ON THE THE GLOBAL STRUCTURE OF THE SQUARE OF THE MODULUS OF THE UPPERCASE XI FUNCTION

JOSEPH DILLON

Abstract. A functional equation for the square of the modulus of the uppercase Xi function, $|\Xi|^2_{(x,y)}$, is defined, where $\Xi(z)$ is a variant of B. Riemann’s zeta function, $\zeta(z)$, that satisfies the symmetric reflection formula. The set of symmetries of $|\Xi|^2_{(x,y)}$ with respect to additive inverses of the variables $x$ and $y$ are cataloged. The larger set of symmetries associated with $|\Xi|^2_{(x,y)}$ is then exploited to derive functional equations pertinent to understanding the relative locations of roots in the nontrivial locus of $\zeta(z)$. In particular, general expressions and explicit functional equations for partial derivatives of $|\Xi|^2_{(x,y)}$ are derived, with symmetry used to prove that $\partial_y|\Xi|^2_{(x,0)} = 0$, $\forall x \in \mathbb{R}$, that $\partial_{yy}|\Xi|^2_{(x,y)} \geq 0$, $\forall (x,y) \in \mathbb{R}^2$, and that $2\pi i(\zeta(0)+y)(\zeta(0-y)) = y \tanh \left( \frac{\pi}{2} \right) \zeta(1+i\nu)\zeta(1-i\nu)$, $\forall y \in \mathbb{R}$. Additionally, minima and saddle points of $|\Xi|^2_{(x,y)}$ are identified, with symmetry used to prove that all reside along the line $y = 0$. Hinging on the collective insights, a Newton-Raphson algorithm to locate nontrivial roots of $\zeta(z)$ is also derived, i.e. $x_{k+1} = x_k - \dfrac{2\pi}{16\pi x_k + \log x} \left( \frac{\Gamma\left( \frac{1}{4} + \frac{x_k^2}{4} \right)}{\Gamma\left( \frac{1}{4} + \frac{y^2}{4} \right)} \right)$, where $\Gamma(z)$ is the Gamma function.

1. Introduction

B. Riemann investigated the Dirichlet series as a function of a complex variable

$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z},$ (1)

to better understand the asymptotic distribution of the prime numbers, [1–5] where $z = x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$ and $i^2 = -1$. [6–11] In particular, he proved that the domain of Eq. 1, which converges for $x > 1$, could be extended to $z \in \mathbb{C}$ having $x < 1$ by re-expressing the infinite series as the contour integral [1, 2]

$\zeta(z) = \frac{\Gamma(-z)}{2\pi i} \int_{-\infty}^{+\infty} \frac{(-x)^x}{e^x - 1} dx.$ (2)

Because Eq. 2 is valid for $\mathbb{C} - \gamma(\zeta)$, it is the unique analytic continuation of Eq. 1 to $\mathbb{C} - \gamma(\zeta)$. [1, 2] Here, $\gamma(\zeta)$ connotes the set of poles of $\zeta(z)$, with $\gamma(\zeta) = \{ z \in \mathbb{C} | \zeta(z) = \infty \} = \{ (1 + i0) \}, |\gamma(\zeta)|| = 1$ and this pole simple because $\lim_{z \to 1}(z - 1)\zeta(z) = 1$. B. Riemann’s work was buoyed by the research of others. Amongst them, L. Euler’s is the most notable, as he established the connection between $\zeta(x)$ and the prime numbers through the identity $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} = \prod_{p_n=1}^{\infty} \frac{1}{1 - \frac{1}{p_n^x}}$, where $p_n$ indexes the primes, in addition to the formulæ $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(2n) = (-1)^{n+1} \frac{B_{2n}}{2(2n)!} \pi^{2n}$, where $B_{2n}$ are Bernoulli numbers. [2–5]

B. Riemann also derived equations relating $\zeta(x)$ to $\zeta(1-x)$ that are referred to as Riemann’s functional equations, in addition to two variants of $\zeta(z)$ known now as the lowercase xi function, $\xi(z)$, and the uppercase Xi function, $\Xi(z)$. Observe that it has become customary to adopt E. Landau’s notation, [12] whereby the function now called $\Xi(z)$ is referred to as $\xi(z)$ in [1], with the change of variable rejected on the premise that it is confusing. [2] Keeping with this naming convention, then, the lowercase Xi function, $\xi(z)$ is defined by the functional equation [1, 2]

$\xi(z) = \frac{1}{2} \pi^{-\frac{z}{2}} z(z - 1) \Gamma\left( \frac{z}{2} \right) \zeta(z)$ (3)

Date: September 1, 2019.

2010 Mathematics Subject Classification. Primary 11M26; Secondary 30C15, 53A04, 58D19.

Key words and phrases. Symmetry; Complex function theory; Riemann’s Xi function; Riemann’s Zeta function.
and obeys $\xi_{1-z} = \xi_z$, $\forall z \in \mathbb{C}$. The importance of this representation lies in that all singularities of $\Gamma(\zeta)$ and $\zeta_{1-z}$ are removed because the polynomial factors $z$ and $(z - 1)$ regularize $\Gamma(\zeta)$ and $\zeta_{1-z}$ at $z = 0$ and $z = 1$, respectively, with $\xi_0 = \xi_1 = \frac{1}{2}$. In other words, $\xi(z)$ is holomorphic $\forall z \in \mathbb{C}$ and $\gamma(\xi) = \mathcal{O}$. [1, 2] The uppercase Xi function, $\Xi(z)$, which is denoted $\xi(z)$ in [1], is obtained from $\xi(z)$ through the change of variable, $z \to \frac{1}{2} + iz$, with $\Xi(z)$ defined by the functional equation [1, 2]

$$\Xi(z) = \frac{1}{\pi} \frac{1}{\Gamma(\zeta) \Gamma(\zeta + iz)} \xi(\zeta) \Gamma(\frac{1}{4} + iz) \xi(\zeta + iz).$$

Since the analyticity of $\xi(z)$ is clearly not altered by this transformation, $\Xi(z)$ is also holomorphic $\forall z \in \mathbb{C}$, with $\gamma(\Xi) = \mathcal{O}$. [1, 2] Here, roots of the polynomial factors $(\frac{1}{2} + iz)$ and $(iz - \frac{1}{2})$ regularize poles in $\gamma(\Gamma)$ of $\Gamma(\zeta)$ and $\gamma(\zeta)$ of $\zeta_{1-z}$, respectively, with $\Xi(\zeta) = \Xi(\zeta + iz) = \frac{1}{2}$. The advantage of this representation relative to $\zeta_{1-z}$ and $\zeta_{1-z}$ is attributable to $\Xi(z)$ possessing symmetric reflection symmetry, which entails that $\Xi(z)$ satisfies $\Xi(-z) = \Xi(z)$, $\forall z \in \mathbb{C}$. [1]

The locations of roots in the nontrivial locus of $\zeta(z)$, denoted $R(\zeta)$, has been a matter of considerable interest, with the entire locus $R(\zeta) = \{z \in \mathbb{C} | \xi(z) = 0\} = \{R(\zeta), R_n(\zeta)\}$ also containing the trivial zeros, denoted $R_0(\zeta) = \{-2n + i0 \in \mathbb{C} | n \in \mathbb{N}\}$. [1–5] A longstanding hypothesis germane to this work is that all roots in $R(\zeta)$ reside on the critical line $z = \frac{1}{2} + iy \in \mathbb{C}$. [1, 2] Progress in this regard is made within by way of an original analysis of the square of the modulus of $\Xi(z)$, i.e. $|\Xi|^2 = \Xi(z) \Xi(z) = 8z^2 + 3iz + t$, that leverages the inbuilt symmetries of $|\Xi|^2_{x,y}$ and so of $\Xi(z)$, in addition to the fact that $R(\Xi) \cong R(\Xi^2) \cong R(\zeta) \cong R_0(\zeta)$. Key observations are that $\partial_y|\Xi|^2_{x,0} = 0$, $\forall x \in \mathbb{R}$, that $\partial_y|\Xi|^2_{x,0} \geq 0$, $\forall x \in \mathbb{R}$ and $\forall n \in \mathbb{N}$, and that $|\Xi|^2_{x,y}$ is bounded away from zero if $y \neq 0$, i.e. $|\Xi|^2_{x,y} > 0$ for $y \neq 0$, with the latter result implying that all roots of $|\Xi|^2_{x,y}$ and so of $\Xi(z)$ arise on the line $y = 0$.

Why is this approach assumed to yield progress? Well, a way to demonstrate that all roots of $\cos(\zeta)$ reside on the line $y = 0$ is to show that the zero set of $\partial_y|\cos|^2_{x,y}$ is the line $y = 0$ and that $\partial_y|\cos|^2_{x,y} \geq 0$, $\forall x \in \mathbb{R}$ and $\forall n \in \mathbb{N}$. That is, linearity in the root locus of $\cos(\zeta)$ and so $|\cos|^2_{x,y} = \cos(\zeta) + \sin(\zeta)$ is a corollary of $\partial_y|\cos|^2_{x,y} = \sinh(2\zeta)$ equaling zero for $y = 0$ and $\partial_y|\cos|^2_{x,y} \geq 2n-1 \cosh(2\zeta)$ not being negative along this line $\forall n \in \mathbb{N}$, as $\sinh(2\zeta) = 1$ and $2n-1 > 0$, $\forall n \in \mathbb{N}$ and $y = 0$.

In §Sec. 2, the symmetries and differential structure of $|\Xi|^2_{x,y}$ are analyzed. Ramifications of the results of §Sec. 2 on the global structure and analytic properties of $|\Xi|^2_{x,y}$ and $\Xi(z)$ are conferred in §Sec. 3, while §Sec. 4 contains a brief summary and concludes.

2. Symmetry and the differential structure of $|\Xi|^2_{x,y}$

A functional equation for the square of the modulus of $\Xi(z) = \Xi(x+iy)$ is defined and its set of symmetries with respect to additive inverses of the variables $x$ and $y$ are cataloged. Afterwards, general and explicit functional equations for partial derivatives of $|\Xi|^2_{x,y}$ are derived, with symmetry featuring prominently in the analysis. The impetus for choosing to examine $|\Xi|^2_{x,y}$ in lieu of $\zeta(z)$, $\zeta(z)$ or their moduli $|\zeta|^2_{x,y}$ and $|\zeta|^2_{x,y}$, which is standard practice, [2] is the larger set of symmetries associated with $|\Xi|^2_{x,y}$ and that command over the structure and properties of $\Xi(z)$ and its root locus can be obtained by analyzing those of $|\Xi|^2_{x,y}$ and its root locus since $|\Xi|^2_{x,y} = 0$ implies $\Xi(x+iy) = 0$, with $R(\Xi) \cong R(|\Xi|^2)$. Consult Appendix A for a 2D representation of subsets of $R_0(\zeta) \cup \gamma(\zeta)$ and $R(\Xi) \cong R(|\Xi|^2)$ and note how roots in $R(\Xi) \cong R(|\Xi|^2) \cong R(\zeta)$ align symmetrically about the line $x = 0$ and along the line $y = 0$.

2.1. The functional equation for $|\Xi|^2_{x,y}$

The approach implemented is made possible by expressing the requisite functional equation for $\Xi(z)$ in terms of the variable $x + iy$. Accordingly, the starting point of the investigation of $|\Xi|^2_{x,y}$ is the definition

**Definition 2.1.** $\Xi(z) = \frac{1}{8} \pi^{-\frac{1}{4} + \frac{z}{2}} (1 + 4z^2) \Gamma(\frac{1}{4} + \frac{z}{2}) \xi(\frac{1}{2} + iz)$ in terms of the variable $x + iy$ is

$$\Xi(x+iy) = \frac{1}{8} \pi^{-\frac{1}{4} + \frac{x}{2} - \frac{y}{2}} (1 + 4x^2 - 4y^2 + 8xy) \Gamma(\frac{1}{4} - \frac{x}{2} + \frac{y}{2}) \xi(\frac{1}{2} - y + ix).$$

Naturally, $\Xi = \Xi(x-iy)$, $\Xi \equiv \Xi(x+iy)$. Definition 2.1 can be used to obtain a functional equation for $|\Xi|^2_{x,y} = 8|\Xi|^2_{x,y} + 3|\Xi|^2_{x,y} + i0$, with the following derived
Lemma 2.2. The functional equation for the square of the modulus of $\Xi(x+iy)$ is

$$|\Xi|^2_{(x,y)} = \frac{1}{64} \pi^{-\frac{1}{2}} \pi(16x^4 + 8x^2(1 + 4y^2) + (1 - 4y^2)^2) \frac{(1 + 4y^2)^2}{(4 - \frac{y}{2} - \frac{x}{2})} \xi_{\frac{1}{2} - y + ix} \xi_{\frac{1}{2} - y - ix} =$$

$$\frac{1}{64} \pi^{-\frac{1}{2}} \pi(16x^4 + 8x^2(1 + 4y^2) + (1 - 4y^2)^2) |\Gamma^2_{(\frac{1}{2} - \frac{y}{2} + \frac{x}{2})} \xi_{\frac{1}{2} - y + ix}|,$$

while $|\Xi|^2_{(x,y)}$ along the line $y = 0$ is

$$|\Xi|^2_{(x,0)} = \frac{1}{64} \pi^{-\frac{1}{2}} \pi(1 + 4x^2)^2 \frac{(1 + 4y^2)^2}{\frac{y}{2}} \xi_{\frac{1}{2} - y + ix} \xi_{\frac{1}{2} - y - ix} =$$

$$\frac{1}{64} \pi^{-\frac{1}{2}} \pi(1 + 4x^2)^2 |\Gamma^2_{(\frac{1}{2} - \frac{y}{2} + \frac{x}{2})} \xi_{\frac{1}{2} - y}|,$$

and $|\Xi|^2_{(y,x)}$ along the line $x = 0$ is

$$|\Xi|^2_{(0,y)} = \frac{1}{64} \pi^{-\frac{1}{2}} \pi(1 - 4y^2)^2 \frac{(1 - 4y^2)^2}{\frac{y}{2}} \xi_{\frac{1}{2} - y + ix} \xi_{\frac{1}{2} - y - ix} =$$

$$\frac{1}{64} \pi^{-\frac{1}{2}} \pi(1 + 4x^2)^2 |\Gamma^2_{(\frac{1}{2} - \frac{y}{2} - \frac{x}{2})} \xi_{\frac{1}{2} - y}|.$$

Proof. Using Eq. 5 and the fact that $|\Xi|^2_{(x+y)} = \Xi_{(x+y)} \Xi_{(x-y)} = \Re_\Xi_{(x,y)} + \Im_\Xi_{(x,y)} + 4$, direct calculation yields Eq. 6 and so Lemma 2.2. Note that $16x^4 + 8x^2(1 + 4y^2) + (1 - 4y^2)^2 = 16y^4 + 8y^2(4y^2 - 1) + (1 + 4y^2)^2 = 0$ at $(0, \pm \frac{1}{2})$, with $16x^4 + 8x^2(1 + 4y^2) + (1 - 4y^2)^2 \geq 0$, $\forall (x, y) \in \mathbb{R}^2$, and that $|\Xi|^2_{(y,x)}$ is a product of a constant, an exponential factor, and the squares of three real-valued functions.

The utility of symmetry as it pertains to Lemma 2.2 is that it gives the following

Lemma 2.3. $|\Xi|^2_{(x,y)}$, as defined in Eq. 6, is symmetric in $x$, $\forall y \in \mathbb{R}$, symmetric in $y$, $\forall x \in \mathbb{R}$, and so symmetric with respect to the origin, with $|\Xi|^2_{(x,-y)} = |\Xi|^2_{(x,y)}$, $|\Xi|^2_{(-x,y)} = |\Xi|^2_{(x,y)}$, $|\Xi|^2_{(-x,0)} = |\Xi|^2_{(x,0)}$ and $|\Xi|^2_{(0,-y)} = |\Xi|^2_{(0,y)}$.

Proof. Lemma 2.3 follows from the known symmetries of $\zeta(z)$ and $\Gamma(z)$ [2,3] and the manner in which the functions are composed. Consider that $\zeta(z)$ and $\Gamma(z)$ do not satisfy a reflection formula, with the only symmetries associated with $\Re_\zeta(z)$ and $\Im_\zeta(z)$ of $\zeta(z)$ and $\Re_\Gamma(z)$ and $\Im_\Gamma(z)$ of $\Gamma(z)$ in the $y$-variable. That is, $\Re_\Gamma(z,-y) = \Re_\Gamma(z,y)$ and $\Im_\Gamma(z,-y) = -\Im_\Gamma(z,y)$ and $\Re_\zeta(z,-y) = \Re_\zeta(z,y)$ and $\Im_\zeta(z,-y) = -\Im_\zeta(z,y)$.

More explicitly, symmetries of $|\Xi|^2_{(x,y)}$ under additive inverses of $x$ and $y$ are inherited from the constraints imposed on $\Re_\Xi_{(x,y)}$ and $\Im_\Xi_{(x,y)}$ by the symmetric reflection formula, $\Xi(-z) = \Xi_{(-x,y)} = \Xi_{(x,-y)} = \Xi_{(x,y)}$. First note that $\Xi(z)$ is symmetric in both $x$ and $y$ about the origin, as $\Xi_{(-x,0)} = \Xi_{(x,0)} = \Xi_{(0,y)}$ and $\Xi_{(0,-y)} = \Xi_{(0,y)}$. Consequences of these identities are that $\Re_\Xi_{(x,y)}$ is symmetric and nonzero in $x$ about the line $y = 0$ and symmetric and nonzero in $y$ about the line $x = 0$ and so symmetric relative to the origin, with $\Re_\Xi_{(-x,0)} = \Re_\Xi_{(x,0)}$, $\Re_\Xi_{(x,-y)} = \Re_\Xi_{(x,y)}$, and $\Re_\Xi_{(x,-y)} = \Re_\Xi_{(x,y)}$, $\forall (x,y) \in \mathbb{R}^2$.
Figure 2. \( gr(\|\Xi\|_2^2_{(x,0,\epsilon)}) \), where \( \epsilon = \|\Xi\|_{(x,0)}^2 \), and \( gr(\|\Xi\|_2^2_{(0,y,\epsilon)}) \), where \( \epsilon = \|\Xi\|_{(0,y)}^2 \).

Given that the square of a symmetric function is a symmetric function, the symmetries of \( \mathbb{R}^2_{\Xi(x,y)} \) are derivable from those of \( \mathbb{R}^2_{\Xi(x,y)} \), with \( \mathbb{R}^2_{\Xi(-x,-y)} = \mathbb{R}^2_{\Xi(x,y)} \), \( \mathbb{R}^2_{\Xi(-x,y)} = \mathbb{R}^2_{\Xi(x,-y)} \) and \( \mathbb{R}^2_{\Xi(x,-y)} = \mathbb{R}^2_{\Xi(x,y)} \), \( \forall (x,y) \in \mathbb{R}^2 \), \( \mathbb{R}^2_{\Xi(-x,0)} = \mathbb{R}^2_{\Xi(x,0)} = \Xi(x), \forall x \in \mathbb{R} \), and \( \mathbb{R}^2_{\Xi(0,-y)} = \mathbb{R}^2_{\Xi(0,y)} \), \( \forall y \in \mathbb{R} \). Similarly, given that the square of an anti-symmetric function is a symmetric function, the symmetries of \( \mathbb{R}^2_{\Xi(x,y)} \) are derivable from those of \( \mathbb{R}^2_{\Xi(x,y)} \), with \( \mathbb{R}^2_{\Xi(-x,-y)} = \mathbb{R}^2_{\Xi(x,y)} \), \( \mathbb{R}^2_{\Xi(-x,y)} = \mathbb{R}^2_{\Xi(x,-y)} \) and \( \mathbb{R}^2_{\Xi(x,-y)} = \mathbb{R}^2_{\Xi(x,y)} \), \( \forall (x,y) \in \mathbb{R}^2 \), \( \mathbb{R}^2_{\Xi(-x,0)} = \mathbb{R}^2_{\Xi(x,0)} = 0 \), \( \forall x \in \mathbb{R} \), and \( \mathbb{R}^2_{\Xi(0,-y)} = \mathbb{R}^2_{\Xi(0,y)} = 0 \), \( \forall y \in \mathbb{R} \).

Now as the sum of symmetric functions yields a symmetric function and \( \|\Xi\|_2^2_{(y)} = \mathbb{R}^2_{\Xi(x,y)} + \mathbb{R}^2_{\Xi(y,x)} + 0 \), with \( \|\Xi\|_2^2_{(x,y)} \geq 0 \), \( \forall (x,y) \in \mathbb{R}^2 \), this gives that \( \|\Xi\|_2^2_{(x,y)} \) is symmetric in \( x \), \( \forall y \in \mathbb{R} \), symmetric in \( y \), \( \forall x \in \mathbb{R} \), and so symmetric relative to the origin, with \( \|\Xi\|_2^2_{(x,-y)} = \|\Xi\|_2^2_{(x,y)} \), \( \|\Xi\|_2^2_{(x,y)} = \|\Xi\|_2^2_{(x,y)} \) and \( \|\Xi\|_2^2_{(x,-y)} = \|\Xi\|_2^2_{(y,-x)} \), \( \forall (x,y) \in \mathbb{R}^2 \), \( \|\Xi\|_2^2_{(x,0)} = \|\Xi\|_2^2_{(0,x)} \), \( \forall x \in \mathbb{R} \), \( \|\Xi\|_2^2_{(0,0)} = \|\Xi\|_2^2_{(0,0)} \), \( \forall y \in \mathbb{R} \).

The graph of \( \|\Xi\|_2^2_{(x,y)} \), denoted \( gr(\|\Xi\|_2^2_{(x,y)}) \) and defined by \( gr(\|\Xi\|_2^2_{(x,y)}) = \{(x,y,\epsilon) \in \mathbb{R}^3 | (x = x = y, \epsilon = \|\Xi\|_{(x,y)}^2) \} \), is provided in Fig. 1. Observe that the 2D surface \( gr(\|\Xi\|_2^2_{(x,y)}) \) is symmetric in both \( x \) and \( y \) about the axes and about the origin. \( gr(\|\Xi\|_2^2_{(x,y)}) \) along the lines \( y = 0 \) and \( x = 0 \) is depicted in Fig. 2, with both 1D curves in \( \mathbb{R}^2 \) symmetric relative to the origin. Note how \( gr(\|\Xi\|_2^2_{(x,y)}) \) oscillates and converges to zero along the line \( y = 0 \) as \( x \rightarrow \pm \infty \) and diverges to infinity along the line \( x = 0 \) as \( y \rightarrow \pm \infty \).

Since it is generally true that roots of an \( f(z) \) and saddle points of \( \Xi(x,y) \) and \( \Xi(y,x) \) arise as minima, a.k.a. zeros or roots, and saddle points of \( gr(\|f\|^2) = gr(\|f\|^2 + 3\|f\|^0) \), respectively, minima and saddle points of \( gr(\|\Xi\|_2^2_{(x,y)}) \) are linked with roots of \( \Xi(z) \), and so nontrivial roots of \( \zeta(z) \), and correlated saddle points of \( \Xi(\xi(x,y)) \) and \( \Xi(\xi(x,y)) \), respectively. [16] Against this background, then, the algebraic problem, ’What set of \( z \in \mathbb{C} \) yield \( \zeta(z) = 0? \)’, has been reformulated as the geometric one, ’Where are minima of \( gr(\|\Xi\|_2^2_{(x,y)}) \) located, i.e. the set of \( (x,y) \) \( \in \mathbb{R}^2 \) such that \( \|\Xi\|_2^2_{(x,y)} = 0? \’

2.2. Partial derivatives of \( \|\Xi\|_2^2_{(x,y)} \) with respect to the \( y \) variable: Odd powers.

Properties of the odd-order derivatives of \( \|\Xi\|_2^2_{(x,y)} \) are characterized in this section. The main outcomes are that the zero set of \( \partial_y\|\Xi\|_2^2_{(x,0)} \) is the line \( y = 0 \), \( i.e. \partial_y\|\Xi\|_2^2_{(x,0)} = 0 \), \( \forall x \in \mathbb{R} \), and that if roots of \( \zeta'_{\frac{1}{2}+ix} \) exist for some \( x \), then they occur at roots of \( \zeta'_{\frac{1}{2}+ix} \) and so at nontrivial roots of \( \zeta(z) \) having multiplicity greater than one. These results will be utilized in §Sec. 2.3 to establish that \( \partial_y\|\Xi\|_2^2_{(x,y)} > 0 \), \( \forall y > 0 \in \mathbb{R} \) and \( \partial_y\|\Xi\|_2^2_{(x,y)} < 0 \), \( \forall y < 0 \in \mathbb{R} \), \( i.e. \partial_y\|\Xi\|_2^2_{(x,y)} \) is bounded away from zero if \( y \neq 0 \).

2.2.1. Symmetries of \( \partial_y\|\Xi\|_{2n+1}^2_{(x,y)} \).

The symmetries of \( \|\Xi\|_2^2_{(x,y)} \) encapsulated in Lemma 2.3 give without calculation that

Lemma 2.4. \( \forall n \in \mathbb{N} \), \( \partial_y\|\Xi\|_{2n+1}^2_{(x,y)} \) is symmetric in \( x \), \( \forall y \in \mathbb{R} \), anti-symmetric in \( y \), \( \forall x \in \mathbb{R} \), and so anti-symmetric relative to the origin, with \( \partial_y\|\Xi\|_{2n+1}^2_{(x,-y)} = -\partial_y\|\Xi\|_{2n+1}^2_{(x,y)} \), \( \partial_y\|\Xi\|_{2n+1}^2_{(x,-y)} = -\partial_y\|\Xi\|_{2n+1}^2_{(y,-x)} \), \( \partial_y\|\Xi\|_{2n+1}^2_{(0,-y)} = -\partial_y\|\Xi\|_{2n+1}^2_{(0,y)} \), \( \partial_y\|\Xi\|_{2n+1}^2_{(-x,-y)} = -\partial_y\|\Xi\|_{2n+1}^2_{(-x,y)} \), \( \partial_y\|\Xi\|_{2n+1}^2_{(-x,-y)} = -\partial_y\|\Xi\|_{2n+1}^2_{(-x,y)} \), \( \partial_y\|\Xi\|_{2n+1}^2_{(-x,0)} = \partial_y\|\Xi\|_{2n+1}^2_{(0,x)} \) and \( \partial_y\|\Xi\|_{2n+1}^2_{(0,-y)} = -\partial_y\|\Xi\|_{2n+1}^2_{(0,y)} \).
Proof. With $|\Xi|_{(x,y)}^2$ symmetric in the $x$ and $y$ coordinates by Lemma 2.3, Lemma 2.4 a consequence of the fact the derivative of a symmetric function is an anti-symmetric function. \qed

Lemma 2.5. $\forall n \in \mathbb{N}$, $\partial_{g^{2n+1}}|\Xi|_{(x,0)}^2 = 0$, $\forall x \in \mathbb{R}$.

Proof. Lemma 2.5 follows from Lemma 2.3. Namely, that $|\Xi|_{(x,y)}^2$ is symmetric in $x$ and $y$ about the origin gives that $\partial_y|\Xi|_{(x,y)}^2$ is anti-symmetric in $y$, $\forall x \in \mathbb{R}$ and so that $\partial_y|\Xi|_{(x,0)}^2 = 0$, $\forall x \in \mathbb{R}$. \qed

2.2.2. General expressions for $\partial_{y^{2n+1}}|\Xi|_{(x,0)}^2$.

To make the claims in Lemmas 2.4 and 2.5 regarding the differential structure exhibited by $|\Xi|_{(x,y)}^2$ more substantive, a general expression for odd-order partial derivatives of $|\Xi|_{(x,y)}^2$ with respect to $y$ and evaluated at $(x,0)$ was derived, where

Lemma 2.6. For $n \geq 1$, $\partial_{y^{2n+1}}|\Xi|_{(x,0)}^2$, are expressable as

\begin{equation}
\partial_{y^{2n+1}}|\Xi|_{(x,0)}^2 = (1 + 4x^2)^2 \partial_{y^{2n+1}}\Omega_{(x,0)} + (4x^2 - 1)(32n^2 + 16n)\partial_{y^{2n-1}}\Omega_{(x,0)} + (256n^4 - 256n^3 - 64n^2 + 64n)\partial_{y^{2n-3}}\Omega_{(x,0)},
\end{equation}

where $\Omega_{(x,y)}$ is

\begin{equation}
\Omega_{(x,y)} = \frac{1}{64\pi} \frac{1}{\sqrt{1 + \frac{y}{2}}} |\Gamma|_{\frac{1}{2} + \frac{x}{2}, \frac{x+y}{2}}^2 (\frac{1}{2} - y + ix).
\end{equation}

Proof. This expression for $\partial_{y^{2n+1}}|\Xi|_{(x,0)}^2$ was obtained using the general Leibniz rule. \cite{15} \qed

2.2.3. The first partial derivatives of $|\Xi|_{(x,y)}^2$ with respect to the $y$ variable.

Characterizing the symmetry and analytic properties of the first partial derivative of $|\Xi|_{(x,y)}^2$ with respect to $y$, i.e. $\partial_y|\Xi|_{(x,y)}^2$, will be an important step in proving that all roots in $R(|\Xi|) \cong R(\Xi) \cong R(\zeta)$ are collinear. Accordingly, the functional equation for $\partial_y|\Xi|_{(x,y)}^2$ was derived, with

Lemma 2.7. The first partial derivative of $|\Xi|_{(x,y)}^2$ with respect to $y$ is defined by the functional equation

\begin{equation}
\partial_y|\Xi|_{(x,y)}^2 = \frac{1}{128\pi} \frac{1}{\sqrt{1 + \frac{y}{2}}} |\Gamma|_{\frac{1}{2} + \frac{x}{2}, \frac{x+y}{2}}^2 \times
\end{equation}

\begin{equation}
(16x^4 + 8x^2(1 + 4y^2) + (1 - 4y^2)^2)(2 \log_e(\pi) - \psi(\frac{1}{4} - \frac{x}{2} + i\frac{y}{2}) - \psi(\frac{1}{4} - \frac{x}{2} + i\frac{y}{2}))|\zeta|_{\frac{1}{4} - y + ix}^2 -
\end{equation}

\begin{equation}
2(16x^4 + 8x^2(1 + 4y^2) + (1 - 4y^2)^2)(\psi(\frac{1}{4} - y + i\frac{x}{2})\psi(\frac{1}{4} - y - i\frac{x}{2}) + \psi(\frac{1}{4} - y + i\frac{x}{2})\psi(\frac{1}{4} - y - i\frac{x}{2})),
\end{equation}

Along the line $x = 0$, the functional equation for $\partial_y|\Xi|_{(x,y)}^2$ simplifies to

\begin{equation}
\partial_y|\Xi|_{(0,y)}^2 = \frac{1}{128\pi} \frac{1}{\sqrt{1 + \frac{y}{2}}} |\Gamma|_{\frac{1}{2}, \frac{y}{2}}^2 \times
\end{equation}

\begin{equation}
((32y(4y^2 - 1) + (1 - 4y^2)^2)(2 \log_e(\pi) - \psi(\frac{1}{4} - \frac{y}{2})\psi(\frac{1}{4} - y) - 2(1 - 4y^2)^2)\psi(\frac{1}{4} - y)\psi(\frac{1}{4} - y) + \psi(\frac{1}{4} - y)\psi(\frac{1}{4} - y)),
\end{equation}

while along the line $y = 0$ the functional equation for $\partial_y|\Xi|_{(x,0)}^2$ simplifies to

\begin{equation}
\partial_y|\Xi|_{(x,0)}^2 = \frac{1}{128\pi} \frac{1}{\sqrt{1 + \frac{x}{2}}} |\Gamma|_{\frac{1}{2} + \frac{x}{2}}^2 \times
\end{equation}

\begin{equation}
((2 \log_e(\pi) - \psi(\frac{1}{4} + i\frac{x}{2}) - \psi(\frac{1}{4} + i\frac{x}{2}))|\zeta|_{\frac{1}{4} + i\frac{x}{2}}^2 - 2(\psi(\frac{1}{4} + i\frac{x}{2})\psi(\frac{1}{4} + i\frac{x}{2}) + \psi(\frac{1}{4} + i\frac{x}{2})\psi(\frac{1}{4} + i\frac{x}{2})),
\end{equation}

where $\psi(z) = \gamma(z)$ is the digamma function.

Proof. The general Leibniz rule \cite{15} was used to obtain Eqs. 11-13. It worth mentioning that $\partial_y|\Xi|_{(x,0)}^2$ defined in Eq. 13 is also expressable as

\begin{equation}
\partial_y|\Xi|_{(x,0)}^2 = \frac{1}{64\pi} \frac{1}{\sqrt{1 + \frac{x}{2}}} |\Gamma|_{\frac{1}{2} + \frac{x}{2}}^2 ((\log_e(\pi) - \Re \psi(\frac{1}{4} + i\frac{x}{2}))|\zeta|_{\frac{1}{4} + i\frac{x}{2}}^2 - 2\Re \zeta(\frac{1}{4} + i\frac{x}{2})\zeta(\frac{1}{4} + i\frac{x}{2})).
\end{equation}

Regarding symmetries of $\zeta(z)$, note that $\Re \zeta(\frac{1}{4} - y) = \Re \zeta(\frac{1}{4} + y)$ and $\Im \zeta(\frac{1}{4} - y) = -\Im \zeta(\frac{1}{4} + y)$.
equal to zero along the line $y = 0$. Note that $(x, y) \in \mathbb{R}^2$ such that $\partial_y |\Xi|^2_{(x,y)} = 0$ are marked in Yellow.

Corollary 2.8. $\partial_y |\Xi|^2_{(x,y)}$, as defined in Eq. 11, is symmetric in $x$, $\forall y \in \mathbb{R}$, anti-symmetric in $y$, $\forall x \in \mathbb{R}$, and so anti-symmetric with respect to the origin, with $\partial_y |\Xi|^2_{(x,-y)} = -\partial_y |\Xi|^2_{(x,y)}$, $\partial_y |\Xi|^2_{(-x,y)} = \partial_y |\Xi|^2_{(x,y)}$ and $\partial_y |\Xi|^2_{(-x,-y)} = -\partial_y |\Xi|^2_{(x,y)}$.

Corollary 2.9. $\partial_y |\Xi|^2_{(0,y)}$, as defined in Eq. 12, is anti-symmetric in $y$, while $\partial_y |\Xi|^2_{(x,0)}$, as defined in Eqs. 13 and 14, is symmetric in $x$, with $\partial_y |\Xi|^2_{(-y,0)} = -\partial_y |\Xi|^2_{(y,0)}$ and $\partial_y |\Xi|^2_{(-x,0)} = \partial_y |\Xi|^2_{(x,0)}$.

The graphs of $\partial_y |\Xi|^2_{(x,y)}$ and $\partial_y |\Xi|^2_{(0,y)}$ are provided in Fig. 3. Observe that the 2D surface $gr(\partial_y |\Xi|^2_{(x,y)})$ is symmetric in $x$ about the axes, in $x$, $\forall y \in \mathbb{R}$, anti-symmetric in $y$ about the axes, $\forall x \in \mathbb{R}$, and so anti-symmetric relative to the origin. Also note that $\partial_y |\Xi|^2_{(x,0)} = 0$ over the subset of $\mathbb{R}^2$ depicted. This is substantiated in $gr(\partial_y |\Xi|^2_{(0,y)})$, as this curve is anti-symmetric about the origin, with $\partial_y |\Xi|^2_{(x,y)} > 0$, $\forall (x, y) \in \mathbb{R}^2$ such that $y > 0$, and $\partial_y |\Xi|^2_{(0,y)} < 0$, $\forall (x, y) \in \mathbb{R}^2$ such that $y < 0$.

2.2.4. The zero set of $\partial_y |\Xi|^2_{(x,y)}$.

Identifying the zero set of $\partial_y |\Xi|^2_{(x,y)}$, i.e. $(x, y) \in \mathbb{R}^2$ such that $\partial_y |\Xi|^2_{(x,y)} = 0$, is important to understanding global structure of $|\Xi|^2_{(x,y)}$, and so of $\Xi_{(x,y)}$. Without calculation, the symmetries of $\partial_y |\Xi|^2_{(x,y)}$ espoused in Lemmas 2.4 and 2.5 yield that

**Lemma 2.10.** $\partial_y |\Xi|^2_{(x,0)} = 0$, $\forall x \in \mathbb{R}$.

**Proof.** The symmetry of $|\Xi|^2_{(x,y)}$ captured in Lemmas 2.3-2.5 gives not only that $\partial_y |\Xi|^2_{(x,y)}$ in an anti-symmetric function in $y$, with $\partial_y |\Xi|^2_{(x,-y)} = -\partial_y |\Xi|^2_{(x,y)}$, $\forall x \in \mathbb{R}$, but also that $\partial_y |\Xi|^2_{(x,y)}$ is identically equal to zero along the line $y = 0$, i.e. $\partial_y |\Xi|^2_{(x,0)} = 0$, $\forall x \in \mathbb{R}$. \hfill $\Box$

Albeit Lemma 2.10 only establishes that $\partial_y |\Xi|^2_{(x,y)}$ is zero along the line $y = 0$, not that this is the only place where $\partial_y |\Xi|^2_{(x,y)}$ is zero and so that the entire zero set is this line $y = 0$, the results of §Sec. 2.3 guarantee this assertion and so that $\partial_y |\Xi|^2_{(x,y)}$ is bounded away from zero if $y \neq 0$.

Given that $\frac{1}{128\sqrt{\pi}}(1 + 4x^2)^2 |\Gamma|^2_{(\frac{x}{2} + \frac{i}{2})} > 0$, $\forall x \in \mathbb{R}$, as $R(\Gamma) = \emptyset$ and $\gamma(\Gamma) = \{ z \in \mathbb{C} | x = -n \in \mathbb{N} \land y = 0 \}$, a noteworthy corollary of Lemma 2.10 and Eq. 13 is the following

**Corollary 2.11.** The zero set of $\partial_y |\Xi|^2_{(x,y)}$ is defined by the equivalent functional equations

\begin{align}
(2 \log_e (\pi) - \psi_1 (\frac{1}{2} - x) - \psi_1 (\frac{1}{2} + x))|\zeta|^2_{(\frac{1}{2} + ix)} &= 2 (\zeta_1 (\frac{1}{2} + ix) \zeta_1 (\frac{1}{2} - ix) + \zeta_1 (\frac{1}{2} + ix) \zeta_1 (\frac{1}{2} - ix)), \\
(\log_e (\pi) - R \psi_1 (\frac{1}{2} + iy))|\zeta|^2_{(\frac{1}{2} + iy)} &= 2 R \xi_1 (\frac{1}{2} + iy) \xi_1 (\frac{1}{2} - iy),
\end{align}

Evaluating Eq. 15 at $x = 0$ yields an expression for $\log_e (\pi)$ in terms of $\zeta(z)$, $\Gamma(z)$ and their derivatives as
Corollary 2.12. \( \log_e(\pi) = \frac{2\zeta(4)}{\zeta(2)} + \frac{\Gamma'(4)}{\Gamma(4)} \).

Eq. 15 can also be exploited to characterize the relative asymptotic behaviors of \( \Re(\zeta(1/2 + it)\zeta'(1/2 - it)) \), \( \Re(\zeta'(1/2 + it)) \), \( |\zeta|^2(1/2 + it) \) and \( |\zeta'|^2(1/2 + it) \), with the following ascertained.

Lemma 2.13. For \( J, K, P > 0 \in \mathbb{R} \), \( \Re(\zeta(1/2 + it)\zeta'(1/2 - it)) \), \( \Re(\zeta'(1/2 + it)) \), \( |\zeta|^2(1/2 + it) \) and \( |\zeta'|^2(1/2 + it) \) behave as \( \Re(\zeta(1/2 + it)\zeta'(1/2 - it)) \sim -J \log_e(|x|)|\zeta|^2(1/2 + it) \) and \( |\zeta'|^2(1/2 + it) \sim P \log^2(|x|)|\zeta|^2(1/2 + it) \), as \( x \to \pm \infty \).

Proof. That \( \Re(\zeta(1/2 + it)\zeta'(1/2 - it)) \) behaves like \( \Re(\zeta(1/2 + it)\zeta'(1/2 - it)) \sim -J \log_e(|x|)|\zeta|^2(1/2 + it) \) as \( x \to \pm \infty \) follows from Eq. 15 and that \( \lim_{x \to \pm \infty} \frac{\log_e(|x|)|\zeta|^2(1/2 + it)}{-J \log_e(|x|)|\zeta|^2(1/2 + it)} = 1 \), since they imply \( \Re(\zeta(1/2 + it)\zeta'(1/2 - it)) \sim \log_e(|x|)|\zeta|^2(1/2 + it) \sim -J \log_e(|x|)|\zeta|^2(1/2 + it) \) and \( \Re(\zeta'(1/2 + it)) \), \( |\zeta|^2(1/2 + it) \) and \( |\zeta'|^2(1/2 + it) \) were determined by calculating \( \partial_y((2 \log_e(\pi) - \psi(1/2 - 1/2 + it) - \psi(1/2 - 1/2 + it + 1/2 + it)|\zeta|^2(1/2 + it) - 2(\zeta(1/2 - y + it)\zeta'(1/2 - y + it) + \zeta'(1/2 - y + it)\zeta(1/2 - y + it))) \). Evaluating the result at \( y = 0 \) yields the relation \( C \log^2(|x|)|\zeta|^2(1/2 + it) \sim \Re(\zeta(1/2 + it)\zeta'(1/2 - it)) \sim |\zeta|^2(1/2 + it) \), and so that \( \Re(\zeta(1/2 + it)\zeta'(1/2 - it)) \sim K \log^2(|x|)|\zeta|^2(1/2 + it) \) and \( |\zeta'|^2(1/2 + it) \sim P \log^2(|x|)|\zeta|^2(1/2 + it) \), where \( K, P, C > 0 \in \mathbb{R} \). Note that each derivative of \( \zeta(z) \) contributes a factor of \( -\log_e(|x|) \) to the asymptotic behavior of \( \Re(\zeta(1/2 + it)\zeta'(1/2 - it)) \), with \( \Re(\zeta(1/2 + it)\zeta'(1/2 - it)) \sim D(-1)^{n + m} \log^m_e(|x|)|\zeta|^2(1/2 + it) \) and \( D > 0 \in \mathbb{R} \).

In support of these estimates for the relative behavior of \( \Re(\zeta(1/2 + it)\zeta'(1/2 - it)) \), \( \Re(\zeta'(1/2 + it)) \), \( |\zeta|^2(1/2 + it) \) and \( |\zeta'|^2(1/2 + it) \) as \( x \to \infty \), graphs of \( \frac{\log_e(|x|)|\zeta|^2(1/2 + it)}{\log_e(|x|)(\zeta(1/2 - y + it)\zeta'(1/2 + it) + \zeta'(1/2 - y + it)\zeta(1/2 + it))} \), \( \frac{\log_e(|x|)|\zeta'|^2(1/2 + it)}{\log_e(|x|)(\zeta(1/2 - y + it)\zeta'(1/2 + it) + \zeta'(1/2 - y + it)\zeta(1/2 + it))} \), \( \frac{\zeta(1/2 + it)\zeta'(1/2 - it) + \zeta'(1/2 + it)\zeta(1/2 + it)}{\zeta(1/2 + it)\zeta'(1/2 - it)} \) are provided in Appendix B. Observe how, over the region depicted, the first ratio converges to 1, while the latter three oscillate, with each bounded as \( 0 < \frac{\log_e(|x|)|\zeta'|^2(1/2 + it)}{|\zeta|^2(1/2 + it)} < 125 \), \( -10 < \frac{\log_e(|x|)(\zeta(1/2 - y + it)\zeta'(1/2 + it) + \zeta'(1/2 - y + it)\zeta(1/2 + it))}{|\zeta(1/2 + it)|} < 25 \) and \( -2 < \frac{\zeta(1/2 + it)\zeta'(1/2 - it) + \zeta'(1/2 + it)\zeta(1/2 + it)}{|\zeta(1/2 + it)|} < 10 \).

Furthermore, Eq. 15 divulges that

Theorem 2.14. If a root of \( \zeta'(z) \) exists on the critical line, then it occurs at a root of \( \zeta(z) \) on the critical line, with this nontrivial root having multiplicity greater than one.

Proof. Theorem 2.14 stems directly from Eq. 15 and known properties of \( \zeta(z), \zeta'(z) \) and \( \psi(z) \). Consider that if \( \exists z = \frac{1}{2} + iy \in \mathbb{C} \) such that \( \zeta'(z) = 0 \), then \( \zeta'(z) = 0 \) at \( z = \frac{1}{2} + iy \in \mathbb{C} \), as \( \Re(\zeta(x - y)) = \Re(\zeta(y - x)) \) and \( \Im(\zeta(x - y)) = -\Im(\zeta(y - x)) \). In this case, Eq. 15 reduces to \((2 \log_e(\pi) - \psi(1/2 - 1/2 + it) - \psi(1/2 - 1/2 + it + 1/2 + it)|\zeta|^2(1/2 + it) = 0, \) with this equality necessitating that both \( |\zeta'|^2(1/2 + it) = 0 \) and \( \zeta(z) = 0 \) at the same \( z = \frac{1}{2} + iy \in \mathbb{C} \). That \( \zeta'(z) = 0 \) and \( \zeta'(z) = 0 \) at the same \( z \) are the defining equations of a root having multiplicity greater than one. In words, Theorem 2.14 indicates that if \( \zeta'(1/2 + iy) \) has roots on the critical line, then they must occur at nontrivial roots of \( \zeta(1/2 + iy) \) on the critical line having multiplicity greater than one. From a different perspective, if all of the roots of \( \zeta(z) \) on the critical line have multiplicity one, then \( \zeta'(1/2 + iy) > 0, \forall y \in \mathbb{R} \).

2.3. Partial derivatives of \( |\Xi|^2(x, y) \) with respect to the \( y \) variable: Even powers.

Properties of the even-order derivatives of \( |\Xi|^2(x, y) \) are characterized in this section. Of note, it is proven through an analysis of the asymptotic properties of \( \partial_{yy}|\Xi|^2(x, y) \) that \( \partial_{yy}|\Xi|^2(x, y) \geq 0 \) along the line \( y = 0 \), which gives not only that \( \partial_{yy}|\Xi|^2(x, y) \) is bounded away from zero if \( y \neq 0 \) but also that \( |\Xi|^2(x, y) \) is bounded away from zero if \( y \neq 0 \). This fact is corroborated by exposing that \( \partial_{yy}|\Xi|^2(x, 0) \geq 0, \forall x \in \mathbb{R} \) and \( \forall n \in \mathbb{N} \). It is then observed that a direct corollary of these results is that all roots of \( |\Xi|^2(x, y) \) reside on the line \( y = 0 \), which with \( R(|\Xi|^2) \cong R(\Xi) \cong R(\zeta) \) implies all nontrivial roots of \( \zeta(z) \) lie on the critical line.
2.3.1. Symmetries of $\partial y^{2n} \| \xi \|^2_{(x,y)}$.

The symmetries of $\| \xi \|^2_{(x,y)}$ discussed in Lemma 2.3 imply the following regarding the symmetries $\partial y^{2n} \| \xi \|^2_{(x,y)}$.

**Lemma 2.15.** $\forall n \in \mathbb{N}$, $\partial y^{2n} \| \xi \|^2_{(x,y)}$ is symmetric in $x$, $\forall y \in \mathbb{R}$, symmetric in $y$, $\forall x \in \mathbb{R}$, and so symmetric relative to the origin, with $\partial y^{2n} \| \xi \|^2_{(x,-y)} = \partial y^{2n} \| \xi \|^2_{(x,y)}$.

**Proof.** That $\partial y^{2n} \| \xi \|^2_{(x,y)}$ is symmetric in $x$ about the line $y = 0$ and symmetric in $y$ about the line $x = 0$, $\forall n \in \mathbb{N}$ and $\forall (x,y) \in \mathbb{R}^2$, follows from the symmetry assignments for $\| \xi \|^2_{(x,y)}$ presented in Lemma 2.3 and the fact that the derivative of symmetric/anti-symmetric function yields an anti-symmetric/symmetric function and so even-order derivatives of symmetric functions are symmetric functions.

Because $\| \xi \|^2_{(x,y)}$ is symmetric in $y$ about the line $y = 0$, $\forall x \in \mathbb{R}$, symmetry dictates that only even-order derivatives of $\| \xi \|^2_{(x,y)}$ in $y$ about any $(x,0)$ are non-zero. This observation, which is central to the subsequent analysis, entails that

**Corollary 2.16.** Taylor series of $\| \xi \|^2_{(x,y)}$ in $y$ about any $(x,0)$ have the form

$$\| \xi \|^2_{(x,y)} = \sum_{n=0}^{\infty} \frac{\partial y^{2n} \| \xi \|^2_{(x,0)}}{(2n)!} y^{2n}. \tag{17}$$

2.3.2. General expressions for $\partial y^{2n} \| \xi \|^2_{(x,0)}$.

Paralleling §Sec. 2.2.2, analogous general expressions for even-order partial derivatives of $\| \xi \|^2_{(x,y)}$ with respect to $y$ and evaluated at $(x,0)$ were derived, with the form for $\partial y^{2n} \| \xi \|^2_{(x,0)}$ determined to be

**Lemma 2.17.** For $n \geq 1$, $\partial y^{2n} \| \xi \|^2_{(x,0)}$ are expressible as

$$\partial y^{2n} \| \xi \|^2_{(x,0)} = (1 + 4x^2)^2 \partial y^{2n} \Omega_{(x,0)} + (4x^2 - 1)(32n^2 - 16n)\partial y^{2n-2} \Omega_{(x,0)} + (256n^4 - 768n^3 + 704n^2 - 192n)\partial y^{2n-4} \Omega_{(x,0)}, \tag{18}$$

where $\Omega_{(x,y)}$ was defined in Eq. 10, and

$$\partial y^{2n} \| \xi \|^2_{(x,0)} = (1 + 4x^2)^2 \log_e(\pi)^2 n \Phi_{(x,0)} + \sum_{j=1}^{n} \log_e(\pi)^2 n - 2j \binom{2n}{2j} \times (384 \binom{2j}{4} \partial y^{2j-4} \Phi_{(x,0)} + 16(4x^2 - 1) \binom{2j}{2} \partial y^{2j-2} \Phi_{(x,0)} + (1 + 4x^2)^2 \binom{2j}{0} \partial y^{2j} \Phi_{(x,0)}), \tag{19}$$

where $\binom{n}{k}$ is a binomial coefficient and $\Phi_{(x,y)}$ is given by

$$\Phi_{(x,y)} = \frac{1}{64\sqrt{\pi}} \left| \frac{\pi}{4} \right| \left( \frac{1}{y + x} + \frac{1}{y - x} \right) \left( \frac{1}{4} - y + x \right) = \pi^{-y} \Omega_{(x,y)}. \tag{20}$$

**Proof.** These expressions for $\partial y^{2n} \| \xi \|^2_{(x,0)}$ were obtained using the general Leibniz rule. [15]

Eqs. 18 and 19 disclose structure inherent to the $\partial y^{2n} \| \xi \|^2_{(x,y)}$ in $x$ and in $n$ along the line $y = 0$ and so structure innate to $\| \xi \|^2_{(x,y)}$. For instance, Eq. 18 can be used to characterize the behavior of $\| \xi \|^2_{(x,y)}$ in $y$ about the the origin

**Lemma 2.18.** At $(x, y) = (0,0)$, $\partial y^{2n} \| \xi \|^2_{(0,0)} > 0$, $\forall n \in \mathbb{N}$, and so $\| \xi \|^2_{(0,y)} > 0$, $\forall y \in \mathbb{R}$.

**Proof.** First note at $x = 0$ that Eq. 18 simplifies to $\partial y^{2n} \| \xi \|^2_{(0,0)} = \partial y^{2n} \Omega_{(0,0)} + (16n - 32n^2)\partial y^{2n-2} \Omega_{(0,0)} + (256n^4 - 768n^3 + 704n^2 - 192n)\partial y^{2n-4} \Omega_{(0,0)}$ and that $\partial y^{2n} \| \xi \|^2_{(0,0)} \sim n^4 \partial y^{2n-4} \Omega_{(0,0)}$ for large $n$. Secondly, note how $\partial y^{2n} \| \xi \|^2_{(0,0)} > 0$ and $\partial y^{2n} \Omega_{(0,0)} > 0$ for $n \leq 5$, see Fig 9, and how sign changes of each $\partial y^{2n} \| \xi \|^2_{(0,0)}$ are dictated by $\partial y^{2n} \Omega_{(0,0)}$, $\partial y^{2n-2} \Omega_{(0,0)}$ and $\partial y^{2n-4} \Omega_{(0,0)}$. In light of these observations, Lemma 2.18 can be proved by induction on $n$ in Eq. 18 at $x = 0$. Now since a zero of $\| \xi \|^2_{(0,y)}$ necessitates that $\exists n$ such that $\partial y^{2n} \| \xi \|^2_{(0,0)} < 0$, it follows that $\| \xi \|^2_{(0,y)}$ is bounded above zero along this line as $y \to \infty$. \qed
2.3.3. The second partial derivatives of $|\Xi|^2_{(x,y)}$ with respect to the y variable.

Characterizing $\partial_{yy}|\Xi|^2_{(x,y)}$ along the line $y = 0$ will be pivotal to understanding the global structure evinced by $|\Xi|^2_{(x,y)}$ and so by $\Xi_{(x,y)}$. Consider that $\partial_{yy}|\Xi|^2_{(x,y)}$ determines the local curvature of $|\Xi|^2_{(x,y)}$ along the line $y = 0$, as $\partial_{yy}|\Xi|^2_{(x,y,0)} = 0$, $\forall x \in \mathbb{R}$, by Lemma 2.10. To make headway, the following functional equations for $\partial_{yy}|\Xi|^2_{(x,y)}$ along the lines $y = 0$ and $x = 0$ were derived

**Lemma 2.19.** The second partial derivative of $|\Xi|^2_{(x,y)}$ with respect to $y$ evaluated along the line $y = 0$ is defined by the functional equation

$$
\partial_{yy}|\Xi|^2_{(x,y,0)} = \frac{1}{128\sqrt{\pi}} \left( (32(4x^2 - 1) + (1 + 4x^2)^2 \psi(\frac{1}{4} - \frac{y}{2}) \right) |\xi|^2_{(\frac{1}{2}+ix)} + \\
(1 + 4x^2)(2\psi(\frac{1}{4}+\frac{y}{2}) + 2\psi(\frac{1}{4} - \frac{y}{2}) - 4\log_e(\pi)) \frac{\psi(\frac{1}{4}+\frac{y}{2})}{\psi(\frac{1}{4} - \frac{y}{2})} + \\
4(1 + 4x^2)^2(\frac{1}{4}+ix) \zeta(\frac{1}{2}+ix) + 4(1 + 4x^2)^2\zeta(\frac{1}{2}+ix),
$$

which is equivalent to

$$
\partial_{yy}|\Xi|^2_{(x,y,0)} = \frac{1}{128\sqrt{\pi}} \left( (32(4x^2 - 1) + (1 + 4x^2)^2 \psi(\frac{1}{4} - \frac{y}{2}) \right) |\xi|^2_{(\frac{1}{2}+ix)} + \\
(1 + 4x^2)(2\psi(\frac{1}{4}+\frac{y}{2}) + 2\psi(\frac{1}{4} - \frac{y}{2}) - 4\log_e(\pi)) \frac{\psi(\frac{1}{4}+\frac{y}{2})}{\psi(\frac{1}{4} - \frac{y}{2})} + \\
4(1 + 4x^2)^2(\frac{1}{4}+ix) \zeta(\frac{1}{2}+ix) + 4(1 + 4x^2)^2\zeta(\frac{1}{2}+ix),
$$

while $\partial_{yy}|\Xi|^2_{(x,y)}$ along the line $x = 0$ is

$$
\partial_{yy}|\Xi|^2_{(0,y)} = \frac{1}{128\sqrt{\pi}} \left( (2\log^2(\pi)(1-4y^2)^2 + 32(12y^2 - 1) + 64y(1 - 4y^2)(\psi(\frac{1}{4} + \frac{y}{2}) - \log_e(\pi)) \right) |\xi|^2_{(\frac{1}{2} - y)} + \\
(1 - 4y^2)^2(\psi(\frac{1}{4} - \frac{y}{2}) + 2\psi(\frac{1}{4} + \frac{y}{2}) - 4\log_e(\pi)) \frac{\psi(\frac{1}{4} - \frac{y}{2})}{\psi(\frac{1}{4} + \frac{y}{2})} + \\
4(1 - 4y^2)^2(\frac{1}{2} - y) \zeta(\frac{1}{2} - y) + 4(1 - 4y^2)^2\zeta(\frac{1}{2} - y),
$$

where $\psi(z)$ is the digamma function and $\psi(z)$ is the trigamma function.

**Proof.** Eqs. 21-23 for $\partial_{yy}|\Xi|^2_{(x,0)}$ and $\partial_{yy}|\Xi|^2_{(0,y)}$ were obtained using the general Leibniz rule, [15] Eqs. 13 and 15 and their derivatives. Similar to $\partial_{yy}|\Xi|^2_{(x,0)}$, $\partial_{yy}|\Xi|^2_{(x,y)}$ can be expressed as

$$
\partial_{yy}|\Xi|^2_{(x,y,0)} = \frac{1}{128\sqrt{\pi}} \left( (1 + 4x^2)^2 |\xi|^2_{(\frac{1}{4} + \frac{y}{2})} \right) + \\
\frac{32(4x^2 - 1)}{(1 + 4x^2)^2} + \Re_{(\frac{1}{4} + \frac{y}{2})} \frac{\psi(\frac{1}{4} + \frac{y}{2})}{\psi(\frac{1}{4} - \frac{y}{2})} + \\
4(\Re_{(\frac{1}{4} + \frac{y}{2})} - \log_e(\pi)) \Re_{(\frac{1}{2} + iy)} \zeta(\frac{1}{2} - iy) + 4 \Re_{(\frac{1}{2} + iy)} \zeta(\frac{1}{2} - iy) + 4 |\zeta|^2_{(\frac{1}{2} + iy)}.
$$

Note, the coefficients of $|\zeta|^2_{(\frac{1}{2} + iy)}$, $\Re_{(\frac{1}{2} + iy)} \zeta(\frac{1}{2} - iy)$, and $|\zeta|^2_{(\frac{1}{2} + iy)}$ behave asymptotically as

$$\lim_{x \to \pm \infty} \frac{32(4x^2 - 1) + (1 + 4x^2)^2}{128\sqrt{\pi} |\xi|^2_{(\frac{1}{4} + \frac{y}{2})}} = \lim_{x \to \pm \infty} \frac{4(1 + 4x^2)^2}{128\sqrt{\pi} \log_e(\frac{1}{2} + iy)} = \lim_{x \to \pm \infty} \frac{4(1 + 4x^2)^2}{64x^2} = 1.
$$

The graph of $\partial_{yy}|\Xi|^2_{(x,y)}$ is provided in Fig. 4, while the graphs of $\partial_{yy}|\Xi|^2_{(x,0)}$ and $\partial_{yy}|\Xi|^2_{(0,y)}$ are depicted in Fig. 5. Akin to $gr(|\Xi|^2_{(x,y,c)})$, the 2D surface $gr(\partial_{yy}|\Xi|^2_{(x,y,c)})$ is symmetric in both $x$ and $y$ about the axes and so symmetric relative to the origin. Also note how $\partial_{yy}|\Xi|^2_{(x,0)} > 0$ and converges to zero as $x \to \pm \infty$, while $\partial_{yy}|\Xi|^2_{(0,y)} > 0$ and diverges to $\infty$ as $x \to \pm \infty$. 

□
That is, of the only points satisfying this set of equations. Corollary 3.2 follows because root multiplicities are from Lemma 2.21.

If $y$ in Eq. 22, is a symmetric function in $\zeta$ from Lemma 2.22.

From the symmetries of $|\Xi|^2_{(x,y)}$ discussed in Lemmas 2.3 and those of $\partial_{y^2}\Xi|_{(x,y)}^2$ discussed in Lemma 2.15, the following symmetries of Eqs. 21-24 follow immediately

**Proof.** This assertion stems directly from $R(|\xi|^2) \equiv R(\Xi) \equiv R(\zeta)$, Lemma 2.10, Theorem 2.14 and Eqs. 21 and 22. Note that in this case $\partial_{y^2y^2}\Xi|_{(x,y)}^2$ reduces to $\partial_{y^2y^2}\Xi|_{(x,y)}^2 = \frac{2}{3}\sqrt{\pi}(1+4x^2)^2|\xi|^2_{(x+ix)}|\zeta''|_{(x+ix)}$. □

**Lemma 2.22.** If $\exists x \in \mathbb{R}$ such that $c'_{(\frac{1}{2}+ix)} = 0$, then $z = x + 0 \in C$ is a root of $\Xi(z)$ and $z = \frac{1}{2} + ix \in C$ is a root of $\zeta(z)$ having multiplicity greater than 1.

**Proof.** Theorem 2.14, Lemma 2.21 and that roots of an $f(z)$ arise as minima of $|f|^2_{(x,y)}$ gives $|\Xi|^2_{(x,y)} = 0$, $\partial_y|\Xi|^2_{(x,y)} = 0$, $\partial_x|\Xi|^2_{(x,y)} = 0$ and $\partial_{xx}|\Xi|^2_{(x,y)} = 0$, with roots having multiplicity larger than one the only points satisfying this set of equations. Corollary 3.2 follows because root multiplicities are preserved by the transformation generating $\Xi(z)$ and $R(|\xi|^2) \equiv R(\Xi) \equiv R(\zeta)$.

Observe that if a root in $R(|\xi|^2)$ exists having $y = 0$ and multiplicity greater than one, then $|\Xi|^2_{(x,0)} = 0$, $\partial_y|\Xi|^2_{(x,0)} = 0$ and $\partial_{yy}|\Xi|^2_{(x,0)} = 0$ by Lemma 2.21. Since $|\Xi|^2_{(x,y)} \geq 0, \forall (x,y) \in \mathbb{R}^2$, and such an $(x,0)$ is a root of $|\Xi|^2_{(x,y)}$, this point is also a minimum of $gr(|\xi|^2)$ and so must have positive curvature in $y$ about this point. That is, $\partial_{y^n}|\Xi|^2_{(x,0)} > 0$, in the vicinity of this $(x,0)$, where $m$ is the smallest $n$ such that $\partial_{y^n}|\Xi|^2_{(x,0)} \neq 0$. □
Accordingly, the existence of nontrivial roots of $\zeta(x)$ having multiplicity greater than one has not been ruled out and results from not yet proving $\zeta'(\frac{1}{4}+ix) \neq 0$, $\forall x \in \mathbb{R}$. All is not lost, though, as the realization that roots of $\zeta'(\frac{1}{2}+ix)$, if they exist, have to occur at roots of $\zeta(\frac{1}{2}+ix)$ proves useful in establishing that $\partial_{yy}|\Xi|^2(x,0) \geq 0$, $\forall x \in \mathbb{R}$, and so that $|\Xi|^2(x,y)$ is bounded above zero if $y \neq 0$. Bear in mind that because of the correspondence between roots of $\zeta'(\frac{1}{2}+ix)$, if they exist, with those of $\zeta(\frac{1}{2}+ix)$, the existence of nontrivial roots of $\zeta(x)$ off the critical line requires $\partial_{yy}|\Xi|^2(x,0) < 0$ over a region $x \in U \subset \mathbb{R}$, bounded by, at least, a pair of adjacent roots of $\zeta(x)$ and so of $\zeta(x)$ on the critical line having multiplicity greater than one.

2.3.4. Asymptotics of $\partial_{yy}|\Xi|^2(x,0)$ and the curvature of $|\Xi|^2(x,y)$ along the line $y = 0$.

To get a handle on whether or not $\partial_{yy}|\Xi|^2(x,0) \geq 0$, $\forall x \in \mathbb{R}$, it is convenient to group the terms of $\partial_{yy}|\Xi|^2(x,0)$ as $\partial_{yy}|\Xi|^2(x,0) = A(x) \cdot B(\partial_{yy}|\Xi|^2(x,0))$, where

\begin{equation}
A(x) = \frac{1}{128\sqrt{\pi}}|\Gamma(\frac{1}{4}+\frac{i\alpha}{2})|
\end{equation}

\begin{equation}
B(\partial_{yy}|\Xi|^2(x,0)) = 32(4x^2-1)|\zeta'(\frac{1}{4}+ix)|^2 \nonumber \\
(1+4x^2)^2\psi(\frac{1}{4}+\frac{i\alpha}{2})|\zeta'(\frac{1}{4}+ix)|^2 \nonumber \\
(1+4x^2)^2(2\psi(\frac{1}{4}+\frac{i\alpha}{2})+2\psi(\frac{1}{4}-\frac{i\alpha}{2})-4\log_e(\pi))|\zeta'(\frac{1}{4}+ix)|^2 \nonumber \\
4(1+4x^2)^2|\zeta'(\frac{1}{4}+ix)|^2 \nonumber \\
4(1+4x^2)^2|\zeta'(\frac{1}{4}+ix)|^2.
\end{equation}

For reference, graphs of the $A(x)$ and $B(\partial_{yy}|\Xi|^2(x,0))$ pieces of $\partial_{yy}|\Xi|^2(x,0) = A(x) \cdot B(\partial_{yy}|\Xi|^2(x,0))$ over a subset of $\mathbb{R}$ are provided in Fig. 6. With this partitioning of $\partial_{yy}|\Xi|^2(x,0)$, the following is proven,

**Theorem 2.23.** $\partial_{yy}|\Xi|^2(x,0) \geq 0$, $\forall x \in \mathbb{R}$.

**Proof.** Characterizing the asymptotic behavior of $A(x)$ is straightforward. From the known properties of $\Gamma(z)$, e.g., $\Gamma(z)$ has no roots and no poles on this line, we have that $A(x) \rightarrow 0$ exponentially as $x \rightarrow \pm \infty$ and that $A(x)$ is bounded from above and below as $0 < A(x) \leq A(0) = \frac{\sqrt{\pi}}{128\sqrt{\pi}} < \infty$, $\forall x \in \mathbb{R}$, that $\lim_{x \rightarrow \pm \infty} A(x) = 0$. $[4]$ These properties of $A(x)$ are readily apparent in its graph, which is provided in Fig. 6. Note that $A(x) \sim \alpha e^{-\frac{\pi}{2}|x|}$ is graphed in Red in the inset of Fig. 6, where $\alpha = \frac{\sqrt{\pi}}{128\sqrt{\pi}}$.

With $0 < A(x) < 0.06 < \infty$, $\forall x \in \mathbb{R}$, demonstrating that $B(\partial_{yy}|\Xi|^2(x,0)) \geq 0$, $\forall x \in \mathbb{R}$, will prove Theorem 2.23. Despite the functional form for $B(\partial_{yy}|\Xi|^2(x,0))$ being more complicated than $A(x)$, $\text{gr}(B(\partial_{yy}|\Xi|^2))$, which is also provided in Fig. 6, illuminates the asymptotic property in question. That is, $B(\partial_{yy}|\Xi|^2(x,0)) \geq 0$, $\forall x \in \mathbb{R}$ and that $\lim_{x \rightarrow \pm \infty} B(\partial_{yy}|\Xi|^2(x,0)) = +\infty$. An approximation to the divergence of $B(\partial_{yy}|\Xi|^2(x,0)) \rightarrow \infty$ is included in Red in Fig. 6, as $B(\partial_{yy}|\Xi|^2(x,0)) \sim 16x^4 \log_{2}(|x|)$.
In order to actually prove that $B(\partial_{yy}|\Xi|^2)_{(x)} \geq 0, \forall x \in \mathbb{R}$, an analysis of the contributions from the terms comprising $B(\partial_{yy}|\Xi|^2)_{(x)}$ and their associated asymptotic behaviors will be undertaken. The following partitioning of $B(\partial_{yy}|\Xi|^2)_{(x)}$ turns out to be effective for these purposes

$$c(x) = 32(4x^2 - 1)|\zeta|^2_{(1 + 4x^2)} + (1 + 4x^2)^2\psi'(\frac{1}{4} - \frac{\pi}{4})|\zeta|^2_{(1 + 4x^2)}$$

$$d(x) = (1 + 4x^2)^2(2\psi'(\frac{1}{4} + \frac{\pi}{4}) + 2\psi'(\frac{1}{4} - \frac{\pi}{4}) - 4\log_2(\pi))|\zeta|^2_{(1 + 4x^2)}|\zeta''|_{(1 + 4x^2)} +$$

$$4(1 + 4x^2)^2|\zeta|^2_{(1 + 4x^2)} + 4(1 + 4x^2)^2|\zeta|^2_{(1 + 4x^2)}$$

with $B(\partial_{yy}|\Xi|^2)_{(x)} = c(x) + d(x)$. Observe that $c(x)$ includes terms multiplied by $|\zeta|^2_{(1 + 4x^2)}$ while $d(x)$ consists of terms multiplied by $\zeta''_{(1 + 4x^2)}$, $\zeta''_{(1 + 4x^2)}$ and $\zeta''_{(1 + 4x^2)}$. Because $B(\partial_{yy}|\Xi|^2)_{(x)} = c(x) + d(x)$ is a real-valued function, only the behavior of the real parts of $c(x)$ and $d(x)$ are analyzed, see Eq. 24.

With this caveat, it is first proven that $c(x) \geq 0, \forall x$, with $c(x) = 0$ at roots of $\zeta''_{(1 + 4x^2)}$. To do so, observe that $|\zeta|^2_{(1 + 4x^2)} \geq 0, \forall x \in \mathbb{R}$, that $32(4x^2 - 1) > 0, \forall x > \frac{1}{2} \in \mathbb{R}$, with $\lim_{x \to \pm\infty} 32(4x^2 - 1) = +\infty$, and that $(1 + 4x^2)^2\psi'(\frac{1}{4} - \frac{\pi}{4}) < 0, \forall x > 0.588797 \in \mathbb{R}$, with $\lim_{x \to \pm\infty}(1 + 4x^2)^2\psi'(\frac{1}{4} - \frac{\pi}{4}) = -\infty$. Accordingly, as $\lim_{x \to \pm\infty} |\zeta|^2_{(1 + 4x^2)} = 1$, we have that $c(x) \sim D|x|^2\zeta|^2_{(1 + 4x^2)} \geq 0$ as $x \to \infty$, with $D > 0 \in \mathbb{R}$.

Graphs of $c(x), 32(4x^2 - 1)|\zeta|^2_{(1 + 4x^2)}$ and $(1 + 4x^2)^2\psi'(\frac{1}{4} - \frac{\pi}{4})|\zeta|^2_{(1 + 4x^2)}$ are depicted in Purple, Cyan and Green, respectively, in the inset plot of Fig. 7. Note, the Purple curve is the sum of the Cyan and Green curves, with $c(x) \geq 0$ because of the relative asymptotic behaviors the coefficient functions.

$d(x)$ is dominated asymptotically by the $|\zeta|^2_{(1 + 4x^2)}$ term. To verify this claim, observe for large $x$ that $4(1 + 4x^2)^2|\zeta'|^2_{(1 + 4x^2)} \geq 0, \forall x \in \mathbb{R}$, that $32(4x^2 - 1) > 0, \forall x \in \mathbb{R}$, that $\lim_{x \to \pm\infty}(1 + 4x^2)^2\psi'(\frac{1}{4} + \frac{\pi}{4}) + 2\psi'(\frac{1}{4} - \frac{\pi}{4}) - 4\log_2(\pi)\zeta''_{(1 + 4x^2)}|\zeta'|^2_{(1 + 4x^2)} \leq 0$, as $\lim_{x \to \pm\infty} 32x^2\psi'(\frac{1}{4} + \frac{\pi}{4}) + 2\psi'(\frac{1}{4} - \frac{\pi}{4}) + 4\log_2(\pi)(1 + 4x^2)^2|\zeta'|^2_{(1 + 4x^2)} \leq 1$, and that $4(1 + 4x^2)^2|\zeta'|^2_{(1 + 4x^2)}|\zeta''|_{(1 + 4x^2)}$ oscillates about zero but is predominantly positive. Additionally, recall from Lemma 2.13 that $R|\zeta|_{(1 + 4x^2)}^2 \sim -J\log_2(|x|)|\zeta|^2_{(1 + 4x^2)} \sim K\log_2(|x|)|\zeta|^2_{(1 + 4x^2)}$ and $|\zeta'|^2_{(1 + 4x^2)} \sim P\log_2(|x|)|\zeta|^2_{(1 + 4x^2)}$. Now because the $\zeta''_{(1 + 4x^2)}$ and $\zeta''_{(1 + 4x^2)}|\zeta'|^2_{(1 + 4x^2)}$ terms largely offset each other relative to the $|\zeta|^2_{(1 + 4x^2)}$ term as $x \to \pm\infty$, with their sum oscillating about zero, these results reveal that $d(x)$ asymptotically behaves like $d(x) \sim F|x|^2\zeta'|^2_{(1 + 4x^2)} \geq 0, \forall x \in \mathbb{R}$. With the $|\zeta|^2_{(1 + 4x^2)}$ term dominating $d(x)$ as $x \to \pm\infty$ and, by Theorem 2.14 and Eq. 28, if $\exists x \in \mathbb{R}$ such that $\zeta''_{(1 + 4x^2)} = 0$, then $\zeta''_{(1 + 4x^2)} = 0$ and so $c(x) = 0$ and $d(x) = 0$, it follows that $d(x) \sim F|x|^2\zeta'|^2_{(1 + 4x^2)} \sim G|x|^2\log_2(|x|)|\zeta'|^2_{(1 + 4x^2)} \geq c(x) \sim D|x|^2\zeta'|^2_{(1 + 4x^2)} \geq 0, \forall x \to \pm\infty$.

The asymptotic behavior of $d(x)$ just described is perceptible in Fig. 7, where $d(x)$ is plotted in Red, the $4(1 + 4x^2)^2|\zeta'|^2_{(1 + 4x^2)}$ term is in Orange and $c(x)$, i.e. the Purple curve, is only visible in the inset figure. Note how the $\zeta''_{(1 + 4x^2)}$ and $\zeta''_{(1 + 4x^2)}$ terms dampen oscillations in the $|\zeta'|^2_{(1 + 4x^2)}$ term.

As $B(\partial_{yy}|\Xi|^2)_{(x)} \sim d(x) \sim 4(1 + 4x^2)^2|\zeta'|^2_{(1 + 4x^2)} \geq c(x) \geq 0, \forall x \in \mathbb{R}$, this analysis also makes known that $4(1 + 4x^2)^2|\zeta'|^2_{(1 + 4x^2)}$ is the asymptotically dominant term of $B(\partial_{yy}|\Xi|^2)_{(x)}$ as $x \to \pm\infty$. Additionally note that $\lim_{x \to \pm\infty} \partial_{yy}|\Xi|^2_{(x, 0)} = A(x) - B(\partial_{yy}|\Xi|^2)_{(x)} = 0$, since $A(x)$ converges to zero more rapidly than $B(\partial_{yy}|\Xi|^2)_{(x)}$ diverges to zero as $x \to \pm\infty$. Taken together, $A(x) > 0$ and $B(\partial_{yy}|\Xi|^2)_{(x)} > 0, \forall x \in \mathbb{R}$, in conjunction with their asymptotic properties, establishes that $0 \leq \partial_{yy}|\Xi|^2_{(x, 0)} \leq \frac{|\Xi|^2_{(x, 0)}}{1 + x^2}, \forall x \in \mathbb{R}$. □

Jointly, Theorem 2.23 and Corollary 2.16 imply the following

**Corollary 2.24.** $\partial_{yy}|\Xi|^2_{(x, 0)}$ is the first nonzero term in Taylor expansions of $|\Xi|^2_{(x, y)}$ in $y$ about, at least, all points on the line $y = 0$.

**Proof.** Corollary 2.24 results from Lemmas 2.3, 2.5 and 2.15, Theorem 2.23 and Corollary 2.16. Note, the almost all stipulation is a consequence of the possibility that $\partial_{yy}|\Xi|^2_{(x, 0)} = 0$ at roots of $|\Xi|^2_{(x, y)}$ on the line $y = 0$ having multiplicity greater than one. In other words, $\partial_{yy}|\Xi|^2_{(x, 0)} > 0, \forall x \in \mathbb{R}$, except possibly at a countable subset of roots of $|\Xi|^2_{(x, y)}$ on the line $y = 0$ having multiplicity greater than one and so roots of $\zeta(z)$ on the critical line having multiplicity greater than one. □
Corollary 2.26. \[ \Omega(x, y) = \frac{\pi}{2} \left| \frac{\pi}{x} \text{tanh}(\frac{\pi}{x}) \right| \text{c}_{(1, y)}. \]

More importantly, though, Theorem 2.23 actually entails that \( \partial_{y^{2n}} |\Omega|^2_{(x,0)} \geq 0 \) for \( \forall x \in \mathbb{R} \) and \( \forall n \in \mathbb{N} \) and so that \( |\Omega|^2_{(x, y)} \) is globally bounded away from zero when \( y \neq 0 \). Consider that if roots of \( \Xi(z) \) existed off of the line \( y = 0 \), then each \( \partial_{y^{2n}} |\Omega|^2_{(x, y)} \) would oscillate along the line \( y = 0 \) and so equal zero for some set of \( x \). To put this another way, either \( \partial_{y^{2n}} |\Omega|^2_{(x,0)} \geq 0 \), \( \forall x \in \mathbb{R} \) and \( \forall n \in \mathbb{N} \), with all precluded from oscillating about zero and having roots, or all \( \partial_{y^{2n}} |\Omega|^2_{(x, y)} \) oscillate and so all have roots.

2.3.5. Taylor approximates and higher-order derivatives of \( |\Omega|^2_{(x, y)} \).

By assessing the quality of quadratic Taylor approximates to \( |\Omega|^2_{(x, y)} \), denoted \( T_2(x, y) \), the relative locations of the roots of \( \Xi^2_{(x, y)} \) in its critical strip, \( i.e. \) \( (x, y) \in \mathbb{R}^2 \) such that \( -\frac{1}{2} < y < \frac{1}{2} \), and so those of \( \Xi(z) \) in its critical strip, \( i.e. \) \( z = x + iy \in \mathbb{C} \) such that \( -\frac{1}{2} < y < \frac{1}{2} \), will be definitively characterized. In particular, that \( |\Omega|^2_{(x, y)} \) and \( \Xi(z) \) only have zeros in this region if \( y = 0 \) is established by examining both deviations between \( \Xi^2_{(x, y)} \) and \( T_2(x, y) \) and the corresponding \( B(|\Omega|^2 - T_2)(x) \) piece along the symmetry equivalent lines \( y = \pm \frac{1}{2} \) that bound the critical strip. This is encapsulated in Lemma 2.25.

Lemma 2.25. \( \forall x \in \mathbb{R} \) and \( 0 < |y| < \frac{1}{2}, |\Omega|^2_{(x, y)} > 0 \).

Proof. With quadratic approximates about the line \( y = 0 \) given by \( T_2(x, y) = |\Omega|^2_{(x,0)} + \frac{1}{2} \partial_{yy} |\Omega|^2_{(x,0)} y^2 + R(x, y) \), where \( R(x, y) = (|\Omega|^2 - T_2)(x, y) = \sum_{n=2}^\infty \frac{\partial_{y^n} |\Omega|^2_{(x,0)}}{(2n)!} y^{2n} \) is the error term, first note that \( |\Omega|^2_{(x,\pm\frac{1}{2})} = \frac{1}{4} (x^2 + x^4) \) and \( T_2(x, \pm\frac{1}{2}) = |\Omega|^2_{(x,0)} + \frac{1}{2} \partial_{yy} |\Omega|^2_{(x,0)} (\pm\frac{1}{2})^2 + R(x, \pm\frac{1}{2}) \). Now because \( A_{(x)} > 0, \forall x \in \mathbb{R}, \) this term can be factored out of \( T_2(x, \pm\frac{1}{2}) \) and \( R(x, \pm\frac{1}{2}) \). As such, the asymptotic analysis employed in §Sec. 2.3.4 can be adopted here. That is, the comparable \( B(|\Omega|^2 - T_2)(x) \) term can be decomposed in tractable pieces and investigated. The outcome is that both \( T_2(x, \pm\frac{1}{2}) > 0 \) and \( R(x, \pm\frac{1}{2}) = \sum_{n=2}^\infty \frac{\partial_{y^n} |\Omega|^2_{(x,0)}}{(2n)!} (\pm\frac{1}{2})^{2n} > 0 \) along the lines \( (x, \pm\frac{1}{2}) \). In other words, this analysis reveals that \( |\Omega|^2_{(x, y)} \) diverges more rapidly than \( T_2_{(x, y)} \) and so can never turn over in the critical strip. To corroborate this assertion, \( gr(R)(x, y, c) = gr(|\Omega|^2 - T_2)(x, \pm\frac{1}{2}, x) \) and \( gr(B(|\Omega|^2 - T_2)(x, \pm\frac{1}{2}, x) \) along the line \( y = \frac{1}{2} \), are provided in Fig. 8. Note how \( gr(B(|\Omega|^2 - T_2)(x, \pm\frac{1}{2}, x) \) diverges as \( x \to \pm\infty \) over the region depicted. \( \square \)

This analysis also discloses how the behavior of \( |\Omega|^2_{(x, y)} \) on the boundaries of its critical strip, \( i.e. \) \( (x, y) \in \mathbb{R}^2 \) such that \(-\frac{1}{2} < y < \frac{1}{2} \), is related to the behavior of \( |\Omega|^2_{(x, y)} \) on the boundaries of its critical strip, \( i.e. \) \( (x, y) \in \mathbb{R}^2 \) such that \( 0 < x < 1 \). Given that \( |\Omega|^2_{(x, \pm\frac{1}{2})} = \frac{2\pi}{x} \text{sech}(\frac{\pi}{x}) \) and \( |\Omega|^2_{(x, \pm\frac{1}{2})} = \frac{\pi}{2} \text{sech}(\frac{\pi}{x}), \) \( |\Omega|^2_{(x, 0)} = |\Omega|^2_{(x, -\frac{1}{2})} \) simplifies to \( 2\pi|\Omega|_{(x, 0)} = \frac{\pi}{x} \text{tanh}(\frac{\pi}{x}) \), with this expression expressible as

Corollary 2.26. \( \forall y \in \mathbb{R}, \left| \Omega_{(0,y)} \right| = \frac{\pi}{2\pi} \text{tanh}(\frac{\pi}{x}) \) \( \left| \Omega_{(1,y)} \right| \).
Corollary 2.30. All nontrivial roots of \( \zeta(z) \) reside on the critical line \( z = \frac{1}{2} + iy \).
Figure 9. \( \text{gr}(\partial_{y^{2n}}|\Xi|^2)_{(x,0,\epsilon)} \), where \( \epsilon = \partial_{y^{2n}}|\Xi|^2_{(x,0)} \), for \( n = \{1(\text{Red}), 2(\text{Orange}), 3(\text{Green}), 4(\text{Purple}), 5(\text{Black})\} \).

Figure 10. \( \text{gr}(B(\partial_{y^{2n}}|\Xi|^2))_{(x,\epsilon)} \), where \( \epsilon = B(\partial_{y^{2n}}|\Xi|^2)_{(x)} \), for \( n = \{0(\text{Cyan}), 1(\text{Red}), 2(\text{Orange}), 3(\text{Green}), 4(\text{Purple}), 5(\text{Black})\} \).
3. Symmetry and the global structures of $|\Xi|^2_{(x,y)}$ and $\Xi(z)$

Below, symmetry and that all roots of $|\Xi|^2_{(x,y)}$ have the form $(x,0) \in \mathbb{R}^2$ are utilized to identify and characterize analytic properties of $|\Xi|^2_{(x,y)}$, i.e. zeros and critical points, and topographical properties of $\text{gr}(|\Xi|^2_{(x,y),e})$, i.e. minima and transition states, whereby the global structures of each and so of $\Xi(z)$ naturally emerges. A practical byproduct of this geometric investigation is the derivation of a Newton-Raphson algorithm that is capable locating any root in the nontrivial locus of $\zeta(z)$.

3.1. Critical points of $|\Xi|^2_{(x,y)}$ and $\Xi(z)$.

**Lemma 3.1.** All minima and saddle points of $|\Xi|^2_{(x,y)}$ and all correlated saddle points of $\Re\Xi(x,y)$ and $\Im\Xi(x,y)$ reside on the line $y = 0$.

**Proof.** That all minima and saddle points of $|\Xi|^2_{(x,y)}$ reside on line $y = 0$ is a consequence of all roots in $\Re(|\Xi|^2)$ residing on the line $y = 0$. Similarly that all roots in $\Re(\Xi)$ have $y = 0$ necessarily implies that all of the correlated saddle points of $\Re\Xi(x,y)$ and $\Im\Xi(x,y)$ also are on this line. Note that saddle points of $\Re\Xi$ and $\Im\Xi$ are correlated by the Cauchy-Riemann equations, with the principle directions of $\Re\Xi(x,y)$ parallel to the axes, while those of $\Im\Xi(x,y)$ are rotated by an angle $\frac{\pi}{4}$ relative to the axes. \(\square\)

Since all critical points of $|\Xi|^2_{(x,y)}$ along the line $y = 0$, symmetry has to be evident in their structure and properties. This is established by calculating the Hessian and so local curvature of the $\text{gr}(|\Xi|^2_{(x,y),e})$ surface at the first few minima and saddle points, i.e. $H(|\Xi|^2_{(x,y)}) = \left(\frac{\partial^2|\Xi|^2}{\partial x^2}, \frac{\partial^2|\Xi|^2}{\partial x \partial y}, \frac{\partial^2|\Xi|^2}{\partial y^2}\right)_{(x,y)}$. At the first three minima, i.e. roots of $|\Xi|^2_{(x,y)}$ and $\Xi(z)$, $H(|\Xi|^2_{(x,y)})$ is approximately equal to

$$H(|\Xi|^2)_{(14.13478,0)} = \begin{pmatrix} 3.82381 \times 10^{-6} & 0 \\ 0 & 3.82381 \times 10^{-6} \end{pmatrix},$$

$$H(|\Xi|^2)_{(21.02204,0)} = \begin{pmatrix} 6.30109 \times 10^{-10} & 0 \\ 0 & 6.30109 \times 10^{-10} \end{pmatrix},$$

$$H(|\Xi|^2)_{(25.01086,0)} = \begin{pmatrix} 3.20148 \times 10^{-12} & 0 \\ 0 & 3.20148 \times 10^{-12} \end{pmatrix},$$

while at the first three saddle points, i.e. correlated saddle points of $\Re\Xi$ and $\Im\Xi$, $H(|\Xi|^2_{(x,y)})$ is approximately equal to

$$H(|\Xi|^2)_{(0,0)} = \begin{pmatrix} 2.28397 \times 10^{-2} & 0 \\ 0 & -2.28397 \times 10^{-2} \end{pmatrix},$$

$$H(|\Xi|^2)_{(15.58570,0)} = \begin{pmatrix} 7.19779 \times 10^{-7} & 0 \\ 0 & -7.19779 \times 10^{-7} \end{pmatrix},$$

$$H(|\Xi|^2)_{(22.09798,0)} = \begin{pmatrix} 1.27394 \times 10^{-10} & 0 \\ 0 & -1.27394 \times 10^{-10} \end{pmatrix}.$$ 

An attribute of $H(|\Xi|^2_{(x,y)})$ at minima and saddle points of $|\Xi|^2_{(x,y)}$ is that each is diagonal. As $\partial_{xy}|\Xi|^2_{(x,y)} = \partial_{yx}|\Xi|^2_{(x,y)} = 0$ along the line $y = 0$, this is a consequence of symmetry and the Cauchy-Riemann equations. Also note that $\partial_{xx}|\Xi|^2_{(x,y)} = \partial_{yy}|\Xi|^2_{(x,y)}$ at minima and so roots of $\Xi(z)$ and $|\Xi|^2_{(x,y)}$ and that $\partial_{xx}|\Xi|^2_{(x,y)} = -\partial_{yy}|\Xi|^2_{(x,y)}$ at saddle points, which again results from symmetry and the Cauchy-Riemann equations.

The structure and properties of $H(|\Xi|^2_{(x,y)})$ observed for these points, persists for all minima and saddle points along the line $y = 0$ because all roots of $|\Xi|^2_{(x,y)}$ reside on the line $y = 0$ and, of course, symmetry.

The structure and properties of the correlated saddle points of the real and imaginary functions, $\Re\Xi(x,y)$ and $\Im\Xi(x,y)$, along the line $y = 0$ must also possess symmetry by Lemma 3.1. Some of these points are characterized in [19]. Graphs of $\Re\Xi(x,y)$ and $\Im\Xi(x,y)$ near a pair of adjacent roots are presented in Fig. 11. Observe how $\text{gr}(\Re\Xi)_{(x,y,e)}$ curves towards and away from the $\epsilon = 0$ or $xy$-plane along the $x$ and $y$ coordinates, respectively, while $\text{gr}(\Im\Xi)_{(x,y,e)} = 0$ along these lines. Symmetry mandates that this local topography about all of the roots along the line $y = 0$ endures as $x \to \pm \infty$. 

Figure 11. $\text{gr}(\Re \Xi)_{(x,y,\epsilon_1)}$ (Green), $\text{gr}(\Im \Xi)_{(x,y,\epsilon_2)}$ (Blue) and $\text{gr}(\Xi)_{(x,y,\epsilon)}$ near a pair of roots of $\Xi(z)$, where $\epsilon_1 = \Re \Xi(x,y)$ and $\epsilon_2 = \Im \Xi(x,y)$. Note that correlated saddle points of $\Re \Xi(x,y)$ and $\Im \Xi(x,y)$ are marked with Black X’s, points where $\Re \Xi(x,y) = \Im \Xi(x,y)$ are in Red and points where $\Re \Xi(x,y) = 0$ or $\Im \Xi(x,y) = 0$ are in Yellow.

3.2. Locating roots in the nontrivial locus of $\zeta(z)$.

Lemma 3.2. The nontrivial roots of $\zeta(z)$ are isomorphic to roots of

\begin{align}
\text{tm}_{(z)} \overset{\text{def}}{=} \Xi(x,0) = -\frac{1}{8} \pi^{-\frac{1}{2}} \Gamma(\frac{1}{4} + \frac{\iota x}{2}) \Gamma(\frac{1}{4} + \frac{\iota x}{2}) \zeta(\frac{1}{2} + \iota x),
\end{align}

which is a real-valued function $\forall x \in \mathbb{R}$.

Proof. Lemma 3.2 follows from the symmetries of $\Xi(x,y)$, that all roots of $\Xi(x,y)$ lie along the line $y = 0$ and that $\Im \Xi(x,y) = 0$ along the line $y = 0$. That is, $\Im \Xi(x,0) = 0$, $\forall x \in \mathbb{R}$ implies $\Xi(x,0) = \Re \Xi(x,0)$, which gives $R(\text{tm}) \cong R(\Xi^2) \cong R(\Xi) \cong R(\zeta)$. \qed
The derivation of an algorithm to locate nontrivial roots of $\zeta(z)$ will begin by reviewing a recently developed Lagrange multiplier procedure for optimizing roots and poles of $f(z)$. Since $a + ib \in \mathbb{R}(f)$, the symmetry structure of $\Xi$ is isomorphic to the symmetry structures of $\cos(z)$, with minor changes to $L(x,y,M,K)$ needed for poles because $\Delta = 0$ and $\Sigma = \infty$ at these points. [14, 16] Regardless, a corollary of $\mathbb{R}(x,y)$ and $\mathbb{Z}(x,y)$ obeying the Cauchy-Riemann equations is that values for the multipliers are $M = K = -1$. [14, 16] With $L(x,y,M,K)$ reducing to $L(x,y,1,1) = \Re(x,y) - \Delta(x,y) - \Sigma(x,y)$, critical points of $L(x,y,1,1)$ satisfy the set of equations: \[ \partial_x \Re - \partial_x \Delta - \partial_x \Sigma = 0, \quad \partial_y \Re - \partial_y \Delta - \partial_y \Sigma = 0, \quad \Delta = 0 \quad \text{and} \quad \Sigma = 0. \] The algorithm is obtained through a quadratic expansion of $L(x,y,1,1)$ and amounts to iteratively solving the pair of coupled equations \[ 14, 16 \]

\[
\begin{aligned}
(x_{k+1}, y_{k+1}) &= \left( x_k, y_k \right) - \frac{1}{j_{(x_k,y_k)}} \left( \frac{\Delta(x_k,y_k)}{\Sigma(x_k,y_k)} \right),
\end{aligned}
\]

where $j_{(x_k,y_k)}$ is defined by the matrix

\[
j_{(x_k,y_k)} = \begin{pmatrix}
\partial_x \Delta & \partial_y \Delta \\
\partial_x \Sigma & \partial_y \Sigma
\end{pmatrix}
\]

and \( (x_k, y_k) \) is the \( k \)th approximate to the root, i.e. $f_{(x_k+y_k)} = \Re(x_k,y_k) + i\Im(x_k,y_k) \approx 0 + i0$. In this way, Eq. \[ 31 \] can be used to optimize all of the roots of $\Xi(z)$ and so all of the nontrivial roots of $\zeta(z)$.

Lemma 3.2 opens up an avenue to simplify this algorithm. Observe that because all roots of $\Xi(x,y)$ have $y = 0$ and $\Im(\Xi(x,0)) = 0$, \( \forall x \in \mathbb{R} \). Eq. \[ 31 \] reduces to the familiar Newton-Raphson method. [14, 16–18] That is, all nontrivial roots of the zeta function can be optimized using the definition for $tm(x)$ in Eq. 30 as the $f_{(x)}$ and iteratively solving the single equation

\[
x_{k+1} = x_k - \frac{1}{sp(x_k)} (tm(x_k)) = x_k - \frac{2t}{16ix_1(1+4x_1^2) + \log_e(\pi) - \psi^{(1)}_{1/2+ix_1} - \frac{2\pi}{\psi^{(1)}_{1/2+ix_1}}},
\]

where $sp(x)$ is defined by the functional equation \[ 19 \]

\[
sp(x) \equiv d_x \Xi(x,0) = d_x (tm(x)) = \frac{1}{16} \pi^{1/4-i\frac{x}{2}} \Gamma^{1/4+\frac{x}{2}} \times ((16ix + (1 + 4x^2) \log_e(\pi) - (1 + 4x^2)\psi^{(1)}_{1/2+ix_1}) - 2(1 + 4x^2)\psi^{(1)}_{1/2+ix_1}).
\]

As $sp(x)$ is the derivative with respect to $x$ of $tm(x)$, which is analytic, real-valued and symmetric relative to the origin, with $\|R(tm)\| = R_0$, $sp(x)$ as defined in Eq. 34 is analytic, real-valued and anti-symmetric relative to the origin, with $sp(0) = 0$ and $\|R(sp)\| = R_0$, where $R_0$ denotes countable infinite. [19]

4. Conclusions

This work went against the convention of spurning B. Riemann’s change of variable $z \to \frac{1}{2} + iz$ which gives $\Xi(z)$ from $\xi(z)$ and, at the expense of simplicity in functional equations, exploited the symmetries with which $\Xi(z)$ and $\Xi_z^2(x,y)$ are imbued to obtain insight into the relative locations of roots in the nontrivial locus of $\zeta(z)$. To wit, all roots in $\mathbb{R}(\zeta)$, regardless of multiplicity, are definitely collinear and so all are obliged to align on the critical line. In this light, and in spite of evidence suggesting B. Riemann himself was confused by the transformation, [2, 20] his motivations for defining the symmetric function now known as $\Xi(z)$ and asserting that ‘it is very likely that all the roots are real’ become less ambiguous.

At the end of the day, symmetry is a harbinger of classifiable structure, with the larger set of symmetries of $\Xi(z)$ relative to $\xi(z)$ and $\zeta(z)$ indispensable to elucidating the linearity of roots in $\mathbb{R}(\Xi(z)) \cong \mathbb{R}(\Xi)$ and, by default, the linearity of roots in $\mathbb{R}(\zeta)$ and $\mathbb{R}(\xi)$. Stated differently, the smaller set of symmetries of $\zeta(z)$ necessarily obfuscates the $D_{\infty \mathbb{C}}$ symmetry of $gr(R(\zeta)) \cong gr(R(\Xi))$. Although not proven within, the symmetry structure of $\Xi(z) = \Xi_z(x+y)$ under additive inverses of $x$ and $y$ of the complex variable $z = x + iy \in \mathbb{C}$ is isomorphic to the symmetry structures of $\cos(z) = \cos(x+iy)$, $\cosh(z) = \cosh(x+iy)$ and $\sinh(z) = \sinh(x+iy)$, with the maximal point group irreducible representation characterizing the sets of symmetries of graphs of these $f_{(z)}$, $\mathbb{R}(x,y)$, $\mathbb{S}(x,y)$ and $\mathbb{R}(f)$, respectively, $C_{2\mathbb{R}[A_1]}$, $D_{2\mathbb{R}[B_1]}$, $D_{2\mathbb{R}[A_2]}$ and $D_{\infty \mathbb{C}}[\Sigma^2]$. [13, 19]
Symmetry, as wide or a narrow as you may define its meaning,
is one idea by which man through the ages has tried to
comprehend and create order, beauty and perfection.

Hermann Weyl
Appendix A. Root and pole loci of $\zeta(z)$, $\Xi(z)$ and $|\Xi|^2_{(x,y)}$

Figure A.1. A subset of $R_t(\zeta) \cup \gamma(\zeta) = R_s(\zeta) \cup R(\zeta) \cup \gamma(\zeta)$, where roots in $R_s(\zeta)$ and $R(\zeta)$ and the pole in $\gamma(\zeta)$ marked with Red +’s, Green +’s and a Black O, respectively. $R(\Xi) \cong R(|\Xi|^2) \cong R(\zeta)$ is depicted separately, with roots marked with Green +’s.

Appendix B. Relative asymptotics of $\mathcal{R}\zeta_{(\frac{1}{2}+ix)}\zeta'_{(\frac{1}{2}-ix)}$, $\mathcal{R}\zeta_{(\frac{1}{2}+ix)}\zeta''_{(\frac{1}{2}-ix)}$, $|\zeta|^2_{(\frac{1}{2}+ix)}$ and $|\zeta'|^2_{(\frac{1}{2}+ix)}$

Figure B.1. Graphs of $-\log_e(|x|)|\zeta|^2_{(\frac{1}{2}+ix)} - \log_e(|x|)|\zeta'_{(\frac{1}{2}+ix)}|$, $\log^2_e(|x|)|\zeta|^2_{(\frac{1}{2}+ix)}$, $\log_e(|x|)|\zeta'|^2_{(\frac{1}{2}+ix)}$, $\log^2_e(|x|)|\zeta'|^2_{(\frac{1}{2}+ix)}$.

Email address: josephjamesdillon@gmail.com
Current address: Nashville TN 37208