochain groups of \( X \). First, each \( U_i \in \mathcal{U} \) is biholomorphic to some disk \( D_i \subseteq \mathbb{C} \) via a map \( z_i \). For \( g \in \mathcal{O}(U_i) \), set \( ||g||_{L^2(U_i)} := ||z_i(g)||_{L^2(D_i)} \). For a 0-cocycle \( \eta = (f_i) \in \mathcal{C}^0(U, \mathcal{O}) \), where \( f_i \in \mathcal{O}(U_i) \), define
\[
||\eta|| := \left( \sum_{i=1}^n ||f_i||_{L^2(U_i)}^2 \right)^{1/2}.
\]
Likewise, for \( \xi = (f_{ij}) \in \mathcal{C}^1(U, \mathcal{O}) \), define
\[
||\xi|| := \left( \sum_{i,j} ||f_{ij}||_{L^2(U_i \cap U_j)}^2 \right)^{1/2}.
\]

Continue in this fashion, we get a norm for every element of \( \mathcal{C}^q(U, \mathcal{O}) \), \( q \geq 0 \). Write \( \mathcal{C}^q_{L^2}(U, \mathcal{O}) \) for the subspace of \( \mathcal{C}^q(U, \mathcal{O}) \) where this norm is finite. Now for the compactly contained covers \( \mathcal{W} \subseteq \mathcal{U} \) and for any \( \varepsilon > 0 \), Lemma 2.4.3 provides a closed, finite-codimension subspace \( A \subseteq \mathcal{Z}^1_{L^2}(U, \mathcal{O}) \) such that for all \( \xi \in A \), \( ||\xi||_{L^2(\mathcal{W})} \leq \varepsilon ||\xi||_{L^2(U)} \).

On the other hand, suppose we have a containment of covers \( \mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{U} \), with compact containments \( W_i \subseteq V_i \subseteq U_i \) for each \( i \).
Lemma 2.4.4. Given compactly contained covers $\mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{U}$, there exists $c > 0$ such that for every cocycle $\xi \in \check{Z}^1_{L^2}(\mathcal{V}, \mathcal{O})$, there is some $\zeta \in \check{Z}^1_{L^2}(\mathcal{U}, \mathcal{O})$ such that

$$\zeta = \xi + d^{(0)}\eta$$

for some $\eta \in \check{C}^0_{L^2}(\mathcal{W}, \mathcal{O})$ with $||\xi||, ||\eta|| \leq c||\xi||$.

Now we play Lemmas 2.4.3 and 2.4.4 against each other. Use Lemma 2.4.4 to “extend” the norm to $\mathcal{U}$ subject to the bound $c > 0$. Let $\varepsilon = \frac{1}{2c}$ and apply Lemma 2.4.3 to produce a closed subspace $A \subseteq \check{Z}^1_{L^2}(\mathcal{U}, \mathcal{O})$ of finite codimension for which $||\xi||_{L^2(\mathcal{V})} \leq \varepsilon||\xi||_{L^2(\mathcal{U})}$ for all $\xi \in A$. Then $S = A^\perp$ is finite dimensional, where $\perp$ is taken with respect to the $L^2$-norm on $\check{Z}^1(\mathcal{U}, \mathcal{O})$.

Consider the restriction map $\check{H}^1(\mathcal{U}, \mathcal{O}) \to \check{H}^1(\mathcal{W}, \mathcal{O})$. We claim the image is equal to the image of $S$. Take $\xi \in \check{Z}^1(\mathcal{U}, \mathcal{O})$ and restrict to a bounded cocycle $\xi|_{\mathcal{V}} \in \check{Z}^1_{L^2}(\mathcal{V}, \mathcal{O})$. Then extend to $\mathcal{U}$ with Lemma 2.4.4 to get $\zeta_0 \in \check{Z}^1_{L^2}(\mathcal{U}, \mathcal{O})$ such that $\zeta_0 = \xi + d^{(0)}\eta$ on $\mathcal{W}$. Write $\zeta = \xi_0 + \sigma_0$ for $\xi_0 \in A$ and $\sigma_0 \in S$. By orthogonal decomposition, $||\xi_0||, ||\sigma_0|| \leq ||\zeta_0||$. Now $\xi_0 + \sigma_0 = \zeta_0 = \xi + d^{(0)}\eta$, so we can restrict $\xi_0$ to $\mathcal{V}$ and extend to $\zeta_1 = \xi_0 + d^{(0)}\eta_1$, with $||\zeta_1|| \leq c\varepsilon||\xi_0|| = \frac{1}{2}||\xi_0||$.

Further decompose $\zeta_1 = \xi_1 + \sigma_1$ for $\xi_1 \in A, \sigma_1 \in S$ and repeat. After $k$ steps, we have

$$\xi_k + (\sigma_0 + \ldots + \sigma_k) = \xi_0 + d^{(0)}(\eta_0 + \ldots + \eta_k)$$

with $||\xi_k|| \leq \frac{1}{2^k}||\xi_0||$

and $||\sigma_i||, ||\eta_i|| \leq \frac{1}{2^i}||\zeta_0||$ for all $i$.

As $k \to \infty$, $\sigma_0 + \ldots + \sigma_k$ converges in $L^2$-norm to some $\sigma \in S$, while $\eta_0 + \ldots + \eta_k$ converges to some $\eta \in \check{C}^0_{L^2}(\mathcal{W}, \mathcal{O})$. Thus we have $\sigma = \xi + d^{(0)}\eta$ on $\mathcal{W}$ and hence the cohomology class of $\xi$ is in the image of $S$.

\(\square\)

Theorem 2.4.5 (Finiteness). If $X$ is a compact Riemann surface with sheaf of analytic functions $\mathcal{O} = \mathcal{O}_X$, then $\dim_{\mathbb{C}} \check{H}^1(X, \mathcal{O}) < \infty$.

Proof. For any open cover $\mathcal{U}$ of $X$, we have a commutative diagram

$$\begin{array}{ccc}
\check{H}^1(X, \mathcal{O}) & \xrightarrow{id} & \check{H}^1(X, \mathcal{O}) \\
\downarrow & & \downarrow \\
\check{H}^1(\mathcal{U}, \mathcal{O}) & \longrightarrow & \check{H}^1(\mathcal{U}, \mathcal{O})
\end{array}$$

By Leray’s theorem (2.4.1), the vertical arrows are isomorphisms but by Proposition 2.4.2, the restriction map along the bottom has finite dimensional image. This implies the identity has finite dimensional image, i.e. $\dim \check{H}^1(X, \mathcal{O}) < \infty$. \(\square\)

As a preliminary step to proving the analytic existence theorem, we have:
Theorem 2.4.6. Let $X$ be a compact Riemann surface. Then for any $a_0 \in X$, there exists a meromorphic function $f \in \mathcal{M}(X)$ that has a pole at $a_0$ and is holomorphic elsewhere.

Proof. Fix $a_0 \in X$ and choose an open cover $\{U_p\}_{p \in X}$ such that $p \in U_p$ and $a_0 \notin U_p$ for any $p \neq a_0$. Since $X$ is compact, there is a finite subcover $X = U_0 \cup U_1 \cup \cdots \cup U_r$ with $a_0 \in U_0 = U_{a_0}$. Set $\mathcal{U} = \{U_i\}_{i=0}^r$. We know the map $U_0 \to B(0,1)$ is biholomorphic.

Now for each $m \in \mathbb{N}$, we define a cocycle $\xi^{(m)} = (f_{ij}^{(m)}) \in \check{Z}^1(\mathcal{U}, \mathcal{O})$ by

\[
\begin{align*}
  f_{ij}^{(m)} &= 0 \quad \text{when } i, j \neq 0 \\
  f_{00}^{(m)} &= 0 \\
  f_{0j}^{(m)} &= -f_{j0}^{(m)} = z^{-m}.
\end{align*}
\]

Note that since $a_0 \in U_0$, $z^{-m}$ is holomorphic. Moreover, we have the restriction map $\check{H}^1(\mathcal{U}, \mathcal{O}) \to \check{H}^1(X, \mathcal{O})$ and $\check{H}^1(X, \mathcal{O})$ is finite dimensional by Theorem 2.4.5, so there is a relation

\[
\sum_{m=1}^{n} c_m \xi^{(m)} = d^{(0)} \eta \tag{1}
\]

where $\eta = (g_i) \in \check{C}^0(\mathcal{U}, \mathcal{O})$ with $g_i \in \mathcal{O}(U_i)$ and $d^{(0)}$ is the 0th boundary operator on the Čech complex. Define $f : X \setminus \{a_0\} \to \mathbb{C}$ by

\[
f(z) = \begin{cases} 
  g_i(z), & z \in U_i \text{ and } i \geq 1 \\
  g_0(z) + \sum_{m=1}^{n} c_m z^{-m}, & z \in U_0.
\end{cases}
\]

By construction, $f(z)$ will be meromorphic so we need only check that the definition is consistent on overlapping elements of the cover $\mathcal{U}$. Suppose $i, j \neq 0$ and take $z \in U_i \cap U_j$. Then by definition the $ij$th part of the left side of (1) is 0 since $(\xi^{(m)})_{ij} = f_{ij}^{(m)} = 0$, whereas on the right it’s $g_i - g_j$ (or the reverse). Hence $g_i = g_j$ on $U_i \cap U_j$. When $j = 0$, we have $\sum_{m=1}^{n} c_m z^{-m} = g_i - g_0$ on $U_i \cap U_0$ so again the definition of $f(z)$ is consistent.

We now prove Theorem 2.2.7.

Proof. Suppose we can find meromorphic functions $h_1, \ldots, h_n \in \mathcal{M}(X)$ such that $h_i(p_j) = \delta_{ij}$ for each $1 \leq i, j \leq n$. Then the function $g(z) = c_1h_1(z) + \cdots + c_nh_n(z)$ will satisfy the desired property. To produce these $h_i$, first use Theorem 2.4.6 to find meromorphic functions $f_1, \ldots, f_n \in \mathcal{M}(X)$ such that for each $i$, $f_i$ has a pole at $p_i$ and is holomorphic everywhere else. Choose $d_{ij} \in \mathbb{C}$ such that $f_i(z) - f_i(p_i) + d_{ij}$ is nonzero whenever we evaluate at $z = p_j$. Then define

\[
h_i(z) = \prod_{j \neq i} \frac{f_i(z) - f_i(p_i)}{f_i(z) - f_i(p_i) + d_{ij}}.
\]

By the choice of $d_{ij}$, these functions are well-defined, meromorphic and clearly $h_i(p_i) = 1$ and $h_i(p_j) = 0$ for all $j \neq i$. Therefore the theorem is proved.
3 Scheme Theory

This chapter follows a short course on scheme theory I gave at the University of Virginia in summer 2017. The topics covered are:

- Affine schemes
- General schemes and their properties
- Fibre products and the fibre functor
- Sheaves of $O_X$-modules
- Quasi-coherent and coherent sheaves
- Differentials
- Group schemes.

The main references for this short course are Szamuely’s *Galois Groups and Fundamental Groups*, Hartshorne’s *Algebraic Geometry* and Vakil’s course notes on algebraic geometry. I have omitted many proofs in favor of covering more material, but these three references contain all the details that were left out.

3.1 Affine Schemes

Hilbert’s Nullstellensatz is an important theorem in commutative algebra which is essentially the jumping off point for classical algebraic geometry (by which we mean the study of algebraic varieties in affine and projective space). We recall the statement here.

**Theorem 3.1.1** (Hilbert’s Nullstellensatz). If $k$ is an algebraically closed field, then there is a bijection

$$\mathbb{A}^n_k \leftrightarrow \text{MaxSpec } k[t_1, \ldots, t_n]$$

$$P = (\alpha_1, \ldots, \alpha_n) \mapsto m_P = (t_1 - \alpha_1, \ldots, t_n - \alpha_n),$$

where $\mathbb{A}^n_k = k^n$ is affine $n$-space over $k$ and MaxSpec denotes the set of all maximal ideals of a ring.

Further, if $f : A \to B$ is a morphism of finitely generated $k$-algebras then we get a map $f^* : \text{MaxSpec } B \to \text{MaxSpec } A$ given by $f^* m = f^{-1}(m)$ for any maximal ideal $m \subset B$. Note that if $k$ is not algebraically closed, $f^{-1}(m)$ need not be a maximal ideal of $A$.

**Lemma 3.1.2.** Let $f : A \to B$ be a ring homomorphism and $p \subset B$ a prime ideal. Then $f^{-1}(p)$ is a prime ideal of $A$.

This suggests a natural replacement for MaxSpec $A$,

$$\text{Spec } A = \{ p \subset A \mid p \text{ is a prime ideal} \}.$$
Definition. An affine scheme is a ringed space with underlying topological space \( X = \text{Spec} \ A \) for some ring \( A \).

In order to justify this definition, I will now tell you the topology on \( \text{Spec} \ A \) and the sheaf of rings making it into a ringed space. For any subset \( E \subseteq A \), define \( V(E) = \{ p \in \text{Spec} \ A \mid E \subseteq p \} \).

**Lemma 3.1.3.** Let \( A \) be a ring and \( E \subseteq A \) any subset. Set \( a = (E) \), the ideal generated by \( E \). Then

(a) \( V(E) = V(a) = V(r(a)) \) where \( r \) denotes the radical of an ideal.

(b) \( V(\{ 0 \}) = \text{Spec} \ A \) and \( V(A) = \emptyset \).

(c) For a collection of subsets \( \{ E_i \} \) of \( A \), \( V(\bigcup E_i) = \bigcap V(E_i) \).

(d) For any ideals \( a, b \subseteq A \), \( V(a \cap b) = V(ab) = V(a) \cup V(b) \).

As a result, the sets \( V(E) \) for \( E \subseteq A \) form the closed sets for a topology on \( \text{Spec} \ A \), called the Zariski topology.

Next, for any prime ideal \( p \subseteq A \), let \( A_p \) denote the localization at \( p \). For any open set \( U \subseteq \text{Spec} \ A \), we define \( \mathcal{O}(U) = \left\{ s : U \to \prod_{p \in U} A_p \mid s(p) \in A_p, \exists p \in V \subseteq U \text{ such that } s(q) = f/g \text{ for all } q \in V, f, g \in A \right\} \).

**Theorem 3.1.4.** \( (\text{Spec} \ A, \mathcal{O}) \) is a ringed space. Moreover,

1. For any \( p \in \text{Spec} \ A \), \( \mathcal{O}_p \cong A_p \) as rings.

2. \( \Gamma(\text{Spec} \ A, \mathcal{O}) \cong A \) as rings.

3. For any \( f \in A \), define the open set \( D(f) = \{ p \in \text{Spec} \ A \mid f \notin p \} \). Then the \( D(f) \) form a basis for the topology on \( \text{Spec} \ A \) and \( \mathcal{O}(D(f)) \cong A_f \) as rings.

**Example 3.1.5.** For any field \( k \), \( \text{Spec} \ k \) is a single point \( \ast \) corresponding to the zero ideal, with sheaf \( \mathcal{O}(\ast) \cong k \).

**Example 3.1.6.** Let \( A = k[t_1, \ldots, t_n] \) be the polynomial ring in \( n \) variables over \( k \). Then \( \text{Spec} \ A = \mathbb{A}^n_k \), the affine \( n \)-space over \( k \). For example, when \( A = k[t] \) is the polynomial ring in a single variable, \( \text{Spec} \ k[t] = \mathbb{A}^1_k \), the affine line.

When \( k = \mathbb{C} \), Hilbert’s Nullstellensatz tells us that all the closed points of \( \mathbb{A}^1_k \) correspond to maximal ideals of the form \( (t - \alpha) \) for \( \alpha \in \mathbb{C} \). But there is also a non-closed, ‘generic point’ corresponding to the zero ideal which was not detected before.

\[
\begin{array}{cccc}
\text{closed points} & -2 & 0 & 1+i \\
\text{Spec} \mathbb{C}[t] & \bullet & \bullet & \bullet \\
\text{generic point} & (t+2) & (t) & (t-(1+i)) & (0)
\end{array}
\]
On the other hand, if \( k = \mathbb{Q} \) or another non-algebraically closed field, the same closed points corresponding to linear ideals \((t - \alpha)\) show up, as well as the generic point corresponding to \((0)\), but there are also points corresponding to ideals generated by higher degree irreducible polynomials like \( t^2 + 1 \). Thus the structure of \( \text{Spec} \mathbb{Q}[t] \) is much different than the algebraically closed case.

Example 3.1.7. Let \( X \) be an algebraic variety over a field \( k \), \( x \in X \) a point and consider the affine scheme \( Y = \text{Spec}(k[\varepsilon]/(\varepsilon^2)) \). We can think of \( Y \) as a “big point” with underlying space * corresponding to the zero ideal, along with a “tangent vector” extending infinitesimally in every direction around *. Then any map \( Y \to X \) determines a unique tangent vector in \( T_xX \), the tangent space of \( X \) at \( x \). This idea is useful in intersection theory. For example, consider the tangency of the \( x \)-axis and the parabola \( y = x^2 \) in \( \mathbb{A}^2_k \):

As a variety, this point \((0, 0)\) corresponds to the quotient of \( k \)-algebras \( k[x, y]/(y, y - x^2) = k[x]/r(x^2) = k[x]/(x) = k \). Thus the information of tangency is lost. However, as an affine scheme, \((0, 0)\) corresponds to \( \text{Spec}(k[x, y]/(y, y - x^2)) = \text{Spec}(k[x]/(x^2)) \) so the intersection information is preserved.

3.2 Schemes

In this section we define a scheme and prove some basic properties resulting from this definition. Recall that a ringed space is a pair \((X, \mathcal{F})\) where \( X \) is a topological space and \( \mathcal{F} \) is a sheaf of rings on \( X \).

**Definition.** A locally ringed space is a ringed space \((X, \mathcal{F})\) such that for all \( P \in X \), there is a ring \( A \) such that \( \mathcal{F}_P \cong A_p \) for some prime ideal \( p \subset A \).

**Example 3.2.1.** Any affine scheme \( \text{Spec} A \) is a locally ringed space by \((1)\) of Theorem 3.1.4. We will sometimes denote the structure sheaf \( \mathcal{O} \) by \( \mathcal{O}_A \).
**Definition.** The category of locally ringed spaces is the category whose objects are locally ringed spaces \((X, \mathcal{F})\) and whose morphisms are morphisms of ringed spaces \((X, \mathcal{F}) \to (Y, \mathcal{G})\) such that for each \(P \in X\), the induced map \(f_P^\# : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}\) is a morphism of local rings, i.e. \((f_P^\#)^{-1}(m_P) = m_{f(P)}\) where \(m_P\) (resp. \(m_{f(P)}\)) is the maximal ideal of the local ring \(\mathcal{O}_{X,P}\) (resp. \(\mathcal{O}_{Y,f(P)}\)).

We are now able to define a scheme.

**Definition.** A scheme is a locally ringed space \((X, \mathcal{O}_X)\) that admits an open covering \(\{U_i\}\) such that each \(U_i\) is affine, i.e. there are rings \(A_i\) such that \((U_i, \mathcal{O}_X|_{U_i}) \cong (\text{Spec } A_i, \mathcal{O}_{A_i})\) as locally ringed spaces.

The category of schemes \(\text{Sch}\) is defined to be the full subcategory of schemes in the category of locally ringed spaces. Denote the subcategory of affine schemes by \(\text{AffSch}\). Also let \(\text{CommRings}\) denote the category of commutative rings with unity.

**Proposition 3.2.2.** There is an isomorphism of categories

\[
\text{AffSch} \xrightarrow{\sim} \text{CommRings}^{\text{op}}
\]

\[
(X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)
\]

\[
(\text{Spec } A, \mathcal{O}) \mapsto A.
\]

**Proof.** (Sketch) First suppose we have a homomorphism of rings \(f : A \to B\). By Lemma 3.1.2 this induces a morphism \(f^* : \text{Spec } B \to \text{Spec } A, p \mapsto f^{-1}(p)\) which is continuous since \(f^{-1}(V(a)) = V(f(a))\) for any ideal \(a \subseteq A\). Now for each \(p \in \text{Spec } B\), define the localization \(f_p : A_{f^{-1}p} \to B_p\) using the universal property of localization. Then for any open set \(V \subseteq \text{Spec } A\), we get a map

\[
f^\#: \mathcal{O}_A(V) \to \mathcal{O}_B((f^*)^{-1}(V)).
\]

One checks that each is a homomorphism of rings and commutes with the restriction maps. Thus \(f^\#: \mathcal{O}_A \to \mathcal{O}_B\) is defined. Moreover, the induced map on stalks is just each \(f_p\), so the pair \((f^*, f^\#)\) gives a morphism \((\text{Spec } B, \mathcal{O}_B) \to (\text{Spec } A, \mathcal{O}_A)\) of locally ringed spaces, hence of schemes.

Conversely, take a morphism of schemes \((\varphi, \varphi^\#) : (\text{Spec } B, \mathcal{O}_B) \to (\text{Spec } A, \mathcal{O}_A)\). This induces a ring homomorphism \(\Gamma(\text{Spec } A, \mathcal{O}_A) \to \Gamma(\text{Spec } B, \mathcal{O}_B)\) but by (2) of Theorem 3.1.4, \(\Gamma(\text{Spec } A, \mathcal{O}_A) \cong A\) and \(\Gamma(\text{Spec } B, \mathcal{O}_B) \cong B\) so we get a homomorphism \(A \to B\). It’s easy to see that the two functors described give the required isomorphism of categories. \(\square\)

**Example 3.2.3.** We saw in Example 3.1.5 that for any field \(k\), \(\text{Spec } k = *\) is a point with structure sheaf \(\mathcal{O}(*) = k\). If \(A = L_1 \times \cdots \times L_r\) is a finite étale \(k\)-algebra, then \(\text{Spec } A = \text{Spec } L_1 \coprod \cdots \coprod \text{Spec } L_r\) is (schematically) a disjoint union of points.

**Example 3.2.4.** Let \(A\) be a DVR with residue field \(k\). Then \(\text{Spec } A = \{0, m_A\}\), a closed point for the maximal ideal \(m\) and a generic point for the zero ideal. There are two open subsets here, \(\{0\}\) and \(\text{Spec } A\), and we have \(\mathcal{O}_A(\{0\}) = k\) and \(\mathcal{O}_A(\text{Spec } A) = A\).

**Example 3.2.5.** If \(k\) is a field and \(A\) is a finitely generated \(k\)-algebra, then the closed points of \(X = \text{Spec } A\) are in bijection with the closed points of an affine variety over \(k\) with coordinate ring \(A\).
Example 3.2.6. Let $A = \mathbb{Z}$ (or any Dedekind domain). Then $\dim A = 1$ and it turns out that $\dim \text{Spec } A = 1$ for some appropriate notion of dimension (see Section 3.3). Explicitly, $\text{Spec } \mathbb{Z}$ has a closed point for every prime $p \in \mathbb{Z}$ and a generic point for $(0)$:

```
  Spec \mathbb{Z}
  \begin{array}{ccccccc}
  \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
  2 & 3 & 5 & 7 & 11 & & (0)
  \end{array}
```

Example 3.2.7. Let $k$ be a field, $X_1 = X_2 = \mathbb{A}^1_k$ two copies of the affine line and $U_1 = U_2 = \mathbb{A}^1_k \setminus \{0\}$, where $0$ is the closed point of $\mathbb{A}^1_k$ corresponding to $(x)$ in $k[x]$. Then we can glue together $X_1$ and $X_2$ along the identity map $U_1 \to U_2$ to get a scheme $X$ which looks like the affine line with the origin “doubled”. Note that $X$ is not affine!

```
X
  \begin{array}{c}
  \mathbb{A}^1_k \setminus \{0\}
  \end{array}
```

3.3 Properties of Schemes

Many definitions in ring theory can be rephrased for schemes. For example:

**Definition.** A scheme $X$ is **reduced** if for all open $U \subseteq X$, $\mathcal{O}_X(U)$ has no nilpotent elements.

**Definition.** A scheme $X$ is **integral** if for all open $U \subseteq X$, $\mathcal{O}_X(U)$ has no zero divisors.

**Lemma 3.3.1.** $X$ is integral if and only if $X$ is reduced and irreducible as a topological space.

**Proof.** ($\implies$) Clearly integral implies reduced, so we just need to prove $X$ is irreducible. Suppose $X = U \cup V$ for open subsets $U, V \subseteq X$. Then $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \times \mathcal{O}_X(V)$ which is not a domain unless one of $\mathcal{O}_X(U), \mathcal{O}_X(V)$ is 0. In that case, $U$ or $V$ is empty, so this shows $X$ is irreducible.

($\impliedby$) Suppose $X$ is reduced and irreducible, but there exists an open set $U \subseteq X$ and $f, g \in \mathcal{O}_X(U)$ with $fg = 0$. Define closed sets

- $C = \{ P \in U \mid f_P \in \mathfrak{m}_P \subset \mathcal{O}_{X,P} \}$
- $D = \{ P \in U \mid g_P \in \mathfrak{m}_P \subset \mathcal{O}_{X,P} \}$

Then by definition of $\mathcal{O}_X$, we must have $C \cup D = U$. By irreducibility, $C = U$ without loss of generality. Thus for any affine open set $U' \subseteq U$ with $U' = \text{Spec } A$, we have $(\mathcal{O}_{X|U'})(D(f)) = 0$ but by (3) of Theorem 3.1.4, $\mathcal{O}_{U'}(D(f)) \cong A_f$, the localization of $A$ at powers of $f$. When $A_f = 0$, $f$ is nilpotent but by assumption this means $f = 0$. Hence $X$ is integral. \qed
Definition. The dimension of a scheme $X$ (or any topological space) is
dim $X = \sup\{n \in \mathbb{N}_0 \mid$ there exists a chain of irreducible, closed sets $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq X\}.$

Proposition 3.3.2. Let $A$ be a noetherian ring. Then $\dim \text{Spec } A = \dim A$, the Krull dimension of $A$.

Be warned that the converse to Proposition 3.3.2 is false in general.

Definition. Let $X$ be any scheme. For a point $P \in X$, we define the codimension of $P$ to be the Krull dimension of the local ring at $P$, that is $\text{codim } P = \dim O_{X,P}.$

Note that by commutative algebra, the codimension of $P$ is equal to the height of the prime ideal $p \subset A$ associated to $P$ for any choice of affine open neighborhood $P \in U = \text{Spec } A$.

Definition. Let $X$ be a scheme. Then
- $X$ is locally noetherian if each stalk $O_{X,P}$ is a local noetherian ring.
- $X$ is noetherian if $X$ is integral and locally noetherian.
- An integral scheme $X$ is normal if each stalk $O_{X,P}$ is integrally closed in its field of fractions.
- $X$ is regular if each $O_{X,P}$ is regular as a local ring, that is, $\dim O_{X,P} = \dim m_P/m_P^2$ as $O_{X,P}/m_P$-vector spaces.

Definition. Let $U \subseteq X$ be an open subset. Then $(U,O_X|_U)$ is a scheme which we call an open subscheme of $X$. The natural morphism $j : U \hookrightarrow X, j^* : O_X \to j_*O_X|_U$ is called an open immersion.

Example 3.3.3. For $X = \text{Spec } A$, let $f \in A$ and recall the open set $D(f)$ defined in Theorem 3.1.4. Then $D(f)$ is an open subscheme of $X$ and the open immersion $D(f) \hookrightarrow X$ corresponds to the natural inclusion of prime ideals $\text{Spec } A_f \hookrightarrow \text{Spec } A$ (this is a property of any localization).

Definition. Let $A \to A/I$ be a quotient homomorphism of rings. Then the induced morphism $\text{Spec}(A/I) \to \text{Spec } A$ is called an affine closed immersion. For a general morphism of schemes $f : X \to Y$, $f$ is called a closed immersion if $f$ is injective, $f(X) \subseteq Y$ is closed and there exists a covering of $X$ by affine open sets $\{U_i\}$ such that each $f|_{U_i} : U_i \to f(U_i)$ is an affine closed immersion. The set $f(X)$ is called a closed subscheme of $Y$.

Definition. Let $X$ be a scheme. The category of schemes over $X$, denoted $\text{Sch}_X$, consists of objects $Y \overset{p}{\to} X$, where $Y$ is a scheme and $p$ is a morphism, and morphisms $Y \to Z$ making the following diagram commute:

$$
\begin{array}{ccc}
Y & \longrightarrow & Z \\
\downarrow & & \downarrow \\
X & \quad & \\
\end{array}
$$
Example 3.3.4. Every scheme $Y$ is a scheme over $\text{Spec} \mathbb{Z}$. Write $Y = \bigcup U_i$ where $U_i \cong \text{Spec} A_i$ for rings $A_i$. Then for each of these there is a canonical homomorphism $\varphi_i : \mathbb{Z} \to A_i$ which induces $\varphi_i^* : \text{Spec} A_i \to \text{Spec} \mathbb{Z}$. Composing these with the isomorphisms $U_i \cong \text{Spec} A_i$, we get a map $Y = \bigcup U_i \to \text{Spec} \mathbb{Z}$.

The fibre of a topological cover $p : Y \to X$ can be interpreted as a fibre product:

$$p^{-1}(x) := \{x\} \times_X Y \rightarrow Y$$

We next construct fibre products in the category $\text{Sch}_X$ and use these to construct the algebraic analogue of a fibre.

**Definition.** Let $X$ be a scheme and $Y, Z$ schemes over $X$. A **fibre product** of $Y$ and $Z$ over $X$, denoted $Y \times_X Z$, is a scheme over both $Y$ and $Z$ such that the diagram

$$
\begin{array}{ccc}
Y \times_X Z & \rightarrow & Y \\
\downarrow & & \downarrow p \\
Y \ltimes & \rightarrow & X \\
\end{array}
$$

commutes and $Y \times_X Z$ is universal with respect to such diagrams, i.e. for any scheme $W$ over both $Y$ and $Z$, the following diagram can be completed uniquely:

$$
\begin{array}{ccc}
W & \rightarrow & Y \times_X Z \\
\downarrow \uparrow \exists! & & \downarrow \rightarrow \\
Y \times_X Z & \rightarrow & Y \ltimes \rightarrow X \\
\end{array}
$$

Given $f : Y \to X$ and any scheme $Z$ over $X$, the induced map $f_Z : Y \times_X Z \to Z$ is called the **base change** of $f$ over $Z$. 

60
Theorem 3.3.5. For any schemes $Y, Z$ over $X$, there exists a fibre product $Y \times_X Z$ which is unique up to unique isomorphism.

Proof. (Sketch) First suppose $X, Y$ and $Z$ are all affine; write $X = \text{Spec } A, Y = \text{Spec } B$ and $C = \text{Spec } Z$. Then $\text{Spec}(B \otimes_A C)$ is a natural candidate for the fibre product in this case. Indeed, the tensor product satisfies the universal property conveyed by the following diagrams:

\begin{center}
\begin{tikzpicture}
\node (A) at (0,0) {$A$};
\node (B) at (1,1) {$B$};
\node (C) at (3,1) {$C$};
\node (D) at (1,2) {$B \otimes_A C$};
\node (E) at (3,2) {$B \otimes_A C$};
\draw[->] (A) to (B);
\draw[->] (A) to (C);
\draw[->] (B) to (D);
\draw[->] (C) to (E);
\draw[->] (D) to (E);
\end{tikzpicture}
\end{center}

Applying the functor $\text{Spec}$ yields the right diagrams with arrows reversed, by Proposition 3.2.2, so the fibre product exists in the affine case.

In general, note that once we construct any fibre product, it will be unique up to unique isomorphism by the universal property, just as in every proof of the solution to a universal mapping problem. Now suppose $X$ and $Z$ are affine and $Y$ is arbitrary. Write $Y$ as a union of affine open subschemes $Y = \bigcup Y_i$. Then the affine case, $Y_i \times_X Z$ exists for each $Y_i$. For each pair of overlapping open sets $Y_i \cap Y_j$, set $U_{ij} = p_{Y_i}^{-1}(Y_i \cap Y_j) \subseteq Y_i \times_X Z$, where $p_{Y_i}$ is the morphism $Y_i \times_X Z \to Y_i$. Then it’s easy to verify that $U_{ij} = (Y_i \cap Y_j) \times_X Z$ (that is, $U_{ij}$ satisfies the definition of the fibre product for $Y_i \cap Y_j$ and $Z$ over $X$), and by the universal property, there are unique isomorphisms $\varphi_{ij} : U_{ij} \to U_{ji}$ for each overlapping pair, commuting with all projections. Therefore we may glue together the fibre products $Y_i \times_X Z$ along the isomorphisms $\varphi_{ij}$ to get a scheme $Y \times_X Z$ which then satisfies the definition of the fibre product for $Y$ and $Z$ over $X$. Now, covering $Z$ by affine open subschemes and repeating this process will construct $Y \times_X Z$ for any schemes $Y, Z$ over an affine scheme $X$.

Finally, let $X$ be an arbitrary scheme and write $X = \bigcup X_i$ for affine open subschemes $X_i$. Let $q : Y \to X$ and $r : Z \to X$ be the given morphisms and for each $X_i$, set $Y_i = q^{-1}(X_i)$ and $Z_i = r^{-1}(X_i)$. By the affine case, each $Y_i \times_{X_i} Z_i$ exists, but any morphisms $f : W \to Y_i$ and $g : W \to Z$ making the diagram
3.3 Properties of Schemes

The fibre product exists in every case.

**Definition.** Let \( p : Y \to X \) be a morphism of schemes, \( x \in X \) a point and \( k(x) = \mathcal{O}_{X,x}/m_x \) the residue field at \( x \), with natural map \( \text{Spec} \ k(x) \to X \). Then the **fibre** of \( p \) at \( x \) is the fibre product \( Y_x := Y \times_X \text{Spec} \ k(x) \).

**Lemma 3.3.6.** Let \( p : Y \to X \) be a morphism of schemes and \( x \in X \) any point. Then

(a) The fibre \( Y_x = Y \times_X \text{Spec} \ k(x) \) is a scheme over the point \( \text{Spec} \ k(x) \).

(b) The underlying topological space of \( Y_x \) is homeomorphic to the set \( p^{-1}(x) \) of preimages of \( x \).

(c) The assignment \( (Y \xrightarrow{p} X) \mapsto Y_x \) is functorial.

**Example 3.3.7.** Let \( A \) be a DVR and consider the affine scheme \( X = \text{Spec} \ A \). We saw in Example 3.2.4 that \( X \) has a closed point \( m = m_A \) and a generic point \( (0) \). For any morphism \( p : Y \to X \), there are two fibres:

- The **generic fibre** \( Y_{(0)} \), which is an open subscheme of \( Y \)
- The **special fibre** \( Y_m \), which is a closed subscheme of \( Y \).

Let \( Y \) be a scheme over \( X \) and define the **diagonal map** \( \Delta : Y \to Y \times_X Y \) coming from the universal property applied to the diagram
Definition. A morphism $Y \to X$ is called separated if the diagonal $\Delta : Y \to Y \times_X Y$ is a closed immersion of schemes. We will say a scheme $Y$ over $X$ is separated if the corresponding morphism $Y \to X$ is separated, and a scheme is simply separated if it is separated as a scheme over $\text{Spec } \mathbb{Z}$.

Example 3.3.8. Let $X = \text{Spec } A$ and $Y = \text{Spec } B$ be affine schemes, with $Y \to X$ a morphism between them. This corresponds to a ring homomorphism $A \to B$ which makes $B$ into an $A$-module. The diagonal $\Delta : Y \to Y \times_X Y$ corresponds to the multiplication map $B \otimes_A B \to B, b \otimes b' \mapsto bb'$, which is a homomorphism of $B$-modules. This map is clearly surjective, so $\Delta$ is a closed immersion. Hence every affine scheme (and morphism of affine schemes) is separated.

Example 3.3.9. One can show that the affine line with the origin doubled (Example 3.2.7) is not separated as a scheme over $\text{Spec } k$.

One perspective on separatedness is that it is a suitable replacement for the Hausdorff condition in algebraic geometry. In the Zariski topology on any scheme, there are always proper open subsets that are dense, so the Hausdorff property usually fails to hold.

Definition. A morphism $f : Y \to X$ is of finite type if there exists an affine covering $X = \bigcup U_i$, with $U_i = \text{Spec } A_i$, such that each $f^{-1}(U_i)$ has an open covering $f^{-1}(U_i) = \bigcup_{j=1}^{n_i} \text{Spec } B_{ij}$ for $n_i < \infty$ and $B_{ij}$ a finitely generated $A_i$-algebra. Further, we say $f$ is a finite morphism if each $n_i = 1$, i.e. $f^{-1}(U_i) = \text{Spec } B_i$ for some finitely generated $A_i$-algebra $B_i$.

Definition. A separated morphism $f : Y \to X$ is proper if it is of finite type and for every morphism $Z \to X$, the base change morphism $Y \times_X Z \to Z$ is closed.

Lemma 3.3.10. Let $X,Y,Z$ be noetherian schemes. Then for any morphism $f : Y \to X$,

(a) If $f$ is an open immersion, then $f$ is separated.

(b) If $f$ is a closed immersion, then $f$ is separated and proper.

(c) If $g : Z \to Y$ is separated (resp. proper) then the composition $f \circ g : Z \to X$ is separated (resp. proper).

(d) If $Z$ is a scheme over $X$, the base change $Y \times_X Z \to Z$ is separated and proper.

(e) If $f$ is finite, then $f$ is proper.

Example 3.3.11. (Projective line over a scheme) Let $X = \text{Spec } A$ be an affine scheme. Then the “affine line” $\mathbb{A}^1_X = \text{Spec } A[t]$ is an affine scheme over $X$. Set $X_1 = \mathbb{A}^1_X = \text{Spec } A[t]$ and $X_2 = A[t^{-1}]$. Then each contains an open subscheme isomorphic to $U = \text{Spec } A[t, t^{-1}]$, coming from applying $\text{Spec}$ to the diagram of $A$-algebras.
3.4 Sheaves of Modules

Gluing along these isomorphic open subschemes gives us a scheme $\mathbb{P}^1_X = X_1 \cup U_2$, called the \textit{projective line over} $X$. In the affine case, we will write $\mathbb{P}^1_A = \mathbb{P}^1_A$.

When $X$ is an arbitrary scheme, $X$ has a covering by open affine subschemes $X = \bigcup U_i$ and a gluing construction defines the projective line $\mathbb{P}^1_X$.

Example 3.3.12. More generally, one defines \textit{projective n-space over} $X$, written $\mathbb{P}^n_X$, by gluing together $n + 1$ copies of affine n-space $\mathbb{A}^n_X = \text{Spec} \ A[t_1, \ldots, t_n]$ along the isomorphic open subsets $X_i = \{ t_i \neq 0 \}$ (when $X$ is affine; in the general case, glue affine subschemes together as in the previous example). The natural morphism $\mathbb{P}^n_X \to X$ generalizes in the following way.

Definition. A morphism of schemes $Y \to X$ is \textbf{projective} if it factors through a closed immersion $Y \to \mathbb{P}^n_X$ for some $n \geq 1$.

Theorem 3.3.13. If $f : Y \to X$ is a projective morphism of noetherian schemes, then $f$ is proper.

The idea behind the proof of Theorem 3.3.13 is to first prove $\mathbb{P}^n_k \to \text{Spec} \ k$ is proper for any $n$, which is a straightforward adaptation of the proof when $\mathbb{P}^n_k$ is considered as a projective algebraic variety. One can then modify this proof for $\mathbb{P}^n_\mathbb{Z} \to \text{Spec} \ \mathbb{Z}$ and then use the properties of proper morphisms in Lemma 3.3.10 to obtain the general result.

3.4 Sheaves of Modules

Through Proposition 3.2.2, we are able to transfer commutative ring theory to the language of affine schemes. In this section, we define a suitable setting for transferring module theory to the language of sheaves and schemes.

Definition. Let $(X, \mathcal{O}_X)$ be a ringed space. A \textbf{sheaf of} $\mathcal{O}_X$-\textbf{modules}, or an $\mathcal{O}_X$-\textbf{module} for short, is a sheaf of abelian groups $\mathcal{F}$ on $X$ such that each $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and for each inclusion of open sets $V \subseteq U$, the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{O}_X(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\
\downarrow & & \downarrow \\
\mathcal{O}_X(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V)
\end{array}
\]
If \( \mathcal{F}(U) \subseteq \mathcal{O}_X(U) \) is an ideal for each open set \( U \), then we call \( \mathcal{F} \) a sheaf of ideals on \( X \).

**Example 3.4.1.** Let \( f : Y \to X \) be a morphism of ringed spaces. Then the pushforward sheaf \( f_* \mathcal{O}_Y \) is naturally an \( \mathcal{O}_X \)-module on \( X \) via \( f^* : \mathcal{O}_X \to f_* \mathcal{O}_Y \). Additionally, the kernel sheaf of \( f^* \), defined on open sets by \( (\ker f^*)(U) = \ker(\mathcal{O}_X(U) \to f_* \mathcal{O}_Y(U)) \), is a sheaf of ideals on \( X \).

Most module terminology extends to sheaves of \( \mathcal{O}_X \)-modules. For example,

- A morphism of \( \mathcal{O}_X \)-modules is a morphism of sheaves \( \mathcal{F} \to \mathcal{G} \) such that each \( \mathcal{F}(U) \to \mathcal{G}(U) \) is an \( \mathcal{O}_X(U) \)-module map. We write \( \text{Hom}_X(\mathcal{F}, \mathcal{G}) = \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \) for the set of morphisms \( \mathcal{F} \to \mathcal{G} \) as \( \mathcal{O}_X \)-modules. This defines the category of \( \mathcal{O}_X \)-modules, written \( \mathcal{O}_X\text{-Mod} \).

- Taking kernels, cokernels and images of morphisms of \( \mathcal{O}_X \)-modules again give \( \mathcal{O}_X \)-modules.

- Taking quotients of \( \mathcal{O}_X \)-modules by \( \mathcal{O}_X \)-submodules again give \( \mathcal{O}_X \)-modules.

- An exact sequence of \( \mathcal{O}_X \)-modules is a sequence \( \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \) such that each \( \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \) is an exact sequence of \( \mathcal{O}_X(U) \)-modules.

- Basically any functor on modules over a ring generalizes to an operation on \( \mathcal{O}_X \)-modules, including \( \text{Hom}, \text{written} \ \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}); \text{direct product} \ \mathcal{F} \otimes \mathcal{G}; \text{tensor product} \ \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}; \text{and exterior powers} \ \wedge^n \mathcal{F} \).

The most important of these constructions for our purposes will be the direct sum operation.

**Definition.** An \( \mathcal{O}_X \)-module \( \mathcal{F} \) is free (of rank \( r \)) if \( \mathcal{F} \cong \mathcal{O}_X^r \) as \( \mathcal{O}_X \)-modules. \( \mathcal{F} \) is locally free if \( X \) has a covering \( X = \bigcup U_i \) such that each \( \mathcal{F}|_{U_i} \) is free as an \( \mathcal{O}_X|_{U_i} \)-module.

**Remark.** The rank of a locally free sheaf of \( \mathcal{O}_X \)-modules is constant on connected components. In particular, the rank of a locally free \( \mathcal{O}_X \)-module is well-defined whenever \( X \) is connected.

**Definition.** A locally free \( \mathcal{O}_X \)-module of rank 1 is called an invertible sheaf.

Let \( A \) be a ring, \( M \) an \( A \)-module and set \( X = \text{Spec} A \). To extend module theory to the language of schemes, we want to define an \( \mathcal{O}_X \)-module \( \mathcal{M} \) on \( X \). To start, for each \( p \in \text{Spec} A \), let \( M_p = M \otimes_A A_p \) be the localization of the module \( M \) at \( p \). Then \( M_p \) is an \( A_p \)-module consisting of ‘formal fractions’ \( \frac{m}{s} \) where \( m \in M \) and \( s \in S = A \setminus p \). For each open set \( U \subseteq X \), define

\[
\mathcal{M}(U) = \left\{ h : U \to \coprod_{p \in U} M_p \mid s(p) \in M_p, \exists p \in V \subseteq U, m \in M, s \in A \text{ with } s(q) = \frac{m}{s} \text{ for all } q \in V \right\}.
\]

(Compare this to the construction of the structure sheaf \( \mathcal{O}_A \) on \( \text{Spec} A \) in Section 3.1. Also, note that necessarily the \( s \in A \) in the definition above must lie outside of all \( q \in V \).)
Proposition 3.4.2. Let $M$ be an $A$-module and $X = \text{Spec } A$. Then $\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-modules on $X$, and moreover,

1. For any $p \in \text{Spec } A$, $\widetilde{M}_p \cong M_p$ as rings.
2. $\Gamma(X, \widetilde{M}) \cong M$ as $A$-modules.
3. For any $f \in A$, $\widetilde{M}(D(f)) \cong M_f = M \otimes_A A_f$ as $A$-modules.

The proof is similar to the proof of Theorem 3.1.4; both can be found in Hartshorne.

Proposition 3.4.3. Let $X = \text{Spec } A$. Then the association

$$A\text{-Mod} \longrightarrow \mathcal{O}_X\text{-Mod}$$

$$M \longmapsto \widetilde{M}$$

defines an exact, fully faithful functor.

Proof. Similar to the proof of Proposition 3.2.2. \hfill \Box

These $\widetilde{M}$ will be our affine model for modules over a scheme $X$. We next define the general notion, along with an analogue of finitely generated modules over a ring.

Definition. Let $(X, \mathcal{O}_X)$ be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if there is an affine covering $X = \bigcup X_i$, with $X_i = \text{Spec } A_i$, and $A_i$-modules $M_i$ such that $\mathcal{F}|_{X_i} \cong \widetilde{M}_i$ as $\mathcal{O}_X|_{X_i}$-modules. Further, we say $\mathcal{F}$ is coherent if each $M_i$ is a finitely generated $A_i$-module.

Example 3.4.4. For any scheme $X$, the structure sheaf $\mathcal{O}_X$ is obviously a coherent sheaf on $X$.

Let $\mathcal{QCoh}_X$ (resp. $\mathcal{Coh}_X$) be the category of quasi-coherent (resp. coherent) sheaves of $\mathcal{O}_X$-modules on $X$.

Theorem 3.4.5. $\mathcal{QCoh}_X$ and $\mathcal{Coh}_X$ are abelian categories.

Example 3.4.6. Let $X = \text{Spec } A$, $I \subseteq A$ an ideal and $Y = \text{Spec } (A/I)$. Then the natural inclusion $i : Y \hookrightarrow X$ is a closed immersion by definition, and it turns out that $i_* \mathcal{O}_Y \cong \widetilde{A/I}$ as $\mathcal{O}_X$-modules, so $i_* \mathcal{O}_Y$ is a quasi-coherent, even coherent, sheaf on $X$.

We next identify the image of the functor $M \mapsto \widetilde{M}$ from Proposition 3.4.3.

Theorem 3.4.7. Let $X = \text{Spec } A$. Then there is an equivalence of categories

$$A\text{-Mod} \cong \mathcal{QCoh}_X.$$ 

Moreover, if $A$ is noetherian, this restricts to an equivalence

$$A\text{-mod} \cong \mathcal{Coh}_X$$

where $A\text{-mod}$ denotes the subcategory of finitely generated $A$-modules.
Proof. (Sketch) The association \( M \mapsto \tilde{M} \) sends an \( A \)-module to a quasi-coherent sheaf on \( X = \text{Spec } A \) by definition of quasi-coherence. Further, one can prove that a sheaf \( F \) on \( X \) is a quasi-coherent \( \mathcal{O}_X \)-module if and only if \( F \cong \tilde{M} \) for an \( A \)-module \( M \). The inverse functor \( \text{QCoh}_X \to A\text{-Mod} \) is given by \( F \mapsto \Gamma(X,F) \).

When \( A \) is noetherian, the above extends to say that \( F \) is coherent if and only if \( F \cong \tilde{M} \) for a finitely generated \( A \)-module \( M \). The rest of the proof is identical. \( \square \)

The following lemma generalizes Example 3.4.6.

**Lemma 3.4.8.** Let \( f : Y \to X \) be a morphism of schemes and let \( \mathcal{G} \) be a quasi-coherent sheaf on \( Y \). Then \( f_*\mathcal{G} \) is a quasi-coherent sheaf on \( X \). Further, if \( \mathcal{G} \) is coherent and \( f \) is a finite morphism, then \( f_*\mathcal{G} \) is also coherent.

Note that the second statement is false in general.

Next, we construct an important example of a quasi-coherent sheaf on a scheme. As always, we begin with a construction on rings.

**Definition.** Let \( A \to B \) be a ring homomorphism. The **module of relative differentials** for \( B/A \) is defined to be

\[
\Omega^1_{B/A} := \mathbb{Z}\langle db | b \in B \rangle/N,
\]

the quotient of the free \( B \)-module generated by formal symbols \( db \) for all \( b \in B \) by the submodule \( N = \langle da, d(b + b') - db - db', d(bb') - b(db') - (db)b' \rangle \). This is the universal \( B \)-module for these three relations.

**Example 3.4.9.** If \( A = k \) is a field and \( B \) is a finitely generated \( k \)-algebra, write \( B = k[t_1, \ldots, t_n]/(f_1, \ldots, f_r) \). Then \( B \) is the coordinate ring of the variety in \( \mathbb{A}^n_k \) cut out by the \( f_i \) and

\[
\Omega^1_{B/k} = k\langle dt_i \rangle / \left\langle \sum_{i=1}^n \frac{\partial f_i}{\partial t_i} dt_i \right\rangle
\]

is the module of total derivatives on this variety.

**Lemma 3.4.10.** Let \( A \to B \) be a ring homomorphism. Then

(a) For any \( A \)-algebra \( C \), \( \Omega^1_{B \otimes_A C/C} \cong \Omega^1_{B/A} \otimes_A C \).

(b) For any multiplicative set \( S \subseteq B \), \( \Omega^1_{S^{-1}B/A} \cong S^{-1}\Omega^1_{B/A} = \omega^1_{B/A} \otimes_B S^{-1}B \).

That is, the functor \( B \mapsto \Omega^1_{B/A} \) commutes with base change and localization. We now give the analogous construction for \( \mathcal{O}_X \)-modules, starting in the affine case.

**Definition.** Let \( A \to B \) be a ring homomorphism. The **sheaf of relative differentials** is the \( \mathcal{O}_B \)-module \( \tilde{\Omega}^1_{B/A} \) on \( \text{Spec } B \) defined by the module \( \Omega^1_{B/A} \).

**Lemma 3.4.11.** Let \( A \to B \) be a ring homomorphism. Then

(a) \( \tilde{\Omega}^1_{B/A} \) is a quasi-coherent sheaf on \( \text{Spec } B \).
(b) For any element \( f \in B \), \( \tilde{\Omega}_{B/A}(D(f)) \cong \Omega_{B_f/A} \) where \( B_f \) is the localization of \( B \) at powers of \( f \).

Now consider the map \( m : B \otimes_A B \to B, m(b_1 \otimes b_2) = b_1 b_2 \) from Example 3.3.8. Let \( I \) be the kernel of \( m \). Since \( m \) is surjective, this means \( B \otimes_A B/I \cong B \). Since \( I \) acts trivially on \( I/I^2 \), there is an induced module action of \( B \otimes_A B/I \) on \( I/I^2 \), and thus a corresponding \( B \)-module structure on \( I/I^2 \). The proof of the following fact can be found in Eisenbud, among other places.

**Lemma 3.4.12.** \( \Omega_{B/A} \cong I/I^2 \).

**Example 3.4.13.** In Example 3.4.9, the isomorphism \( \Omega_{B/k} \cong I/I^2 \) is induced by the map

\[
B \to \Omega_{B/k}, \\
t_i \mapsto dt_i.
\]

Let \( Y \to X \) be a separated morphism of schemes and let \( \Delta : Y \to Y \times_X Y \) be the corresponding diagonal. This induces a morphism of sheaves \( \Delta^\#: \mathcal{O}_{Y \times_X Y} \to \Delta_* \mathcal{O}_X \) which has kernel sheaf \( \mathcal{I} \) (a sheaf on \( Y \times_X Y \)). This \( \mathcal{I} \) in fact defines the closed subscheme \( \Delta(Y) \subseteq Y \times_X Y \).

**Lemma 3.4.14.** For \( Y \to X \), \( \Delta \) and \( \mathcal{I} \) as above,

(a) \( \mathcal{O}_{\Delta(Y)} \cong \mathcal{O}_{Y \times_X Y}/\mathcal{I} \) as sheaves on \( \Delta(Y) \).

(b) \( \mathcal{I}/\mathcal{I}^2 \) is an \( \mathcal{O}_{\Delta(Y)} \)-module.

Identifying \( Y \) with its image \( \Delta(Y) \) in the fibre product \( Y \times_X Y \) allows us to define a sheaf analogue of the module of differentials by pulling back \( \mathcal{I}/\mathcal{I}^2 \).

**Definition.** For a separated morphism \( Y \to X \), the **sheaf of relative differentials** \( \Omega_{Y/X} \) is the pullback:

\[
\begin{array}{ccc}
\Omega_{Y/X} & \longrightarrow & \mathcal{I}/\mathcal{I}^2 \\
\downarrow & & \downarrow \\
Y & \leftarrow & \Delta(Y)
\end{array}
\]

**Remark.** \( \Omega_{Y/X} \) is a sheaf of \( \mathcal{O}_Y \)-modules on \( Y \). Moreover, on an affine patch \( \text{Spec } B \subseteq Y \), the sheaf of relative differentials restricts to \( \tilde{\Omega}_{B/A} \cong I/I^2 \) for some rings \( A \to B \). In particular, \( \Omega_{Y/X} \) is quasi-coherent.

We finish the section by discussing some applications of relative differentials. Let \( k \) be a field and \( X \) a connected scheme over \( \text{Spec } k \) of finite type and dimension \( d \).

**Definition.** A \( k \)-scheme \( X \) is **smooth** over \( k \) if the sheaf of relative differentials \( \Omega_{X/\text{Spec } k} \) is locally free of rank \( d \).
Theorem 3.4.15. Assume $k$ is algebraically closed and $\dim X = d$. Then the following are equivalent:

1. $X$ is smooth over $k$.
2. For every affine open subset $U \cong \text{Spec}(k[t_1, \ldots, t_n]/(f_1, \ldots, f_m))$, the Jacobian matrix
   \[ J_P = \left( \frac{\partial f_i}{\partial t_j}(P) \right) \]
   has rank $n - d$ at all closed points $P \in X$.
3. For every closed point $P \in X$, the stalk $\mathcal{O}_{X,P}$ is a regular local ring.

This amazing result unites geometry ($\Omega_X/\text{Spec}k$ being locally free), algebra ($\mathcal{O}_{X,P}$ being regular) and analysis (the vanishing of partial derivatives in $J_P$) into one concept of smoothness. Unfortunately, the theorem fails when $k$ is not algebraically closed, but it still hints at a deep intersection between all three areas of math.

Theorem 3.4.16. Let $X$ be any variety over an algebraically closed field. Then there is a dense open subset which is smooth.

Recall that a finitely generated $k$-algebra $A$ is finite étale if $A = L_1 \times \cdots \times L_r$ for finite, separable extensions $L_i/k$.

Proposition 3.4.17. A finitely generated $k$-algebra $A$ is finite étale if and only if $\Omega^1_{A/k} = 0$.

Proof. See Eisenbud.

Finally, we give an important construction relating the geometry and algebra of a smooth scheme. Let $X$ be smooth over a field $k$ and let $n = \dim X$.

Definition. The canonical sheaf of $X$ is the $n$th exterior power sheaf
\[ \omega_X := \bigwedge^n \Omega^1_{X/\text{Spec}k}. \]

Here are some interesting facts about the canonical sheaf:

- $\omega_X$ is an invertible sheaf on $X$.
- One can define the geometric genus of $X$ by $g(X) := \dim_k \Gamma(X, \omega_X)$. Then when $X$ is a curve (a smooth scheme of dimension 1), $g(X)$ is equal to the arithmetic genus of $X$, another important algebraic invariant. These genera are not equal in general.
- If $X$ is a curve over $\mathbb{C}$, then the genus of the corresponding Riemann surface $X(\mathbb{C})$ is precisely $g(X)$, so the canonical bundle carries important topological information about $X(\mathbb{C})$.  

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3.5 Group Schemes

Recall that a group is a set $G$ together with three maps,

$$
\begin{align*}
\mu : G \times G &\to G \quad \text{(multiplication)} \\
e : \{e\} &\hookrightarrow G \quad \text{(identity)} \\
i : G &\to G \quad \text{(inverse)}
\end{align*}
$$

satisfying associativity, identity and and inversion axioms. This generalizes to the notion of a group object in an arbitrary category $\mathcal{C}$. We state the definition for scheme categories here.

**Definition.** A group scheme over a scheme $X$ is a scheme $G \to X$ with morphisms

$$
\begin{align*}
\mu : G \times G &\to G \\
e : \{e\} &\hookrightarrow G \\
i : G &\to G
\end{align*}
$$

satisfying the following axioms:

1. (Associativity) $\mu \circ (id \times \mu) = \mu \circ (\mu \times id)$.
2. (Identity) $\mu \circ (id \times e) = id = \mu \circ (e \times id)$.
3. (Inversion) $\mu \circ (id \times i) = e \circ p = \mu \circ (i \times id)$.

**Definition.** A group scheme $G$ over $X$ is finite if $p : G \to X$ is a finite morphism, and flat if $p : G \to X$ is a flat morphism, i.e. $p_* \mathcal{O}_G$ is a sheaf of flat $\mathcal{O}_X$-modules.

**Remark.** When $G$ is a finite group scheme over $X$, flat is equivalent to locally free.

The following describes an equivalent, and equally important, perspective on group schemes using the language of functors.

**Proposition 3.5.1.** Let $G$ be a scheme over $X$. Then a choice of group scheme structure on $G$ is equivalent to a compatible choice of group structure on the sets $\text{Hom}_X(Y, G)$ for all schemes $Y$ over $X$. That is, a group scheme structure is a functor $\text{Sch}_X \to \text{Groups}$ such that the composition with the forgetful functor $\text{Groups} \to \text{Sets}$ is representable.

**Proof.** This is a basic application of Yoneda’s Lemma.

**Example 3.5.2.** Let $G$ be a finite group of order $n$ and let $X$ be any scheme. The constant group scheme on $G$ over $X$ is defined as $G_X := \prod_{i=1}^n X$, with projection map induced by the identity on each disjoint copy of $X$. Multiplication $G_X \times_X G_X \to G_X$ is given by sending $(P, Q)$, where $P = x \in X$ in the $g_i$th component of $\prod_{i=1}^n X$ and $Q = x \in X$ in the $g_j$th component $(g_i, g_j \in G)$ to the corresponding point $PQ = x$ in the $g_ig_j$th component of the disjoint union. (Note that $P$ and $Q$ must correspond to the same point $x \in X$ by definition of the fibre product – draw the diagram!) Similarly, the identity is the morphism taking $X$ onto the copy of $X$ indexed by the identity element $e_G \in G$, $e : X \to X_{e_G} \subseteq \prod_{i=1}^n X$. Finally, the
inversion morphism \( i : G_X \to G_X \) takes \( P \) in the \( g_i \)th component to the corresponding point \( P^{-1} \) in the \( g_{i^{-1}} \)th component.

This definition can be extended to an arbitrary group \( G \). Note that when \( G \) is finite, \( G_X \) is a finite (étale) group scheme over \( X \). A special case of this is the trivial group scheme \( \{1\}_X = X \). Thus every scheme is a group scheme.

**Example 3.5.3.** Let \( X = \text{Spec } A \) be affine and recall the affine line \( A_X^1 = \text{Spec } A[t] \) constructed in Example 3.3.11. Then \( A_X^1 \) is an affine group scheme over \( X \), denoted \( \mathbb{G}_a \), called the **additive group scheme** over \( X \), with morphisms induced by the following ring homomorphisms:

\[
\begin{align*}
  t &\longmapsto t \otimes 1 + 1 \otimes t \\
e^* : A[t] &\longrightarrow A \\
  t &\longmapsto 0 \\
  t &\longmapsto -t.
\end{align*}
\]

Notice that these are just the axioms of a Hopf algebra! This construction generalizes to the affine line over a non-affine scheme as well.

**Example 3.5.4.** For \( X = \text{Spec } A \), the **multiplicative group scheme** over \( X \) is \( \mathbb{G}_m := \text{Spec } A[t, t^{-1}] \) with morphisms induced by

\[
\begin{align*}
\mu^* : A[t, t^{-1}] &\longrightarrow A[t, t^{-1}] \otimes A[t, t^{-1}] \\
  t &\longmapsto t \otimes t \\
  t^{-1} &\longmapsto t^{-1} \otimes t^{-1} \\
e^* : A[t, t^{-1}] &\longrightarrow A \\
  t, t^{-1} &\longmapsto 1 \\
i^* : A[t, t^{-1}] &\longrightarrow A[t, t^{-1}] \\
  t &\longmapsto t^{-1} \\
  t^{-1} &\longmapsto t.
\end{align*}
\]

Note that when \( A = k \) is a field, these are just the schematic versions of the algebraic groups \( \mathbb{G}_{a,k} \) and \( \mathbb{G}_{m,k} \). This shows that group schemes are a direct generalization of algebraic groups.

**Example 3.5.5.** For \( X = \text{Spec } A \), the **\( n \)th roots of unity** form a group scheme defined by \( \mu_n = \text{Spec}(A[t, t^{-1}]/(t^n - 1)) \). This is a finite group subscheme of \( \mathbb{G}_m \).

**Example 3.5.6.** If \( \text{char } A = p > 0 \), then \( \alpha_p = \text{Spec}(A[t]/(t^p)) \) defines a group scheme over \( \text{Spec } A \) which is isomorphic as a scheme to \( \text{Spec } A \), but not as a group scheme!

**Example 3.5.7.** The Jacobian of a curve is a group scheme. In particular, an elliptic curve (a dimension 1 scheme with a specified point \( O \)) is a group scheme with identity \( O \).
Definition. Let $G \to X$ be a finite, flat group scheme. A left $G$-torsor is a scheme $Y \to X$ with $q$ finite, locally free and surjective, together with a group action $\rho : G \times_X Y \to Y$, which satisfies:

1. $\rho \circ (e \times id_Y)$ is equal to the projection map $X \times_X Y \to Y$.
2. $\rho \circ (id_G \times \rho) = \rho \circ (\mu \times id_Y) : G \times_X Y \to G \times_X Y \to Y$.
3. $\rho \times id_Y : G \times_X Y \to Y \times_X Y$ is an isomorphism of $X$-schemes.

Right $G$-torsors are defined similarly.

Remark. The idea is that a $G$-torsor is exactly the same as $G$, except we have forgotten the “identity point” $e$ (which is a morphism, not a point).

Example 3.5.8. Left multiplication defines a $G$-torsor structure on $G$ itself.

Example 3.5.9. Let $k$ be a field and $m$ an integer such that $\text{char } k \nmid m$. Let $\mu_m = \text{Spec}(k[t, t^{-1}] / (t^m - 1))$ be the group scheme of $m$th roots of unity over $\text{Spec } k$. Take $a \in k^\times / (k^\times)^m$ (that is, $a$ is not an $m$th power in $k$) and set $L = k(\sqrt[m]{a})$, which is a finite field extension of $k$. We claim $Y = \text{Spec } L$ is a $\mu_m$-torsor over $\text{Spec } k$.

Let $\zeta$ be a primitive $m$th root of unity. Up to finite base change, we may assume $\zeta \in k$. Define

$$\rho^* : k(\sqrt[m]{a}) \to k[t, t^{-1}] / (t^m - 1) \otimes_k k(\sqrt[m]{a})$$

$$\sqrt[m]{a} \mapsto \zeta \otimes \sqrt[m]{a}.$$ 

This defines a morphism $\rho : \mu_m \times_{\text{Spec } k} Y \to Y$ and one can prove that it satisfies axioms (1) and (2) of a torsor by checking the corresponding properties for $\rho^*$. When char $k \nmid m$, $\mu_m$ is a reduced scheme over $\text{Spec } k$ and it’s easy to see that $\mu_m \cong (\mathbb{Z}/m\mathbb{Z})_{\text{Spec } k}$, the constant group scheme on $\mathbb{Z}/m\mathbb{Z}$ over $\text{Spec } k$. Moreover, $L/k$ is a Galois extension with $\text{Gal}(L/k) \cong \mathbb{Z}/m\mathbb{Z}$, $L \otimes_k L \cong \prod_{i=1}^m L$ and this has a corresponding Galois action which induces the isomorphism $L \otimes_k L \cong k[t, t^{-1}] / (t^m - 1) \otimes_k L$. Applying $\text{Spec}$ again, we get the isomorphism $\mu_m \times_{\text{Spec } k} Y \cong Y \times_{\text{Spec } k} Y$ so $Y$ is indeed a $\mu_m$-torsor.

Using Artin-Schreier theory, one can show that every $\mu_m$-torsor arises in this way, i.e. as $\text{Spec } L$ for $L = k(\sqrt[m]{a})$. 

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4 Fundamental Groups of Schemes

In this chapter we define the étale fundamental group of a scheme. This will generalize the construction for curves in Section 1.5 and allow us to fully translate the topological theory of covering spaces to an algebraic setting. We first recall the so-called Galois theory for covering spaces.

Let \( p: Y \rightarrow X \) be a topological cover and let \( \text{Aut}(Y/X) \) be the group of automorphisms of the cover, i.e. the homeomorphisms \( Y \rightarrow Y \) making the diagram commute. If \( p \) is a finite, degree \( n \) cover, then \( \# \text{Aut}(Y/X) \leq n \). When \( \# \text{Aut}(Y/X) = n \), we call \( p \) a regular, or Galois cover. In this case, we write \( \text{Gal}(Y/X) = \text{Aut}(Y/X) \).

**Proposition 4.0.1.** For a covering space \( p: Y \rightarrow X \) of degree \( n \), there is a bijective correspondence

\[
\{ \text{subgroups of } \text{Gal}(Y/X) \} \longleftrightarrow \{ \text{intermediate covers } Y \rightarrow Y' \rightarrow X \}
\]

\[
H \mapsto (Y/H \xrightarrow{p} X)
\]

\[
\text{Aut}(Y/Y') \mapsto (Y' \rightarrow X).
\]

**Example 4.0.2.** Let \( X = Y = \mathbb{C} \setminus \{0\} \), the punctured complex plane, with complex parameters \( x \) and \( y \), respectively. Then an example of a Galois cover of degree \( n \) is the \( n \)th power map \( f: Y \rightarrow X, y \mapsto y^n \). For example, when \( n = 2 \) the real coordinates of this cover take the form of a parabola covering the line in all but one point:

In general, \( f: y \mapsto y^n \) is a Galois cover with Galois group \( \text{Gal}(Y/X) \cong \mathbb{Z}/n\mathbb{Z} \) generated by the automorphism \( \varphi: Y \rightarrow Y, y \mapsto \zeta_n y \), where \( \zeta_n \) is a primitive \( n \)th root of unity. Viewing \( X \) and \( Y \) as complex varieties, \( f \) and \( \varphi \) are in fact morphisms of varieties which determine corresponding field embeddings:

\[
\mathbb{C}(x) \hookrightarrow \mathbb{C}(y) \quad \text{and} \quad \mathbb{C}(y) \hookrightarrow \mathbb{C}(y), \quad x \mapsto y^n \quad y \mapsto \zeta_n y.
\]

We want to generalize this to any variety or scheme.
4.1 Galois Theory for Schemes

Let $X$ and $Y$ be schemes.

**Definition.** A finite morphism $\varphi : Y \to X$ is **locally free** if the direct image sheaf $\varphi_* \mathcal{O}_Y$ is locally free of finite rank as an $\mathcal{O}_X$-module.

**Definition.** Suppose $\varphi : Y \to X$ is finite and locally free. If the fibre $Y_P$ over a point $P \in X$ is equal to the ring spectrum $\text{Spec} \ A$ of some finite étale $\kappa(P)$-algebra $A$, we say $\varphi$ is **étale** at $P$. If this holds for each point $P \in X$, we call $\varphi$ a **finite étale morphism**. Finally, if $\varphi$ is also surjective, it is called a **finite étale cover** of schemes.

**Remark.** Note that if $A$ is a local ring, a finitely generated $A$-module $M$ is free if and only if it is flat. It follows that $\varphi : Y \to X$ is locally free if and only if $\varphi_* \mathcal{O}_Y$ is a sheaf of flat $\mathcal{O}_X$-modules.

**Lemma 4.1.1.** Let $\varphi : Y \to X$ be a finite étale morphism. Then

(a) If $\psi : Z \to Y$ is finite étale, then so is $\varphi \circ \psi : Z \to X$.

(b) If $\psi : Z \to Y$ is any morphism, then the base change $Y \times_X Z \to Z$ is finite étale.

The category $\text{Fét}_X$ is defined to be the full subcategory of $\text{Sch}_X$ consisting of finite étale covers $Y \to X$. To compare to the topological case of a covering space (see Section 0.1), we define:

**Definition.** A **geometric point** of $X$ is a morphism $\bar{x} : \text{Spec} \ \Omega \to X$ for some algebraically closed field $\Omega$.

Concretely, the image of $\bar{x}$ is some point $x \in X$ for which $\kappa(x) \subseteq \Omega$.

**Definition.** Let $\varphi : Y \to X$ be a morphism and $\bar{x} : \text{Spec} \ \Omega \to X$ a geometric point. Then the **geometric fibre** of $\bar{x}$ is the fibre product $Y_{\bar{x}} := Y \times_X \text{Spec} \ \Omega$.

**Proposition 4.1.2.** For a morphism $\varphi : Y \to X$, $\varphi$ is étale at each $P \in X$ if and only if every geometric fibre $Y_{\bar{x}}$ of $\varphi$ is of the form $\text{Spec}(\Omega \times \cdots \times \Omega)$, where $\Omega \supseteq \kappa(P)$ is an algebraically closed field.

**Proof.** This comes from the algebraic fact that for any field $k$, a finitely generated $k$-algebra $A$ is finite étale if and only if $A \otimes_k \bar{k}$ is isomorphic to a finite direct sum of copies of $\bar{k}$, where $\bar{k}$ is an algebraic closure of $k$. \qed

**Example 4.1.3.** The cover $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}, y \mapsto y^n$ in Example 4.0.2 is a finite étale cover.

**Example 4.1.4.** Let $X$ be a normal scheme of dimension 1 and let $\varphi : Y \to X$ be a finite morphism. In the affine case, with $X = \text{Spec} \ A$ and $Y = \text{Spec} \ B$, this corresponds to an extension of Dedekind rings $\varphi^* : A \to B$. Then points in the fibre $Y_P$ of a point $P \in X$ correspond to a prime factorization of ideals:

$$PB = \prod_{i=1}^{r} Q_i^{e_i}$$
where \( Q_1, \ldots, Q_r \) are prime ideals of \( B \). Then \( Y_P \) is a finite étale algebra if and only if \( e_i = 1 \), i.e. the prime \( P \) is unramified in the language of algebraic number theory. In the general case, a finite morphism of normal schemes of dimension 1 is étale if and only if each affine piece corresponds to a finite, unramified extension of Dedekind rings.

**Proposition 4.1.5.** If \( \varphi : Y \to X \) is a finite, locally free morphism of schemes then the following are equivalent:

1. \( \varphi \) is étale.
2. \( \Omega_{Y/X} = 0 \).
3. The diagonal \( \Delta : Y \to Y \times_X Y \) induces an embedding \( Y \hookrightarrow \Delta(Y) \) as an open and closed subscheme of \( Y \times_X Y \).

**Proof.** (1) \( \iff \) (2) When \( X \) and \( Y \) are affine, the equivalence is implied by Proposition 3.4.17. Then the general case is obtained by base change and localization properties of differentials (Lemma 3.4.10).

(2) \( \Rightarrow \) (3) Since \( \varphi \) is finite, it is separated and thus by definition the diagonal \( \Delta : Y \to Y \times_X Y \) is a closed immersion. By Lemma 3.4.14, \( \Delta(Y) \) corresponds to a sheaf of ideals \( I \) on \( Y \times_X Y \) such that \( \Omega_{Y/X} \) is naturally identified with the pullback of \( I/I^2 \) along \( \Delta \). By assumption \( \Omega_{Y/X} = 0 \), so it follows that \( I_P = 0 \) for all \( P \in \Delta(Y) \) and consequentially \( I = 0 \) on an open subset of \( Y \times_X Y \). Finally, this means \( \Delta(Y) \) is open and closed in \( Y \times_X Y \).

(3) \( \Rightarrow \) (1) We will use the criterion in Proposition 4.1.2. Fix a geometric point \( \bar{x} : \text{Spec } \Omega \to X \) and consider the base change of the geometric fibre \( Y_{\bar{x}} \):

\[
Y_{\bar{x}} \times_X Y = (\text{Spec } \Omega \times_X Y) \times_{\text{Spec } \Omega} \text{Spec } \Omega \times_X Y = Y_{\bar{x}} \times_{\text{Spec } \Omega} Y_{\bar{x}}.
\]

This comes equipped with a morphism \( \Delta_{\bar{x}} : Y_{\bar{x}} \to Y_{\bar{x}} \times_X Y \). Since (3) is consistent under base change, \( \Delta_{\bar{x}} \) is then an isomorphism from \( Y_{\bar{x}} \) to an open and closed subscheme of \( Y_{\bar{x}} \times_{\text{Spec } \Omega} Y_{\bar{x}} \).

Now \( Y_{\bar{x}} \) is concretely the spectrum of a finite dimensional \( \Omega \)-algebra so schematically it has finitely many points, each having residue field \( \Omega \). For any of these points \( \bar{y} : \text{Spec } \Omega \to Y_{\bar{x}} \), one more base change with \( \bar{y} \) as above yields an open and closed immersion \( \text{Spec } \Omega \hookrightarrow Y_{\bar{x}} \). But since \( \text{Spec } \Omega \) is connected, the image of this map is a connected component of \( Y_{\bar{x}} \). Hence \( Y_{\bar{x}} \) is a finite disjoint union of schematic points, and thus the criterion in Proposition 4.1.2 for \( \varphi \) to be étale is satisfied.

We next translate the ‘local triviality’ condition from covering space theory to schemes.

**Proposition 4.1.6.** Let \( X \) be a connected scheme and \( \varphi : Y \to X \) an affine surjective morphism. Then \( \varphi \) is a finite étale cover if and only if there exist a locally free, surjective morphism \( \psi : Z \to X \) such that \( Y \times_X Z \) is a trivial cover of \( Z \), i.e. a disjoint union of copies of \( Z \) with the morphism \( Y \times_X Z \to Z \) equal to the identity on each piece.

**Proof.** (\( \implies \)) The assumption that \( X \) is connected means every fibre of \( \varphi \) has the same cardinality, say \( n \). If \( n = 1 \), the property is trivial, so we may induct on \( n \). By (3) of Proposition 4.1.5, the diagonal \( Y \to Y \times_X Y \) induces a decomposition \( Y \times_X Y = \Delta(Y) \coprod Y' \) for \( Y' \) an open and closed subscheme of \( Y \times_X Y \). The maps \( Y' \hookrightarrow \Delta(Y) \coprod Y' \) (obvious)
and $Y \times_X Y \to Y$ (Lemma 4.1.1(b)) are finite étale. Thus Lemma 4.1.1(a) implies their composition $Y' \to Y$ is finite étale as well. But by construction the fibres of $Y' \to Y$ have cardinality $n-1$ so by induction, there exists a finite, locally free, surjective morphism $\psi' : Z \to Y$ such that $Y' \times_Y Z$ consists of $n-1$ disjoint copies of $Z$. Then finally the composition $\psi = \varphi \circ \psi' : Z \to X$ is finite, locally free and surjective (it is the composition of two such maps) and $(Y \times_X Y) \times_Y Z \cong Y \times_X Z$ is the disjoint union of $n$ copies of $Z$.

$(\Leftarrow)$ Since $\psi$ is locally free, each $P \in X$ has an open neighborhood $U = \text{Spec} \ A$ such that $\psi^{-1}(U) = \text{Spec} \ C$ for some finitely generated free $A$-algebra $C$. Let $\varphi^{-1}(U) \subseteq Y$ be $\psi^{-1}(U) = \text{Spec} \ B$ for $B$ an $A$-module. Then the base change of $\varphi$ to $\psi$ over $\varphi^{-1}(U)$ is given by $\text{Spec} (B \otimes_A C) \to \text{Spec} \ B$. By linear algebra, $B \otimes_A C$ is a finitely generated, free $C$-module so it is also finitely generated and free as an $A$-module. Hence $B$ must be finitely generated and free as an $A$-module, too.

We next show $\varphi$ is étale using Proposition 4.1.2. Let $\bar{z} : \text{Spec} \Omega \to Z$ be a geometric point of $Z$; then $\bar{x} = \psi \circ \bar{z} : \text{Spec} \Omega \to X$ is a geometric point of $X$. There is a natural isomorphism of fibres

$$Y_{\bar{x}} = \text{Spec} \Omega \times_X Y \cong \text{Spec} \Omega \times_Z (Y \times_X Z) = (Y \times_X Z)_{\bar{z}}$$

but the hypothesis implies $(Y \times_X Z)_{\bar{z}}$ is a disjoint union of copies of $\text{Spec} \Omega$. Thus $Y_{\bar{x}}$ is a disjoint union of points, and since $\psi$ is surjective, every geometric fibre of $\varphi$ is as well. Hence Proposition 4.1.2 implies $\varphi$ is étale.

Recall from Section 3.3 that the codimension of a point $P \in X$ is defined as the dimension of the local ring $\mathcal{O}_{X,P}$ (or the height of the prime associated to $P$ in any affine neighborhood). The powerful Zariski-Nagata purity theorem says that to show a map is étale, it suffices to check the étale property at all codimension 1 points of the base scheme.

**Theorem 4.1.7** (Zariski-Nagata Purity Theorem). If $\varphi : Y \to X$ is a finite surjective morphism of integral schemes, with $Y$ normal and $X$ regular, such that the fibre $Y_P$ of $\varphi$ above each codimension 1 point $P \in X$ is étale over $\kappa(P)$, then $\varphi$ itself is étale.

**Corollary 4.1.8.** Let $X$ be a regular integral scheme and $U \subseteq X$ an open subscheme consisting of points of codimension at least 2. Then there is an equivalence of categories

$$\text{Fét}_X \longrightarrow \text{Fét}_U$$

$$Y \longmapsto Y \times_X U.$$

**Proposition 4.1.9.** Let $\varphi : Y \to X$ be a finite étale cover, $\bar{z} : \text{Spec} \Omega \to z$ a geometric point and $Z$ a connected scheme over $X$. If $f, g : Z \to Y$ are two $X$-morphisms such that $f \circ \bar{z} = g \circ \bar{z}$, then $f = g$.

**Proof.** By Lemma 4.1.1(b), we may assume $X = Z$, so that $f$ and $g$ are two sections of $\varphi : Y \to X$. It follows that $f$ and $g$ are finite étale morphisms and each induces an isomorphism of $Z = X$ with an open and closed subscheme of $Y$. Since $Z$ is connected, the images of $f$ and $g$ are determined by the images of any geometric point, hence $f \circ \bar{z} = g \circ \bar{z}$ implies $f = g$. □

**Corollary 4.1.10.** Let $\varphi : Y \to X$ be a finite étale cover. Then $\text{Aut}(Y/X)$ is finite.
Proof. Take $\sigma \in \text{Aut}(Y/X), \sigma \neq 1$, and set $f = \varphi$ and $g = \varphi \circ \sigma$. Then by Proposition 4.1.9, $f$ and $g$ send some geometric point of $Y$ to different points. In other words, the action of $\text{Aut}(Y/X)$ on any geometric fibre is free, or the permutation representation of $\text{Aut}(Y/X)$ on any of the geometric fibres is faithful. Now since the map is finite étale, each geometric fibre is finite as a set, so this implies $\text{Aut}(Y/X)$ is itself finite.

Now let $\varphi : Y \rightarrow X$ be a finite étale cover and let $G$ be a group scheme over $X$ such that $Y$ is a left $G$-torsor (see Section 3.5). Let $Y/G$ be the quotient space with projection map $\pi : Y \rightarrow Y/G$. We define a sheaf on $Y/G$ by $\mathcal{O}_{Y/G} := (\pi_* \mathcal{O}_Y)^G$, the subsheaf of $G$-invariants of the pushforward of $\mathcal{O}_Y$ to $Y/G$ along $\pi$. This makes $Y/G$ into a ringed space.

**Proposition 4.1.11.** The ringed space $(Y/G, \mathcal{O}_{Y/G})$ is a scheme over $X$. Moreover, $\varphi : Y \rightarrow X$ factors through a finite morphism $\psi : Y/G \rightarrow X$.

The following are analogues of the basic Galois theory of covering spaces (see Section 0.1, e.g.).

**Proposition 4.1.12.** If $\varphi : Y \rightarrow X$ is a connected, finite étale cover and $G \leq \text{Aut}(Y/X)$ is any finite subgroup of automorphisms, then $\pi : Y \rightarrow Y/G$ is a finite étale cover.

**Definition.** A connected, finite étale cover $\varphi : Y \rightarrow X$ is a **Galois cover** if $\text{Aut}(Y/X)$ acts transitively on every geometric fibre of $\varphi$.

**Theorem 4.1.13.** Let $\varphi : Y \rightarrow X$ be a Galois cover and suppose $\psi : Z \rightarrow X$ is a connected, finite étale cover such that $Z$ is a scheme over $Y$ and the diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & Z \\
\downarrow & & \downarrow \\
X & \searrow & \\
& \psi & \\
\end{array}
$$

commutes. Then

1. $Y \rightarrow Z$ is a Galois cover and $Z \cong Y/G$ for some subgroup $G \leq \text{Aut}(Y/X)$.
2. There is a bijection

   $$
   \{\text{subgroups } G \leq \text{Aut}(Y/X)\} \longleftrightarrow \{\text{intermediate covers } Y \rightarrow Z \rightarrow X\}.
   $$

3. The correspondence is bijective on normal subgroups of $\text{Aut}(Y/X)$ and Galois covers $Z \rightarrow X$, and in this case $\text{Aut}(Z/X) \cong \text{Aut}(Y/X)/G$ as groups.
4.2 The Étale Fundamental Group

Let $X$ be a scheme and $\text{F}\acute{e}t_X$ the category of finite étale covers of $X$. Fix a geometric point $\bar{x} : \text{Spec} \Omega \to X$ and let

$$\text{Fib}_{\bar{x}} : \text{F}\acute{e}t_X \to \text{Sets} \quad Y \mapsto Y_{\bar{x}} = Y \times_X \text{Spec} \Omega$$

be the fibre functor over $\bar{x}$. By Lemma 3.3.6(c), $\text{Fib}_{\bar{x}}$ is indeed a functor. Using the definition of the automorphism group of a functor from Section 0.1, we can now define the algebraic, or étale, fundamental group of a scheme.

**Definition.** The algebraic, or étale fundamental group of a scheme of $X$ at a geometric point $\bar{x} : \text{Spec} \Omega \to X$ is the automorphism group of the fibre functor over $\bar{x}$:

$$\pi_1(X, \bar{x}) := \text{Aut}(\text{Fib}_{\bar{x}}).$$

By Theorem 0.1.3, this coincides with the topological fibre functor and fundamental group. We will also see below that this definition for schemes coincides with the definition given for curves in Section 1.5.

**Theorem 4.2.1** (Grothendieck). Let $X$ be a connected scheme and $\bar{x} : \text{Spec} \Omega \to X$ a geometric point. Then

1. $\pi_1(X, \bar{x})$ is a profinite group and its action on $\text{Fib}_{\bar{x}}(Y)$ is continuous for all $Y \in \text{F}\acute{e}t_X$.
2. $\text{Fib}_{\bar{x}}$ induces an equivalence of categories

$$\text{F}\acute{e}t_X \sim \{\text{finite, continuous, left } \pi_1(X, \bar{x})\text{-sets}\}$$

$$Y \mapsto \text{Fib}_{\bar{x}}(Y).$$

**Example 4.2.2.** Let $k$ be a field and consider $X = \text{Spec} k$. Then finite étale covers $Y \to \text{Spec} k$ are precisely $Y = \text{Spec} A$ for $A = L_1 \times \cdots \times L_r$ a finite étale $k$-algebra. Here, the fibre functor over any $\bar{x} : \text{Spec} \bar{k} \to \text{Spec} k$ (equivalent to a choice of algebraic closure $\bar{k}$ of $k$) is exhibited by

$$\text{Fib}_{\bar{x}}(Y) = \text{Spec}(A \otimes_k \bar{k}) = \text{Spec}(\Omega \times \cdots \times \Omega),$$

and $\text{Spec}(\Omega \times \cdots \times \Omega)$ is a finite set of $r$ closed points indexed by the homomorphisms $A \to \bar{k}$. Indeed, as we saw in Section 0.2, $\text{Fib}_{\bar{x}}(Y) \cong \text{Hom}_k(A, k_s)$, where $k_s$ is the separable closure of $k$ in $\bar{k}$ via the embedding $k \hookrightarrow \bar{k}$. The action of $\text{Aut}(\text{Hom}_k(-, k_s))$ on any given $\text{Hom}_k(A, k_s)$ is given by $T \cdot \sigma = \sigma \circ T$, or precisely the action of $G_k = \text{Gal}(k_s/k)$ on $\text{Hom}_k(A, k_s)$ from Example 0.4.2. Therefore

$$\pi_1(X, \bar{x}) = \text{Aut}(\text{Fib}_{\bar{x}}) \cong \text{Aut}(\text{Hom}_k(-, k_s)) \cong \text{Gal}(k_s/k).$$

Moreover, there is an identification (via Prop. 3.2.2) $\text{Hom}_k(A, k_s) = \text{Hom}_{\text{Spec} k}(\text{Spec} k_s, \text{Spec} k)$. 

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The above shows that although $\text{Fib}_{\overline{x}}$ is not a representable functor – $\text{Spec} \, k_s$ is not a finite étale $k$-scheme – it is pro-representable in $\text{F\acute{e}ts}_{\text{Spec} \, k}$. Explicitly,

$$\text{Fib}_{\overline{x}}(\text{Spec} \, A) = \lim_{\to} \text{Hom}_{\text{Spec} \, k}(Y', \text{Spec} \, A)$$

where the direct limit is over all finite étale Galois covers $Y' \to \text{Spec} \, k$ ordered by the existence of a $\text{Spec} \, k$-morphism $Y' \to Y''$.

In fact, there was nothing special about $X$ being $\text{Spec} \, k$ or even affine in the last paragraph.

**Proposition 4.2.3.** For a connected scheme $X$ and any geometric point $\overline{x} : \text{Spec} \, \Omega \to X$, the fibre functor $\text{Fib}_{\overline{x}}$ is pro-representable. Explicitly, for any finite étale cover $Y \to X$,

$$\text{Fib}_{\overline{x}}(Y) = \lim_{\to} \text{Hom}_{X}(Y', Y)$$

where the direct limit is over all finite étale Galois covers $Y' \to Y$.

**Lemma 4.2.4.** Every automorphism of $\text{Fib}_{\overline{x}}$ is determined by a unique automorphism of the direct system $(Y' \to X)$ of finite Galois covers of $X$.

We can now give the proof of Theorem 4.2.1.

**Proof.** (1) By Proposition 4.2.3, $\text{Fib}_{\overline{x}}$ is pro-representable by the direct system of $\text{Hom}_{X}(Y', Y)$ where $Y'$ ranges over all finite Galois covers of $X$. Now Lemma 4.2.4 implies

$$\text{Aut}(\text{Fib}_{\overline{x}}) = \lim_{\leftarrow} \text{Aut}(Y'/X)$$

where again $Y'$ ranges over all finite Galois covers. Since Corollary 4.1.10 says that each $\text{Aut}(Y'/X)$ is a finite group, we see that $\pi_1(X, \overline{x}) = \text{Aut}(\text{Fib}_{\overline{x}})$ is a profinite group. Note that $\pi_1(X, \overline{x})$ has a natural action on each $\text{Fib}_{\overline{x}}(Y)$ for any finite étale cover $Y \to X$. It remains to show this action is continuous. Take a geometric point $\overline{y} \in \text{Fib}_{\overline{x}}(Y)$ lying over $\overline{x} : \text{Spec} \, \Omega \to X$. If $\overline{y}$ comes from an element of $\text{Hom}_{X}(Y', Y)$ by the direct limit in Proposition 4.2.3, then the action of $\pi_1(X, \overline{x})$ factors through $\text{Aut}(Y'/X)$. This implies continuity.

(2) Take a finite, continuous, left $\pi_1(X, \overline{x})$-set $S$. The action of $\pi_1(X, \overline{x})$ is transitive on each orbit of $S$, so we may assume it is transitive on $S$ to begin with. Let $H$ be the stabilizer of any point in $S$. Then $H$ is an open subgroup of $\pi_1(X, \overline{x})$, so it contains the open normal subgroup corresponding to the kernel of $\pi_1(X, \overline{x}) \to \text{Aut}(Y'/X)$ for some finite Galois cover $Y' \to X$. Let $\overline{H}$ be the image of $H$ in $\text{Aut}(Y'/X)$. Then one proves $S \cong Y'/\overline{H}$. This proves essential surjectivity; fully faithfulness is routine.

The following shows that our definition of the fundamental group for schemes properly captures the notion we described for curves in Section 1.5.

**Theorem 4.2.5.** Let $X$ be an integral normal scheme with function field $K$ and fix a separable closure $K_s/K$. Let $K_X$ be the compositum of all finite subextensions $L/K$ in $K_s$ such that the normalization $X_L$ of $X$ in $L$ is étale over $X$. Then
(1) $K_X/K$ is Galois.

(2) For any geometric point $\bar{x}: \text{Spec } K \to X$, we have $\pi_1(X, \bar{x}) \cong \text{Gal}(K_X/K)$.

Proof. The proof of (1) is basically the same as the proof of Proposition 1.5.1. Then (2) follows from (2) of Theorem 4.2.1.

Example 4.2.6. Let $X = Y = \mathbb{C} \setminus \{0\}$ and define the finite étale cover

$$f: Y \longrightarrow X$$

$$y \longmapsto \pi y^n.$$  

(Here, we use $\pi$ to ensure there are no algebraic relations on this map.) As in Example 4.1.3, $f$ is a finite étale Galois cover of degree $n$, with Galois group $G = \langle \sigma \rangle \cong \mathbb{Z}/n\mathbb{Z}$, where $\sigma$ is the automorphism $\sigma: y \mapsto \zeta_n y$ for $\zeta_n$ a primitive $n$th root of unity. Notice that by setting $z = \pi^{1/n}y$, we may define $f$ over $\mathbb{Q}$. We make use of the following important result.

Theorem 4.2.7. If a scheme $X/\mathbb{C}$ is defined over $\mathbb{Q}$, then any finite étale cover of $X$ is also defined over $\mathbb{Q}$.

In particular, the finite cover $f: y \mapsto z^n$ is defined over some number field $K/\mathbb{Q}$. A classic question asks for a description of $K$ in terms of the topology of this cover.

Set $G_\mathbb{Q} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and fix an automorphism $\omega \in G_\mathbb{Q}$. Let $G$ be any group and suppose $Y \to X$ is a $G$-Galois cover which is defined over a number field $K$. Then $\text{Gal}(\overline{\mathbb{Q}}/K) \leq G_\mathbb{Q}$ and if $\omega$ lies in $\text{Gal}(\overline{\mathbb{Q}}/K)$, then $\omega$ carries this cover $Y \to X$ to itself (the action on $Y$ being the action of $\omega$ on the coefficients of the map). The field of moduli for the cover $Y \to X$ is defined to be the subfield $\mathcal{M}$ of $\mathbb{Q}$ fixed by all $\omega$ taking $Y \to X$ to itself.

Proposition 4.2.8. If $G$ is an abelian group, then the unique maximal field of definition for $Y \to X$ is precisely $\mathcal{M}$.

Note that Proposition 4.2.8 is completely open when $G$ is nonabelian. In any case, the consequences for abelian covers is that we can see the Galois theory of finite étale covers in two parallel ways:

- (Geometric) Every Galois cover of schemes has a group of automorphisms and a corresponding field extension for which this is the Galois group.
- (Arithmetic) Every cover of schemes has a field of definition over $\mathbb{Q}$, and in the abelian case there is even a unique minimal such field of definition.

As in Section 1.6, these related pieces are governed by a short exact sequence

$$1 \to \pi_1(\overline{X}) \to \pi_1(X) \to \text{Gal}(\overline{k}/k) \to 1$$

(here, the base point can be omitted and $k$ can be any field). Accordingly, $\pi_1(\overline{X})$ is called the geometric fundamental group of $X$, $\pi_1(X)$ the algebraic fundamental group and $\text{Gal}(\overline{k}/k)$ the arithmetic fundamental group. As in Corollary 1.6.4, the sequence splits if and only if $X$ contains a $k$-rational point.
4.3 Properties of the Étale Fundamental Group

Let $X$ be a connected scheme and take two geometric points $\bar{x} : \text{Spec } \Omega \to X$ and $\bar{x}' : \text{Spec } \Omega' \to X$. In the topological case (Section 0.1), there was a nice notion of a path between two points. We now describe an analogue for schemes.

**Proposition 4.3.1.** For any two geometric points of $X$, $\bar{x} : \text{Spec } \Omega \to X$ and $\bar{x}' : \text{Spec } \Omega' \to X$, there is a natural isomorphism $\text{Fib}_x \cong \text{Fib}_{x'}$.

**Proof.** We saw in Proposition 4.2.3 that each fibre functor is pro-representable by an inverse system $(P_x)$ of finite étale Galois covers of $X$, but the choice of $X$-morphisms in each case depends on the choice of basepoints in each $P_x$ lying over $\bar{x}$ or $\bar{x}'$. Let the covering morphisms be denoted $\varphi_{\alpha\beta}, \psi_{\alpha\beta} : P_\beta \to P_\alpha$ for $\bar{x}, \bar{x}'$, respectively. Then by Proposition 4.2.3, for any $Y \in \text{FÉt}_X$, $\text{Fib}_x(Y) \cong \lim_{\longrightarrow} \text{Hom}(P_\alpha, Y)$ where the connecting homomorphisms in the direct limit are those induced by the $\varphi_{\alpha\beta}$; and similarly $\text{Fib}_{x'}(Y) \cong \lim_{\longrightarrow} \text{Hom}(P_\alpha, Y)$ with the maps induced by the $\psi_{\alpha\beta}$. Therefore it is sufficient to exhibit an isomorphism of inverse systems $(P_\alpha, \varphi_{\alpha\beta}) \cong (P_\alpha, \psi_{\alpha\beta})$.

Let $\alpha \leq \beta$ and take $\lambda_\beta \in \text{Aut}_X(P_\beta)$. Let $x_\alpha \in P_\alpha$ and $x_\beta \in P_\beta$ be the specified points over $\bar{x}$ and set $x'_\alpha = \psi_{\alpha\beta} \circ \lambda_\beta(x_\beta)$:

\[
\begin{array}{ccc}
\xymatrix{ & P_\beta \ar[r]^{\lambda_\beta} & P_\beta \\
\varphi_{\alpha\beta} \ar[u] & x_\alpha \ar[r]_{\lambda_\alpha} & x'_\alpha \\
\ar[d]_{\psi_{\alpha\beta}} & & \ar[d]_{\psi_{\alpha\beta}} \\
P_\alpha & P_\alpha & P_\alpha \ar[u]_{\lambda_\beta}
}\end{array}
\]

Since $P_\alpha$ is a Galois cover of $X$, there exist a (unique) automorphism $\lambda_\alpha \in \text{Aut}_X(P_\alpha)$ such that $\lambda(x_\alpha) = x'_\alpha$. It follows from Proposition 4.1.9 that this $\lambda_\alpha$ makes the diagram above commute. Define $\rho_{\alpha\beta} : \text{Aut}_X(P_\beta) \to \text{Aut}_X(P_\alpha)$ by sending $\lambda_\beta \in \text{Aut}_X(P_\beta)$ to the $\lambda_\alpha$ described above. This gives an inverse system $(\text{Aut}_X(P_\alpha, \rho_{\alpha\beta})$ in which each $\text{Aut}_X(P_\alpha)$ is finite and nonempty, so the corresponding inverse limit $\lim_{\leftarrow} \text{Aut}_X(P_\alpha)$ is nonempty. That is, there is at least one system of isomorphisms of $(P_\alpha)$ compatible with the $\varphi_{\alpha\beta}$ and $\psi_{\alpha\beta}$. □

**Corollary 4.3.2.** For any pair of geometric points $\bar{x} : \text{Spec } \Omega \to X$ and $\bar{x}' : \text{Spec } \Omega' \to X$, there is a continuous isomorphism of profinite groups $\pi_1(X, \bar{x}') \to \pi_1(X, \bar{x})$.

**Proof.** Any natural isomorphism $\lambda : \text{Fib}_x \cong \text{Fib}_{x'}$ from Proposition 4.3.1 determines an isomorphism $\lambda^* : \pi_1(X, \bar{x}') \to \pi_1(X, \bar{x})$ by $\lambda^*(\varphi) = \lambda^{-1} \circ \varphi \circ \lambda$. □

**Definition.** A natural isomorphism $\lambda : \text{Fib}_x \to \text{Fib}_{x'}$ is called a *path* from $\bar{x}$ to $\bar{x}'$.

Note that the isomorphism $\lambda^* : \pi_1(X, \bar{x}') \to \pi_1(X, \bar{x})$ depends on the choice of path $\lambda$, but only up to inner automorphism.
Now let $X$ and $X'$ be two connected schemes, take a geometric point in each, $\bar{x}: \text{Spec } \Omega \to X$ and $\bar{x'}: \text{Spec } \Omega \to X'$, and suppose there is a covering morphism $\varphi: X' \to X$ such that $\varphi \circ \bar{x'} = \bar{x}$. Then $\varphi$ induces a base change functor

$$B_{\bar{x}, \bar{x'}}: \text{Fét}_X \longrightarrow \text{Fét}_{X'},$$

$$Y \longmapsto Y \times_X X'.$$

Recall from the proof of Proposition 4.1.6 that there is a natural isomorphism $\text{Fib}_{\bar{x}} \cong \text{Fib}_{\bar{x'}} \circ B_{\bar{x}, \bar{x'}}$. This induces a map

$$\varphi_*: \pi_1(X', \bar{x'}) \longrightarrow \pi_1(X, \bar{x}).$$

**Lemma 4.3.3.** $\varphi_*: \pi_1(X', \bar{x'}) \to \pi_1(X, \bar{x})$ is a continuous homomorphism of profinite groups.

**Proposition 4.3.4.** Let $\varphi: X' \to X$ be a cover of connected schemes. Then

1. $\varphi_*$ is trivial if and only if for all connected $Y \in \text{Fét}_X$, the base change $Y \times_X X' \to X'$ is a trivial cover.

2. $\varphi_*$ is surjective if and only if for all connected $Y \in \text{Fét}_X$, the base change $Y \times_X X' \to X'$ is a connected cover.

**Proof.** (1) First note that an $X'$-cover is trivial if and only if $\pi_1(X', \bar{x'})$ acts trivially on its fibre over $\bar{x'}$. Thus if, for any $\lambda \in \pi_1(X', \bar{x'})$ and $Y \in \text{Fét}_X$, $\lambda$ acts trivially on $\text{Fib}_{\bar{x}}(Y \times_X X') \cong \text{Fib}_{\bar{x}}(Y)$, then $\varphi_*(\lambda)$ must be trivial. Conversely, if some $Y \times_X X'$ is a nontrivial cover, there exists $\lambda \in \pi_1(X', \bar{x'})$ acting nontrivially on $\text{Fib}_{\bar{x}}(Y) \cong \text{Fib}_{\bar{x'}}(Y \times_X X')$. Hence $\varphi_*(\lambda)$ is nontrivial.

(2) Suppose $Y \in \text{Fét}_X$ (connected) is such that $Y \times_X X'$ is not connected. Then there exist $x_1, x_2 \in \text{Fib}_{\bar{x}}(Y)$ such that $\varphi_*(\lambda)(x_1) \neq x_2$ for any $\lambda \in \pi_1(X', \bar{x'})$. However, the fact that $Y$ is connected implies there exists $\mu \in \pi_1(X, \bar{x})$ such that $\mu(x_1) = x_2$. Thus $\mu$ is not in the image of $\varphi_*$, so $\varphi_*$ is not surjective. Going the other direction, if $\varphi_*$ is surjective then $\text{im } \varphi_*$ is a proper, closed subgroup of the profinite group $\pi_1(X, \bar{x})$, so there exists a proper, open subgroup $U \subseteq \pi_1(X, \bar{x})$ such that $U \supseteq \text{im } \varphi_*$. Thus $\pi_1(X', \bar{x'})$ acts trivially on the coset space $\pi_1(X, \bar{x})/U$, or in other words, the connected cover $Y \to X$ corresponding to $\pi_1(X, \bar{x})/U$ from Theorem 4.1.13 base changes to a trivial cover $Y \times_X X' \to X'$. But if $U \neq \pi_1(X, \bar{x})$, then $Y \times_X X' \neq X'$ and is therefore a disconnected cover.

**Remark.** Suppose $Y \to X$ is connected, $\bar{y}$ is a geometric point over $\bar{x}$ and $U = \text{Stab}_{\pi_1(X, \bar{x})}(\bar{y})$. Then $U$ is an open subgroup of $\pi_1(X, \bar{x})$ and $\text{Fib}_{\bar{x}}(Y) \cong \pi_1(X, \bar{x})/U$ as left $\pi_1(X, \bar{x})$-sets.

As above, let $\varphi: X' \to X$ be a cover of connected schemes such that $\varphi \circ \bar{x'} = \bar{x}$.

**Proposition 4.3.5.** Let $U \subseteq \pi_1(X, \bar{x})$ be any open subgroup and let $Y \to X$ be the corresponding connected cover; let $\bar{y} \in \text{Fib}_{\bar{x}}(Y)$ be the geometric point such that $U = \text{Stab}(\bar{y})$. Then $U \supseteq \text{im } \varphi_*$ if and only if the base change $Y \times_X X' \to X'$ has a section such that $y' := \bar{x}'(\text{Spec } \Omega) \subseteq X'$ maps to $y := \bar{y}(Y \times_X \text{Spec } \Omega) \subseteq Y \times_X X'$.
Proof. Note that

\[ U \supseteq \text{im } \varphi_* \iff \text{every element of } \pi_1(X, \bar{x}) \text{ fixes } \bar{y} \]
\[ \iff \text{the connected component of } y \text{ is fixed under the } \pi_1(X, \bar{x})\text{-action} \]
\[ \iff \text{the connected component of } y \text{ is isomorphic to } X'. \]

Therefore under \( Y \times_X X' \to X' \), the component of \( y \) maps isomorphically to \( X' \), so we get a section \( X' \to Y \times_X X' \). Conversely, any such section determines a \( \pi_1(X', \bar{x}')\)-equivariant map \( \text{Spec } \Omega \to \text{Fib}_{\bar{x}'}(Y \times_X X') \) whose image must be fixed. Hence \( U \supseteq \text{im } \varphi_* \). \( \square \)

Proposition 4.3.6. Let \( U'' \supseteq \pi_1(X', \bar{x}') \) be an open subgroup, \( Y' \to X' \) the corresponding connected cover and \( \bar{y}' \) the geometric point such that \( \text{Stab}_{\pi_1(X', \bar{x}')}(\bar{y}') = U' \). Then \( U'' \supseteq \ker \varphi_* \) if and only if there exists a connected cover \( Y \to X \) and an \( X'\)-morphism \( Y_0 \to Y' \) such that \( Y_0 \) is a connected component of \( Y \times_X X' \to X' \).

Proof. ( \( \implies \) ) Suppose \( U'' \supseteq \ker \varphi_* \). We know \( \text{im } \varphi_* \) is a subgroup in \( \pi_1(X, \bar{x}) \) and \( V' := \varphi_*(U') \) is open in \( \text{im } \varphi_* \), so by profinite group theory, there exists an open subgroup \( V \subseteq \pi_1(X, \bar{x}) \) such that \( V \cap \text{im } \varphi_* = V' \). This \( V \) corresponds to a cover \( Y \to X \). Let \( Y_0 \) be the connected component of \( Y \times_X X' \to X' \) containing \( y = \bar{y}(Y \times_X \text{Spec } \Omega) \subseteq Y \times_X X' \). Then \( \bar{y} \in \text{Fib}_{\bar{x}'}(Y_0) \) and there is an isomorphism of \( \pi_1(X', \bar{x}') \)-sets

\[ \text{Fib}_{\bar{x}'}(Y_0) \cong \pi_1(X', \bar{x}')/U'' \quad \text{where } U'' = \text{Stab}(\bar{y}). \]

Now an \( X'\)-morphism \( Y_0 \to Y' \) exists if and only if there is an equivariant map \( \pi_1(X', \bar{x}')/U'' \to \pi_1(X', \bar{x}')/U' \). This in turn occurs if and only if \( U'' \subseteq U' \). If \( \lambda \in \ker \varphi_* \), then \( \lambda \) fixes \( \bar{y} \) in \( \text{Fib}_{\bar{x}'}(Y) = \text{Fib}_{\bar{x}'}(Y \times_X X') \), hence \( \lambda \) fixes \( \bar{y} \) in \( \text{Fib}_{\bar{x}'}(Y_0) \). This means \( \lambda \in U'' \) and so we see that \( U'' \supseteq \ker \varphi_* \). By hypothesis, \( U'' \supseteq \ker \varphi_* \) as well, so by the correspondence theorem, \( U'' \subseteq U' \) if and only if \( \varphi_*(U'') \subseteq \varphi_*(U') \), but this holds by construction. Therefore an \( X'\)-morphism \( Y_0 \to Y' \) exists.

( \( \iff \) ) We must show \( U'' \supseteq \ker \varphi_* \). Applying the fibre functor \( \text{Fib}_{\bar{x}'} \) to the morphism \( Y_0 \to Y' \) gives a map \( \text{Fib}_{\bar{x}'}(Y_0) \to \text{Fib}_{\bar{x}'}(Y') \). Note that \( \bar{y}' \in \text{Fib}_{\bar{x}'}(Y') \); choose any lift \( \bar{y} \in \text{Fib}_{\bar{x}'}(Y_0) \) and set \( U = \text{Stab}(\bar{y}) \). Identifying \( Y_0 \to Y' \) with a subgroup \( U'' \subseteq \pi_1(X', \bar{x}') \), we get \( U'' \subseteq U' \) by the above. Then once again, \( \ker \varphi_* \) has to fix \( \bar{y} \), so \( \ker \varphi_* \subseteq U'' \subseteq U' \). \( \square \)

Corollary 4.3.7. \( \varphi_* \) is injective if and only if for all connected covers \( Y' \to X' \), there exists a cover \( Y \to X \) and a morphism \( Y_0 \to Y' \) over \( X' \) such that \( Y_0 \) is isomorphic to a connected component of \( Y \times_X X' \).

Corollary 4.3.8. If every connected cover \( Y' \to X' \) arises as a base change \( Y' = Y \times_X X' \) of some \( Y \to X \), then \( \varphi_* \) is injective.

Corollary 4.3.9. Let \( X'' \xrightarrow{\varphi} X' \xrightarrow{\psi} X \) be a pair of morphisms of connected schemes and let \( \bar{x} : \text{Spec } \Omega \to X, \bar{x}' : \text{Spec } \Omega \to X' \) and \( \bar{x}'' : \text{Spec } \Omega \to X'' \) be geometric points such that \( \varphi \circ \bar{x}' = \bar{x} \) and \( \psi \circ \bar{x}'' = \bar{x}'. \) Then the sequence of profinite groups

\[ \pi_1(X', \bar{x}') \xrightarrow{\psi_*} \pi_1(X', \bar{x}') \xrightarrow{\varphi_*} \pi_1(X, \bar{x}) \]

is exact if and only if both of the following conditions hold:
(1) For all covers $Y \to X$, the base change $Y \times_X X'' \to X''$ is a trivial cover.

(2) For all connected covers $Y' \to X'$ such that $Y' \times_{X'} X'' \to X''$ admits a section $X'' = Y' \times_{X'} X''$, there exists a connected cover $Y \to X$ and an $X'$-morphism $Y_0 \to Y'$ for $Y_0$ a connected component of $Y \times_X X'$.

**Remark.** It suffices to check condition (2) on Galois covers, since $\ker \varphi_* = \bigcap_{U \supset \ker \varphi_*} U$.

### 4.4 Structure Theorems

In this section we prove several fundamental results about $\pi_1(X)$ for schemes over $\mathbb{C}$, and then generalize. In particular, we will state an analogue of Theorem 1.5.5. First, let us show that for any smooth, projective, integral scheme $X$ over $\mathbb{C}$, $\pi_1(X)$ is finitely generated. To do this, we need:

**Theorem 4.4.1 (Bertini).** Let $k$ be an algebraically closed field and let $X \subseteq \mathbb{P}^n_k$ be a smooth, closed subscheme. Then there exists a hyperplane $H \subseteq \mathbb{P}^n_k$ not containing $X$ such that $X \cap H$ is regular.

**Proof.** Hartshorne, II.8.1.8.

**Lemma 4.4.2.** Let $k$ be algebraically closed and let $X \subseteq \mathbb{P}^n_k$ be a smooth, connected, closed subscheme of dimension $\dim X \geq 2$ and let $Y \to X$ be a connected, finite étale cover. Then there exists a hyperplane $H \subseteq \mathbb{P}^n_k$ not containing $X$ so that $X \cap H$ is smooth and connected and $Y \times_X (X \cap H)$ is connected.

**Proof.** Let $H$ be as in Bertini’s theorem. When $\dim X \geq 2$, one can use sheaf cohomology (cf. Hartshorne, III.7.9.1) to show that $X \cap H$ is not just regular but also smooth and connected. Then the same proof applies to the closed immersion $Y \times_X (X \cap H) \hookrightarrow Y$ to show that $Y \times_X (X \cap H)$ is connected.

**Theorem 4.4.3.** For any smooth, projective, integral scheme $X$ over $\mathbb{C}$ and every geometric point $\bar{x} : \text{Spec} \mathbb{C} \to X$, $\pi_1(X, \bar{x})$ is finitely generated.

**Proof.** Suppose $\dim X \geq 2$. Then by Lemma 4.4.2, for any connected $Y \to X$ there exists a hyperplane $H \subseteq \mathbb{P}^n_k$ such that $X \cap H$ is smooth, connected and $Y \times_X (X \cap H)$ is connected. By Lemma 4.1.1(b), $Y_\times (X \cap H) \to X \cap H$ is a (connected) finite étale cover, so Proposition 4.3.4 implies $\pi_1(X \cap H, \bar{x}) \to \pi_1(X, \bar{x})$ is surjective. Now note that $X \cap H$ is smooth, projective and $\dim X \cap H < \dim X$, so repeating this process, we eventually obtain a surjection $\pi_1(C, \bar{x}) \to \pi_1(X, \bar{x})$ for some smooth projective curve $C$ over $\mathbb{C}$. By Theorem 1.5.5, we know $\pi_1(C, \bar{x})$ is finitely generated so this implies $\pi_1(X, \bar{x})$ is as well.

Even stronger than the surjections $\pi_1(X \cap H) \to \pi_1(X)$ in the above proof, we have the following result of Lefschetz.
**Theorem 4.4.4** (Lefschetz’s Hyperplane Theorem). Let $X$ be a smooth, closed projective scheme over $\mathbb{C}$ and $H \subseteq \mathbb{P}^n_{\mathbb{C}}$ a hyperplane such that $X \cap H$ is smooth. Then

$$H_k(X \cap H) \rightarrow H_k(X)$$
$$H^k(X \cap H) \rightarrow H^k(X)$$
$$\pi_k(X \cap H) \rightarrow \pi_k(X)$$

are all isomorphisms for $k < \dim X - 1$ and are surjective for $k = \dim X - 1$.

To compare $\pi_1$ and $\pi^\top_1$ as we did in Chapter 2, we associate to any $\mathbb{C}$-scheme $X$ of finite type a complex analytic space $X^{an}$ as follows. Write $X = \bigcup \text{Spec } A_i$ for $\mathbb{C}$-algebras $A_i$ of finite type. Then $A_i \cong \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ for polynomials $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$. Then each $f_j$ may be regarded as a holomorphic function on $\mathbb{C}^n$, so the zero set of $\{f_1, \ldots, f_r\}$ in $\mathbb{C}^n$ is a ringed space $(Y_i, \mathcal{O}_{Y_i})$, where $\mathcal{O}_{Y_i}$ is the ring of holomorphic functions on $Y_i$ (in the sense of complex geometry). We regard this $(Y_i, \mathcal{O}_{Y_i})$ as the basic model of a complex analytic space. Now glue the $Y_i$ together using the same gluing as for $\bigcup \text{Spec } A_i$ to get a ringed space $(X^{an}, \mathcal{O}_X^{an})$. Note that for any morphism of schemes $Y \rightarrow X$, there is a natural induced morphism of complex analytic spaces $Y^{an} \rightarrow X^{an}$. The complex analytic space $X^{an}$ has the following properties relative to $X$.

**Proposition 4.4.5.** Let $X/\mathbb{C}$ be a scheme of finite type. Then

1. $X$ is separated if and only if $X^{an}$ is Hausdorff.
2. $X$ is connected if and only if $X^{an}$ is connected.
3. $X$ is reduced if and only if $X^{an}$ is reduced.
4. $X$ is smooth if and only if $X^{an}$ is a complex manifold.
5. A morphism $Y \rightarrow X$ is proper (i.e. of finite type and the base change is closed, as in Section 3.3) if and only if $Y^{an} \rightarrow X^{an}$ is proper (i.e. compact sets pull back to compact sets).
6. In particular, $X$ is proper as a $\mathbb{C}$-scheme if and only if $X^{an}$ is compact.
7. There is a morphism of locally ringed spaces $(\varphi, \varphi^\#) : (X^{an}, \mathcal{O}_X^{an}) \rightarrow (X, \mathcal{O}_X)$ sending $X^{an}$ bijectively to the closed points of $X$.

As in Chapter 2 for curves, Grothendieck proved the following equivalence of categories.

**Theorem 4.4.6** (Grothendieck). Let $X$ be a connected scheme over $\mathbb{C}$ of finite type. Then there is an equivalence of categories

$$F\text{ét}_X \rightarrow F\text{Cov}_{X^{an}} \quad (\text{finite-sheeted topological covers})$$

$$(Y \rightarrow X) \leftrightarrow (Y^{an} \rightarrow X^{an}).$$

Therefore, for any geometric point $\bar{x} : \text{Spec } \mathbb{C} \rightarrow X$ with image $x = \bar{x}(\text{Spec } \mathbb{C})$, the induced map

$$\pi^\top_1(X^{an}, x) \rightarrow \pi_1(X, \bar{x})$$

is an isomorphism.
Just like in Chapter 2, the difficult part of this proof is establishing the essential surjectivity of the functor \( Y \mapsto Y^{an} \). This is essentially the main idea behind Serre’s GAGA principle.

**Question** (Serre). Does there exist a scheme \( X \) over \( \mathbb{C} \) of finite type such that \( \pi_1(X) = 1 \) but \( \pi_1^{top}(X^{an}) \neq 1 \)?

To extend Theorem 4.4.6 to any algebraically closed field \( k \) of characteristic 0, we need the following generalization of Proposition 1.6.1.

**Proposition 4.4.7.** Let \( X \) be a noetherian integral scheme, \( k \) an algebraically closed field, \( \varphi : Y \to X \) a proper flat morphism with geometrically integral fibres and suppose \( \bar{x} : \text{Spec} \ k \to X \) is a geometric point such that \( k \) is the algebraic closure of the residue field \( \kappa(x) \), where \( x = \text{im} \bar{x} \). Fix \( \bar{y} \in Y_{\bar{x}} \). Then there is an exact sequence

\[
\pi_1(Y_{\bar{x}}, \bar{y}) \to \pi_1(Y, \bar{y}) \to \pi_1(X, \bar{x}) \to 1.
\]

**Corollary 4.4.8.** Let \( k \) be algebraically closed and suppose \( X, Y \) are noetherian, connected schemes over \( k \), with \( X \) proper and geometrically integral. Then for any geometric points \( \bar{x} : \text{Spec} \ k \to X, \bar{y} : \text{Spec} \ k \to Y \), the natural map \( \pi_1(X \times_k Y, (\bar{x}, \bar{y})) \to \pi_1(X, \bar{x}) \times \pi_1(Y, \bar{y}) \) is an isomorphism.

**Proof.** From Proposition 4.4.7, we get a diagram (with basepoints suppressed):

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(X \times_k Y) & \longrightarrow & \pi_1(Y) & \longrightarrow & 1 \\
\downarrow & & \downarrow \text{id} & & \downarrow & & \downarrow \text{id} & & \\
1 & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(X) \times \pi_1(Y) & \longrightarrow & \pi_1(Y) & \longrightarrow & 1
\end{array}
\]

with the 1 on the left in the top row coming from the natural section \( X \subseteq X \times_k Y \) of the projection morphism \( X \times_k Y \to X \), and the entire bottom row representing the direct product of profinite groups. Since the left and right vertical arrows are identities, the Five Lemma implies the middle vertical arrow is an isomorphism.

**Corollary 4.4.9.** Let \( K \supseteq k \) be an extension of algebraically closed fields, \( X \) a proper integral scheme over \( k \) and \( X_K \) its base change. Then for any geometric points \( \bar{x} : \text{Spec} \ k \to X, \bar{x}' : \text{Spec} \ K \to X_K \), the induced map \( \pi_1(X_K, \bar{x}') \to \pi_1(X, \bar{x}) \) is an isomorphism.

**Proof.** We first show that \( \pi_1(X_K, \bar{x}') \to \pi_1(X, \bar{x}) \) is surjective. Suppose \( Y \to X \) is connected. Then since \( k \) is algebraically closed, the \( K \)-algebra \( k(Y) \otimes_k K \) is a field, but since this is equal to the function field of \( Y_K \), we see that \( Y_K \) is connected. Therefore by Proposition 4.3.4, the induced map on fundamental groups is surjective.

For injectivity, suppose \( Y \to X_K \) is a connected cover. It is possible (see Szamuely) to find a subfield \( k' \subseteq K \) such that \( k'/k \) is finitely generated, as well as an integral affine \( k \)-scheme \( T \) such that \( k(T) = k' \) and there is a connected finite étale cover \( Y' \to X \times_k T \). Fix a geometric point \( \bar{t} : \text{Spec} \ k \to T \). By Corollary 4.4.8, there is an isomorphism \( \pi_1(X \times_k T, (\bar{x}, \bar{t})) \cong \pi_1(X, \bar{x}) \times \pi_1(T, \bar{t}) \), so Corollary 4.3.7 implies there exist connected covers \( Z \to X \) and
4.5 Specialization and Characteristic \( p \) Results

To study fundamental groups of schemes in characteristic \( p > 0 \), we give a brief account of Grothendieck’s theory of specialization of the fundamental group. Suppose \( A \) is a complete DVR with fraction field \( K \) and residue field \( k \) of characteristic \( p > 0 \). Set \( X = \text{Spec} \, A \) and let \( \eta : \text{Spec} \, K \hookrightarrow X \) and \( x : \text{Spec} \, k \hookrightarrow X \) be the generic and closed points of \( X \), respectively. Also let \( \bar{\eta} : \text{Spec} \, \overline{K} \hookrightarrow \text{Spec} \, K \hookrightarrow X \) and \( \bar{x} : \text{Spec} \, \overline{k} \hookrightarrow \text{Spec} \, k \hookrightarrow X \) be the corresponding geometric points. For any \( X \)-scheme \( Y \to X \), set \( Y_{\eta} = Y \times_X \text{Spec} \, K, Y_x = Y \times_X \text{Spec} \, k, Y_{\bar{\eta}} = Y \times_X \text{Spec} \, \overline{K} \) and \( Y_{\bar{x}} = Y \times_X \text{Spec} \, \overline{k} \). Grothendieck proves:

**Theorem 4.5.1.** If \( \varphi : Y \to X \) is a proper map, then

1. For any geometric point \( \bar{y}' \in Y_x \), the natural map

\[
\pi_1(Y_x, \bar{y}') \to \pi_1(Y, \bar{y}')
\]

is an isomorphism.

2. Further, if \( k \) is algebraically closed, \( \varphi \) is flat and \( Y_{\bar{\eta}} \) and \( Y_x \) are geometrically reduced, then for any geometric point \( \bar{y} \in Y_{\bar{\eta}} \), the natural map

\[
\pi_1(Y_{\bar{\eta}}, \bar{y}) \to \pi_1(Y, \bar{y})
\]

is an isomorphism.

For the moment, assume \( k \) is algebraically closed, \( \varphi : Y \to X \) is a morphism of schemes and \( \bar{y} \in Y_{\bar{\eta}} \) and \( \bar{y}' \in Y_x \) are geometric points as above. By Corollary 4.3.2, there is a noncanonical isomorphism \( \pi_1(Y, \bar{y}) \cong \pi_1(Y, \bar{y}') \) determined by picking a path \( \lambda : \text{Fib}_\bar{y} \to \text{Fib}_{\bar{y}'} \). The isomorphisms in Theorem 4.5.1 allow us to make the following definition.

**Definition.** The **specialization map** from \( \bar{y} \) to \( \bar{y}' \) (associated to \( \varphi : Y \to X \)) is the composite

\[
\text{sp} : \pi_1(Y_{\bar{\eta}}, \bar{y}) \to \pi_1(Y, \bar{y}) \cong \pi_1(Y, \bar{y}') \cong \pi_1(Y_x, \bar{y}').
\]
For any profinite group $G$ and prime integer $p$, we denote by $G^{(p)}$ the maximal quotient of $G$ whose order (as a profinite group) is relatively prime to $p$. This is typically called the prime-to-$p$ part of $G$. We will prove an important result of Grothendieck which says that specialization induces an isomorphism on prime-to-$p$ parts of the fundamental groups of $Y_\eta$ and $Y_\zeta$. First, we need:

**Lemma 4.5.2** (Abhyankar). Let $(A, m, k)$ be a DVR with fraction field $K$ and $L/K$ and $M/K$ finite Galois extensions in which $A$ has integral closures $A_L$ and $A_M$, respectively, and integral closure $B$ in the compositum $LM$. Suppose $p_L \subset A_L$ and $p_M \subset A_M$ are maximal ideals lying over $m$ such that the inertia degrees $e(p_L | m)$ and $e(p_M | m)$ are relatively prime to $\text{char } k$ and $e(p_L | m) | e(p_M | m)$. Then $\text{Spec } B \to \text{Spec } A_M$ is a finite étale morphism of affine schemes.

**Proof.** Set $p = \text{char } k$. Fix a maximal ideal $\mathfrak{P} \subset B$ lying over $m$. We may assume $\mathfrak{P} \cap A_L = p_L$ and $\mathfrak{P} \cap A_M = p_M$. Then the injection $\text{Gal}(LM/K) \hookrightarrow \text{Gal}(L/K) \times \text{Gal}(M/K)$ restricts to an injection of inertia groups $I_{\mathfrak{P}} \hookrightarrow I_{p_L} \times I_{p_M}$. Moreover, the induced projections $I_{\mathfrak{P}} \to I_{p_L}$ and $I_{\mathfrak{P}} \to I_{p_M}$ are surjective. Since $|I_{\mathfrak{P}}| = e(p | m)$ for any prime $p$ lying over $m$, then by hypothesis the inertia groups $I_{\mathfrak{P}}, I_{p_L}$ and $I_{p_M}$ all have prime-to-$p$ order and hence are cyclic. Further, since $|I_{p_L}|$ divides $|I_{p_M}|$, every element of $I_{\mathfrak{P}} \subseteq I_{p_L} \times I_{p_M}$ has order dividing $e(p_M | m)$, but since there is a surjection $I_{\mathfrak{P}} \to I_{p_M}$ and the target has order $e(p_M | m)$, this implies the map is really an isomorphism $I_{\mathfrak{P}} \cong I_{p_M}$. By transitivity of inertia degrees, that is $e(\mathfrak{P} | m) = e(\mathfrak{P} | p_M)e(p_M | m)$, we get $e(\mathfrak{P} | p_M) = 1$, so $\mathfrak{P}$ is unramified over $M$. Since $\mathfrak{P}$ was arbitrary, it follows from Proposition 1.4.6 that $\text{Spec } B \to \text{Spec } A_M$ is étale.

**Theorem 4.5.3** (Grothendieck). If $k$ is algebraically closed of characteristic $p > 0$, $\varphi : Y \to X = \text{Spec } A$ is smooth, proper and has geometrically connected fibres, then for any geometric points $\bar{y} \in Y_\eta$ and $\bar{y}' \in Y_\zeta$, the specialization map descends to an isomorphism of prime-to-$p$ parts

$$\pi_1(Y_\eta, \bar{y})^{(p')} \cong \pi_1(Y_\zeta, \bar{y}')^{(p')}.$$ 

**Proof.** By definition of $\text{sp}$, it’s enough to show $\pi_1(Y_\eta, \bar{y})^{(p')} \to \pi_1(Y, \bar{y})^{(p')}$ is an isomorphism of prime-to-$p$ profinite groups. To begin, suppose $Z \to Y$ is a connected finite étale cover with base change $Z_\eta = Z \times_X \text{Spec } K$ over the generic point. Then

$$Z_\eta = \lim Z \times_X \text{Spec } L$$

where the inverse limit is over all finite extensions $L/K$ contained in a fixed algebraic closure $\overline{K}$ corresponding to $\overline{\eta}$. Then each $Z \times_X \text{Spec } L$ is an open subscheme of the base change $Z_L = Z \times_X X_L$, where $X_L = \text{Spec } A_L$ and $A_L$ is the integral closure of $A$ in $L$. We claim $Z_L$ is connected. If $Z_L = Z_1 \cup Z_2$ for two closed subsets $Z_1, Z_2$, then the special fibre $Z_x = Z \times_X \text{Spec } k$ lies in one of them, say $Z_1$. Under the map $Z_L \to X_L$, $Z_x$ is taken to $Y_x$, so $Z_2$ must be open, contradicting the assumption that $Y \to X$, and therefore also $Z_L \to X_L$, is proper and in particular closed. Thus $Z_L$ is connected, so $Z \times_X \text{Spec } L$ is as well. Since $L$ was arbitrary and the inverse limit of connected spaces is connected, we have shown that $Z_\eta$ is connected. Now (2) of Proposition 4.3.4 implies that $\pi_1(Y_\eta, \bar{y}) \to \pi_1(Y, \bar{y})$ is surjective and in particular it is surjective on prime-to-$p$ parts.

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4.5 Specialization and Characteristic $p$ Results

For injectivity (on prime-to-$p$ parts), Corollary 4.3.7 says that it’s enough to show that every finite étale cover $Z' \rightarrow Y_\bar{q}$ of prime-to-$p$ degree is the base change of some cover $Z \rightarrow Y$ of prime-to-$p$ degree. In fact, the remark at the end of Section 4.3 implies that we can check the condition for Galois covers $Z' \rightarrow Y_\bar{q}$. Applying Theorem 4.5.1, we have an isomorphism

$$\pi_1(Y \times_X X_L, z) \cong \pi_1(Y, z)$$

so replacing $X$ with $X_L$, we may assume $Z'$ is obtained by base change from a finite Galois cover $Z_\bar{q} \rightarrow Y_\bar{q}$. Let $\overline{Y}$ be the normalization of $Y$ in the function field $K(\bar{Z})$ of $\bar{Z}$. Then $K(\bar{Z})/K(Y)$ is a finite Galois extension, say of degree $d$, and by the Zariski-Nagata purity theorem (4.1.7), it’s enough to find a finite Galois extension $K'/K$ such that $\overline{Y} \times_X X_{K'} \rightarrow Y \times_X X_{K'}$ is étale over all codimension 1 points. Further, since all codimension 1 points of $Y$ lie in $Y_\bar{q}$ except for the generic point $\xi$ of $Y$, we need only check that $\overline{Y} \times_X X_{K'}$ is étale over every point of $Y \times_X X_{K'}$ lying over $\xi$. If $A$ has maximal ideal $m = (\pi)$, then $\pi$ also generates the maximal ideal of the local ring $\mathcal{O}_{Y, \bar{\xi}}$. It follows from Kummer theory that $K' = K(Y)(\sqrt[d]{\pi})$ is a finite Galois extension of $K(Y)$ of degree $d$ and since $A$ is a DVR, ramification theory implies that there is a unique point $\xi' \in Y_{K'}$ lying above $\xi$, with inertia group $I(\xi' \mid \xi) \cong \mathbb{Z}/d\mathbb{Z}$. Now Abhyankar’s lemma shows that $Y_{K(\bar{Z})K'}$ is étale over $\xi'$. Moreover, $Y_{K(\bar{Z})K'} \cong Y \times_X X_{K'}$ by construction. Finally, this shows that the base change of $Z := \overline{Y} \times_X X_{K'} \rightarrow Y$ to $Y_\bar{q}$ is precisely $Z'$, completing the proof. \square

Now let $X$ be any locally noetherian scheme and $Y \rightarrow X$ a proper morphism with connected geometric fibres. Suppose $x_0, x_1 \in X$ such that $x_0$ is a specialization of $x_1$, i.e. $x_0 \in \{x_1\}$. Let $\bar{x}_0, \bar{x}_1$ be geometric points of $X$ with images $x_0, x_1$, respectively, and fix geometric points $\bar{y}_0, \bar{y}_1 \in Y$ lying over $\bar{x}_0, \bar{x}_1$, respectively. Set $Z = \{x_1\}$, consider the local ring $\mathcal{O}_{Z, x_0}$ and note that the geometric fibres $Y_{x_0}$ and $Y_{x_1}$ over $X$ remain the same over $\text{Spec} \mathcal{O}_{Z, x_0}$. Denote by $A$ the completion of the localization at a height 1 prime ideal of $\mathcal{O}_{Z, x_0}$. Then $A$ is a complete DVR and we have a specialization map

$$\text{sp}_{\bar{y}_1} : \pi_1(Y_{\bar{q}}, \bar{y}_1) \longrightarrow \pi_1(Y_{\bar{x}_1}, \bar{y}_1)$$

for the generic point $\bar{\eta}$ of $\text{Spec} A$. On the other hand, by Corollary 4.4.9, the morphism $Y_{\bar{q}} \rightarrow Y_{\bar{x}_0}$ induces an isomorphism

$$\pi_1(Y_{\bar{q}}, \bar{y}_0) \cong \pi_1(Y_{\bar{x}_0}, \bar{y}_0).$$

**Definition.** The specialization map from $\bar{y}_0$ to $\bar{y}_1$ is the composition

$$\pi_1(Y_{\bar{x}_1}, \bar{y}_1) \cong \pi_1(Y_{\bar{q}}, \bar{y}_0) \longrightarrow \pi_1(Y_{\bar{x}_0}, \bar{y}_0).$$

**Theorem 4.5.4.** Suppose $\varphi : Y \rightarrow X$ is a proper flat morphism over a locally noetherian scheme $X$ with connected geometric fibres. Then for any geometric points $\bar{y}_0, \bar{y}_1$ of $Y$ lying over $x_0, x_1 \in X$ with $x_0 \in \{x_1\}$, the specialization map from $\bar{y}_0$ to $\bar{y}_1$ descends to an isomorphism of prime-to-$p$ parts

$$\pi_1(Y_{\bar{x}_1}, \bar{y}_1) \cong \pi_1(Y_{\bar{x}_0}, \bar{y}_0).$$

Grothendieck used this specialization result to prove the following analogue of Theorem 4.4.6.
4.5 Specialization and Characteristic $p$ Results

**Theorem 4.5.5.** Let $X$ be an integral, proper, normal curve of genus $g$ over an algebraically closed field $k$ of characteristic $p > 0$. Then for any geometric point $\bar{x} : \text{Spec} k \to X$,

1. $\pi_1(X, \bar{x})$ is finitely generated as a profinite group.

2. $\pi_1(X, \bar{x})^{(p)}$ is isomorphic to the profinite prime-to-$p$ completion of the topological fundamental group

\[ \Pi_g = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle. \]

**Corollary 4.5.6.** Let $k$ be an algebraically closed field of characteristic $p > 0$, $X$ a smooth, connected, projective scheme over $k$ and $\bar{x} : \text{Spec} k \to X$ a geometric point. Then $\pi_1(X, \bar{x})$ is finitely generated.

**Proof.** Same as the proof in characteristic 0, using the same reduction to the case of curves as in Theorem 4.4.3. \qed

**Example 4.5.7.** This result is false in characteristic $p$ when $X$ is not assumed to be proper. For example, Raynaud proved that for $X = A^1_k = \mathbb{P}^1_k \setminus \{\infty\}$, $\pi_1(X, \bar{x})$ is infinitely generated. (Compare this to Example 1.5.10.) Any Artin-Schreier equation $y^p - y = f(x)$, where $f(x) \in k[x]$ has prime-to-$p$ degree, defines a nontrivial Galois cover $Y_f \to \mathbb{P}^1_k$ that is étale over $A^1_k$ and has Galois group $\mathbb{Z}/p\mathbb{Z}$. If $k$ is algebraically closed, this gives an infinitely family of finite Galois covers of $A^1_k$. More generally, $\pi_1(X, \bar{x})$ is infinitely generated for any affine curve $X$ over $k$.

**Definition.** Let $\varphi : Y \to X$ be a finite separable morphism of normal integral schemes. Then $\varphi$ is tamely ramified at a codimension 1 point $P \in X$ if $\varphi$ is ramified at $P$ and for each $Q \in \varphi^{-1}(P)$, the ramification index $e(Q \mid P)$ of $\varphi$ at $Q$ is relatively prime to the characteristic of $k(P)$. We say $\varphi$ is tamely ramified, or simply tame, if it is tamely ramified at every codimension 1 point of $X$.

Let $X$ be an integral normal scheme, let $U \subseteq X$ be an open subscheme and denote by $\text{Fét}_U^t$ the full subcategory of $\text{Fét}_U$ consisting of finite separable covers $\varphi : Y \to X$ which are tamely ramified over $X \setminus U$. Note that the fibre functor $\text{Fib}_x$ restricts to a functor on $\text{Fét}_U^t$.

**Definition.** For a geometric point $\bar{x} : \text{Spec} \Omega \hookrightarrow X$ with image $x \in U$, the tame fundamental group of $U$ at $\bar{x}$ is

\[ \pi^t_1(U, \bar{x}) := \text{Aut}(\text{Fib}_x|_{\text{Fét}_U^t}). \]

**Theorem 4.5.8.** Let $\bar{x}$ be a geometric point in $X$ and $U \subseteq X$ an open subscheme containing $\bar{x}$. Then

1. $\pi^t_1(U, \bar{x})$ is a quotient of $\pi_1(U, \bar{x})$ and in particular a profinite group and its action on $\text{Fib}_x(Y)$ is continuous for all $Y \in \text{Fét}_U^t$.

2. The restriction of $\text{Fib}_x$ to $\text{Fét}_U^t$ induces an equivalence of categories

\[ \text{Fét}_U^t \xrightarrow{\sim} \{\text{finite, continuous, left } \pi^t_1(U, \bar{x})\text{-sets}\} \]

\[ Y \mapsto \text{Fib}_x(Y). \]
(3) Let $K_U^t$ be the compositum inside a fixed algebraic closure of $K = k(U)$ of all finite extensions $L/K$ such that the normalization $X_L$ of $X$ in $L$ is étale over $U$ and tamely ramified over $X \setminus U$.

(4) The tame fundamental group $\pi_1^t(U, \bar{x})$ is finitely generated as a profinite group.

Statements (1) – (3) can be derived from similar arguments as for the full étale fundamental group. The proof of (4) relies on the deep fact that tame covers in characteristic $p$ lift to $p$-tame covers in characteristic 0, where a $p$-tame cover is one defined over a field of characteristic 0 in which $p$ does not divide the order of any inertia groups of the cover. However, this more or less allows one to compute the tame fundamental group by lifting to characteristic 0.

In contrast, the pro-$p$-part of the fundamental group in characteristic $p$ is not completely understood and is the subject of much current research. But we will end the discussion here.