1. Background

My research program touches on a wide variety of problems in algebraic geometry, number theory and topology, but the unifying theme is the theory of algebraic stacks, which have a short but colorful history. A moduli problem is a type of classification problem in math which admits a natural algebraic parametrization – the corresponding space of solutions is a variety or scheme. Unfortunately, many important moduli problems cannot be solved directly by studying the points of a moduli space, due to the presence of nontrivial automorphisms. For example, every elliptic curve admits a nontrivial order 2 automorphism, the ‘hyperelliptic involution’, which makes it impossible to perfectly classify elliptic curves as points on a scheme. That is, elliptic curves form a stack with nontrivial automorphisms at every point.

This new perspective began to take hold after Deligne and Mumford [DM] proved that the moduli space of curves of genus \( g \) is irreducible. Deligne and Mumford’s approach to the issue of automorphisms in a moduli problem was to encode nontrivial automorphisms into the space itself. This strategy, fully realized by Grothendieck, Artin and their peers (cf. [Art], [SGA IV]), leads one to define algebraic stacks – for a modern account, see [Ols].

Beyond their utility in algebraic geometry itself, algebraic stacks are essential for understanding the structure of modular curves and modular forms (cf. [Ser70], [DR], [Beh], [VZB]), one of the central topics in modern number theory. My work centers on Deligne–Mumford stacks, whose points have no infinitesimal automorphisms, making their study more manageable. A guiding problem in the field is to classify the structure of Deligne–Mumford stacks of a given dimension over a field \( k \). Note that the goal is not necessarily to classify all Deligne–Mumford stacks up to isomorphism – for algebraic varieties, this is already a major undertaking. However, in some cases it is possible to classify the stack structures that occur, as we illustrate now.

Over the complex numbers, much is known about smooth Deligne–Mumford stacks: they have a coarse moduli space parametrizing their objects up to isomorphism [KM] (in the original sense of moduli problems); locally, they admit the structure of a complex orbifold [AV]; and when there is a dense subset of points with no nontrivial automorphisms, the remaining automorphism groups are essentially all cyclic [GS]. The classification problem is completely solved for smooth, separated Deligne–Mumford stacks of finite type over \( k = \mathbb{C} \) (loc. cit.). For example, when \( n = 1 \), a complex stacky curve is uniquely determined by its underlying Riemann surface and a finite list of numbers corresponding to the cyclic automorphism groups at finitely many marked points. These numbers are called the orders of the stacky points.

In contrast, such a clean classification fails in characteristic \( p > 0 \) for several reasons. Let \( \mathcal{X} \) be a stacky curve and \( x \) a point of \( \mathcal{X} \) with nontrivial automorphism group \( G \). If \( p \) divides \( |G| \), then \( G \) may not even be abelian, although [Ser79] places restrictions on which groups may occur. Even in the simplest case when \( G \) is cyclic of order \( p \), there are infinitely many non-isomorphic stacky curves with the same underlying scheme and marked point with automorphisms \( G \) – this is described further in Section 2.1. Thus any approach to the classification problem in positive characteristic requires finer invariants than the order of the automorphism group.
2. Stacky Curves in Positive Characteristic

2.1. My Work. We saw that complex stacky curves are completely classified by their underlying Riemann surface and a finite list of positive integers. The following core results in my research classify the structure of stacky curves in characteristic $p$ when the stacky points on the curve have order $p$. This is done by taking into account an arithmetic invariant known as the ramification jump. First, I show that cyclic $p$-covers of curves $Y \rightarrow X$ yield quotient stacks that have an Artin–Schreier root stack structure. Such a stack, first appearing in [Kob] and denoted by $\varphi^{-1}_m((L, s, f)/X)$, is a way of encoding $p$th roots of line bundles in characteristic $p$ and is built from certain geometric data on $X$. The notation and construction are described further in Section 2.2.

**Theorem 2.1 ([Kob], Thm. 1.1).** Suppose $\varphi : Y \rightarrow X$ is a finite separable Galois cover of curves over $k$ and $y \in Y$ is a ramified point with inertia group $\mathbb{Z}/p\mathbb{Z}$. Then étale-locally, $\varphi$ factors through an Artin–Schreier root stack $\varphi^{-1}_m((L, s, f)/X)$.

Theorem 2.1 also implies, using Artin–Schreier theory, that there are infinitely many nonisomorphic stacky curves over $\mathbb{P}^1$ with a single stacky point of order $p$. This phenomenon only arises in characteristic $p$ and illustrates the rich geometric structures in that setting.

Next, if a stacky curve has an order $p$ stacky point, then locally about this point the stacky curve is isomorphic to an Artin–Schreier root stack.

**Theorem 2.2 ([Kob], Thm. 1.2).** Let $\mathcal{X}$ be a stacky curve over a field $k$ of characteristic $p > 0$. If $\mathcal{X}$ contains a stacky point $x$ of order $p$, there is an open substack $\mathcal{Z} \subseteq \mathcal{X}$ containing $x$, an open subscheme $Z$ of the coarse space of $\mathcal{X}$ and a triple $(L, s, f)$ on $Z$ as above such that $\mathcal{Z} \cong \varphi^{-1}_m((L, s, f)/Z)$ for some $(m, p) = 1$.

Moreover, if the coarse space of $\mathcal{X}$ is $\mathbb{P}^1$, then this holds nearly globally:

**Theorem 2.3 ([Kob], Thm. 1.3).** Suppose all the nontrivial stabilizers of $\mathcal{X}$ are cyclic of order $p$. If $\mathcal{X}$ has coarse space $X = \mathbb{P}^1$, then $\mathcal{X}$ is the fibre product of finitely many Artin–Schreier root stacks of the form $\varphi^{-1}_m((L, s, f)/\mathbb{P}^1)$.

If the coarse space $X$ of $\mathcal{X}$ is not $\mathbb{P}^1$, then Theorem 2.3 fails in general when the genus of the curve $X$ is at least 1 ([Kob], Ex. 6.21).

2.2. Methods. Here I describe an approach to proving Theorems 2.1, 2.2 and 2.3, and I provide a description of the stacks $\varphi^{-1}_m((L, s, f)/X)$ in their statements.

The main feature of tame stacky curves (those with no order $p$ automorphisms) that makes classification possible is that they are locally a root stack: a modification of the coarse space obtained by replacing a point with a stacky point of “fractional order”. This construction, due independently to Cadman [Cad] and Abramovich–Graber–Vistoli [AGV], is an essential tool in the study of algebraic stacks. Given a line bundle $L$ on the base $X$ and a section $s$ of $L$, the $n$th root stack $\sqrt[n]{(L, s)/X}$ is a stack parametrizing $n$th roots of $(L, s)$, or pairs $(M, t)$ such that $M^{\otimes n} \cong L$ and $t^n$ is identified with $s$. Alternatively, $\sqrt[n]{(L, s)/X}$ can be constructed functorially by pulling back the universal $n$th root stack $[\mathbb{A}^1/\mathbb{G}_m] \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$, which is induced by $x \mapsto x^n$.

One might try to classify so-called wild stacky curves in characteristic $p$ in a similar fashion. However, the above pullback method fails because the map $x \mapsto x^p$ has much different behavior in characteristic $p$. To remedy this, I utilized a geometric version of the theory of cyclic field extensions of order $p$, known as Artin-Schreier theory.


A cyclic extension $L/k$ with Galois group $\mathbb{Z}/p\mathbb{Z}$ is called an Artin–Schreier extension. These have an explicit form (cf. [Ser79], Ch. X) and one can extract an arithmetic invariant of $L/k$ called the ramification jump. It is known that different ramification jumps yield non-isomorphic $\mathbb{Z}/p\mathbb{Z}$-extensions, a phenomenon which does not occur in characteristic 0. Moreover, Artin–Schreier extensions of a local field are completely classified up to isomorphism by their ramification jump ([Ser79], Ch. IV), so this discrete invariant is important to understanding $\mathbb{Z}/p\mathbb{Z}$-extensions in characteristic $p$.

To study a stacky curve with automorphism group $\mathbb{Z}/p\mathbb{Z}$, I used the following construction [Kob], based on a suggestion of David Rydh. For an integer $m$ coprime to $p$, the universal Artin–Schreier root stack with ramification jump $m$ is the cover of stacks $\mathcal{P}^{-1}_m((L,s,f)/X)$ induced by the order $p$ map $[x : y] \mapsto [x^p : y^p - yx^{m(p-1)}]$. 

**Definition.** For a scheme $X$, a line bundle $L \to X$ and sections $s$ of $L$ and $f$ of $L^\otimes m$ determine a map $X \to \mathbb{P}(1,m)/\mathbb{G}_a$. Then the Artin–Schreier root stack of $X$ along the data $(L,s,f)$, denoted $\varphi_{m}^{-1}((L,s,f)/X)$, is defined by the normalized pullback of the universal Artin–Schreier root stack along this map.

Artin–Schreier root stacks fit seamlessly into the existing theory: they are functorial and Deligne–Mumford; they can be iterated easily; and they admit étale neighborhoods given by affine equations such as those in Theorem 2.1. This last feature in particular allowed me to reduce many proofs to the key case of a one-point cover of curves $Y \to X$, whose theory is well-studied.

2.3. Generalization. Theorems 2.1, 2.2 and 2.3 answer the classification problem for stacky curves with $\mathbb{Z}/p\mathbb{Z}$ automorphism groups. For curves with larger automorphism groups, the classification problem can be attacked using a vast generalization of the theory described in Section 2.2.

I am currently developing the theory for higher order cyclic $p$-groups. Here is a glimpse into the needed tools. Cyclic field extensions $L/k$ are described by Artin–Schreier–Witt theory, which associates to $L/k$ an element in the ring of Witt vectors $\mathbb{W}_n(k)$ of $k$ (cf. [Wit], [Lan]). In [Gar], Garuti constructed an equivariant compactification $\mathbb{W}_n$ of $\mathbb{W}_n$ that classifies $\mathbb{Z}/p^n\mathbb{Z}$-covers of algebraic varieties. By studying maps from the base of a $\mathbb{Z}/p^n\mathbb{Z}$-cover of curves to $\mathbb{W}_n$, one can read off a sequence of ramification jumps which appear in the classification of such covers.

Garuti’s perspective dovetails perfectly with the geometric approach I took in the Artin–Schreier case. Namely, I replace $\mathbb{G}_a$ by $\mathbb{W}_n$ and $\mathbb{P}(1,m)$ by a suitable stacky compactification of the Witt vectors which I denote $\mathbb{W}(m_1, \ldots, m_n)$ for a sequence of positive integers $(m_i)$. These compactifications are constructed as an iterated root stack over $\mathbb{W}_n$ and, while seemingly unwieldy, contain a wealth of arithmetic information not present in $\mathbb{W}_n$. My immediate objectives are to determine the structure of $\mathbb{W}(m_1, \ldots, m_n)$ as an algebraic stack and use this information to construct an Artin–Schreier–Witt root stack of a scheme $X$ along the appropriate data. Then I plan to attack the classification of stacky curves with more general automorphism groups as I did in [Kob].

3. Applications in Arithmetic Geometry

3.1. Canonical Rings. Stacky curves are currently of particular interest in birational arithmetic geometry. One direction of study concerns the canonical ring $R(\mathcal{X}) = \bigoplus_{n=0}^{\infty} H^0(\mathcal{X}, \Omega_{\mathcal{X}}^n)$. 

Here, $\Omega_X^{\otimes n}$ are tensor powers of the cotangent bundle of $X$, which encode information about projective embeddings of the curve. The authors in [VZB] give explicit generators and relations for the canonical ring when $X$ is a tame stacky curve, but the problem remains open in the wild case.

Here are some preliminary results in the direction of answering this problem in positive characteristic. For a point $x$ of a stacky curve $X$ with automorphism group $G_x$, one can define a ramification filtration $(G_{x,i})_{i \geq 0}$ of $G_x$ using the geometry of $X$, in the sense of [Ser79].

**Proposition 3.1** (Stacky Riemann–Hurwitz; [Kob], Prop. 7.1). For a stacky curve $X$ with coarse moduli space $\pi : \mathcal{X} \to X$, the following defines a canonical divisor of $X$:

$$K_X = \pi^* K_X + \sum_{x \in X(k)} \sum_{i \geq 0} (|G_{x,i}| - 1) x.$$

Next, using a Riemann–Roch formula adapted for stacks (cf. [VZB] or [Beh]), one can compute formulas for global sections of line bundles on $X$. For example, if $X$ is a stacky $\mathbb{P}^1$ with a single stacky point of order $p$ and ramification jump $m$ prime to $p$, then the number of linearly independent sections of the cotangent bundle depends on $m$:

**Proposition 3.2** ([Kob], Ex. 7.8). For $X$ described above, $\dim H^0(X, K_X) = m - \left\lfloor \frac{m}{p} \right\rfloor - 1$.

When $n \geq 2$, there are more complicated formulas for $\dim H^0(X, nK_X)$ in which the integer parts of $m$ and $n$ interact, and these illustrate how the ramification jump contributes to the canonical ring in new ways. I intend to compute these formulas in larger classes of examples to further understand the patterns that arise.

### 3.2. Modular Forms.

I am most excited about applying the theory in Section 3.1 to modular forms in characteristic $p$. Typically encountered in complex analysis as certain holomorphic functions on the complex upper half-plane, modular forms in fact appear in a wide number of contexts in number theory. For me, they are best studied as differential forms on a modular curve, a natural compactification of the quotient of upper half-plane by a given subgroup of $SL_2(\mathbb{Z})$. This curve possesses the structure of a Deligne–Mumford stack, which leads to the beautiful formulas for the dimensions of modular forms of weight $k$ found in most introductory texts (cf. [Ser70], [DI]).

Katz’s theory of geometric modular forms [Kat] allows one to construct models of modular curves over a field of characteristic $p$, but the resulting covers of curves often have wild ramification. In characteristic $p = 3$, for example, the quotient stack $\mathcal{X} = [X(\ell)/PSL_2(\mathbb{F}_\ell)]$ of the modular curve $X(\ell)$ for $\ell \neq p$ prime has two stacky points with automorphism groups $\mathbb{Z}/\ell\mathbb{Z}$ and $S_3$, the latter of which has higher ramification invariants that contribute to the canonical ring of $\mathcal{X}$ in nontrivial ways [BCG]. I am beginning to study these examples using the techniques described above, with the goal of providing a stack-theoretic interpretation of the ring of Katz modular forms in characteristic $p > 0$.

### 4. $\mathbb{A}^1$-Enumerative Geometry

In a different direction, algebraic stacks have applications in a fast-expanding area of research known as $\mathbb{A}^1$-enumerative geometry, a relatively recent outgrowth of Morel and Voevodsky’s program uniting algebraic geometry and homotopy theory, known as $\mathbb{A}^1$-homotopy theory [MV].
4.1. **A Short History.** In a growing number of publications ([KW17a], [KW17b], [Lev], [SW], [LV]), researchers have applied techniques from $\mathbb{A}^1$-homotopy theory to produce enriched versions of classical enumerative geometry theorems, such as the count of lines on a complex cubic surface ([KW17b]), that make sense over an arbitrary field $k$ (possibly of characteristic not 2). The authors in [KW17a] and [KW17b] develop a technique for solving enumerative problems over $k$ using arithmetic enrichments of classical invariants of vector bundles such as the Euler class, which in turn can be computed by summing the local indices of a section of the bundle as one does in manifold topology. In this case the enriched Euler class is an element of the Grothendieck–Witt group $GW(k)$, the free abelian group generated by isomorphism classes of finite dimensional quadratic spaces over $k$. Voevodsky’s proof of the Milnor conjecture ([Voe1], [Voe2]), which identified $GW(k)$ with the Milnor $K$-theory of $k$, paved the way for these types of enumerative problems to be studied over arbitrary fields using $\mathbb{A}^1$-homotopy theory.

4.2. **My Work.** Together with Libby Taylor in a recent paper [KT], I explain certain obstacles to this enrichment process using descent theory and then remove the obstacles by passing to a suitable root stack. This technique produces **new enriched formulas** that better explain existing phenomena. Moreover, our approach is unique in that it unites the parallel notions of the topology and algebraic geometry of the original $\mathbb{A}^1$-homotopy theory program, the language of quadratic forms used to state the enriched formulas, and the modern theory of algebraic stacks. In the future, we plan to use our approach to shed light on other enriched problems in enumerative geometry.

**References**


