Stochastic ice stream dynamics

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Ice streams are narrow corridors of fast-flowing ice that constitute the arterial drainage network of ice sheets. Therefore, changes in ice stream flow are key to understanding paleoclimate, sea level changes, and rapid disintegration of ice sheets during deglaciation. The dynamics of ice flow are tightly coupled to the climate system through atmospheric temperature and snow recharge, which are known exhibit stochastic variability. Here we focus on the interplay between stochastic climate forcing and ice stream temporal dynamics. Our work demonstrates that realistic climate fluctuations are able to (i) induce the coexistence of dynamic behaviors that would be incompatible in a purely deterministic system and (ii) drive ice stream flow away from the regime expected in a steady climate. We conclude that environmental noise appears to be crucial to interpreting the past behavior of ice sheets, as well as to predicting their future evolution.

ice streams | noise-induced phenomena | climate variability | Hopf bifurcation

Ice sheets are large bodies of ice that spread over continents under their own weight, the most prominent contemporary examples being the Greenland and Antarctic ice sheets. Ice sheets are built by snow accumulation over hundreds of thousands of years, and they constantly exchange mass and energy with the ocean and the atmosphere. As such, ice sheets play an active role in the global climate system, to which they are tightly coupled. Ice sheets exhibit rich spatiotemporal dynamics associated with the development of ice streams; these are narrow corridors (tens of kilometers wide) of fast-flowing ice (hundreds to thousands of meters per year) bordered by the slowly moving ice sheet, which stretch from the margin of the ice sheet toward the interior for hundreds of kilometers. Because ice streams convey most ice flux to the ocean, exceeding 80% of the overall discharge in present-day Antarctica (1), their variability has dramatic potential impact on sea level change.

Temporal variability of ice stream flow spans decadal to multimillennial time scales. Episodes of large-scale ice discharge from ice streams in the North American Laurentide ice sheet are inferred from deep marine sediment records spanning the last glacial period (2, 3). These Heinrich events, which occurred repeatedly with periodicity of 5,000–10,000 yr, are hypothesized to have been large enough to impact ocean and atmospheric circulation (4), and to raise sea level several meters (5). Centennial- and subcentennial-scale variability has taken place in Antarctica over the last 10,000 yr (6). Variability on a decadal time scale includes the ongoing deceleration and thinning of Whillans Ice Stream (7, 8) and velocity changes in its tributaries (9, 10), as well as subsequent deceleration and speedup of MacAyeal and Bindschadler Ice Streams (11), and centennial-scale stagnation and reactivation cycles have been inferred for Whillans, Kamb, and MacAyeal Ice Streams (12). The slowdown of Whillans, along with the shutdown of Kamb Ice Stream ca. 160 yr ago (13), is responsible for the currently positive mass balance of the Ross region of West Antarctica (14).

Ice streams also display a complex spatial dynamics. Self-organization of ice sheet flow into distinct streams, regardless of strong topographic control (6, 15), is evident in the contemporary Ross ice streams, and is suggestive of a flow instability. Moreover, geomorphological evidence supports the occurrence of similar spatiotemporal dynamics in the past during periods of apparently stable climate (16, 17), thus suggesting that significant changes in ice flow patterns can occur even with little or no external forcing, and over short periods of time.

The present work focuses on ice sheet temporal variability and, in particular, on the variety of concurrent processes leading to stagnation and activation cycles in ice stream flow. The coupling between ice sheets and the climate system, which features a strong stochastic component (18), motivates our question about whether internal climate variability affects the dynamics of ice stream flow. In fact, a similar pattern of interaction has been identified for mountain glaciers, where fluctuations in the surface mass balance due to climate variability drive fluctuations in glacier length (19–21). However, to our knowledge, no attempt has been made so far to characterize the role of interannual climate variability with respect to ice streams, which are known to undergo variability that can contribute significantly to global sea level.

The peculiar flow regime of ice streams originates from a force balance where the bed of the ice stream supports only a small portion of the gravitational driving stress, and the narrow regions that separate slow- and fast-moving ice (the so-called “shear margins”) contribute significantly to the force balance (22, 23). This regime can be explained by thermal feedbacks that enhance basal meltwater production (24–27). How meltwater modulates ice stream velocity depends on the nature of the ice–bed contact: If the ice stream is underlain by a layer of sediment, meltwater infiltrates the sediment, which, in turn, deforms more easily as a result of the shear stress imposed by the ice (27–29). If, instead, ice is in direct contact with bedrock, the lubrication provided by water at the ice–rock interface controls the rate at which ice slips over the bed (25). Water transport and ponding along ice stream beds have been also observed (30–33), and model studies (34–39) also point to the subglacial drainage system as a control on the spatial and temporal dynamics of ice streams.

Significance

Ice streams form the backbone of the flow field of ice sheets. They are known to exhibit a complex spatiotemporal dynamics, which is largely not well understood. Understanding the controls on such dynamics is crucial to sea level change projections as well as to the interpretation of paleorecords. Our contribution unravels the impact of climate variability on the temporal dynamics of ice streams. In particular, we show that minimal, and at the same time realistic, random fluctuations due to climate variability are capable of producing unprecedented behaviors in ice stream dynamics. These results may open a new perspective on the past and future behavior of ice sheets.

Author contributions: E.M., M.B.B., and L.R. designed research, performed research, and wrote the paper.

The authors declare no conflict of interest.

This article is a PNAS Direct Submission.

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This article contains supporting information online at www.pnas.org/lookup/suppl/doi:10.1073/pnas.1600362113/-/DCSupplemental.
Despite different levels of complexity, a common feature of ice stream models is oscillatory behavior in the form of stagnation and activation cycles (24–27, 29, 40). Recent work (29) has proven that this behavior emerges through a Hopf bifurcation in the dynamics of steady ice stream flow under changes in the control parameters, which noticeably include atmospheric temperature and snow recharge. This kind of bifurcation, which consists of a transition from a stable fixed point to a stable limit cycle, is widespread in environmental systems, and the literature suggests that it is sensitive to stochastic forcing (41, 43).

Stochastic forcings were generally associated with additive disordered random fluctuations around the deterministic behavior of dynamical systems. In recent years, it has been shown that random fluctuations can induce temporal and spatial behaviors that do not exist in the presence of purely deterministic dynamics (44, 45). This novel perspective on the significance of randomness has become essential to the environmental sciences, where stochastic components are pervasive (45–48). Even though noise-driven phenomena such as stochastic resonance, noise-controlled patterning, and noise-enhanced shift precursors have been recently identified in this context, unraveling how environmental fluctuations can cause structural changes in the dynamics is still a largely unexplored subject.

In this framework, we focus on the random fluctuations occurring as key external, climate-related, drivers of ice stream dynamics for which time series spanning the whole Holocene are available from climate proxies. We explore the impact of fluctuations in the forcing on the underlying Hopf bifurcation with a modeling approach and find that, in certain parametric regimes, realistic climate fluctuations are able to induce the coexistence of dynamic behaviors that would be incompatible in a purely deterministic system. We investigate the differences between atmospheric temperature forcing and accumulation rate forcing, and find the former to dominate the stochastic dynamics under realistic levels of noise. We also see that stochastic forcings yield temporal variability of ice stream flow with a relatively short characteristic time scale, which could explain some aspects of observations (6, 12) that are reproduced by existing models only partially.

**Unforced Ice Stream Dynamics**

Observations identify at least two fundamental flow regimes for ice streams: steady streaming and activation/deactivation cycles. The simplest model able to capture these dynamics is proposed by Robel et al. (29). They develop a physically based, spatially lumped model where time is the only independent variable, to understand the physical processes underpinning temporal ice stream dynamics. The mathematical simplicity of this model allows us to use stochastic climate forcing, and is thus instrumental in exploring the role of environmental randomness with respect to ice stream dynamics. In this section, we review the physical features of the model relevant to our study. The governing equations are stated in *Mathematical Model*, and the reader is referred to ref. 29 for further detail.

Robel et al. (29) consider an ice stream of fixed width underlain by a layer of subglacial sediment (known as till). The key ingredients of the model are (i) the ice stream mass balance and (ii) the basal energy budget. Ice accumulates from snowfall and moves purely by basal sliding accompanied by water-laden subglacial till. Subglacial till displays a Coulomb plastic rheology, with yield strength dependent on water content (49). The nature of this dependency ensures that the higher the till water content, the lower the shear strength of the till. The dynamics of water content are driven by the basal energy budget, which is the sum of frictional heating, geothermal heating, and conductive cooling toward the ice surface. Excess meltwater beyond till saturation is assumed to evacuate via subglacial drainage, which is not resolved explicitly. In this study, we will vary accumulation rate and ice surface temperature, as they relate to climate variability. Geothermal heating and geometrical and physical parameters are kept constant and equal to ref. 29, so as to resemble the Siple Coast region.

The model predicts two distinct flow regimes. Steady streaming is attained if conductive cooling toward the ice surface is balanced by geothermal heat flux and frictional heating at the bed. This balance yields a constant till water content, \( w \), and, subsequently, a constant ice thickness, \( h \), and ice velocity, \( u \). If, instead, conductive cooling dominates, we observe periodic activation and deactivation cycles, known as binge–purge mode. This pattern is interpreted as a relaxation–oscillation between a stagnant phase, where the basal energy budget is dominated by conductive cooling, and a surge phase, where fast flow is sustained by frictional heating. Changes in surface temperature, \( T_s \), which sets the strength of conductive cooling along with \( h \), yield a transition between these two regimes. Fig. 1 displays the bifurcation diagram, i.e., the amplitude of ice thickness oscillations, \( \Delta h_{\text{det}} \), against one control parameter (specifically \( T_s \), which decreases from left to right): The thick red branch of the curve corresponds to steady streaming (\( \Delta h_{\text{det}} = 0 \)), which we observe for relatively warm \( T_s \), whereas the thick blue branch corresponds to binge–purge (\( \Delta h_{\text{det}} \neq 0 \)). Both branches are stable, i.e., each regime is maintained in time. Differently, the black dashed branch represents unstable oscillation that collapses onto either steady streaming or binge–purge, depending on the initial conditions. Note that the two stable branches overlap in the region with dashed background, whereas, in the white- and gray-shaded regions, a single stable branch exists (further detail is provided in the legend of Fig. 1).

We now consider three specific values of \( T_s \), labeled a–c on the abscissa of Fig. 1, one for each region. Fig. 1, *Insets a–c* display schematic representations of the phase portrait for these cases, with orbits in black and arrows marking the direction of evolving time. We observe a transition from a fixed point (red dot in *Inset b*) to a stable limit cycle (blue circle in *Inset c*) as \( T_s \) decreases, with the two coexisting in *Inset b*. These phase portraits depict a subcritical Hopf bifurcation taking place at \( T_s = T_{\text{hopping}} \), and a saddle-node bifurcation at \( T_s = T_{\text{saddle}} \). Such dynamical structure leads to bistability and hysteresis, in the sense that (i) steady streaming and binge–purge modes coexist for \( T_s > T_{\text{saddle}} \) and (ii) the transitions are abrupt, i.e., the switch is from steady flow to large-amplitude oscillations. The same results hold also for changes in the accumulation rate, \( a_c \) (not shown for the sake of brevity).

**Results**

In the model described in *Unforced Ice Stream Dynamics*, ice stream dynamics are coupled to the climate via atmospheric temperature at the ice surface and snow accumulation rate. We now investigate the effects of climate variability on ice stream dynamics by stochastically forcing these parameters. Details about the simulation techniques are provided in *Materials and Methods*.

The key result of our study is that noise causes structural changes in the dynamics, inducing widespread irregular flow in parametric regimes where, in a steady climate, we would expect steady flow. This behavior is illustrated in Fig. 2. Because the system exhibits irregular fluctuations, we now consider the mean oscillation amplitude over each realization, \( \overline{\Delta h} \), against the mean value of the forcing parameters, \( \overline{T_s} \) and \( \overline{a_c} \). Theoretical results (45) suggest that the effect of the initial condition is smoothed out by the stochastic dynamics after a short transient. This transient is typically shorter than \( 10^3 \) y in the present case; thus this first portion of the output signal has been disregarded when averaging. The time-dependent forcing has fixed variance \( \sigma \) and
integral scale $I$ (as defined in Materials and Methods) that reflect natural variability as recorded by the ice core GISP (Greenland Ice Sheet Project) 2 (50) in Greenland (further details are provided in Table 1 and Fig. S1). We then allow for variations of $T_s$ in the interval $[-22, -32] \, ^\circ C$ and variations in $\tau$ in the interval [0.2, 0.3] m/y. Despite being physically plausible, such intervals are not intended to represent a specific setting but rather to capture a spectrum of potentially feasible situations.

We first consider each parameter separately, with results for stochastically forced $T_s$ reported in Fig. 24. Here we explore the effect of changes in the mean climate, described by the behavior of $\Delta h$ against $T_s$ (red curve). We refer to this curve as a stochastic bifurcation diagram, as opposed to the deterministic bifurcation diagram plotted in black for reference. A wide and smooth transition region now substitutes the hysteresis region, where steady streaming and binge–purge branches overlap in the absence of noise (dashed background in Fig. 24). The red curve approaches the black curve for the extreme values of $T_s$, because the deterministic dynamics are recovered only far from the bifurcation. This behavior proves that noise not only forces fluctuations about stable deterministic solutions but also yields unprecedented behaviors. In particular, the amplitude of ice thickness fluctuations in the transition region is much larger than the intensity of noise. This pattern implies that stochastic forcing excites stable modes of the dynamical system, whose response is, however, internally driven. In this sense, noise has a constructive role, as it is capable of inducing behaviors that are absent when stochastic fluctuations are suppressed.

The constructive effect of noise is more evident when examining the three time series displayed in Fig. 24: For $T_s < T_{\text{Hopf}}$ (Top Inset), stochastic (red) and deterministic (black dashed) oscillation amplitudes are substantially similar, with only a slight shift between the two time series. Hence we conclude that the limit cycle is preserved in the stochastic dynamics. For $T_{\text{Saddle}} < T_s < T_{\text{Hopf}}$ (Middle Inset), stochastic forcing suppresses the dependence on initial conditions, and the system switches between the two deterministic stable solutions (black: solid is the fixed point, and dashed is the limit cycle solution). Once again, the limit cycle appears to dominate the stochastic dynamics, as demonstrated by the similarity of oscillation amplitudes with and without stochastic forcing. However, the coexistence with the fixed point leads to (i) alternation between bursts in the velocity and steady streaming periods (Fig. S2) and (ii) a modulation of oscillation amplitudes (note the modulation of the peaks’ height of the red curve in Fig. 24, Middle Inset) that does not take place in the deterministic dynamics. Finally, Fig. 24, Bottom Inset shows the stochastic

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**Fig. 1.** Deterministic dynamics. Bifurcation diagram with $T_s$ as control parameter ($T_s$ decreasing from left to right, $a_s = 0.23 \, \text{m/y}$); thick solid lines denote stable branches (steady streaming in red, binge–purge in blue). The black dashed line marks the unstable branch (unstable oscillations). A subcritical Hopf bifurcation occurs at $T_{\text{Hopf}} = -29.3 \, ^\circ C$, and a saddle–node bifurcation at $T_{\text{Saddle}} = -27.9 \, ^\circ C$. Insets a–c show phase portraits for three specific values of $T_s$ (labeled a–c on the abscissa of the main diagram), namely, $T_s = -18 \, ^\circ C$, $-28.5 \, ^\circ C$, and $-32 \, ^\circ C$, respectively. In these insets, the red dot marks a fixed point, the blue circle marks a stable limit cycle, and the dashed black circle marks an unstable limit cycle; orbits are in solid black, with the arrows showing the direction of evolving time. Insets b1 and b2 show time series of $u$ and $w$ for case b, with differing initial conditions ($h, w$). Inset b1 ($(th, w) = (750 \, \text{m}, 0.6 \, \text{m})$) displays a steady streaming solution, and Inset b2 ($(h, w) = (600 \, \text{m}, 0.6 \, \text{m})$) displays a binge–purge solution. These two solutions correspond to the two trajectories in the phase diagram in Inset b. Two further characteristic points must be labeled in the parameter space, namely, $T_{u} (-28.1 \, ^\circ C)$ and $T_{r} (-31 \, ^\circ C)$. $T_{u}$ denotes a transition between till water content at ($w = w_{u}$) or below ($w < w_{u}$) the saturation threshold $w_{u}$; we recall that excess meltwater is evacuated when $w$ attains $w_{u}$. When $T_{s} < T_{u}$, all of the available meltwater has been frozen ($w = 0$), and basal temperature $T_{b}$ drops below the melting point. This is depicted in Inset c, where $w$ spans positive abscissas and $T_{b}$ spans the negative ones. Hence, when the orbit crosses the y axis, a basal cooling cycle is initiated, and the state of the system at each instant in time is uniquely defined by $h$ and $T_{b}$ rather than $h$ and $w$. Note that the bifurcation occurs always in the regime $w_{u} < w < 0$. 
dynamics for $T_s > T_{saddle}$: Even though the intensity of noise is significantly lower than the width of the hysteresis region (Table 1), we observe high-intensity bursts in the velocity with intervening variable-duration periods of small-amplitude oscillations about the fixed point (black line).

Analogous behavior is displayed by time series forced with stochastically varying accumulation rate. In this case noise has a less wide-range effect (Fig. S3) because of the additive nature of the accumulation forcing. In fact, accumulation appears as an additive term in the mass balance (Eq. SI5), as opposed to the multiplicative nature of surface temperature forcing, which instead appears as coefficient of one of the dependent variables, $h$ (Eq. SI5). Stochastic differential equations (SDEs) are known to be more sensitive to multiplicative forcing, as, in the effect of noise is modulated by the state of the system itself (45).

In Fig. 2B, we explore the role of simultaneous fluctuations in surface temperature and accumulation rate, motivated by the fact that $T_s$ and $a_c$ are both time-dependent forcings and exhibit variability on a comparable time scale. Because the GISP2 data show only weak cross-correlation of the two signals (Materials and Methods), we consider uncorrelated $T_s$ and $a_c$ forcings. A contour plot of $\Delta h$ as a function of $T_s$ and $a_c$ is displayed in Fig. 2B, with the white line marking the saddle--node bifurcation in the deterministic dynamics. In a steady climate, steady streaming would be expected on the left of the white line. However, the contour plot of $\Delta h$ shows widespread, large-amplitude fluctuations in this region. This observation implies that the behavior outlined in Fig. 2A is preserved under the more realistic scenario with both $a_c$ and $T_s$ undergoing stochastic fluctuations.

A further perspective on the effects of climate forcing is provided by the probability density function (pdf) of the amplitude of ice thickness fluctuations reported in Fig. 3. Insets. Here we consider again stochastic forcing of $T_s$ and $a_c$, separately, and investigate the effect of changes in the mean climate on the pdfs. For each realization, the pdf of the ice thickness oscillation range is computed, and its mode is tracked throughout the parameter regime [blue curve in Fig. 3; $\Delta h_{det}$ (black curve) and $\Delta h$ (red curve) are also reported for reference]. The pdfs can be classified in three paradigmatic behaviors regardless of the forcing parameter under consideration ($T_s$ in Fig. 3A, $a_c$ in Fig. 3B): When the mean of the control parameter is sufficiently distant from the bifurcation (white background), the pdfs are unimodal, whereas bimodal pdfs are observed in the region across the bifurcation (gray background).

We first focus on bimodal pdfs. Bimodality is the stochastic counterpart of hysteresis in the deterministic bifurcation diagram, the key differences being that (i) the parametric region exhibiting bimodality is wider than the deterministic hysteresis region (gray dashed region), and (ii) the height of the peaks in

Table 1. Statistics of the GISP2 time series of accumulation rate and surface temperature

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$CV$</th>
<th>$l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Accumulation rate</td>
<td>0.23 m/y</td>
<td>0.01 m/y</td>
<td>0.04</td>
<td>816 y</td>
</tr>
<tr>
<td>Surface temperature</td>
<td>$-30.6^\circ$ C</td>
<td>0.75 $^\circ$ C</td>
<td>0.02</td>
<td>307 y</td>
</tr>
</tbody>
</table>

Data from ref. 50.
the pdfs (Fig. 3, Insets) varies across the parameter space, mirroring different ways of interaction between stochastic forcing and deterministic dynamics. This pattern provides further evidence of the constructive effect of noise, as well as of its ability to cause structural changes in the dynamics.

The ratio between the width of the bimodality region (gray background in Fig. 3) and the width of the hysteresis region (gray dashed background in Fig. 3) can be used to quantify the importance of changes brought about by either additive noise in $a$, or multiplicative noise in $T_s$. Although the coefficient of variation ($CV = \sigma / \mu$, with $\mu$ being the mean) is similar for the GISP2 accumulation and surface temperature time series (Table 1), this ratio of the widths is significantly larger for Fig. 3A than for Fig. 3B (4.95 and 1.09, respectively). We thus understand that fluctuations in surface temperature have a stronger effect than fluctuations in the accumulation rate, with the former resulting in a wide and smooth transition region, as opposed to a very sharp transition for the latter.

We now consider more closely the regions of Fig. 3 with unimodal pdfs. In the leftmost region of each panel, we observe a shift between mode $\Delta h_{\text{det}}$ (red curve) and $\Delta h_{\text{det}}$ that is absent in the rightmost region (see also Fig. 3B, Lower Inset). An explanation for such behavior is that noise induces oscillations about the fixed point whose amplitude is a positive quantity. This results necessarily in an asymmetric pdf (Upper Left Inset in Fig. 3 A and B), and a shift between the nonzero mode of the distribution and $\Delta h_{\text{det}}$, which is zero. Where steady streaming would occur under a steady climate, the rate of decay to zero of stochastic fluctuations is remarkably slower in Fig. 3A than in Fig. 3B. This behavior may reflect either the different integral time scales of temperature and accumulation rate signals (Table 1) or the profound difference between the additive and multiplicative nature of the stochastic forcings; alternatively, it might depend on a difference in strength between the limit cycle and fixed point attractors.

**Discussion and Conclusions**

The present study addresses the interaction between climate variability and ice stream temporal dynamics from a modeling perspective. Based on past climate data recorded in ice cores spanning the Holocene, we are able to simulate the variability of atmospheric temperature and accumulation rate of a specific site in Greenland as colored Gaussian noise, thus obtaining realistic climatic forcing for a simple but physically based model of ice stream temporal dynamics. Our analysis puts ice stream behavior in a wider perspective. In fact, it shows that minimal and, at the same time, realistic climate fluctuations can drive ice stream flow away from the regime expected in a steady climate. Most importantly, we find that, even if an ice stream currently appears to be in a steady state, it may not remain so even if the mean climate does not change. As a result, climate variability emerges as a prominent control on the future behavior of ice streams, suggesting that incorporation of noisy forcings in prognostic ice sheet models may affect predictions of ice sheet mass balance and sea level change.

We have examined forcings of two external drivers, namely, atmospheric temperature and snow accumulation rate. Our results suggest that different levels of complexity in the dynamic response are to be expected, depending on which of the control parameters is forced, with atmospheric temperature playing by far the most significant role. In contrast to the deterministic case, we find that ice streams transition gradually from a steady streaming to a binge–purge mode as a result of changes in the mean climate, with short time scale fluctuations consistently observed throughout the parameter space. The absence of a lower bound on the amplitude of fluctuations is peculiar to the stochastic model [for the deterministic model, this lower bound...
is \( \sim 800 \) y with the current parameter choice (29)], and is consistent with observations of ice stream temporal variability on centennial time scales in West Antarctica (6, 12).

Our approach has limitations, and raises new questions. Firstly, the model used is highly idealized and simplified. More complex, spatially extended mechanical and thermal models of ice streams, possibly including the effects of complex basal topography and subglacial drainage, may provide a more realistic description of the stochastic ice stream dynamics. However, Monte Carlo techniques would be required to simulate stochastic forcing in spatially extended models, at the expense of increased computational cost and less straightforward interpretation of the physical mechanisms relevant to ice stream variability. We do not exclude the possibility that more complex models may partially dampen the effects of stochastic forcing, but we are reassured by investigations carried out with a flowline extension of the present model (51) that demonstrate that the Hopf bifurcation is a robust feature. Hence, we expect the bulk of our results to hold in spatially extended systems, but we are also aware that more sophisticated models should be used to predict the specific details of stochastic ice stream flow. A second point that deserves further inquiry is an extensive investigation of the stochastic dynamics, especially concerning the more realistic case of simultaneous accumulation rate and surface temperature forcing. Based on the weak cross-correlation of the GISP2 data, here we have considered uncorrelated forcings. Fig. S4 presents time series of ice thickness obtained with different levels of correlation between temperature and accumulation. The pdfs experience little changes as the strength of cross-correlation (measured by the cross-correlation coefficient) rises from 0 to 0.9, thus suggesting that the core of the stochastic response observed in the absence of cross-correlation is robust. This result may be due to the marginal role of the accumulation forcing, which appears not to affect the system significantly, even in the case of strong correlation with temperature. Finally, we wish to emphasize that no significant trend is observed in the stochastic forcing, and therefore the results we presented are not closely related to anthropogenic climate change scenarios.

In summary, our results show that the internal dynamics of ice streams are affected by the stochastic variability of climate, but the extent and reversibility of this effect is still to be quantified. However, this behavior potentially holds implications for the interpretation of observations and geological records, and may shed a new light on the potential for ice sheets to contribute abruptly to sea level change.

Materials and Methods

To investigate the impact of random forcings on ice sheet dynamics, we focus on two key external parameters involved in the model: accumulation rate, \( a_c \), and ice surface temperature, \( T_s \). There are three reasons for this choice: (i) These parameters are closely tied to atmospheric circulation, which is known to exhibit stochastic variability. (ii) These parameters feature in the model with a different level of complexity. (iii) Data are widely available. In fact, long time series of \( a_c \) and \( T_s \) can be compiled thanks to climate proxies recorded in ice cores (52).

The last point is essential to constructing realistic estimates of random fluctuations. To this aim, we consider data from the GISP2 ice core in Greenland (50), whose main statistics, along with the integral scale, are displayed in Table 1, being \( l = \int_0^\infty \rho(r) \, dr \), where \( \rho(r) \) is the autocorrelation function and \( r \) is the time delay of the corresponding time series. The autocorrelation function refers to the time series obtained by linear interpolation of the unevenly spaced original data and resampling the thus constructed time series with a time step equal to the mean sampling step of the original time series. When the variance of the sampling times is moderate, as is the case for the GISP2 time series, this is considered the standard procedure (e.g., refs. 53–55) because it is simple and does not introduce a strong bias in the data.

Data show that (i) the random component is not negligible (\( CV_{a_c} = 0.04 \) m/y and \( CV_{T_s} = 0.02 \) \(^\circ\)C, where CV is the coefficient of variation) and (ii) time series are significantly autocorrelated, namely, they exhibit a memory component.

Supported by the features of the GISP2 data set, we describe the accumulation rate and surface temperature time series as

\[
ad_c(t) = \bar{a}_c + \xi_{a_c}(t), \quad T_s(t) = \bar{T}_s + \xi_{T_s}(t),
\]

where \( \bar{a}_c \) and \( \bar{T}_s \) are average values, and \( \xi_{a_c} \) and \( \xi_{T_s} \) are the random components, which we model as colored Gaussian noises, coherently to the dominant character of core records variability (56). In addition, colored Gaussian noise has a mathematically simple structure, is adopted frequently to simulate autocorrelated processes (45), and can be simulated in a numerically efficient way. We are aware of the debate about modeling noise in climate records (e.g., ref. 57), but, to date, no general recipe is available. Because we do not wish to reproduce specific settings, but we rather aim to illustrate the ability of noise to introduce structural changes in the deterministic ice stream dynamics, we adopt the more conservative colored Gaussian noise.

We force the deterministic model with \( a_c(t) \) and \( T_s(t) \) given by relations [1]. We then simulate the random component as an Ornstein–Uhlenbeck process (58) and adopt the numerical scheme proposed by ref. 59. Low cross-correlation between the precipitation rate and surface temperature time series (\( r_{ac} = 0.28 \), with \( r \) as the cross-correlation coefficient) allows the two noise components \( \xi_{a_c} \) and \( \xi_{T_s} \) to be considered reasonably as independent.

Acknowledgments. E.M. thanks A. Robel, C. Schoof, and V. Radić for discussion and suggestions during completion of this project, and K. Unglert for comments on the manuscript. We are grateful to C. Camporeale and two anonymous reviewers for insightful comments and constructive criticism that helped to improve our work.

Supporting Information

Mantelli et al. 10.1073/pnas.1600362113

Mathematical Model

We summarize here briefly the mathematical model by Robel et al. (29) to support the interpretation of our results. The reader is referred to the original paper for further detail.

The model is spatially lumped, with time \( t \) the only independent variable. The ice stream is considered to have length \( L \), width \( W \), and ice thickness \( h \). Mass conservation therefore requires

\[
\frac{dh}{dt} = a_c - \frac{u_b h}{L} \tag{S1}
\]

where \( a_c \) is the accumulation rate (positive for accumulation, negative for ablation), and \( u_b \) is the basal sliding velocity. The constitutive relation for basal velocity is obtained under the assumption that sliding by till deformation is dominant, and reads

\[
u_b = \frac{A_b W^{n+1}}{4\pi (n+1) h^n} \max[\tau_d - \tau_b, 0]^n. \tag{S2}
\]

Here \( \tau_d = \rho gh^2 / L \) is an approximation of the driving stress (\( \rho \) is the ice density and \( g \) is the acceleration due to gravity), and \( \tau_b \) is the basal shear stress. \( A_b \) is a creep parameter, whereas \( n \) is the exponent of Glen’s law for ice rheology. Till strength is modeled as a Coulomb friction law, which can be expressed in terms of void ratio \( e = w / Z_i \) (where \( Z_i \) is the thickness that unfrozen till would occupy if it was reduced to zero porosity, and \( w \) is the till water content) as

\[
\tau_b = \begin{cases} 
 a' \exp(-b(e-e_c)) & \text{if } w > 0 \\
\infty & \text{if } w = 0 \end{cases}, \tag{S3}
\]

where \( a \) and \( b \) are constants, and \( e_c \) is a lower bound of the void ratio.

Energy conservation reduces to a basal meltwater budget that controls the temporal evolution of till water content \( w \) as

\[
\frac{dw}{dt} = m - \frac{Q_d}{LW} \text{ if } w > 0, \tag{S4}
\]

where \( m \) is the basal melt rate and \( Q_d \) is the flow rate discharged by the subglacial drainage network. The basal melt rate is related to geothermal heat flux \( G \), heat conduction into the ice, and heat dissipation by friction at the bed as

\[
m = \frac{1}{\rho_i L_f} \left[ G + \frac{k_i (T_s - T_b)}{h} + \tau_b u_b \right]. \tag{S5}
\]

where \( T_s \) is a fixed surface temperature, \( T_b \) is the basal temperature, and \( k_i \) is the ice thermal conductivity. We note that \( T_s \), one of the stochastically varying parameters in our analysis, multiplies one of the dependent variables, i.e., the ice thickness \( h \).

As for drainage, once the saturation threshold \( w_i \) is reached in the till, further meltwater is assumed to be drained by an active subglacial system. Mathematically, we have

\[
Q_b = \begin{cases} 
0 & \text{if } w < w_i, \text{ or } m < 0 \\
\text{otherwise} & \text{otherwise}
\end{cases}. \tag{S6}
\]

Finally, at low void ratio, a frozen water fringe can propagate within sediments, until the unfrozen sediment thickness \( Z_i \) is reduced to zero. This is assumed to occur at \( e = e_c \), and is governed by

\[
\frac{dZ_i}{dt} = \begin{cases} 
0 & \text{if } e > e_c, \text{ or } Z_i = 0 \\
\frac{m}{e_c} & \text{if } e = e_c, \text{ and } Z_i > 0
\end{cases}, \tag{S7}
\]

where \( Z_i \) is the total sediment thickness. Once all of the sediments have frozen, basal ice may be cooled as a result of heat conduction toward the surface; that is,

\[
\frac{dT_b}{dt} = \frac{\rho_i L_f m}{L C_i h_b} \text{ if } w = 0,
\]

and either \( T_b = T_m \) and \( m < 0 \) or \( T_b < T_m \),

where \( C_i \) is the heat capacity of ice and \( h_b \) is the thickness of temperate basal ice available for cooling. Otherwise, basal temperature is at the melting point \( T_m \).

From the standpoint of our analysis, the relevant parameters are the accumulation rate \( a_c \) and the surface temperature \( T_s \), which enter the dynamics through Eqs. S1 and S5, respectively.
**Fig. S1.** Time series of (A) accumulation rate and (B) surface temperature from the GISP2 ice core. (C) Scatter plot of the two variables, and (D) the cross-correlation function. Note the fast decay of the cross-correlation.

**Fig. S2.** Ice thickness (Left) and velocity (Right) time series for three characteristic values of mean surface temperature (namely, from top to bottom, $T_s = -32, -28.5, -25 \degree C$), with constant accumulation rate $a_c = 0.23 \text{ m/yr}$. Stochastic time series are in red, and deterministic ones are in black (solid and dashed lines refer to the steady streaming and oscillatory deterministic solution, respectively). The time series of ice thickness are identical to those in Fig. 2, and are reported here for comparison. The stochastic velocity time series (Bottom Right) highlights that the stochastic forcing produces extended periods of stagnation with intervening bursts in velocity in correspondence to the steady streaming stochastic solution.
Fig. S3. Ice thickness (Left) and velocity (Right) time series for three characteristic values of mean accumulation rate (namely, from top to bottom, $\bar{a}_c = 0.24, 0.26, 0.28$ m/y), with constant surface temperature $T_s = -30.6^\circ$C. Stochastic time series are in red, and deterministic ones are in black (solid and dashed line refer to the steady streaming and oscillatory deterministic solution, respectively).

Fig. S4. Effect of cross-correlation between accumulation rate and surface temperature. (Left, from top to bottom) Time series of ice thickness with (i) stochastically forced $T_s (T_s = -29^\circ$C) and constant $a_c = 0.23$ m/y and (ii–iv) stochastically forced $T_s$ and $a_c (a_c = -0.23$ m/y), with cross-correlation coefficient $r_{at} = 0$, 0.4, and 0.9, respectively. (Right) The pdfs of the ice thickness for the same cases. Note that the pdf remains essentially unchanged.