ON THE SYMMETRIES AND GLOBAL STRUCTURE OF THE UPPERCASE XI FUNCTION

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Abstract. The known sets of symmetries of B. Riemann’s zeta function, \( \zeta(z) \), the uppercase Xi function, \( \Xi(z) \), and the square of the modulus of the uppercase Xi function, \( |\Xi|^2(z,y) \), are characterized using point groups and their irreducible representations, with the schema employed related to the classification of real-valued functions of a real variable as either symmetric, a.k.a. even, anti-symmetric, a.k.a. odd, or asymmetric. Although nothing new is gained from cataloging their symmetries in this way, intuition garnered from symmetry analyses of these functions is exploited to derive equations pertinent to understanding the relative locations of roots in the nontrivial locus of \( \zeta(z) \). In particular, general expressions and explicit functional equations for partial derivatives of \( |\Xi|^2(z,y) \) are derived, with symmetry used to show that \( \partial_y|\Xi|^2(z,0) = 0 \).

\( \forall x \in \mathbb{R}, \text{ that } \partial_y|\Xi|^2(z,y) \geq 0, \forall (x,y) \in \mathbb{R}^2, \text{ and that } \log_\gamma(\pi) = \frac{2\zeta(0)}{\gamma(\frac{1}{2})} + \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} \text{, minima and saddle points of } |\Xi|^2(z,y) \text{ are optimized, with all proven to reside along the line } y = 0, \text{ and the principal, Gauss and mean curvatures of the surface comprising the graph of } |\Xi|^2(z,y) \text{ are calculated, with the lines } y = 0 \text{ and } x = 0 \text{ demonstrated to be normal sections of this surface, i.e. geodesics. Hinging on the collective insights, algorithms to locate roots in the nontrivial locus of } \zeta(z) \text{ are also presented.}

1. Introduction

B. Riemann investigated the Dirichlet series as a function of a complex variable

\begin{equation}
\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z},
\end{equation}

to better understand the asymptotic distribution of the prime numbers, [1–7] where \( z = x + iy \in \mathbb{C}, x, y \in \mathbb{R} \) and \( y^2 = -1 \). [8–14] In particular, he proved that the domain of Eq. 1, which converges for \( x > 1 \), could be extended to \( z \in \mathbb{C} \) having \( x < 1 \) by re-expressing the infinite series as the contour integral [1–4]

\begin{equation}
\zeta(z) = \frac{\Gamma\left(-z\right)}{2\pi i} \int_{-\infty}^{+\infty} \frac{(-x)^z}{e^x - 1} \, dx.
\end{equation}

Because Eq. 2 is valid for \( \mathbb{C} - \gamma(\zeta) \), it is the unique analytic continuation of Eq. 1 to \( \mathbb{C} - \gamma(\zeta) \). [1–4] Here, \( \gamma(\zeta) \) connotes the set of poles of \( \zeta(z) \), with \( \gamma(\zeta) = \{ z \in \mathbb{C} | \zeta(z) = \infty \} = \{(1 + i0)\} \), \( ||\gamma(\zeta)|| = 1 \) and this pole simple because \( \lim_{z \to -1}(z - 1)\zeta(z) = 1 \). B. Riemann’s work was buoyed by the research of others. Amongst them, L. Euler’s is the most notable, as he established the connection between \( \zeta(x) \) and the prime numbers through the identity \( \zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-x}} \), where \( p_n \) indexes the primes, in addition to the formulas \( \zeta(2) = \frac{\pi^2}{6} \) and \( \zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{(2n)!} \), where \( B_{2n} \) are Bernoulli numbers, [2,5–7]

B. Riemann also derived equations relating \( \zeta(z) \) to \( \zeta(1-z) \) that are referred to as Riemann’s functional equations (RFEs), in addition to two other variants of \( \zeta(z) \) known now as the lowercase xi function, \( \xi(z) \), and the uppercase Xi function, \( \Xi(z) \), since it has become customary to adopt E. Landau’s swap in definitions, [1,2] The lowercase Xi function, \( \xi(z) \), which is denoted \( \Xi(z) \) in [1], is defined by the functional equation [1,2]

\begin{equation}
\xi(z) = \frac{1}{2} \pi^{-\frac{z}{2}} z (z - 1) \Gamma\left(\frac{z}{2}\right) \zeta(z)
\end{equation}

and obeys \( \xi(1-z) = \xi(z) \). The importance of this representation lies in that all singularities of \( \Gamma\left(\frac{z}{2}\right) \) and \( \zeta(z) \) are removed because the polynomial factors \( z \) and \( (z - 1) \) regularize \( \Gamma\left(\frac{z}{2}\right) \) and \( \zeta(z) \) at \( z = 0 \) and \( z = 1 \).

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respectively, with \( \xi(0) = \xi(1) = \frac{1}{2} \). In other words, \( \xi(z) \) is holomorphic \( \forall z \in \mathbb{C} \) and \( \gamma(\xi) = \emptyset [1,2] \). The uppercase Xi function, \( \Xi(z) \), which is denoted \( \xi(z) \) in [1], is obtained from \( \xi(z) \) through the transformation \( z \to \frac{1}{2} + iz \), with \( \Xi(z) \) defined by the functional equation [1,2] \[
(4) \quad \Xi(z) = \frac{1}{2} \pi i^{-\frac{1}{2} - \frac{|z|}{2}} (i z - \frac{1}{2}) \Gamma(\frac{i}{4} + \frac{|z|}{2}) \zeta(\frac{i}{2} + iz).
\]
Since the analyticity of \( \xi(z) \) is clearly not altered by this transformation, \( \Xi(z) \) is also holomorphic \( \forall z \in \mathbb{C} \), with \( \gamma(\Xi) = \emptyset [1,2] \). Here, roots of the polynomial factors \( (\frac{1}{2} + iz) \) and \( (iz - \frac{1}{2}) \) regularize poles in \( \gamma(\Gamma) \) of \( \Gamma(\frac{i}{4} + \frac{|z|}{2}) \) and \( \gamma(\zeta) \) of \( \zeta(\frac{i}{2} + iz) \), respectively, with \( \Xi(z) = \Xi(\overline{z}) = \frac{1}{2} \). The advantage of this representation relative to \( \xi(z) \) and \( \xi(z) \) is attributable to \( \Xi(z) \) possessing symmetric reflection symmetry, with \( \Xi(-z) = \Xi(z) \) [1].

The locations of roots in the nontrivial locus of \( \xi(z) \), denoted \( \Re(\xi) \), has been a matter of considerable interest, with the entire locus \( \Re_\xi(\xi) \) \( \{ z \in \mathbb{C} | \xi(z) = 0 \} \) \( \{ \Re(\xi), \Re_\xi(\xi) \} \) also containing the trivial zeros, denoted \( \Re(\xi) = \{ -2n + 0 \} \in \mathbb{C}[n \in \mathbb{N}] \). [1,2,5-7] A longstanding hypothesis germane to this work is that all roots in \( \Re(\xi) \) reside on the critical line \( z = \frac{1}{2} + iy \in \mathbb{C} [1,2] \). Despite much effort, though, it has only been proven that \( \Re(\xi) \) must inhabit a critical strip \( z = \left( \frac{1}{2} + \alpha \right) + iy \) for \( |\alpha| < \frac{1}{2} [2,5,6] \). Progress in this regard is made within by way of an original analysis of the square of the modulus of \( \Xi(z) \), i.e. \( \Xi(z) \Xi(\overline{z}) = \Xi(z) \Xi(\overline{z}) = \iff \Xi \geq 0 \geq 0 \), that leverages the unsuitably integrated properties of \( \Xi(z) \) and so \( \Xi(\overline{z}) \), in addition to the fact that \( \Re(\Xi) \Re(\Xi) = \Re(\Xi(\overline{z})) \Re(\Xi) \neq \Re(\xi) \). Key observations are that \( \partial_\text{y}_0 \Xi(\overline{z}) = 0 \forall x \in \mathbb{R} \), that \( \partial_\text{y}_0 \Xi \geq 0 \) \( \forall x \in \mathbb{R} \) and so that \( \Xi(\overline{z}) \) is bounded away from zero if \( y \neq 0 \), i.e. \( \Xi(\overline{z}) > 0 \) for \( y \neq 0 \), with the latter result giving that all roots of \( \Xi(\overline{z}) \) arise on the line \( y = 0 \).

Why is this approach assured to yield progress? Well, a way to demonstrate that all roots of \( \cos(z) \) reside on the line \( y = 0 \) is to show that the zero set of \( \partial_\text{y}_0 \cos \Xi \geq 0 \), \( \forall x \in \mathbb{R} \) and \( \forall n \in \mathbb{N} \). That is, linearity in the root locus of \( \cos(z) \) and so \( \cos \Xi \Xi = \cos^2(x) + \sinh^2(y) \) is a corollary of \( \partial_\text{y}_0 \cos \Xi = \sinh(2y) \) equaling zero for \( y = 0 \) and \( \partial_\text{y}_0 \cos \Xi \Xi = 2^n \sinh(2y) \) not being negative along this line \( \forall n \in \mathbb{N} \), as \( \sinh(2y) = 0 \), \( \cosh(2y) = 1 \) and \( 2^n > 0 \).\( \forall n \in \mathbb{N} \) and \( y = 0 \).

In §Sec. 2, the distinct sets of symmetries of \( \xi(z), \Xi(z) \) and \( \Xi(\overline{z}) \) are characterized using point groups. The differential structure of \( \Xi(\overline{z}) \) is analyzed in §Sec. 3. Ramifications of the results of §Secs. 2 and 3 on the global structure and analytic properties of \( \Xi(\overline{z}) \) and \( \Xi(z) \) are conferred in §Sec. 4, while §Sec. 5 contains a brief summary and concludes.

2. Characterizing the symmetries of functions with point groups

The classification of a real-valued function of a real variable, i.e. \( f(x) \) where \( x \in \mathbb{R} \), as symmetric, anti-symmetric or asymmetric depends on the effects of the transformation \( x \to -x \), i.e. under additive inverses of the variable \( x \), with \( f(x) \) symmetric if \( f(x) = f(-x) = 0 \), \( \forall x \), anti-symmetric if \( f(x) + f(-x) = 0 \), \( \forall x \) and asymmetric otherwise. Knowing the parity of an \( f(x) \) under such a transformation, is useful because symmetry considerations alone dictate properties evinced by these functions and their root and pole loci, \( \Re(f) \) and \( \gamma(f) \). For instance, given a differentiable, nonzero even[odd] \( f(x) \) over \( \mathbb{R} \), then \( f(x) \) is nonzero[zero] at the origin, \( dx_f \equiv \frac{df}{dx} \equiv \text{odd}[\text{even}] \overset{\text{over } \mathbb{R}}{\overset{x}{\longrightarrow}} \overset{\text{even}[\text{odd}] \text{ terms contribute to Maclaurin series of } f(x)}. \)

Below, the symmetries of \( f(x) \)’s are alternatively characterized with point groups and their irreducible representations(\textit{irreps}). [15–17] The classification schema is then applied to \( f(x), \Re(x), \Im(x) \) and \( \Re(f) \cup \gamma(f) \) of \( \xi(z) \) and \( \Xi(z) \), as well as \( \Xi(\overline{z}) \). Albeit no new knowledge is acquired from characterizing the symmetries of these functions using point groups, understanding the implications of the assignments derived on their properties and how they manifest in their graphs is essential to appreciating the analysis in §Secs. 3 and 4.

2.1. Algebraic preliminaries.

As a matter of notation, \( G = \{ g_1, g_2, \ldots, g_n \} \) will represent a point group of order \( \|G\| = n \), \( \rho_G = \{ \rho_{g_1}, \rho_{g_2}, \ldots, \rho_{g_n} \} \) a representation of \( G \), with the action of \( \rho_G \) on a generic set \( \alpha \) given by \( \rho_G \times \alpha = \{ \rho_{g_1} \times \alpha, \rho_{g_2} \times \alpha, \ldots, \rho_{g_n} \times \alpha \} = \chi_{G|\alpha} \chi_{g_1} \chi_{g_2} \ldots, \chi_{g_n} \alpha = \chi_{G|\alpha} \times f \), where \( \chi_{g_n} \) represents the parity of \( \chi_{g_n} \alpha \), while \( \chi_{G|\alpha} = \{ \chi_{g_1}, \chi_{g_2}, \ldots, \chi_{g_n}, \alpha \} \). [18–20] For the purposes of this work, \( \alpha \) will be an \( f(x), f(x,y), f(z) = f(x+iy) = \Re(x), \Im(x), \Re(y), \Re(f) \cup \gamma(f) \) or one of their graph in \( \mathbb{R}^3 \).

With these designations, the \( TSG \) of an \( \alpha \) is defined as
Definition 2.1. The TSG of \( \alpha \) is the largest point group under which \( \alpha \) transforms as an irrep, with \( \text{TSG}(\alpha) = \{ \text{largest } G | x_{G\{\alpha} \cong \Lambda \} \} \), where \( \Lambda \) denotes an irrep of \( G \).

Note that character tables for all finite point groups referenced can be found in the Appendix A. [16,17]

2.2. The TSG for \( f(x), R(f) \cup \gamma(f) \) and their graphs.

In this subsection, the transformation properties \( f(x) \), their loci of roots and poles, \( R(f) \cup \gamma(f) \), in addition to their graph counterparts are characterized using point groups and their irreps.

Theorem 2.2. The TSG of symmetric[anti-symmetric] \( f(x), R(f) \cup \gamma(f) \) and their graphs \( gr(f) \) and \( gr(R(f) \cup \gamma(f)) \) are the \( C_i, C_{1v}, C_{2v} \) and \( C_{2v} \) point groups, respectively, with \( x_{Ci[f]} = A_g[A_u], x_{C_{1v}[R(f) \cup \gamma(f)]} = A_g[A_g], x_{C_{2v}[R(f) \cup \gamma(f)]} = A_1[A_1] \).

Proof. With the 2D inversion point group \( C_i = \{ E, I \} \), where \( E \) and \( I \) are the identity and inversion symmetry elements, the parity of the action of \( \rho_{C_i} \) on symmetric and anti-symmetric \( f(x) \) yields \( \rho_{C_i} \times f(x) = [\rho_{E}f(x), \rho_{I}f(x)] = [f(x), f(-x)] = [x_{E}f(x), x_{I}f(x)] = x_{Ci[f x]} \), with \( x_{Ci[f]} = \{ \{x, x\} \} = \{+, +\} \) and \( x_{C_{1v}[f]} = \{+, -, -\} \), respectively. That is, \( x_{Ci[f]} \) for symmetric and anti-symmetric \( f(x) \) correspond with the \( A_g \) and \( A_u \) irreps of \( C_i \), respectively, and asymmetric \( f(x) \) transform as the \( A \) irrep of \( C_1 \).

Regarding the symmetries of the set of root and poles in \( R(f) \cup \gamma(f) = \{ x \in \mathbb{R} | f(x) = 0 \} \cup \{ x \in \mathbb{R} | f(x) = \infty \} \), with \( gr(f) \subset \mathbb{R}^2 \), comparable analyses reveal that \( gr(f) \) for symmetric and anti-symmetric \( f(x) \) transform as \( B_2 \) and \( A_2 \) in \( C_{2v} = \{ E, C_2, \sigma_x, \sigma_y \} \), respectively, while asymmetric \( gr(f) \) transform as \( A'' \) in \( C_4 \). More explicitly, \( \rho_{C_{2v}} \times gr(f) \) for symmetric \( f(x) \) yields \( \rho_{C_{2v} \times gr(f)} = \{ \rho_{E}gr(f), \rho_{I}gr(f) \} = \{ gr(f), -gr(f) \} \), while \( \rho_{C_{2v}} \times gr(f) \) for symmetric \( f(x) \) yields \( \rho_{C_{2v} \times gr(f)} = \{ gr(f), gr(f), -gr(f) \} \). To make these assignments, observe that \( \rho_{C_{2v}} \) on a generic point \( (x, \epsilon) \in \mathbb{R}^2 \) gives \( \rho_{C_{2v}} \times (x, \epsilon) = [\rho_{E}x, \rho_{I}x, \rho_{E}x, \rho_{I}x](x, \epsilon) = [(x, \epsilon), (-x, \epsilon), (x, \epsilon), (-x, \epsilon)] \) and so the action of \( \rho_{C_{2v}} \) on a point on the curve in \( gr(f(x)) \), i.e. \( (x, \epsilon = f(x)) \in \mathbb{R}^2 \), gives \( \rho_{C_{2v}} \times (x, f(x)) = [(x, f(x)), (x, f(x)), (x, f(x)), (x, f(x))] \).

Summarizing, all real-valued functions of a real variable, \( f(x) \), satisfy \( C_i \geq \text{TSG}(f) \geq C_1 \) and \( 2 \geq ||TSG(f)|| \geq 1 \), with \( TSG(f) = C_i \) if and only if \( f(x) \) is symmetric or anti-symmetric. Similarly, all \( gr(f) \) satisfy \( C_{2v} \geq \text{TSG}(f) \geq C_s \) and \( 4 \geq ||TSG(gr(f))|| \geq 2 \), with \( TSG(gr(f)) = C_{2v} \) if and only if \( f(x) \) is symmetric or anti-symmetric.

2.3. The TSG’s of \( \Xi(z), \mathbb{R}, \mathfrak{E}, \mathfrak{S}, \mathbb{R}(\Xi) \) and their graphs.

In this subsection, the transformation properties \( \Xi = \mathbb{R} \cup \mathfrak{S} \mathbb{R}, \mathbb{R}, \mathfrak{S}, \mathbb{R}(\Xi) \), and their graph counterparts are characterized using point groups and their irreps.

Theorem 2.3. The TSG of \( \Xi(z), \mathbb{R}, \mathfrak{S}, \mathbb{R}(\Xi), gr(\mathbb{R}), gr(\mathfrak{S}), gr(\mathbb{R}(\Xi)) \) are \( C_i, C_{2v}, C_{2v}, C_{2v}, C_{2v}, D_{2h}, D_{2h}, D_{2h}, D_{2h} \), respectively, with \( x_{C_i[z]} = A_g, x_{C_{2v}[\mathbb{R}]} = A_1, x_{C_{2v}[\mathfrak{S}]} = A_2, x_{C_{2v}[\mathbb{R}(\Xi)]} = A_1, x_{C_{2v}[gr(\mathbb{R})]} = A_1, x_{C_{2v}[gr(\mathfrak{S})]} = B_{1u}, x_{C_{2v}[gr(\mathbb{R}(\Xi))]} = A_u \) and \( x_{D_{2h}[\mathbb{R}(\Xi)]} = \Sigma^+ \).

Proof. The TSG and irreps of \( \Xi(z), \mathbb{R}, \mathfrak{S} \) and \( \mathbb{R}(\Xi) \) are derivable from the constraints imposed on these functions by the symmetric reflection formula, i.e. \( \Xi(-z) = \Xi(-x+iy) = \Xi(x+iy) = \Xi(z) \). First note that \( \Xi(z) \) is a symmetric function in both \( x \) and \( y \) about the origin, as \( \Xi(-x+0) = \Xi(x+0) = \Xi(x) \) and \( \Xi(0-y) = \Xi(0+y) = \Xi(0) \). Consequences of these identities are that \( \mathbb{R}_{\Xi(z,y)} \) is symmetric and nonzero in both \( x \) and \( y \) about the origin, with \( \mathbb{R}_{\Xi(-x,0)} = \mathbb{R}_{\Xi(x,0)} = \Xi(z) \) and \( \mathbb{R}_{\Xi(0,-y)} = \mathbb{R}_{\Xi(0,y)} = \Xi(z) \), while \( \mathfrak{S}_{\Xi(z,y)} \) is anti-symmetric about the origin and zero in both \( x \) and \( y \), with \( \mathfrak{S}_{\Xi(-x,0)} = \mathfrak{S}_{\Xi(x,0)} = 0 \) and \( \mathfrak{S}_{\Xi(0,-y)} = \mathfrak{S}_{\Xi(0,y)} = 0 \).

These symmetries of \( \mathbb{R}_{\Xi(z,y)} \) and \( \mathfrak{S}_{\Xi(z,y)} \) in the \( x \) and \( y \) variables with respect to the origin dictate that each of these \( f(x,y) \) maximally transforms as an irrep of \( C_{2v} = \{ E, I, \sigma_x, \sigma_y \} \) point group, with \( TSG(\mathbb{R}) = TSG(\mathfrak{S}) = TSG(\Xi(z)) = C_{2v} \). More specifically, the action of the operators on \( \rho_{C_{2v}} \) on the \( \mathbb{R} \) function gives \( \rho_{C_{2v}} \times \mathbb{R}_{\Xi(z,y)} = \{ \rho_{E}, \rho_I, \rho_{E}, \rho_I \} \mathbb{R}_{\Xi(z,y)} = \{ \mathbb{R}_{\Xi(z,y)}, \mathbb{R}_{\Xi(-x,-y)}, \mathbb{R}_{\Xi(-x,y)}, \mathbb{R}_{\Xi(x,-y)} \} = \{ \mathfrak{S}_{\Xi(z,y)}, \mathfrak{S}_{\Xi(x,y)} \} \mathbb{R}_{\Xi(z,y)} = \mathfrak{S}_{\Xi(z,y)} \mathbb{R}_{\Xi(z,y)} \), where \( \mathfrak{S}_{\Xi(z,y)} = \{+, +, +, +\} = A_1 \). The exact same analysis on \( \mathfrak{S}_{\Xi(z,y)} \) yields \( \rho_{C_{2v}} \times \mathfrak{S}_{\Xi(z,y)} = \{+, +, -, -\} \mathfrak{S}_{\Xi(z,y)} \), with \( \mathfrak{S}_{\Xi(z,y)} = \{+, +, -, -\} = A_2 \).
As a result of $\chi_{C_{2v}[\mathbb{R}_2]} \neq \chi_{C_{2v}[\mathbb{S}_2]}$, the uppercase Xi function $\Xi_{(x+y)}$ maximally transforms as an irrep of a subgroup of $TSG(\mathbb{R}_2) = TSG(\mathbb{S}_2) = C_{2v}$. This is easily appreciated by noting that $\rho_{C_i, x} \Xi_{(x+y)} = \chi_{C_i}[\Xi_{(x+y)}]_{(x+y)}$, where $\chi_{C_i}[\Xi_{(x+y)}] = [+,-] = A_y$. Alternatively, observe how $TSG(\Xi)$ arises as the normal subgroup of $TSG(\mathbb{R}_2)$ whose actions on $\mathbb{R}_2$ and $\mathbb{S}_2$ have equivalent parities. In equations this amounts to $TSG(\Xi) = \{ g_n \in TSG(\mathbb{R}_2) = TSG(\mathbb{S}_2) = C_{2v}, \rho_{g_n}[\Xi_{(x+y)}] = \pm \Xi_{(x+y)} \}$ as $g_n \in TSG(\mathbb{R}_2) = TSG(\mathbb{S}_2) = C_{2v}, \chi_{g_n}[\mathbb{R}_2] = \pm 1 = I \neq C_{2v}$, with the elements of symmetry of $\Xi(\Xi) = \Xi_{(x+y)}$ evidenced to correspond with the $+$'s in the direct product of $\chi_{C_{2v}[\mathbb{R}_2]}$ and $\chi_{C_{2v}[\mathbb{S}_2]}$.

Since $\chi_{C_{2v}[\mathbb{R}_2]} \otimes \chi_{C_{2v}[\mathbb{S}_2]} = [+,-,-,-] = A_2$, this gives $TSG(\Xi) = C_i \otimes C_{2v}, \chi_{C_i}[\Xi] = [+,-] = A_y$ and $TSG(\Xi) \otimes TSG(\mathbb{R}_2) = TSG(\mathbb{S}_2)$.

As $R(\Xi)$ are symmetrically distributed about the $x$ and $y$ axes, irrespective if roots in $R(\Xi)$ exist off the line $y = 0$ and so if $R(\zeta)$ exist off the critical line, it is straightforward to prove that the $TSG$ associated with the locus of roots in $R(\Xi)$ is also $C_{2v}$, with $\chi_{C_{2v}[R(\Xi)]} = [+,-,-,-]$. For this reason, $R(\Xi)$ can be said to transform under $C_{2v}$ as the totally symmetric irrep, $A_1$.

The symmetry structure of the $\mathbb{R}_2(\mathbb{X}, y)$ and $\mathbb{S}_2(\mathbb{X}, y)$ functions propagates into their graphs as surfaces in $R^3$ in a regular manner, with $gr(\mathbb{R}_2) = \{(x, y, \epsilon_1) \in R^3|x = x = y = \epsilon_1 = \mathbb{R}_2(\mathbb{X}, y)\}$ and $gr(\mathbb{S}_2) = \{(x, y, \epsilon_2) \in R^3|x = x = y = \epsilon_2 = \mathbb{S}_2(\mathbb{X}, y)\}$. For these functions, it was determined that $\chi(\mathbb{X})$ and $\chi(\mathbb{S}_2)$ transform as irreps of $D_{2h} = \{E, C_2', C_2'' \sigma_x, \sigma_y, \sigma_x \sigma_y\}$. Unambiguously, the action of $D_{2h}$ on $gr(\mathbb{R}_2)$ and $gr(\mathbb{S}_2)$, i.e. $\rho_{D_{2h}} \times gr(\mathbb{R}_2)$ and $\rho_{D_{2h}} \times gr(\mathbb{S}_2)$, yields $\chi_{D_{2h}[gr(\mathbb{R}_2)]} = [+,-,-,-,+] = B_{1u}$ and $\chi_{D_{2h}[gr(\mathbb{S}_2)]} = [+,-,-,-,-] = A_4$.

The graph of $\Xi(\Xi)$ will be defined as the vector $(gr(\mathbb{R}_2), gr(\mathbb{S}_2))$ and as so a pair of intersecting surfaces in $R^3$, with $gr(\Xi) = \{(x, y, \epsilon_1, (x, y, \epsilon_2)) \in R^3|x = x = y = \epsilon_1 = \mathbb{R}_2(\mathbb{X}, y)\} \bigwedge (x = x = y = \epsilon_2 = \mathbb{S}_2(\mathbb{X}, y))$. Because of the distinct symmetries of $gr(\mathbb{R}_2)$ and $gr(\mathbb{S}_2)$, i.e. $\chi_{D_{2h}[gr(\mathbb{R}_2)]} \neq \chi_{D_{2h}[gr(\mathbb{S}_2)]}$, $\Xi(\Xi)$ transforms as an irrep of the normal subgroup of $TSG(\mathbb{R}_2) = TSG(\mathbb{S}_2) = D_{2h}$ having equivalent parities. The explicit $TSG$ and irrep associated with $gr(\Xi)$ were determined from the direct product of $\chi_{D_{2h}[gr(\mathbb{R}_2)]}$ and $\chi_{D_{2h}[gr(\mathbb{S}_2)]}$, which gives $TSG(\Xi) = C_{2h} \in D_{2h}$.

The graph of the root locus is denoted $gr(\Xi)$ and defined by $gr(\Xi) = \{(x, y, \epsilon_1) \in R^3|x = x = y = \epsilon_1 = \mathbb{R}_2(\mathbb{X}, y) = 0\} \bigwedge (x = x = y = \epsilon_2 = \mathbb{S}_2(\mathbb{X}, y) = 0)$. As such, $gr(\Xi)$ is a set of $O$D points in $R^3$. Using the same methods as above, it is easy to see that $gr(\Xi)$ transform as totally symmetric in $D_{2h}$ irrespective if all roots are reside on the line $y = 0$, with $\chi_{D_{2h}[gr(\Xi)]} = [+,-,-,-,+]=[A_y]$. Although this true, $gr(\Xi)$ actually transforms as the totally symmetric in the larger $D_{2h}$ point group. That is, $TSG(\Xi) = D_{2h}$, with $\chi_{D_{2h}[gr(\Xi)]} = \Sigma_{y}^{1}$; because it will be proved later on in app. $3$ that the set of roots in $D_{2h}$ is collinear.

2.4. The TSG’$s$ of $\zeta(\zeta), R_2, \mathbb{S}_2, \zeta(\zeta) \cup \gamma(\gamma)$ and their graphs.

In this subsection, the transformation properties $\zeta(\zeta) = R_2(\mathbb{X}, y) + i \zeta(\zeta, y), R_2(\mathbb{Y}, y), \zeta(\zeta, y), R(\Xi) \cup \gamma(\gamma)$, and their graph counterparts are characterized using point groups and their irreps.

Theorem 2.4. The TSG of $\zeta(\zeta), R_2, \mathbb{S}_2, \zeta(\zeta) \cup \gamma(\gamma), gr(\zeta), gr(R_2), gr(\mathbb{S}_2), gr(\zeta \cup \gamma(\gamma))$ are $C_1, C_s, C_s, C_s, C_s, C_2v, C_2v, \text{ and } C_2v, \text{ respectively, with } \chi_{C_1} = A, \chi_{C_s[\mathbb{R}_2]} = A, \chi_{C_s[\mathbb{S}_2]} = B, \chi_{C_s[\mathbb{S}_2]} = A, \chi_{C_s[\mathbb{R}_2]} = A, \chi_{C_2v[\mathbb{R}_2]} = B, \chi_{C_2v[\mathbb{S}_2]} = A, \chi_{C_2v[\mathbb{S}_2]} = A, \chi_{C_2v[gr(\zeta)]} = A_1, \chi_{C_2v[gr(\zeta)]} = A_1$.

Proof. Since $\zeta$ does not satisfy reflection formula, i.e. $\zeta(\zeta') \neq \pm \zeta(\zeta)$, the TSG for $\zeta(\zeta), R_2, \mathbb{S}_2, \zeta(\zeta) \cup \gamma(\gamma)$ and their graphs $gr(\zeta), gr(R_2), gr(\mathbb{S}_2)$, and $gr(\zeta(\zeta) \cup \gamma(\gamma))$ turns out to be a subgroup of the $D_{2h}$. Of note, the only symmetries of $R_2$ and $\mathbb{S}_2$ are in the $y$-variable with respect to the origin, with $R_2(\mathbb{X}, y) = \mathbb{R}_2(\mathbb{X}, y)$ and $\zeta(\zeta) = -\zeta(\zeta')$. With these observations, the assignments made in Theorem 2.4 follow immediately.

Although beyond the scope of this work, the symmetries of any $f(z)$ satisfy $C_1 \leq TSG(f) \leq TSG(\mathbb{R}_2) \leq TSG(\mathbb{S}_2) \leq TSG(R(f) \cup \gamma(f)) \leq C_{2v}$ and $C_2 \leq TSG(gr(f)) \leq TSG(gr(\mathbb{R}_2)) = TSG(gr(\mathbb{S}_2)) \leq TSG(gr(R(f) \cup \gamma(f)))$, with $gr(\mathbb{R}_2 \cup \gamma(f))$ obliged to transform as totally symmetric in both $TSG(gr(f))$ and $TSG(gr(R(f)))$. Since all TSG's are interconnected in this manner, $gr(R(f) \cup \gamma(f))$ is seen to dictate the TSG's and transformation properties of $R_2(y)$ and $\mathbb{S}_2(\mathbb{X}, y)$ and so of $f(z).$ [21, 22]
2.5. The TSG’s of $|\Xi(x,y)|^2$, $R(|\Xi|^2)$ and their graphs.

In this subsection, the transformation properties $|\Xi(x,y)|^2 = \Re^2\Xi(x,y) + 3\Im^2\Xi(x,y) + i0$, $R(|\Xi|^2)$, and their graph counterparts are characterized using point groups and their irreps.

**Theorem 2.5.** The TSG of $|\Xi|^2$, $R(|\Xi|^2)$, $gr(|\Xi|^2)$, and $gr(R(|\Xi|^2))$ are $C_{2v}$, $C_{2h}$, $D_{2h}$ and $D_{2h}$, respectively, with $\chi_{C_{2v}}|\Xi|^2 = A_1$, $\chi_{C_{2h}}|R(\Xi)|^2] = A_1$, $\chi_{D_{2h}}[gr(|\Xi|^2)] = B_{1u}$, and $\chi_{D_{2zh}}[gr(R(|\Xi|^2))] = \Sigma^+$. 

**Proof.** As a result of $\Xi(z)$ obeying $\Xi(-z) = \Xi(z)$, $|\Xi|^2$ is symmetric in both $x$ and $y$ with respect to the origin, with $|\Xi(z)|^2 = |\Xi(x,y)|^2$, $|\Xi(-x,y)|^2 = |\Xi(x,y)|^2$ and $|\Xi(-x,-y)|^2 = |\Xi(x,y)|^2$. Given that $R(\Xi) \cong R(\zeta) \cong R(|\Xi|^2)$ and $gr(R(\Xi)) \cong gr(R(\zeta)) \cong gr(R(|\Xi|^2))$, the determination of the assignments of Theorem 2.5 is straightforward using the same techniques developed above. 

3. Symmetry and the differential structure of $|\Xi(x,y)|^2$

Implications of the assignments made in §2.5 on properties of the square of the modulus of $\Xi(z)$ will now be addressed. The impetus for choosing to analyze $|\Xi|^2$ in lieu of $\zeta(z)$, $\zeta(z)$ or their moduli $|\zeta|^2$ and $|\xi|^2$, which is standard practice, [2] is the larger set of symmetries associated with $|\Xi(x,y)|^2$ and that command over the structure and properties of $\Xi(z)$ can be obtained through an examination of $|\Xi(x,y)|^2$ since $|\Xi|^2(x,y) = 0$ implies $\Xi(x+iy) = 0$, with $R(\Xi) \cong R(|\Xi|^2)$. Consult Appendix B for a 2D representation of subsets of $R_t(\zeta) \cup \gamma(\zeta)$ and $R(\Xi) \cong R(|\Xi|^2)$ and note how roots in $R(\Xi) \cong R(|\Xi|^2) \cong R(\zeta)$ align symmetrically about the line $x = 0$ and along the line $y = 0$.

3.1. The functional equation for $|\Xi(x,y)|^2$

The analytic approach implemented is made possible by expressing the requisite functional equation for $\Xi(z)$ in terms of the variable $x + iy$. Accordingly, the starting point of the investigation of $|\Xi|^2$ is the definition

**Definition 3.1.** $\Xi(z) = -\frac{1}{8}\pi^{-\frac{1}{4} + \frac{\nu}{2}}(1 + 4x^2)^\Gamma\left(\frac{1}{4} + \frac{\nu}{2}\right)\zeta^{(\frac{1}{2} + iz)}$ in terms of the variable $x + iy$ is

$$\Xi(x+iy) = -\frac{1}{8}\pi^{-\frac{1}{4} + \frac{\nu}{2}}(1 + 4x^2 - 4y^2 + 8ixy)^\Gamma\left(\frac{1}{4} - \frac{\nu}{2} + \frac{i\nu}{2}\right)\zeta^{\left(\frac{1}{2} - y + iz\right)}.$$ 

Naturally, Eq. 5 satisfies $\Xi(-x+iy) \equiv \Xi(x+iy)$. Definition 3.1 can be used to obtain a functional equation for $|\Xi|^2(x,y) = \Re^2\Xi(x,y) + 3\Im^2\Xi(x,y) + i0$, with the following derived
Lemma 3.2. The square of the modulus of B. Riemann’s \( \Xi \) function is given by the functional equation:

\[
\|\Xi\|^2_{(x,y)} = \frac{1}{64} \pi^{-\frac{1}{2}+y}(16x^4 + 8x^2(1 + 4y^2) + (1 - 4y^2)^2)\Gamma(\frac{1}{4} - \frac{y}{2} + \frac{y}{4})\Gamma(\frac{1}{4} - \frac{y}{2} - \frac{y}{4})\xi(\frac{1}{2} - y - ix) = 0,
\]

(6)

while \( \|\Xi\|^2_{(x,0)} \) along the line \( y = 0 \) is

\[
\|\Xi\|^2_{(x,0)} = \frac{1}{64\sqrt{\pi}}(1 + 4x^2)^2\Gamma(\frac{1}{4} + \frac{y}{2})\Gamma(\frac{1}{4} - \frac{y}{2})\xi(\frac{1}{2} + ix),
\]

and \( \|\Xi\|^2_{(0,y)} \) along the line \( x = 0 \) is

\[
\|\Xi\|^2_{(0,y)} = \frac{1}{64} \pi^{-\frac{1}{2}+y}(1 - 4y^2)^2\Gamma(\frac{1}{4} - \frac{y}{2} + \frac{y}{4})\Gamma(\frac{1}{4} - \frac{y}{2} - \frac{y}{4})\xi(\frac{1}{2} - y)\xi(\frac{1}{2} + y).
\]

Proof. Using Eq. 5 and the fact that \( \|\Xi\|^2_{(x,y)} = \Xi_{x + iy}\Xi_{x - iy} = \Re^2(\Xi(x,y) + \Im^2(\Xi(x,y)) - \xi(\frac{1}{2} - y)\xi(\frac{1}{2} + y)) \)

the utility of symmetry as it pertains to Lemma 3.2 is that it gives, without calculation, the following

Lemma 3.3. \( \|\Xi\|^2_{(x,y)} \) as defined in Eq. 6, is symmetric in \( x \) and \( y \) about the origin, with \( \|\Xi\|^2_{(-x,-y)} = \|\Xi\|^2_{(x,y)} \),

\[
\|\Xi\|^2_{(-x,-y)} = \|\Xi\|^2_{(x,y)}, \|\Xi\|^2_{(x,-0)} = \|\Xi\|^2_{(0,x)}, \|\Xi\|^2_{(-x,0)} = \|\Xi\|^2_{(0,x)} \text{ and } \|\Xi\|^2_{(0,-y)} = \|\Xi\|^2_{(0,y)}.
\]

Proof. Lemma 3.3 follows from Theorem 2.5 and the known analytic and symmetry properties of \( \xi(z) \) and \( \Gamma(z) \). More explicitly, Eqs. 6-8 are invariant to substitutions of \(-x\) for \( x \) and/or \(-y\) for \( y \). [2, 5]

What’s more, symmetry will be used below to prove that \( \partial_x\|\Xi\|^2_{(x,0)} \) and \( \partial_y\|\Xi\|^2_{(0,y)} \) are anti-symmetric functions and so equal to zero at the origin, \( \partial_x\|\Xi\|^2_{(0,y)} = 0, \forall y \in \mathbb{R}, \partial_y\|\Xi\|^2_{(x,0)} = 0, \forall x \in \mathbb{R}, \partial_{xy}\|\Xi\|^2_{(x,0)} = \partial_{xy}\|\Xi\|^2_{(0,y)} = 0, \forall x \in \mathbb{R}\), and \( \partial_{yx}\|\Xi\|^2_{(0,y)} = \partial_{yx}\|\Xi\|^2_{(0,y)} = 0, \forall y \in \mathbb{R} \).

The graph of \( \|\Xi\|^2_{(x,y)} \) over a subset of \( \mathbb{R}^2 \) is provided in Fig. 1, with \( gr(\|\Xi\|^2_{(x,y)}) \) a smooth 2D surface in \( \mathbb{R}^3 \), while the graphs of \( \|\Xi\|^2_{(x,y)} \) along the lines \( y = 0 \) and \( x = 0 \) are provided in Fig. 2, with both 1D curves in \( \mathbb{R}^2 \). Observe how \( gr(\|\Xi\|^2) \) oscillates along the line \( y = 0 \) and rapidly diverges to infinity along the line \( x = 0 \) as \( y \to \infty \). As it is generally true that correlated saddle points of \( \Re(\Xi(y,x)) \) and \( \Im(\Xi(x,y)) \) and roots of an \( f(z) \) arise as local saddle points and local minima and zeros of the graph of \( |f|^2 = \Re^2 + \Im^2 + a0 \), respectively, roots of \( \Xi(x,y) \), and so nontrivial roots of \( \xi(z) \), are minima and zeros of \( gr(\|\Xi\|^2) \), while correlated saddle points of \( gr(\Re(\Xi)) \) and \( gr(\Im(\Xi)) \) are linked with the saddle points of \( gr(|\Xi|^2) \). [28] By symmetry, all of these critical points of \( gr(|\Xi|^2) \) are guaranteed to be symmetrically distributed about the line \( x = 0 \), with \( |y| \leq \frac{1}{2} \).
3.2. Partial derivatives of $|\Xi|^2_{(x,y)}$ with respect to the $y$ variable: Odd powers.

Properties of the odd-order derivatives of $|\Xi|^2$ are characterized, with symmetry facilitating the subsequent analysis. The main outcomes are that the zero set of $\partial_y|\Xi|^2_{(x,0)}$ is the line $y = 0$, i.e. $\partial_y|\Xi|^2_{(x,0)} = 0$, $\forall x \in \mathbb{R}$, and that if roots of $\zeta'(\frac{1}{2}+ix)$ exist for some $x$, then they occur at roots of $\zeta(\frac{1}{2}+ix)$ and so at nontrivial roots of $\zeta(z)$ having multiplicity greater than one. These results will be used in §Sec. 3.3 to establish that $\partial_y|\Xi|^2_{(x,y)} > 0$, $\forall y > 0 \in \mathbb{R}$ and $\partial_y|\Xi|^2_{(x,y)} < 0$, $\forall y < 0 \in \mathbb{R}$ and so that $\partial_y|\Xi|^2_{(x,y)}$ is bounded away from zero for $y \neq 0$.

3.2.1. Symmetries of $\partial_{y^{2n+1}}|\Xi|^2_{(x,y)}$.

The symmetries of $|\Xi|^2$ encapsulated in Theorem 2.5 and Lemma 3.3 gives without calculation that

\begin{equation}
\forall n \in \mathbb{N}, \partial_{y^{2n+1}}|\Xi|^2_{(x,y)} \text{ is symmetric in } x \text{ about the origin and anti-symmetric in } y \text{ about the origin, with } \partial_{y^{2n+1}}|\Xi|^2_{(-x,y)} = \partial_{y^{2n+1}}|\Xi|^2_{(x,-y)} = -\partial_{y^{2n+1}}|\Xi|^2_{(x,y)}, \partial_{y^{2n+1}}|\Xi|^2_{(-x,-y)} = -\partial_{y^{2n+1}}|\Xi|^2_{(x,y)},
\end{equation}

\begin{equation}
\partial_{y^{2n+1}}|\Xi|^2_{(x,-y)} = -\partial_{y^{2n+1}}|\Xi|^2_{(x,y)}, \partial_{y^{2n+1}}|\Xi|^2_{(-x,0)} = \partial_{y^{2n+1}}|\Xi|^2_{(x,y)} \text{ and } \partial_{y^{2n+1}}|\Xi|^2_{(0,-y)} = -\partial_{y^{2n+1}}|\Xi|^2_{(x,y)}.
\end{equation}

Proof. With $|\Xi|^2_{(x,y)}$ is symmetric in the $x$ and $y$ coordinates by Theorem 2.5 and Lemma 3.3, Lemma 3.4 a consequence of the fact the derivative of a symmetric function is an anti-symmetric function.

Lemma 3.5. $\forall n \in \mathbb{N}, \partial_{y^{2n+1}}|\Xi|^2_{(x,0)} = 0$, $\forall x \in \mathbb{R}$.

Proof. Lemma 3.5 follows from Theorem 2.5 and Lemma 3.3. Namely, that $|\Xi|^2_{(x,y)}$ is symmetric in $x$ and $y$ about the origin gives that $\partial_y|\Xi|^2_{(x,y)}$ is anti-symmetric in $y$, $\forall x \in \mathbb{R}$ and so that $\partial_y|\Xi|^2_{(x,0)} = 0$, $\forall x \in \mathbb{R}$.

3.2.2. General expressions for $\partial_{y^{2n+1}}|\Xi|^2_{(x,0)}$.

To get at the global structure exhibited by $|\Xi|^2_{(x,y)}$, in addition to making the claims in Lemmas 3.4 and 3.5 more substantive, a general expression for odd-order partial derivatives of $|\Xi|^2_{(x,y)}$ with respect to $y$ and evaluated at $(x,0)$, i.e. $\partial_{y^{2n+1}}|\Xi|^2_{(x,0)}$, was derived

Lemma 3.6. For $n \geq 1$, $\partial_{y^{2n+1}}|\Xi|^2_{(x,0)}$, are expressible as

\begin{equation}
\partial_{y^{2n+1}}|\Xi|^2_{(x,0)} = (1 + 4x^2)^2\partial_{y^{2n+1}}\Omega_{(x,0)} + (4x^2 - 1)(32n^2 + 16n)\partial_{y^{2n-1}}\Omega_{(x,0)} + (256n^4 - 256n^3 - 64n^2 + 64n)\partial_{y^{2n-3}}\Omega_{(x,0)},
\end{equation}

where $\Omega_{(x,y)}$ is

\begin{equation}
\Omega_{(x,y)} = \frac{1}{64}\pi^{-\frac{1}{2}+y}|\Gamma|^2_{(\frac{1}{2}-\frac{y}{2}+\frac{\epsilon}{4})}\zeta^2_{(\frac{1}{2}-y+ix)}.
\end{equation}

Proof. The expression for $\partial_{y^{2n+1}}|\Xi|^2_{(x,y)}$ for $n \geq 1$ was derived using the general Leibniz rule.

3.2.3. The first partial derivatives of $|\Xi|^2_{(x,y)}$ with respect to the $y$ variable.

Characterizing the symmetry and analytic properties of the first partial derivative of $|\Xi|^2_{(x,y)}$ with respect to $y$, i.e. $\partial_y|\Xi|^2_{(x,y)}$, will be an important step in proving that all roots in $R(|\Xi|^2) \cong R(\Xi) \cong R(\zeta)$ are collinear. Accordingly, the functional equation for $\partial_y|\Xi|^2_{(x,y)}$ was derived, with

Lemma 3.7. The first partial derivative of $|\Xi|^2_{(x,y)}$ with respect to $y$ is

\begin{equation}
\partial_y|\Xi|^2_{(x,y)} = \frac{1}{128}\pi^{-\frac{1}{2}+y}|\Gamma|^2_{(\frac{1}{2}-\frac{y}{2}+\frac{\epsilon}{4})} \times (32y(4x^2 + 4y^2 - 1)|\zeta|^2_{(\frac{1}{2}-y+ix)} + (16x^4 + 8x^2(1 + 4y^2) + (1 - 4y^2)^2)(2\log(\pi) - \psi(\frac{1}{2}-\frac{y}{2}+\frac{\epsilon}{4}) - \psi(\frac{1}{2}-\frac{y}{2}+\frac{\epsilon}{4})|\zeta|^2_{\frac{1}{2}+ix} - 2(16x^4 + 8x^2(1 + 4y^2) + (1 - 4y^2)^2)(\zeta_{\frac{1}{2}-y+ix}\zeta'_{\frac{1}{2}-y-ix} + \zeta'_{\frac{1}{2}+ix}\zeta_{\frac{1}{2}-y+ix})},
\end{equation}
Along the lines $x = 0$ and $y = 0$, $\partial_y|\Xi|^2_{(x,y)}$ simplifies to

$$\partial_y|\Xi|^2_{(x,y)} = \frac{1}{128\sqrt{\pi}}(1 + 4x^2)^2|\Gamma|^2_{(\frac{1}{4} + \frac{i}{x})} \times \left((32y(4y^2 - 1) + (1 - 4y^2)^2)(2\log_2(\pi) - 2\psi_1(\frac{1}{4} - \frac{i}{y})\zeta^2_{\frac{1}{2} - y}) - 2(1 - 4y^2)^2(\zeta_{\frac{1}{2} - y}^2 - 2(\zeta_{\frac{1}{2} - y} + \zeta_{\frac{3}{2} + i(0, y)}\zeta_{\frac{3}{2} - y}^2))\right),$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function.

Proof. The general Leibniz rule [23] was used to obtain this equation for $\partial_y|\Xi|^2_{(x,y)}$. Note that $\zeta_{\frac{1}{2} - y + i\pi}^2 + \zeta_{\frac{1}{2} - y - i\pi}^2 = \partial_y|\Xi|^2_{(x,y)}$ and that the real-valued function $\partial_y|\Xi|^2_{(x,0)}$ defined in Eq. 13 is expressible as

$$\partial_y|\Xi|^2_{(x,0)} = \frac{1}{64\sqrt{\pi}}(1 + 4x^2)^2|\Gamma|^2_{(\frac{1}{4} + \frac{i}{x})} \times \left((2\log_2(\pi) - \psi_1(\frac{1}{4} - \frac{i}{x})\zeta^2_{\frac{1}{2} - y}) - 2(\zeta_{\frac{1}{2} - y} + \zeta_{\frac{3}{2} + i\pi}^2)\right).$$

Regarding symmetries of $\zeta(z)$, it is worth mentioning here that $\Re\zeta_{\frac{1}{2} - y} + \Im\zeta_{\frac{1}{2} - y} = \Re\zeta_{\frac{1}{2} + y} - \Im\zeta_{\frac{1}{2} + y}$.

Theorem 2.5 and Lemmas 3.3 and 3.4 immediately imply the following for the functions defined in Eqs. 11-14

**Corollary 3.8.** $\partial_y|\Xi|^2_{(x,y)}$, as defined in Eqs. 11, is symmetric in the $x$ variable and anti-symmetric in the $y$ variable, with $\partial_y|\Xi|^2_{(-x,y)} = -\partial_y|\Xi|^2_{(x,y)}$, $\partial_y|\Xi|^2_{(x,-y)} = -\partial_y|\Xi|^2_{(x,y)}$, and $\partial_y|\Xi|^2_{(0,y)} = 0$.

**Corollary 3.9.** $\partial_y|\Xi|^2_{(0,y)}$, as defined in Eq. 12, is anti-symmetric in $y$, while $\partial_y|\Xi|^2_{(x,0)}$, as defined in Eqs. 13 and 14, is symmetric in $x$, with $\partial_y|\Xi|^2_{(0,-y)} = -\partial_y|\Xi|^2_{(0,y)}$ and $\partial_y|\Xi|^2_{(x,0)} = 0$.

The graph of $\partial_y|\Xi|^2_{(x,y)}$ along the line $x = 0$ is provided in Fig. 3. Observe that $gr(\partial_y|\Xi|^2_{(0,y,\epsilon)})$ is an anti-symmetric curve in $\mathbb{R}^2$ about the origin, with $\partial_y|\Xi|^2_{(0,0)} = 0$, $\partial_y|\Xi|^2_{(0,y)} > 0$ for $y > 0$ and $\partial_y|\Xi|^2_{(0,y)} < 0$ for $y < 0$ over the subset of $\mathbb{R}$ depicted.

3.2.4. The zero set of $\partial_y|\Xi|^2_{(x,y)}$.

Identifying the zero set of $\partial_y|\Xi|^2_{(x,y)}$, i.e. $(x, y) \in \mathbb{R}^2$ such that $\partial_y|\Xi|^2_{(x,y)} = 0$, is important to understanding global structure of $|\Xi|^2_{(x,y)}$ and so of $\Xi_{(x,y)}$. Without calculation, Lemma 3.5 yields that

**Lemma 3.10.** $\partial_y|\Xi|^2_{(x,0)} = 0$, $\forall x \in \mathbb{R}$.

Proof. The symmetry of $|\Xi|^2_{(x,y)}$ captured in Theorem 2.5 and Lemmas 3.3-3.5 gives not only that $\partial_y|\Xi|^2_{(x,y)}$ in an anti-symmetric function in $y$, with $\partial_y|\Xi|^2_{(x,-y)} = -\partial_y|\Xi|^2_{(x,y)}$, $\forall x \in \mathbb{R}$, but also that $\partial_y|\Xi|^2_{(x,y)}$ is identically equal to zero along the line $y = 0$, i.e. $\partial_y|\Xi|^2_{(x,0)} = 0$, $\forall x \in \mathbb{R}$. □
Corollary 3.11. The zero set of $\partial_y \Xi(x,y)$ along the line $y = 0$ is defined by the following equivalent equations

\begin{equation}
(2 \log_e(\pi) \Gamma(\frac{1}{4} + iy) \Gamma(\frac{1}{4} - iy) - \Gamma'(\frac{1}{4} + iy) \Gamma(\frac{1}{4} - iy) - \Gamma'(\frac{1}{4} - iy) \Gamma(\frac{1}{4} + iy)) |\zeta|^2_{\frac{1}{4} + iy} =
\end{equation}

\begin{equation}
2\Gamma(\frac{1}{4} + iy) \Gamma(\frac{1}{4} - iy) (\zeta_{\frac{1}{4} + iy})' \zeta_{\frac{1}{4} - iy} + \zeta_{\frac{1}{4} + iy}' \zeta_{\frac{1}{4} - iy}.
\end{equation}

Proof. Given Lemmas 3.4 and 3.5, that $\psi(z) = \frac{1}{\Gamma(z)}$ and that $A(z) = \frac{1}{128\sqrt{\pi}} |\Gamma(\frac{1}{4} + iy)| > 0 \forall y \in \mathbb{R}$ from known properties of the $\Gamma$ function, e.g. $R(\Gamma) = \mathcal{O}$, [6] a byproduct of Eq. 13 is that Eqs. 15 and 16 are valid $\forall x \in \mathbb{R}$. Note Eq. 16 can also be expressed as $(\log_e(\pi) - R \psi(\frac{1}{4} + iy)) |\zeta|^2_{\frac{1}{4} + iy} = 2R \zeta_{\frac{1}{4} + iy}' \zeta_{\frac{1}{4} - iy}$. \qed

Evaluating Eq. 15 and Eq. 16 at $x = 0$ yields an expression for $\log_e(\pi)$.

Corollary 3.12. $\log_e(\pi) = \frac{2\zeta_{\frac{1}{4}}}{\eta_{\frac{1}{4}}} + \frac{r_{\frac{1}{4}}}{r_{\frac{1}{4}}}$.

Because $\lim_{x \to \pm \infty} \frac{\log_e(\pi) - R \psi(\frac{1}{4} + iy)}{\log_e(\frac{1}{4})} = 1$, Eq. 16 implies

Corollary 3.13. $\lim_{x \to \pm \infty} \frac{\log_e(\frac{1}{4}) |\zeta|^2_{\frac{1}{4} + iy}}{R \zeta_{\frac{1}{4} + iy}' \zeta_{\frac{1}{4} - iy}} = 1$.

More importantly, though, Corollary 3.11 divulges that

Theorem 3.14. If a root of $\zeta(z)$ exists at a critical line, then it occurs at a root of $\zeta(z)$ on the critical line, with this nontrivial root necessarily having multiplicity greater than one.

Proof. From the known analytic and symmetry properties of $\zeta(z)$, $\zeta'(z)$, $\Gamma(z)$, and $\psi(z)$, [2, 6] Eqs. 15 and 16 give that if $\zeta'(z) = 0$ for $z = \frac{1}{2} + iy \in \mathbb{C}$, then $\zeta(z) = 0$ at the same $y$, with these the defining equations for a root having multiplicity greater than one. In words, Theorem 3.14 indicates that if $\zeta'(\frac{1}{2} + iy)$ has roots on the critical line, then they necessarily occur at a nontrivial root of $\zeta(\frac{1}{2} + iy)$ on the critical line having multiplicity greater than one. From a different perspective, observe that if all of the roots of $\zeta(z)$ on the critical line have multiplicity one, then $\zeta'(\frac{1}{2} + iy) > 0 \forall y \in \mathbb{R}$. These considerations yield Theorem 3.14. \qed

Even though it has only been proven that $\partial_{y} |\Xi|^2$ is zero along this line not that this is the only place where $\partial_{y} |\Xi|^2$ is zero and so that the entire zero set is this line $y = 0$, the results of §Sec. 3.3 will confirm that $\partial_{y} |\Xi|^2$ is bounded away from zero if $y \neq 0$.

3.3. Partial derivatives of $|\Xi|^2(x,y)$ with respect to the $y$ variable: Even powers.

Properties of the even-order derivatives of $|\Xi|^2$ are characterized. Of note, it is proven that $\partial_{yy} |\Xi|^2(x,0) \geq 0$ along the line $y = 0$ through an analysis of its asymptotic properties. This gives not only that $\partial_{y} |\Xi|^2(0,y) > 0$, $\forall y > 0 \in \mathbb{R}$, and $\partial_{y} |\Xi|^2(0,y) < 0$, $\forall y < 0 \in \mathbb{R}$, but also that $|\Xi|^2(x,y)$ is locally bounded from zero when $y \neq 0$. Later on, it is exposed that $\partial_{n}^2 |\Xi|^2(x,0) \geq 0$, $\forall x \in \mathbb{R}$ and $\forall n \in \mathbb{N}$, and so that $|\Xi|^2(x,y)$ is globally bounded away from zero if $y \neq 0$. It immediately follows from this that $|\Xi|^2(x,y)$ and $\partial_{y} |\Xi|^2(x,y)$ are bounded away from zero if $y \neq 0$. A direct corollary of these observations is that all roots of $|\Xi|^2(x,y)$ reside on the line $y = 0$, which with $R(|\Xi|^2) \equiv R(\Xi) \equiv R(\zeta)$ implies all nontrivial roots of $\zeta(z)$ lie on the critical line.

3.3.1. Symmetries of $\partial_{y}^n |\Xi|^2(x,y)$

Theorem 2.1 and Lemma 3.3 gives implies the following regarding $\partial_{y}^n |\Xi|^2(x,y)$.

Lemma 3.15. $\forall n \in \mathbb{N}$, $\partial_{y}^n |\Xi|^2(x,y)$ is symmetric in $x$ about the origin and symmetric in $y$ about the origin, with $\partial_{y}^2 |\Xi|^2(x,-y) = \partial_{y}^2 |\Xi|^2(x,y)$, $\partial_{y}^2 |\Xi|^2(x,-y) = \partial_{y}^2 |\Xi|^2(x,y)$, $\partial_{y}^2 |\Xi|^2(x,-y) = \partial_{y}^2 |\Xi|^2(x,y)$, $\partial_{y}^2 |\Xi|^2(x,-y) = \partial_{y}^2 |\Xi|^2(x,y)$. \qed
Corollary 3.16. \( \partial_y^{n^2}|\Xi|^2_{(x,y)} \) are the only terms that contribute to Taylor series of \(|\Xi|^2\) in \( y \) about any \( x_0 \), with

\[
|\Xi|^2_{(x_0,y)} = \sum_{n=1}^{\infty} \frac{\partial_y^{n^2}|\Xi|^2_{(x_0,0)}}{(2n)!} y^{2n}.
\]

Proof. Corollary 3.16 is consequence of \( \partial_y^{n^2}|\Xi|^2_{(x,y)} \) being symmetric in \( x \) and \( y \) about the origin. \( \square \)

3.3.2. General expressions for \( \partial_y^{n^2}|\Xi|^2_{(x,y)} \) being symmetric in \( x \) and \( y \) about the origin.

Paralleling §Sec. 3.2.2, analogous general expressions for even-order partial derivatives of \(|\Xi|^2_{(x,y)}\) with respect to \( y \) and evaluated at \((x,0)\), i.e. \( \partial_y^{n^2}|\Xi|^2_{(x,0)} \) were derived

Lemma 3.17. For \( n \geq 1 \), even-order partial derivatives of \(|\Xi|^2_{(x,y)}\) with respect to \( y \) and evaluated at \((x,0)\), i.e. \( \partial_y^{n^2}|\Xi|^2_{(x,0)} \), are expressible as

\[
\partial_y^{n^2}|\Xi|^2_{(x,0)} = (1 + 4x^2)^2 \partial_y^{n^2} \Omega_{(x,0)} +
(4x^2 - 1)(32n^2 - 16n) \partial_y^{n^2-2} \Omega_{(x,0)} +
(256n^4 - 768n^3 + 704n^2 - 192n) \partial_y^{n^2-4} \Omega_{(x,0)},
\]

where \( \Omega_{(x,y)} \) was defined in Eq. 10, and

\[
\partial_y^{n^2}|\Xi|^2_{(x,0)} = (1 + 4x^2)^2 \log_e(\pi)^{2n} \Phi_{(x,0)} + \sum_{j=1}^{n} \log_e(\pi)^{2n-2j} \left(\begin{array}{c} 2j \\not(x,0) \end{array}\right) \left(\begin{array}{c} 2n \not(x,0) \end{array}\right) \times
(384 \left(\begin{array}{c} 2j \\not(x,0) \end{array}\right) \partial_y^{2j-4} \Phi_{(x,0)} + 16(4x^2 - 1) \left(\begin{array}{c} 2j \\not(x,0) \end{array}\right) \partial_y^{2j-2} \Phi_{(x,0)} + (1 + 4x^2)^2 \left(\begin{array}{c} 2j \\not(x,0) \end{array}\right) \partial_y^{2j} \Phi_{(x,0)}),
\]

where \( \left(\begin{array}{c} a \\not(x,0) \end{array}\right) \) is a binomial coefficient and \( \Phi_{(x,y)} \) is given by

\[
\Phi_{(x,y)} = \frac{1}{64\sqrt{\pi}} |\Gamma|^{2} \xi^{-x+iy} |\zeta|^{2} \xi^{x-iy} = \pi^{x} \Omega_{(x,y)}.
\]

Proof. These expressions for \( \partial_y^{n^2}|\Xi|^2_{(x,y)} \) for \( n \geq 1 \) were derived using the general Leibniz rule. [23] Note that the resulting expressions for \( \partial_y^{n^2}|\Xi|^2_{(x,y)} \) can be simplified by identifying equations defining zero sets of Eqs. 9 and their derivative. Subsequently, the zero set of Eq. 11 defined by Eqs. 15 and 16 will used to derive a manageable expressions for \( \partial_y^{n^2}|\Xi|^2_{(x,y)} \).

Eq. 18 discloses structure in \( x \) and in \( n \) inherent to the \( \partial_y^{n^2}|\Xi|^2_{(x,y)} \) and so structure in \( |\Xi|^2_{(x,y)} \), which is captured in the following Lemma.

Lemma 3.18. At \( x = 0 \), \( \partial_y^{n^2}|\Xi|^2_{(0,y)} > 0 \) \forall \( y \) and so \( |\Xi|^2_{(0,y)} > 0 \) \forall \( y \in \mathbb{R} \).

Proof. Given that sign changes of each \( \partial_y^{n^2}|\Xi|^2_{(x,0)} \) are dictated by \( \Omega_{(x,0)} \) and its derivatives, that \( \Omega_{(x,0)} \geq 0 \) \forall \( x \in \mathbb{R} \) and that, at \( x = 0 \) and large \( n \), \( \partial_y^{n^2}|\Xi|^2_{(0,0)} \sim n^{4} \partial_2^{n^2} \Omega_{(0,0)} \), it follows that \( |\Xi|^2_{(0,y)} \) is bounded above zero along this line as \( y \to \infty \), since a zero of \( |\Xi|^2_{(0,y)} \) necessitates that \( \exists n \) such that \( \partial_y^{n^2}|\Xi|^2_{(0,y)} < 0 \). Lemma 3.18 also can be proved using induction on \( n \) in Eq. 18 at \( x = 0 \) and as \( n \to \infty \).

Some consequences of Lemma 3.18 are

Corollary 3.19. At \( x = 0 \), \( |\Xi|^2_{(0,y)} \) is monotonic in \( y \) on \([0,\infty)\) and \((-\infty,0]\).

Proof. Given the definition of complete monotonicity of a function \( f(y) \) on half-line as \((-1)^{n} \partial_y^{n} f(y) \geq 0 \forall n \in \mathbb{N} \) and \( \forall y \in \mathbb{R} \), Corollary 3.19 follows directly from Lemma 3.18.

Corollary 3.20. At \( x = 0 \), \( |\Xi|^2_{(0,y)} \) is a mixture of exponential functions.

Proof. This is a direct application of Bernstein’s Theorem on monotone functions. [24]
3.3.3. The second partial derivatives of $|\Xi|^2_{(x,y)}$ with respect to the $y$ variable.

Characterizing $\partial_{yy}|\Xi|^2_{(x,y)}$ along the line $y = 0$ will be pivotal to understanding the global structure evinced by $|\Xi|^2_{(x,y)}$ and so by $\Xi_{(x,y)}$. Consider that $\partial_{yy}|\Xi|^2_{(x,y)}$ determines the local curvature of $|\Xi|^2$ along the line $y=0$, as $\partial_y|\Xi|^2_{(x,0)}=0 \forall x \in \mathbb{R}$ by Lemma 3.10. To make headway, the following functional equations for $\partial_{yy}|\Xi|^2_{(x,y)}$ along the lines $y=0$ and $x=0$ were derived.

**Lemma 3.21.** The second partial derivative of $|\Xi|^2_{(x,y)}$ with respect to $y$ evaluated along the line $y = 0$ is

$$\partial_{yy}|\Xi|^2_{(x,0)} = \frac{1}{128\sqrt{\pi}} |\Gamma|^2_{\frac{1}{4}+\frac{x}{y}} \times$$

$$((32(4x^2-1)+(1+4x^2)^2\psi_{x}^{'}(\frac{1}{4}+\frac{x}{y})))|\zeta|^2_{\frac{1}{4}+ix} +$$

$$(1+4x^2)^2(2\psi_{x}(\frac{1}{4}+\frac{x}{y})+2\psi_{y}(\frac{1}{4}-\frac{y}{x})-4\log_e(\pi))\zeta(\frac{1}{4}+ix)\zeta_{x}^{''}(\frac{1}{2}-ix) +$$

$$4(1+4x^2)^2\zeta_{x}^{''}(\frac{1}{4}+ix)\zeta_{y}^{''}(\frac{1}{4}+ix) +$$

$$4(1+4x^2)^2|\zeta|^2_{\frac{1}{4}+ix},$$

which is equivalent to

$$\partial_{yy}|\Xi|^2_{(x,0)} = \frac{1}{128\sqrt{\pi}} |\Gamma|^2_{\frac{1}{4}+\frac{x}{y}} \times$$

$$((32(4x^2-1)+(1+4x^2)^2\psi_{x}^{'}(\frac{1}{4}+\frac{x}{y})))|\zeta|^2_{\frac{1}{4}+ix} +$$

$$(1+4x^2)^2(2\psi_{x}(\frac{1}{4}+\frac{x}{y})+2\psi_{y}(\frac{1}{4}-\frac{y}{x})-4\log_e(\pi))\zeta(\frac{1}{4}+ix)\zeta_{x}^{''}(\frac{1}{2}+ix) +$$

$$4(1+4x^2)^2\zeta_{x}^{''}(\frac{1}{4}+ix)\zeta_{y}^{''}(\frac{1}{4}+ix) +$$

$$4(1+4x^2)^2|\zeta|^2_{\frac{1}{4}+ix},$$

while $\partial_{yy}|\Xi|^2_{(x,y)}$ along the line $x=0$ is

$$\partial_{yy}|\Xi|^2_{(0,y)} = \frac{1}{128\pi^{-\frac{1}{4}+y}\Gamma^2_{\frac{1}{4}-\frac{y}{x}}} \times$$

$$((2\log^2(\pi)(1-4y^2)^2 + 32(12y^2 - 1) + 64y(1-4y^2)(\psi(\frac{1}{4}-\frac{y}{x}) - \log_e(\pi))\zeta(\frac{1}{2}-y) +$$

$$(1-4y^2)^2(2\psi_{x}(\frac{1}{4}-\frac{x}{y})+2\psi_{y}(\frac{1}{4}-\frac{y}{x})-4\log_e(\pi))\zeta_{y}^{''}(\frac{1}{2}-y) +$$

$$(128y(1-4y^2) + 8(1-4y^2)^2(\psi(\frac{1}{4}-\frac{x}{y}) - \log_e(\pi))\zeta(\frac{1}{4}-y)\zeta_{x}^{''}(\frac{1}{2}-y) +$$

$$4(1-4y^2)^2\zeta_{x}^{''}(\frac{1}{4}-y)\zeta_{y}^{''}(\frac{1}{4}-y) +$$

$$4(1-4y^2)^2|\zeta|^2_{\frac{1}{4}-y},$$

where $\psi(z)$ is the digamma function and $\psi_{x}^{'}(z)$ is the trigamma function.

Proof. Eqs. 21-23 for $\partial_{yy}|\Xi|^2_{(x,0)}$ and $\partial_{yy}|\Xi|^2_{(0,y)}$ were obtained using the general Leibniz rule, [23] Eqs. 14 and 15 and their derivatives. Similar to $\partial_{yy}|\Xi|^2_{(x,0)}$, $\partial_{yy}|\Xi|^2_{(x,0)}$ can be expressed as

$$\partial_{yy}|\Xi|^2_{(x,0)} = \frac{1}{128\sqrt{\pi}}(1+4x^2)^2|\Gamma|^2_{\frac{1}{4}+\frac{x}{y}} \times$$

$$((\frac{32(4x^2-1)}{(1+4x^2)^2} + \Re\psi_{x}^{'}(\frac{1}{4}+\frac{x}{y})))|\zeta|^2_{\frac{1}{2}+ix} +$$

$$4(\Re\psi_{x}^{'}(\frac{1}{4}+\frac{x}{y}) - \log_e(\pi))\Re\zeta_{y}^{''}(\frac{1}{2}+ix)\zeta_{x}^{''}(\frac{1}{2}-ix) +$$

$$4\Re\zeta_{x}^{''}(\frac{1}{4}+ix)\zeta_{y}^{''}(\frac{1}{4}+ix) +$$

$$4|\zeta|^2_{\frac{1}{4}+ix}. $$
Note, the asymptotic behavior of the coefficients is captured by \(\lim_{x \to \pm \infty} \frac{4(1+4x^2)^2(\log_2(\pi \cdot x) - \log_2(\pi))}{128x^2 \log_2(x)} = \lim_{x \to \pm \infty} \frac{4(1+4x^2)^2}{64x^2} = 1.\)

The graph of \(\partial_{yy}\|_1^2(x,0)\) and \(\partial_{yy}\|_1^2(0,y)\) are depicted in Fig. 4. Over the subset of \(\mathbb{R}\) presented, observe that \(\partial_{yy}\|_1^2(x,0) > 0\) and appears to rapidly converge to zero as \(x \to \pm \infty\), while \(\partial_{yy}\|_1^2(0,y) > 0\) and appears to rapidly diverge to \(\infty\) as \(x \to \pm \infty\). Some obvious corollaries of Lemma 3.21 are

**Corollary 3.22.** \(\partial_{yy}\|_1^2(x,0)\) and \(\partial_{yy}\|_1^2(0,y)\), as defined in Eqs. 21 and 22, are symmetric in \(x\) and \(y\), respectively, with \(\partial_{yy}\|_1^2(-x,0) = \partial_{yy}\|_1^2(x,0)\) and \(\partial_{yy}\|_1^2(0,-y) = \partial_{yy}\|_1^2(0,y)\).

**Proof.** These symmetries of \(\partial_{yy}\|_1^2(x,y)\) follow from those of \(\|_1^2(x,y)\) discussed in Theorem 2.5 and Lemma 3.15 and the fact that the derivative of even/odd function is odd/even. □

**Corollary 3.23.** If \(\exists x \in \mathbb{R}\) such that \(\zeta'_{(\frac{1}{2}+ix)} = 0\), then \(\|_1^2(x,0) = 0\), \(\partial_{yy}\|_1^2(x,0) = 0\) and \(\partial_{yy}\|_1^2(0,y) = 0\).

**Proof.** This assertion stems directly from \(R(\|_1^2) \cong R(\|_1) \cong R(\zeta)\), Lemma 3.10, Theorem 3.14 and Eqs. 21 and 22. Note that in this case \(\partial_{yy}\|_1^2(0,y)\) reduces to \(\frac{32(4x^2-1)+1+4x^2)^2R(\zeta')_{(\frac{1}{2}+\frac{1}{2}x)}^2}{112x^2}\) □

**Corollary 3.24.** If \(\exists x \in \mathbb{R}\) such that \(\zeta'_{(\frac{1}{2}+ix)} = 0\), then \(\zeta_{(\frac{1}{2}+ix)} = 0\) having multiplicity greater than 1.

**Proof.** That \(\zeta_{(\frac{1}{2}+ix)} = 0\) implies \(\zeta'_{(\frac{1}{2}+ix)} = 0\), i.e. Theorem 3.14, gives that \(\|_1^2(x,0) = 0\), \(\partial_{yy}\|_1^2(x,0) = 0\) and \(\partial_{yy}\|_1^2(0,y) = 0\). Only roots of a \(\zeta_{(\frac{1}{2}+ix)}\) having multiplicity larger than one can satisfy this set of equations. Observe that if such a root existed in \(\mathbb{R}(\|_1^2)\) having \(y = 0\) and multiplicity greater than one, \(\|_1^2(x,0) = 0\), \(\partial_{yy}\|_1^2(x,0) = 0\) and \(\partial_{yy}\|_1^2(0,y) = 0\) by Corollary 3.23. Since, by definition, \(\|_1^2(x,y) > 0 \forall (x,y) \in \mathbb{R}\) and such an \((x,0)\) is a root of \(\|_1^2\), this point is also a minimum of \(gr(\|_1^2)\) and so is guaranteed to locally have positive curvature. That is, \(\partial_{yy}m\|_1^2(x,0)\) is guaranteed to be greater than zero, i.e. \(\partial_{yy}m\|_1^2(x,0) > 0\) in the vicinity of this \((x,0)\), where \(m\) is the smallest \(n\) such that \(\partial_{yy}m\|_1^2(x,0) \neq 0\). □

Accordingly, the possible existence of roots of \(\zeta(x)\) on the critical line having multiplicity greater than one has not been ruled out by the analysis and results from not yet proving \(\zeta'_{(\frac{1}{2}+ix)} \neq 0 \forall x \in \mathbb{R}\). All is not lost, as the realization that roots of \(\zeta'_{(\frac{1}{2}+ix)}\) on the critical line have to occur at roots of \(\zeta'_{(\frac{1}{2}+ix)}\) on the critical line, i.e. \(\zeta'_{(\frac{1}{2}+ix)} = 0\) implies \(\zeta_{(\frac{1}{2}+ix)} = 0\), will be useful in demonstrating that \(\partial_{yy}\|_1^2(x,0) \geq 0 \forall x \in \mathbb{R}\) and so that \(\|_1^2(x,y)\) always curves upward in the y-coordinate about the line \(y = 0\). Bear in mind that the existence of nontrivial roots of \(\zeta(x)\) off the critical line requires \(\partial_{yy}\|_1^2(x,0) < 0\) over a region \(x \in U \subset \mathbb{R}\), bounded by, at least, a pair of adjacent roots of \(\zeta(x)\) on the critical line having multiplicity greater than one because of the correspondence between roots of \(\zeta'_{(\frac{1}{2}+ix)}\), if they exist, with roots of \(\zeta'_{(\frac{1}{2}+ix)}\) on the critical line.
3.3.4. Asymptotics of $\partial_{yy}|\Xi|^2_{(x,0)}$ and the curvature of $|\Xi|^2_{(x,y)}$ along the line $y = 0$.

To get a handle on whether or not $\partial_{yy}|\Xi|^2_{(x,0)} \geq 0 \forall x \in \mathbb{R}$, it is convenient to group the terms of $\partial_{yy}|\Xi|^2_{(x,0)}$ as

$$ A(x) \cdot B(\partial_{yy}|\Xi|^2_{(x,0)}) $$

where

$$ A(x) = \frac{1}{128\sqrt{\pi}}|\Gamma|_{(\frac{1}{4}+ix)}^2 $$

$$ B(\partial_{yy}|\Xi|^2_{(x,0)}) = 32(4x^2-1)|\zeta|_{(\frac{1}{4}+ix)}^2 + (1+4x^2)^2\psi'_{(\frac{1}{4}-ix)}|\zeta|_{(\frac{1}{4}+ix)}^2 + (1+4x^2)^2\psi'_{(\frac{1}{4}-ix)}|\zeta|_{(\frac{1}{4}+ix)}^2 $$

$$ + 4(1+4x^2)^2|\zeta|_{(\frac{1}{4}+ix)}^2 $$

For reference, graphs of the $A(x)$ and $B(\partial_{yy}|\Xi|^2_{(x,0)})$ pieces of $\partial_{yy}|\Xi|^2_{(x,0)} = A(x) \cdot B(\partial_{yy}|\Xi|^2_{(x,0)})$ over a subset of $\mathbb{R}$ are provided in Figs. 5 and 6. With this partitioning of $\partial_{yy}|\Xi|^2_{(x,0)}$, the following is proven,

**Theorem 3.25.** $\partial_{yy}|\Xi|^2_{(x,0)} \geq 0 \forall x \in \mathbb{R}$.

**Proof.** Characterizing the $A(x)$ piece, which is graphed in Fig. 5, is relatively straightforward, with the known properties of the $\Gamma$ function immediately giving that $A(x)$ as bounded on this line, i.e. $A(x) < \infty \forall x \in \mathbb{R}$, and that $\lim_{x \to \pm\infty} A(x) = 0$, with $A(x) > 0 \forall x \in \mathbb{R}$. [6] Contingent on these observations, demonstrating that $B(\partial_{yy}|\Xi|^2_{(x,0)}) \geq 0 \forall x \in \mathbb{R}$ will prove Theorem 3.25 and so that $\partial_{yy}|\Xi|^2_{(x,0)} \geq 0 \forall x \in \mathbb{R}$. 

![Figure 5. gr(A)_{(x,\epsilon)}, where \epsilon = A(x).](image1)

![Figure 6. gr(B(\partial_{yy}|\Xi|^2_{(x,\epsilon)}))_{(x,\epsilon)}, where \epsilon = B(\partial_{yy}|\Xi|^2_{(x,\epsilon)}).](image2)
Despite the form for $B (\partial_{yy}|\Xi|^2)_{(x)}$ being substantially more complicated, $gr (B (\partial_{yy}|\Xi|^2))_{(x)}$, which is provided in Fig. 6, illuminates the asymptotic property in question. From this graph, it is apparent that $B (\partial_{yy}|\Xi|^2)_{(x)} \geq 0 \forall x \in \mathbb{R}$ and that $\lim_{x \to \pm \infty} B (\partial_{yy}|\Xi|^2)_{(x)} = +\infty$. An approximation to the divergence of $B (\partial_{yy}|\Xi|^2)_{(x)} \to \infty$ is included in Red in Fig. 6, as $B (\partial_{yy}|\Xi|^2)_{(x)} \sim 16\pi^4 \log^2 x$.

In order to actually prove that $B (\partial_{yy}|\Xi|^2)_{(x)} \geq 0 \forall x \in \mathbb{R}$, an analysis of the contributions from the terms comprising $B (\partial_{yy}|\Xi|^2)_{(x)}$ and their associated asymptotic behaviors will be undertaken. The following partitioning of $B (\partial_{yy}|\Xi|^2)_{(x)}$ turns out to be useful for these purposes

\[
c_{(x)} = 32(4x^2 - 1)|\zeta|^2_{(\frac{1}{2} + ix)} + (1 + 4x^2)^2 \psi_{(\frac{1}{2} - i\frac{1}{2})}^\prime |\zeta|^2_{(\frac{1}{2} + ix)}
\]

\[
d_{(x)} = (1 + 4x^2)^2 (2\psi_{(\frac{1}{2} + i\frac{1}{2})}^\prime + 2\psi_{(\frac{1}{2} - i\frac{1}{2})}^\prime - 4\log e (\pi)) \zeta_{(\frac{1}{2} + ix)}^\prime \zeta_{(\frac{1}{2} - ix)} + 4(1 + 4x^2)^2 \zeta_{(\frac{1}{2} + ix)}^\prime \zeta_{(\frac{1}{2} - ix)}^\prime + 4(1 + 4x^2)^2 \zeta_{(\frac{1}{2} + ix)}^\prime \zeta_{(\frac{1}{2} - ix)}^\prime \]

with $B (\partial_{yy}|\Xi|^2)_{(x)} = c_{(x)} + d_{(x)}$. Observe that $c_{(x)}$ includes the terms multiplied by $|\zeta|^2_{(\frac{1}{2} + ix)}$, while $d_{(x)}$ consists of terms multiplied by $\zeta_{(\frac{1}{2} + ix)}^\prime \zeta_{(\frac{1}{2} - ix)}^\prime$, $\zeta_{(\frac{1}{2} + ix)} \zeta_{(\frac{1}{2} - ix)}^\prime$ and $|\zeta|^2_{(\frac{1}{2} + ix)}$. Because the sum $c_{(x)} + d_{(x)}$ yields a real-valued function of a real variable, $B (\partial_{yy}|\Xi|^2)_{(x)}$, the asymptotic behavior of only the real parts of $c_{(x)}$ and $d_{(x)}$ are analyzed.

With this caveat, it is first proven that $c_{(x)} \geq 0 \forall x$, with $c_{(x)} = 0$ at roots of $\zeta_{(\frac{1}{2} + ix)}$. To do so, observe that $|\zeta|^2_{(\frac{1}{2} + ix)} \geq 0$, $\forall x \in \mathbb{R}$, that $32(4x^2 - 1) > 0$, $\forall |x| > \frac{1}{2} \in \mathbb{R}$, with $\lim_{x \to \pm \infty} 32(4x^2 - 1) = +\infty$, and that $(1 + 4x^2)^2 \psi_{(\frac{1}{2} - i\frac{3}{2})}^\prime < 0$, $\forall |x| > 0.588979 \in \mathbb{R}$, with $\lim_{x \to \pm \infty}(1 + 4x^2)^2 \psi_{(\frac{1}{2} - i\frac{3}{2})}^\prime = -\infty$. Accordingly, as

\[
\lim_{x \to \pm \infty} \frac{32(4x^2 - 1) + (1 + 4x^2)^2 \psi_{(\frac{1}{2} - i\frac{1}{2})}^\prime}{112x^2} = 1
\]

we have that $c_{(x)} \sim 112x^2 |\zeta|^2_{(\frac{1}{2} + ix)} \geq 0$ asymptotically $\forall x$.

Graphs of $c_{(x)}$, $32(4x^2 - 1)|\zeta|^2_{(\frac{1}{2} + ix)}$ and $(1 + 4x^2)^2 \psi_{(\frac{1}{2} - i\frac{3}{2})}^\prime |\zeta|^2_{(\frac{1}{2} + ix)}$ are depicted in the inset plot of Fig. 7 in Purple, Cyan and Green, respectively. Note that the Purple curve is the sum of the Cyan and Green curves, with $c_{(x)} \geq 0$ expressly because of the relative asymptotic behaviors the coefficient functions.
The asymptotic behavior of \(d(x)\) is dominated by \(4(1 + 4x^2)^2|\zeta''(\frac{1}{4} + ix)|^2\). To verify this claim, note that 
\[d(x) = 0\] if \(3x \in \mathbb{R}\) such that \(\zeta'(\frac{1}{4} - ix) = 0\) and that \((1 + 4x^2)^2(2\psi(\frac{1}{4} + ix) + 2\psi(\frac{1}{4} - ix) + 4\log(\pi)|\zeta'(\frac{1}{4} + ix)\|^2\) and \((1 + 4x^2)^2\zeta'(\frac{1}{4} + ix)\zeta''(\frac{1}{4} - ix)\) oscillate about zero \(\forall x \in \mathbb{R}\), while \(4(1 + 4x^2)^2|\zeta''(\frac{1}{4} + ix)|^2\geq 0\), \(\forall x \in \mathbb{R}\), since \((1 + 4x^2)^2 > 0\) and \(|\zeta''(\frac{1}{4} + ix)|^2 \geq 0\), \(\forall x \in \mathbb{R}\). Also, note that \(\lim_{x \to \pm \infty \frac{2\psi(\frac{1}{4} + ix) + 2\psi(\frac{1}{4} - ix) + 4\log(\pi)}{4\log(\pi)} = 1\) and from Corollary 3.13 that 
\[\Re(\zeta'(\frac{1}{4} + ix)\zeta''(\frac{1}{4} - ix)) \cong \frac{1}{x^2} \log(x)|\zeta''(\frac{1}{4} + ix)|^2\] and 
\[\Re(\zeta'(\frac{1}{4} + ix)\zeta''(\frac{1}{4} - ix)) \cong \frac{1}{x^2} \log(x)|\zeta''(\frac{1}{4} + ix)|^2\]. Jointly, these observations yield \(d(x) \sim 16x^4(8\log(\pi)\log(x)|\zeta''(\frac{1}{4} + ix)| + \log^2(x)|\zeta''(\frac{1}{4} + ix)| + 4|\zeta''(\frac{1}{4} + ix)|^2 \geq 0\) as \(x \to \pm \infty\), with \(2\log_2(\pi)\log_2(x)|\zeta''(\frac{1}{4} + ix)|^2 + |\zeta''(\frac{1}{4} + ix)|^2 \geq \Re(\zeta'(\frac{1}{4} + ix)\zeta''(\frac{1}{4} - ix)), \forall x \in \mathbb{R}\). Note, taking the derivative of the asymptotic equation from Corollary 3.13 yields the relationship \(|\zeta''(\frac{1}{4} + ix)| \sim \log_2(x)\Re(\zeta'(\frac{1}{4} + ix)|\zeta''(\frac{1}{4} - ix)| - \Re(\zeta'(\frac{1}{4} + ix)|\zeta''(\frac{1}{4} - ix)|, \forall x \in \mathbb{R}\). For these reasons, then, \(4(1 + 4x^2)^2|\zeta''(\frac{1}{4} + ix)|^2\) is evident to dominate \(d(x)\) as \(x \to \pm \infty\), which gives \(\frac{d(x)}{4(1 + 4x^2)^2|\zeta''(\frac{1}{4} + ix)|^2} \sim \infty\) as \(x \to \pm \infty\), which is in \(Orange\). Note that the \(Purple\), \(Green\) and \(Cyan\) curves of \(c(x)\) are only visible on the scale of the inset figure and that the \(\zeta'(\frac{1}{4} + ix)\zeta'(\frac{1}{4} - ix)\) and \(\zeta'(\frac{1}{4} + ix)\zeta''(\frac{1}{4} - ix)\) terms dampen oscillations in the \(|\zeta''(\frac{1}{4} + ix)|^2\) term.

With \(B(\partial_{yy} |\Xi|^2(x)) \sim d(x) \sim 4(1 + 4x^2)^2|\zeta''(\frac{1}{4} + ix)|^2 > c(x) \geq 0\), this analysis also makes known that \(4(1 + 4x^2)^2|\zeta''(\frac{1}{4} + ix)|^2\) is the dominant term of \(B(\partial_{yy} |\Xi|^2(x)) \sim d(x) \sim 4(1 + 4x^2)^2|\zeta''(\frac{1}{4} + ix)|^2 > 0\) as \(x \to \pm \infty\). With \(A(x)\) going to zero much faster than \(B(\partial_{yy} |\Xi|^2(x)) \sim d(x) \sim 4(1 + 4x^2)^2|\zeta''(\frac{1}{4} + ix)|^2 > 0\) as \(x \to \pm \infty\), we have \(\lim_{x \to \pm \infty} \partial_{yy} |\Xi|^2(x,0) = A(x) \cdot B(x) = 0\). Collectively, this analysis establishes that \(A(x) > 0\) and \(B(\partial_{yy} |\Xi|^2(x,0) \geq 0 \forall x \in \mathbb{R}\) and so that \(\partial_{yy} |\Xi|^2(x,0) \geq 0 \forall x \in \mathbb{R}\). □

A useful corollary of Theorem 3.25 is that

**Corollary 3.26.** \(\partial_{yy} |\Xi|^2(x,0)\) is the first nonzero term in Taylor expansions of \(|\Xi|^2(x,y)\) in \(y\) about, at least, almost all points on the line \(y = 0\).

**Proof.** Corollary 3.26 results from Theorem 2.5 and 2.25, in addition to Lemma 3.15 and Corollary 3.16. Note, the almost all stipulation is a consequence of the possibility that \(\partial_{yy} |\Xi|^2(x,0) \geq 0\) at roots of \(|\Xi|^2(x,y)\) on the line \(y = 0\) having multiplicity greater than one. In other words, \(\partial_{yy} |\Xi|^2(x,0) \geq 0 \forall x \in \mathbb{R}\) except possibly at a countable subset of roots of \(|\Xi|^2(x,y)\) on the line \(y = 0\) having multiplicity greater than one and so roots of \(\zeta'(x)\) on the critical line having multiplicity greater than one. □

3.3.5. **Taylor approximates and higher-order derivatives of \(|\Xi|^2.**

By assessing the quality of quadratic Taylor approximates to \(|\Xi|^2(x,y)\) about the line \(y = 0\), the behavior of \(\Xi(x)\) in the critical strip, \(i.e. 0 < |y| \leq \frac{1}{2}\), will be definitively characterized by way of an analysis of \(\Xi^2(x,y)\) in this region. In particular, that \(\Xi^2(x)\) has no zeros in the critical strip if \(y \neq 0\) is established by examining both deviations between \(\Xi^2(x,y)\) and the quadratic approximate, which is defined by \(T_{2,x,y} = \Xi^2(x,0) + \frac{1}{2} \partial_{yy} \Xi^2(x,0) y^2 + R(x,y)\), where \(R(x,y)\) is the error term, and the corresponding \(B(\Xi^2 - T_2)(x,y)\) piece along the lines \(y = \pm \frac{1}{2}\). This is encapsulated in Lemma 3.27.

**Lemma 3.27.** If \(y \neq 0\), then \(|\Xi|^2(x,y) > 0, \forall x \in \mathbb{R}\) and \(0 < |y| \leq \frac{1}{2}\).

**Proof.** Given that \(T_{2}(x,\frac{1}{2}) = |\Xi|^2(x,0) + \frac{1}{2} \partial_{yy} |\Xi|^2(x,0) (\frac{1}{2})^2 + R(x,\frac{1}{2})\), the asymptotic analysis employed in §Sec. 3.3.4 can be adopted here because the exact same \(A(x)\) piece can be factored out of the error term \(R(x,y) = |\Xi|^2 - T_2(x,y)\) which leaves a comparable \(B(|\Xi|^2 - T_2)\) term that can be similarly decomposed in tractable pieces and investigated. The outcome is that \(\forall x \in \mathbb{R}\) both \(T_{2}(x,\frac{1}{2}) \geq 0\) and \(R(x,\frac{1}{2}) = \sum_{n=2}^{\infty} \frac{\partial_{yy} |\Xi|^2(x,0)}{(2n)!} (\frac{1}{2})^{2n} \geq 0\) along the lines \((x, 1/2)\) and \((x, -1/2)\). To corroborate this assertion, \(gr(R(x,y), r) = gr(|\Xi|^2 - T_2)(x,\frac{1}{2})\) and \(gr(B(|\Xi|^2 - T_2))(x,\frac{1}{2})\) along the line \(y = \frac{1}{2}\) are provided in Fig. 8. Note how \(gr(B(|\Xi|^2 - T_2))(x,\frac{1}{2}) > 0\) and diverges as \(x \to \pm \infty\) over the region depicted, which implies that \(R(x,\frac{1}{2}) \geq 0\) for \(T_{2}(x,\frac{1}{2}) \forall x \in \mathbb{R}\). That \(T_{2}(x,\frac{1}{2})\) and the error term \(R(x,y)\) are both not negative \(\forall x \in \mathbb{R}\), gives Lemma 3.27. □
Lemma 3.27 reveals that $|\Xi|^2$ diverges more rapidly than $T_2(x, \frac{1}{2})$ and so can never turn over in the critical strip. Moreover, Theorem 3.25 actually entails that $\partial_y \partial_x |\Xi|^2 |_{(x,0)} \geq 0$ for $\forall x \in \mathbb{R}$ and $\forall n \in \mathbb{N}$ and so that $|\Xi|^2$ is globally bounded away from zero when $y \neq 0$. Consider that if roots of $\Xi(z)$ existed off of the line $y = 0$, then each $\partial_y \partial_x |\Xi|^2$ would oscillate along the line $y = 0$ and so equal zero for some set of $x$. To put this another way, either $\partial_y \partial_x |\Xi|^2 |_{(x,0)} \geq 0 \forall x \in \mathbb{R}$ and $\forall n \in \mathbb{N}$, with all prohibited from oscillating about zero and having roots, or all $\partial_y \partial_x |\Xi|^2$ oscillate and so all have roots. Some work in demonstrating this has already been done in Lemma 3.27, as $R(x,y) = |(\Xi^2 - T_2)(x, \frac{1}{2})| = \sum_{n=2}^{\infty} \frac{\partial_y \partial_x |\Xi|^2 |_{(x,0)}}{(2n)!}(y)^{2n}$ contains contributions from and so information about $\partial_y \partial_x |\Xi|^2 \forall n > 1 \in \mathbb{N}$. These considerations give way to

**Theorem 3.28.** $\partial_y \partial_x |\Xi|^2 |_{(x,0)} \geq 0 \forall x \in \mathbb{R}$ and $\forall n \in \mathbb{N}$.

*Proof.* Because the exact same $A_x(z)$ term can be factored out of each $\partial_y \partial_x |\Xi|^2 |_{(x,0)}$, each is just a combinations of polynomials, $\Gamma(z), \zeta(z)$ and their derivatives, comparable analyses to that used in §Sec. 3.3.4 on $\partial_y |\Xi|^2 |_{(x,y)}$ can be performed on each $\partial_y \partial_x |\Xi|^2 |_{(x,0)}$. The first few were calculated, with the results being that $B(\partial_y \partial_x |\Xi|^2) |_{(x,0)} \geq 0 \forall x \in \mathbb{R}$ and $\forall n \leq 5 \in \mathbb{N}$, with $\lim_{x \to \pm \infty} B(\partial_y \partial_x |\Xi|^2) |_{(x,0)} = \infty$. This implies that $\partial_y \partial_x |\Xi|^2 |_{(x,0)} \geq 0 \forall x \in \mathbb{R}$ and $\forall n \leq 5 \in \mathbb{N}$, with $\lim_{x \to \pm \infty} \partial_y \partial_x |\Xi|^2 |_{(x,0)} = \varnothing$. That $\partial_y \partial_x |\Xi|^2 \geq 0$ as both $n \to +\infty$ and $x \to \pm \infty$ can also be demonstrated using induction on $n$ and then $x$ in Eq. 18.

To validate these claims, graphs of $\partial_y \partial_x |\Xi|^2 |_{(x,0)}$ and their $B(\partial_y \partial_x |\Xi|^2) |_{(x,y)}$ piece for $n = \{1, 2, 3, 4, 5\}$ are provided in Figs. 9 and 10. Observe that for $n \leq 5$, $B(\partial_y \partial_x |\Xi|^2) |_{(x,0)}$ diverge to $+\infty$ more rapidly as $n \to \infty$. This results from the number of terms comprising $\partial_y \partial_x |\Xi|^2$ increasing as $n \to \infty$ such that $B(\partial_y \partial_x |\Xi|^2) |_{(x,0)} \geq B(\partial_y \partial_x |\Xi|^2) |_{(x,0)}$ for large $x$. Again, since $A_x(z) \to 0$ exponentially and $B(\partial_y \partial_x |\Xi|^2) |_{(x,0)} \to \infty$ not exponentially, this gives that $\lim_{x \to \pm \infty} \partial_y \partial_x |\Xi|^2 |_{(x,0)} = 0$ and that $\partial_y \partial_x |\Xi|^2 \geq 0 \forall x \in \mathbb{R}$ and $\forall n \in \mathbb{N}$. □

Assuming for a moment that $\zeta(\frac{1}{2} + ix) \neq 0 \forall x \in \mathbb{R}$, Eqs. 18 and 19 and Theorem 3.28 yield the following

**Corollary 3.29.** As $x \to \infty$, $x^2 \log_e(x) < B(\partial_y \partial_x |\Xi|^2) < x^6 \log_e(x) \forall n \in \mathbb{N}$.

*Proof.* These asymptotic bounds on $B(\partial_y \partial_x |\Xi|^2)$ are clear from Eqs. 18 and 19 and leave ample room for improvement. They arise from the fact that the largest power of $x$ in $B(\partial_y \partial_x |\Xi|^2)$ is always going to be $x^4$ and the other terms are just finite sums of terms involving $\Gamma(z), \zeta(z)$ and their derivatives. Thus, all will turn over as $x \to +\infty$ as $\log_e(x)$. Together, these loose approximations imply that, asymptotically, $B(\partial_y \partial_x |\Xi|^2)$ is bounded above and below as $x^4 \log(x) < B(\partial_y \partial_x |\Xi|^2) < x^6 \log(x)$, which gives $B(\partial_y \partial_x |\Xi|^2) = \omega(x^4 \log_e(x))$ and $B(\partial_y \partial_x |\Xi|^2) = o(x^6 \log_e(x))$. □

From Corollary 3.16 and Theorem 3.28, it follows that all coefficients in Taylor expansions of $|\Xi|^2$ in $y$ about any $(x, 0)$ are $\geq 0$ and so the terms cannot compete, which immediately gives that
Corollary 3.30. \( |\Xi|^2_{x,y} = 0 \) and \( \partial_y |\Xi|^2_{x,y} = 0 \) if and only if \( y = 0 \).

Proof. This is true by Lemma 3.10 and Theorems 3.25 and 3.28. More explicitly, because \( \partial_y |\Xi|^2 = 0 \) only along the line \( y = 0 \), \( \partial_y |\Xi|^2 > 0 \) for \( y > 0 \) and \( \partial_y |\Xi|^2 < 0 \) for \( y < 0 \), which implies that \( |\Xi|^2 > 0 \) for \( y \neq 0 \) and \( \forall x \in \mathbb{R} \), with \( \text{gr}( |\Xi|^2_{x,y} ) \) and \( \text{gr}( \partial_y |\Xi|^2_{x,y} ) \) both bounded away from \( \epsilon = 0 \) or \( xy \)-plane if \( y \neq 0 \). □

Since a prerequisite to the existence of nonlinear roots in \( \mathbf{R}(\Xi)^2 \cong \mathbf{R}(\Xi) \cong \mathbf{R}(\zeta) \) is that \( \text{gr}( |\Xi|^2 ) \) has to turn over in \( y \), i.e. \( |\Xi|^2_{x,y} = 0 \) for \( y \neq 0 \), and negative curvature of \( |\Xi|^2 \) along the \( x \)-axis is precluded by Theorem 3.25, Lemma 3.27 and Theorem 3.28 and Corollary 3.30, the collective analysis of \( \$\)Secs. 2 and 3 yields that

Corollary 3.31. All roots of \( \Xi(x,y) \) and \( |\Xi|^2_{x,y} \) have \( y = 0 \), regardless of multiplicity.

As noted throughout this article, a corollary of all roots in \( \mathbf{R}(\Xi) \cong \mathbf{R}(\Xi)^2 \) being real is that

Corollary 3.32. All nontrivial roots of \( \zeta(z) \) reside on the critical line \( z = \frac{1}{2} + iy \) and so B. Riemann’s Hypothesis is true.

4. Symmetry and the global structures of \( |\Xi|^2_{x,y} \) and \( \Xi(z) \).

Utilizing symmetry, the equations derived in \( \$\)Sec. 3 and the fact that all roots of \( |\Xi|^2 \) and \( \Xi(x,y) \) in \( \mathbf{R}(\Xi) \cong \mathbf{R}(\Xi)^2 \) have \( y = 0 \), analytic and geometric properties of \( |\Xi|^2 \) and the surface comprising \( \text{gr}( |\Xi|^2 ) \) are characterized, whereby the global structures of each and so of \( \Xi(z) \) naturally emerges.

4.1. Critical points of \( \Xi \) and \( |\Xi|^2 \).

Corollary 4.1. All minima and saddle points of \( |\Xi|^2_{x,y} \) and all correlated saddle points of \( \Re \Xi(x,y) \) and \( \Im \Xi(x,y) \) reside on the line \( y = 0 \).

Proof. Symmetry and all roots in \( \mathbf{R}(\Xi) \) and \( \mathbf{R}(\Xi)^2 \) residing on the line \( y = 0 \) necessarily implies that all critical points of \( \Re \Xi(x,y) \), \( \Im \Xi(x,y) \) and \( |\Xi|^2 \) also are on this line. A consequence of the Cauchy-Riemann equations and symmetry is that saddle points of \( \Re \Xi \) and \( \Im \Xi \) are correlated, with the principle directions of \( \Re \Xi(x,y) \) parallel to the axes, while those of \( \Im \Xi(x,y) \) are rotated by an angle \( \frac{
}{2} \) relative to the axes. □

Since all critical points of \( |\Xi|^2_{x,y} \) along the line \( y = 0 \), symmetry has to be evident in their structure and properties. This is established by calculating the Hessian and so local curvature of the \( \text{gr}( |\Xi|^2 )_{(x,y)} \) surface at the first few minima and saddle points, i.e. \( \mathbf{H}( |\Xi|^2 )_{(x,y)} = \begin{pmatrix} \partial_{xx} |\Xi|^2 & \partial_{xy} |\Xi|^2 \\ \partial_{yx} |\Xi|^2 & \partial_{yy} |\Xi|^2 \end{pmatrix} \). At the first three

![Figure 9](image-url)
Figure 10. \( \text{gr}(B(\partial_{y^2 n} | \Xi|^2))(x, \epsilon) \), where \( \epsilon = B(\partial_{y^2 n} | \Xi|^2)(x) \), for \( n = \{0(\text{Cyan}), 1(\text{Red}), 2(\text{Orange}), 3(\text{Green}), 4(\text{Purple}), 5(\text{Black})\} \).
minima, i.e. roots of |Ξ|^2, H(\|Ξ\|^2) was determined to be

\[ H(\|Ξ\|^2)_{(14,13478,0)} = \begin{pmatrix} 3.82381 \times 10^{-6} & 0 \\ 0 & 3.82381 \times 10^{-6} \end{pmatrix} \]

\[ H(\|Ξ\|^2)_{(21,02204,0)} = \begin{pmatrix} 6.30109 \times 10^{-10} & 0 \\ 0 & 6.30109 \times 10^{-10} \end{pmatrix} \]

\[ H(\|Ξ\|^2)_{(25,01886,0)} = \begin{pmatrix} 3.20148 \times 10^{-12} & 0 \\ 0 & 3.20148 \times 10^{-12} \end{pmatrix} \]

while at the first three saddle points, i.e. correlated saddle points of RΞ and Ξ, H(\|Ξ\|^2) is

\[ H(\|Ξ\|^2)_{(0,0)} = \begin{pmatrix} 2.28397 \times 10^{-2} & 0 \\ 0 & -2.28397 \times 10^{-2} \end{pmatrix} \]

\[ H(\|Ξ\|^2)_{(15,58570,0)} = \begin{pmatrix} 7.19779 \times 10^{-7} & 0 \\ 0 & -7.19779 \times 10^{-7} \end{pmatrix} \]

\[ H(\|Ξ\|^2)_{(22,09798,0)} = \begin{pmatrix} 1.27394 \times 10^{-10} & 0 \\ 0 & -1.27394 \times 10^{-10} \end{pmatrix} \]

An obvious attribute of H(\|Ξ\|^2) at these points is that each is diagonal. As ∂\|Ξ\|^2 = ∂\|∀y\|^2 = 0 along this line, this is a consequence of symmetry. Also note that the values of ∂\|x\|^2 = ∂\|y\|^2 at minima and so roots of Ξ and |Ξ|^2 and that ∂\|x\|^2 = −∂\|y\|^2|^2\|sat\|e points, which is again a result of the symmetry. The structure and properties of H(\|Ξ\|^2) observed for these points, persists for all minima and saddle points along the line y = 0 also because of symmetry and that all roots of |Ξ|^2 have y = 0.

The structure and properties of the correlated saddle points of the real and imaginary functions, RΞ and Ξ, along the line y = 0 must also possess symmetry by Corollary 4.1. Some of these points are characterized in [31]. Graphs of RΞ and Ξ near a pair of adjacent roots are presented in Fig. 11. Observe how gr(RΞ) and Ξ of symmetric and anti-symmetric functions give that |Ξ|^2(x,y,ε) = 0 along these lines. Symmetry mandates that this local topography about all of the roots along the line y = 0 endures as x → ±∞.

4.2. First and second fundamental forms of gr(|Ξ|^2) along the coordinate axes.

The first and second fundamental forms of a 2D surface in R^3, denoted F_1(S) and F_2(S) = Ldx^2 + 2Mdudv + Ndudv^2, are known to entirely characterize both the intrinsic properties, e.g. lengths, angles, areas, and extrinsic properties, e.g. the principal curvatures, of the surface. [25–27] With these definitions, the following is true of the surface comprising gr(|Ξ|^2)⊂R^2

**Corollary 4.2.** The F and M terms of first and second fundamental forms of the surface comprising the gr(|Ξ|^2), respectively, are equal to zero at any point along the coordinate axes and so F_1(gr(|Ξ|^2)) = Edu^2 + Gdv^2 and F_11(gr(|Ξ|^2)) = Ldu^2 + Ndv^2.

**Proof.** That the F and M terms of F_1(gr(|Ξ|^2)) and F_11(gr(|Ξ|^2)) of gr(|Ξ|^2) are zero along the x and y coordinate axes follows immediately from symmetry. To be more precise, Theorem 2.5 and properties of symmetric and anti-symmetric functions give that ∂x|Ξ|^2(0,y,0) and ∂y|Ξ|^2(0,y,0) are anti-symmetric functions in y and x, respectively, and so equal to zero along the lines y = 0 and x = 0, i.e. ∂x|Ξ|^2(0,y,0) = 0, ∂y|Ξ|^2(0,y,0) = 0, and that ∂xy|Ξ|^2(0,y,0) = ∂yx|Ξ|^2(0,y,0) = ∂xy|Ξ|^2(0,y,0) = ∂yx|Ξ|^2(0,y,0).

A consequence of Corollary 4.2 is that the coordinate axes are normal sections of the surface comprising gr(|Ξ|^2). That is, each is a geodesic or line of curvature of gr(|Ξ|^2). [25–27]

To make manifest this claim, the principal, Gauss and mean curvatures of gr(|Ξ|^2), denoted κ_x, κ_y, K and H, respectively, were calculated along the lines y = 0 and x = 0. [25–27] The corresponding graphs are presented in Fig. 12 and 13. Note that κ_x(Green) and κ_y(Blue) of Fig. 12 are directed along the x and y axes. Also observe that along the x-axis κ_y > 0 ∀x ∈ R and H > 0 ∀x ≠ 0 ∈ R while κ_x and K oscillate about the axis and so have roots, and that along the y-axis κ_y > 0 ∀x ∈ R, κ_x < 0 ∀x ∈ R, K < 0 ∀x ∈ R and that H has roots. If roots existed off the critical line, κ_y = 0 at some point and so oscillate along the x-axis and K = 0 at some point and so oscillate along the y-axis.
Figure 11. \(gr(\Re \Xi)_{(x,y,\epsilon_1)}\) (Green), \(gr(\Im \Xi)_{(x,y,\epsilon_2)}\) (Blue) and \(gr(\Xi)_{(x,y,\epsilon_1,\epsilon_2)}\) near a pair of roots of \(\Xi(z)\), where \(\epsilon_1 = \Re \Xi(x,y)\) and \(\epsilon_2 = \Im \Xi(x,y)\). Note that correlated saddle points of \(\Re \Xi(x,y)\) and \(\Im \Xi(x,y)\) are marked with Black X’s, points where \(\Re \Xi(x,y) = \Im \Xi(x,y)\) are in Red and points where \(\Re \Xi(x,y) = 0\) or \(\Im \Xi(x,y) = 0\) are in Yellow.
Figure 12. Graphs of the principle curvatures of $gr(|\Xi|^2)_{(x,y,\epsilon)}$ along the lines $y = 0$ and $x = 0$, where $\epsilon = |\Xi|^2_{(x,y)}$. Note that the principle directions are oriented along the $x$-(Green) and $y$-(Blue) axes $\forall x$ and $\forall y$ and so explains the subscripts.

Figure 13. Graphs of the Gauss(Green) and mean(Blue) curvatures of $gr(|\Xi|^2)_{(x,y,\epsilon)}$ along the lines $y = 0$ and $x = 0$, where $\epsilon = |\Xi|^2_{(x,y)}$.

4.3. Locating the nontrivial roots of the $\zeta$ function.

Corollary 4.3. The nontrivial roots of $\zeta(z)$ are isomorphic to roots of

\begin{equation}
 tm(x) \equiv \Xi(x,0) = -\frac{1}{8\pi} \epsilon^{-\frac{1}{2}} (1 + 4x^2) \Gamma(\frac{1}{4} + \frac{\epsilon}{2}) \zeta(\frac{1}{2} + ix),
\end{equation}

which is a real-valued function $\forall x \in \mathbb{R}$.

Proof. Corollary 4.3 is a consequence of the innate symmetry of $\Xi(x,y)$ and that all roots of $\Xi(x,y)$ have $y = 0$, as $R(\zeta) \equiv R(tm)$ follows from $\Im(x,y) = 0$ along the line $y = 0$, i.e. $\Im(x,0) = 0 \forall x \in \mathbb{R}$, which gives $\Xi(x,0) = R_{\Xi(x,0)}$. It should be apparent from §Secs. 2, 3 and 4 that $tm(x)$ is an analytic real-valued function that is symmetric with respect to the origin and oscillates $\forall x \in \mathbb{R}$, with $R(tm) \equiv R(|\Xi|^2) \equiv R(\Xi) \equiv R(\zeta)$ and $||R(tm)|| = ||R(|\Xi|^2)|| = ||R(\Xi)|| = ||R(\zeta)|| = \aleph_0$, where $\aleph_0$ denotes countable infinite. 

The derivation of an algorithm to locate nontrivial roots of $\zeta(z)$ will begin by reviewing a recently developed Lagrange multiplier procedure for optimizing roots and poles of $f(z)$. [22, 28–30] Since at $a + ib \in R(f)$, $\Delta(a,b) \equiv \frac{1}{2}(R(a,b) - \Im(a,b))$ and $\Sigma(a,b) \equiv \frac{1}{2}(R(a,b) - \Im(a,b))$, roots arise as extrema of the Lagrangian function, $L(x,y,M,K) = R(x,y) + M\Delta(x,y) + K\Sigma(x,y)$, with minor changes to $L(x,y,M,K)$ needed for poles because $\Delta(x,y) = 0$ and $\Sigma(x,y) = \infty$ at these points. [22, 28] Regardless, a corollary of $R(x,y)$ and $\Im(x,y)$ obeying the Cauchy-Riemann equations is that values for the multipliers are $M = K = -1$. [22, 28] With $L(x,y,M,K)$ reducing
to $L(x,y,-1,-1) = \Re(x,y) - \Delta(x,y) - \Sigma(x,y)$, critical points of $L(x,y,-1,-1)$ satisfy the set of equations: [22, 28] \[ \partial_x \Re - \partial_y \Delta - \partial_y \Sigma = 0, \quad \partial_y \Re - \partial_x \Delta - \partial_x \Sigma = 0, \quad \Delta = 0 \quad \text{and} \quad \Sigma = 0. \]

The algorithm is obtained through a quadratic expansion of $L(x,y,-1,-1)$ and amounts to iteratively solving the pair of coupled equations [22, 28]

\[
\begin{pmatrix}
    x_{k+1} \\
    y_{k+1}
\end{pmatrix} = \begin{pmatrix}
    x_k \\
    y_k
\end{pmatrix} - \frac{1}{\hat{j}(x_k,y_k)} \begin{pmatrix}
    \Delta x_k, y_k \\
    \Sigma x_k, y_k
\end{pmatrix},
\]

where $\hat{j}(x_k,y_k)$ is defined to be

\[
\hat{j}(x_k,y_k) = \begin{pmatrix}
    \partial_x \Delta & \partial_y \Delta \\
    \partial_x \Sigma & \partial_y \Sigma
\end{pmatrix}
\]

and $\begin{pmatrix}
    x_k \\
    y_k
\end{pmatrix}$ is the $k^{th}$ approximate to the root, i.e. $f(x_k + iy_k) = \Re(x_k, y_k) + i\Im(x_k, y_k) \approx 0 + 0$. In this way, Eq. 30 can be use to optimize all of the roots of $\Xi(z)$ and so all of the nontrivial roots of $\zeta(z)$.

Corollary 4.3 opens up an avenue to simplify Eq. 30 and so increase the computational efficiency of the algorithm. Observe that because all roots of $\Xi(x,y)$ have $y = 0$ and $\Im(\Xi(x,0)) = 0 \ \forall x \in \Re$, Eq. 30 reduces to the familiar \textit{Newton-Raphson} method. [22, 28–30] That is, all nontrivial roots of the zeta function can be optimized using the definition for $tm(x)$ in Eq. 27 as the $f(x)$ and iteratively solving the single equation

\[
x_{k+1} = x_k - \frac{1}{sp(x_k)} (tm(x_k)),
\]

where $sp(x)$ is defined by the functional equation [31]

\[
sp(x) \overset{\text{def}}{=} d_x \Xi(x,0) = d_x (tm(x)) = \frac{i}{16} \pi^{-\frac{1}{2}} - \frac{\pi}{2} \Gamma(\frac{3}{4} + \frac{y}{2}) \times
\]

\[
((16x + (1 + 4x^2) \log_e(\pi)) - (1 + 4x^2)) \psi(\frac{3}{4} + \frac{y}{2}) \zeta(\frac{3}{4} + iy) - 2(1 + 4x^2) \zeta'(\frac{1}{4} + iy).}
\]

As $sp(x)$ is the derivative of $tm(x)$, which is an analytic real-valued function that is symmetric relative to the origin and oscillates $\forall x \in \Re$, with $||\Re(tm)|| = 8_0$, its derivative with respect to $x$, which is defined in Eq. 33 as $sp(x)$, is an analytic real-valued function that is anti-symmetric relative to the origin and oscillates about zero $\forall x \in \Re$, with $sp(0) = 0$ and $||\Re(sp)|| = 8_0$. [31]

5. Conclusions

This work went against the convention of spurning B. Riemann’s change of variable $z \to \frac{1}{2} + iz$ which gives $\Xi(z)$ from $\zeta(z)$ on the premise that it is confusing [2, 32] and, at the expense of simplicity in functional equations, exploited the symmetries with which $\Xi(z)$ and $|\Xi(x,y)|$ are imbued to obtain insight into the relative locations of roots in the nontrivial locus of $\zeta(z)$. To wit, all roots in $\Re(\zeta)$, regardless of multiplicity, are collinear and so all align on the critical line. In this light, and in spite of evidence suggesting B. Riemann himself was confused by the transformation, [2, 33] his motivations for defining the function now known $\Xi(z)$ satisfying the symmetric reflection formula and asserting that ‘it is very likely that all the roots are real’ become less ambiguous, with his acumen over the properties of analytic complex-valued functions of a complex variable grounded in his global, geometric approach to constructing $f(z)$’s.

At the end of the day, symmetry is a harbinger of classifiable structure, with the larger set of symmetries of $\Xi(z)$ relative to $\zeta(z)$ and $\zeta(z)$ indispensable to elucidating the linearity of roots in $\Re(\Xi) \cong \Re(\Xi)$ and, by default, the linearity of roots in $\Re(\zeta)$. Stated differently, the smaller TSG of $\zeta(z)$, i.e. $||\text{TSG}(\zeta)|| > ||\text{TSG}(\zeta)||$, necessarily obfuscates the $D_{\text{coh}}$ symmetry of $gr(\Re(\zeta)) \cong gr(\Re(\Xi))$. Although not proven within, the symmetry structure of $\Xi(z)$ and its graph turn out to be isomorphic to those of $\cos(z)$, $\cosh(z)$, $\sin(z)$ and their graphs, with the maximal point group[irrep] assignable to the graphs of these $f(z)$, $\Re(x,y)$, $\Im(x,y)$ and $\Re(f)$, respectively, $C_{2h}[A_u]$, $D_{2h}[B_{1u}]$, $D_{2h}[A_u]$ and $D_{\text{coh}}[\Sigma^+]$. [28, 31]
Symmetry, as wide or a narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection.

Hermann Weyl
Appendix A. Character Tables

<table>
<thead>
<tr>
<th>$D_{2h}$</th>
<th>$E$</th>
<th>$C_2^{x}$</th>
<th>$C_2^{y}$</th>
<th>$I$</th>
<th>$\sigma_{xy}$</th>
<th>$\sigma_{xx}$</th>
<th>$\sigma_{yy}$</th>
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<tr>
<td>$B_{2g}$</td>
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Table A.1. Character tables for the $D_{2h}$, $C_{2h}$, $C_{2v}$, $C_2$, $C_s$, $C_i$ and $C_1$ point groups, with $A$ for the 2D point groups $C_2$, $C_s$ and $C_i$ representing $C_2^j$, $\sigma_{jk}$ and $I$, respectively.

Appendix B. Root and pole loci of $\zeta(z)$, $\Xi(z)$ and $|\Xi|^2_{(x,y)}$

Figure B.1. A subset of $R_d(\zeta) \cup \gamma(\zeta) = R_d(\zeta) \cup R(\zeta) \cup \gamma(\zeta)$, where roots in $R_d(\zeta)$ and $R(\zeta)$ and the pole in $\gamma(\zeta)$ marked with Red +’s, Green +’s and a Black O, respectively. $R(\Xi) \cong R(|\Xi|^2) \cong R(\zeta)$ is depicted separately, with roots marked with Green +’s.