Supraglacial channel inception: Modeling and processes

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Abstract Supraglacial drainage systems play a key role in glacial hydrology. Nevertheless, physical processes leading to spatial organization in supraglacial networks are still an open issue. In the present work we thus address from a quantitative point of view the question of what is the physics leading to widely observed patterns made up of evenly spaced channels. To this aim, we set up a novel mathematical model describing a condition antecedent channel formation, i.e., the down-glacier flow of a distributed meltwater film. We then perform a linear stability analysis to assess whether the ice-water interface undergoes a morphological instability compatible with observed patterns. The instability is detected, its features depending on glacier surface slope, ice friction factor, and water as well as ice thermal conditions. By contrast, in our model channel spacing is solely hydrodynamically driven and relies on the interplay between pressure perturbations, flow depth response, and Reynolds stresses. Geometrical features of the predicted pattern are quantitatively consistent with available field data. The hydrodynamic origin of supraglacial channel morphogenesis suggests that alluvial patterns might share the same physical controls.

1. Introduction

During the summer, supraglacial channelization is widespread in the ablation region of glaciers and ice sheets (Figures 1a and 1b). Channels develop along the steepest direction as a consequence of meltwater thermal erosion, conveying water toward moulins, lakes, and crevasses and eventually feeding englacial and subglacial drainage networks.

Supraglacial hydrology plays a key role in the glacial drainage system, as it establishes a direct connection among glacier surface, englacial and subglacial networks. However, limited knowledge is available about supraglacial channelization, and physical processes leading to spatial organization in supraglacial drainage systems are still an open issue. Field studies addressed, on the one hand, the relation between discharge, channel geometry, and incision rate (e.g., see the review by Irvine-Fynn et al. [2011]) and, on the other hand, channel meandering behavior [Knighton, 1972; Ferguson, 1973]. Regular spacing of supraglacial channels has been often observed [e.g., Marston, 1983; Knighton, 1981, 1985; Kostrewski and Zwolinski, 1995], but only one set of spacing data is currently available [Karlstrom et al., 2014]. Theoretical efforts include the stability analysis by Parker [1975], who demonstrated meandering channels to result from a combination of hydrodynamic and thermal processes, while Karlstrom et al. [2013] extended Parker’s work taking into consideration channel curvature effects.

From a wider perspective, recent work has pointed out that the feedback between enhanced surface melt and ice sheets acceleration primarily relies on short-term and spatially localized meltwater increase [Joughin et al., 2008; Schoof, 2010]. Knowledge of processes leading to spatial organization in supraglacial drainage systems might thus improve the understanding of surface meltwater-basal lubrication feedbacks and, possibly, of warming climate effects on glacier ice sheet mass loss.

Information about the structure of supraglacial networks may be relevant to characterize the englacial network as well. Although a variety of processes can potentially form englacial passages [Fountain and Walder, 1998], Gulley et al. [2009a, 2009b] showed field evidence that englacial networks on high-latitude polythermal and cold glaciers can originate from the supraglacial one by a “cut and closure” mechanism driven by creep deformation, and the same process has been reproduced numerically by Jarosch and Gudmundsson [2012].

Patterns with geometrical features analogous to those of supraglacial systems are known to occur in alluvial environments as a consequence of rainfall-induced erosion (Figure 1c). Erosional rills have been extensively...
studied in the framework of drainage basin formation [e.g., Smith and Bretherton, 1972; Montgomery and Dietrich, 1992; Izumi and Parker, 1995, 2000; Perron et al., 2008], and the question about the physical drivers of rill spacing has been posed since the seminal work by Smith and Bretherton [1972]. The simplest model of rill inception producing scale selection is the one by Izumi and Parker [1995], who proved via a stability analysis that a rainfall-induced, distributed overland flow down a flat and erodible hillslope tends to self-organize into a system of discrete channels. In their model rill spacing ensues from turbulent open-channel flow hydrodynamics, independently of the evolution equation for the erodible bed. In view of these findings, and also considering that differences between alluvial and glacial settings mostly concern the mechanism of erosion (sediment transport versus thermal erosion), we suggest that the analogy between supraglacial channelization and erosional rills can be drawn further, and that, at least in some settings, supraglacial channel spacing might be hydrodynamically driven as well.

The present study addresses supraglacial channelization with the objective of shedding light on physical processes underlying pattern formation. To this aim, we develop an idealized mathematical model for the distributed flow of meltwater over ice (sections 2 and 3), and investigate the resulting Stefan-like problem in the framework of linear stability (section 4). We demonstrate that even channel spacing in supraglacial drainage networks can be explained as a morphological instability, with free-surface flow hydrodynamics playing a major role (section 5). Efforts are devoted to unravel the physical processes underlying the instability and their interaction (section 6). The results of such an investigation can be possibly extended to the alluvial environment as well. We conclude our work by comparing our results to available literature data, which are found to support model predictions (section 7).

2. Physical Background

The net energy flux from the atmosphere to the surface of glaciers drives ablation, either in the form of melting or sublimation. When surface ice is at the melting point, as is the case for midlatitude glaciers and Greenland ice sheet ablation areas, ablation entails melting. Once formed, meltwater drains down-glacier along the steepest slope direction. The interaction between subaerial, gravity-driven meltwater flow and the ice surface is the subject of the present work.
Meltwater flow occurring in the ablation area is here thought of as a distributed water film flowing down a flat ice surface inclined at an angle \( \varphi \) with respect to the horizontal. In this respect, our approach is analogous to what previously done by Izumi and Parker [1995] for erosional rills. For simplicity the ice surface will be considered impermeable, as in the works by Arnold et al. [1998], Banwell et al. [2012], and Jarosch and Gudmundsson [2012]. Nevertheless, in some settings partially melt, weathered ice may allow a subsurface porous flow to exist [Irvine-Fynn et al., 2011; Karlstrom et al., 2014]. We will demonstrate in section 7 how a permeable wall can be accounted for, and we will also show that ice permeability does not affect the present analysis.

The flow regime is assumed to be turbulent, as a consequence of surface slope and small ice roughness (ice Manning coefficient is \( O(10^{-2}) \) m\(^{-1/3}\) s [Fountain and Walder, 1998]). A justification for this is provided by the following argument: consider a meltwater film originated by local surface melt only flowing over a flat ice surface with uniform, small-scale roughness, and assume a uniform and steady melt rate. A typical melt rate is not easily defined, as it depends on the local energy budget. This is in turn affected by altitude and meteorological variables and exhibits marked diurnal and seasonal cycles. However, a daily melt in the range 0.01–0.1 m/d \(^{-1}\) w.eq. can be taken as a rough estimate for a midlatitude glacier [Cuffey and Paterson, 2010, pp. 160–170]. Then, in typical conditions (\( \varphi = 10^\circ \), \( n = 0.01 \) m\(^{-1/3}\) ), critical Reynolds number \( Re_c = 2000 \) is already attained 200 m downstream of the beginning of bare ice, with a flow depth of 1.2 mm. The laminar region extent, if any, is overestimated by this computation, as the flow rate produced by snowpack melt upstream of the bare ice region has not been taken into consideration. Since the extent of the ablation region greatly exceeds the computed laminar region length, the latter will be disregarded.

A further motivation to investigate turbulent regime is provided by the work by Camporeale and Ridolfi [2012], who showed the morphological instability driven by a laminar film flow (i) to produce cross-flow wavelengths of the order of \( 10^{-1} \) m (see Figure 8 therein), not compatible with channelization data, and (ii) to be suppressed on slopes steeper than \( 10^\circ \). Conversely, supraglacial channels have been observed on steeper slopes as well [Knighton, 1981]. Their findings, together with the analogy to the alluvial setting, point in the direction of turbulence being key to pattern formation.

Some considerations on spatial scales allow for a simplification of the mathematical description. The characteristic length scale of the meltwater film is the flow depth \( D_0 \). Given snowpack meltwater contribution, the large amount of drag associated to the turbulent regime, and, possibly, curvature effects routing water toward topographic lows, a realistic estimate is \( D_0 = O(10^{-2}–10^{-3}) \) m. A second, intrinsic scale is the spacing between incipient adjacent channels, \( \lambda \). Field evidence of regular channel spacing is reported by Karlstrom et al. [2014], with \( \lambda \) in the range 1–10 m. Such a spacing is expected to remain constant until the amplitude of interface perturbations is small, i.e., while the linear theory developed in the following is valid. The small aspect ratio \( D_0/\lambda \) enables the shallow-water approximation to be adopted for the mathematical description of the liquid domain. Flow field and temperature in the film can be thus described by depth-averaged velocity and temperature. Large-scale topographic features, such as glacier curvature, and melt-induced discharge increase with downstream length will not be considered, as they occur on length scales much longer than the characteristic ones. Spatially uniform roughness, ice slope, and flow depth are accordingly assumed.

As a result, we consider a meltwater film spanning an infinite domain in the horizontal plane, wherein gravity acceleration is balanced by bed friction and a uniform, depth-integrated velocity is attained. As for the solid domain, the thermal structure of near-surface ice is to be described. Temperature distribution within the surface conductive layer is mostly determined by seasonal fluctuations of atmospheric temperature, and is thus inherently local. Two are the possible configurations: either temperate surface ice (i.e., temperature at melting point), as in the case of low-latitude entirely temperate glaciers, or cold surface ice (i.e., temperature below melting point), as observed in Antarctica cold glaciers, in perennial polythermal glaciers, and, seasonally, in temperate glaciers before the winter cold wave has faded out [Greve and Blatter, 2009; Cuffey and Paterson, 2010]. Since polythermal glaciers are widespread both at high [Blatter, 1987; Blatter and Kappenberger, 1988; Pettersson et al., 2003] and midlatitudes [Blatter and Haeberli, 1984; Gilbert et al., 2012; Ryser et al., 2013], and information about channelization inception over temperate ice can be obtained as a limit case of the cold one, the more general case of cold surface ice will be primarily addressed in the present study.

From a mathematical standpoint, the solid domain is modeled as an infinite half space. This is because the surface conductive layer’s thickness is of the order of tens of meters [Hutter, 1983; Cuffey and Paterson, 2010], 3 orders of magnitude larger than the characteristic length scale, \( D_0 \). The heat equation for ice is
solved therein requiring the temperature gradient to match surface layer’s one in the far-field and fixed, melting-point temperature at the ice-water interface.

Last, some considerations about temporal scales are required. The spatio-temporal dynamics of the ice-water interface is described by the so-called Stefan equation, which states the interface energy balance. More specifically, the rate of solidification/melting is set by the difference between the heat flux convectively transferred by meltwater to the wall and the heat flux diffusively removed by ice. The chosen characteristic time scale, \( \tau \), is the morphological one. i.e., the one controlling channel formation. Given that we are considering turbulent regime, this time scale is set by convective heat exchange between meltwater and the ice wall.

Computed values of \( \tau \) are typically of the order of few hours, so that selected waveforms are likely to be associated to the first significant stages of the ablation period. Once channelization is initiated, meltwater is routed preferentially toward troughs, and the linearly selected pattern is further deepened as a consequence of thermal erosion. Diurnal solar forcing is thus unlikely to affect wavelength selection at the inception of channelization, and will be disregarded in the following. The same is not true for long-term trends in ablation intensity or heavy rainfall events, which are instead expected to govern network remodeling on a several day time scale [Karlstrom et al., 2014]. However, modeling the temporal evolution of the drainage network is far beyond the scope of the present work, and the relevant controls will not be taken into account.

3. Model

We consider the idealized case of a water film flowing down an infinitely wide and infinitely long ice wall at the melting point. In the absence of corrugations, the ice surface is inclined at the angle \( \varphi \) with respect to the horizontal (Figure 2). Let us introduce the right-handed Cartesian reference frame, \( \mathbf{x} = \{x, y, z\} \), where the \( \hat{x} \) axis is tangent to the base plane and parallel to the direction of the maximum slope, while the \( \hat{z} \) axis is orthogonal to the base plane and points upward, with the tilde denoting dimensional variables. In the following, \( \hat{x} \) and \( \hat{y} \) directions will be referred to as streamwise and spanwise, respectively. The solid domain spans the half-space \( -\infty < \hat{z} < \tilde{\eta}, \tilde{\eta}(x, y, t) \) being the ice-water interface displacement. Henceforth, superscripts \( L \) and \( S \) mark the liquid and the solid phase, while subscripts \( F \) and \( I \) denote free surface and liquid-solid interface.

The model is scaled with the unperturbed uniform flow depth, \( \tilde{D}_0 \), the corresponding velocity, \( \tilde{U}_0 \), the difference between the bulk film temperature and the melting point, \( \tilde{\Delta} \), and a morphological time scale, \( \tau \), characterizing the spatiotemporal evolution of the ice-water interface. The reader is referred to Appendix A for a derivation of employed scales.

Dimensionless variables are defined as follows:

\[
(D, \eta) = \left( \frac{\hat{D}, \tilde{\eta}}{\tilde{D}_0} \right), \quad (U, V) = \left( \frac{\hat{U}, \tilde{V}}{\tilde{U}_0} \right), \quad \Theta^L(x, y, z, t) = \tilde{\eta}, \quad \Theta^S(x, y, t) = \tilde{T}_t \]

where \( D(x, y, t) \) is the flow depth, \( U(x, y, t) = \{U, V\} \) the depth-averaged flow field, \( \Theta^L(x, y, t) \) the depth-averaged liquid temperature, and \( \Theta^S(x, y, z, t) \) the solid temperature.

Froude, Nusselt, Stefan, and Stanton numbers are defined as

\[
Fr = \frac{\tilde{U}_0}{\sqrt{g \tilde{D}_0}}, \quad (1a)
\]
Nu = \frac{h_i D_0}{\kappa L}, \quad (1b)

Ste = \frac{\Lambda}{\kappa_0 \Delta}, \quad (1c)

St = \frac{h_i}{\rho_c q U_0}, \quad (1d)

where \( h_i = B U_0 \) is the heat transfer coefficient between a fully turbulent water stream and an ice wall, with \( B = 2.64 \times 10^3 \text{ J/m}^3 \text{ K} \) [Isenko et al., 2005, and references therein], while \( g \) is gravity acceleration, \( \kappa \) the thermal conductivity, \( \Lambda \) the latent heat of solidification, and \( \rho \) the density. From a physical standpoint, \( Nu \) compares convective to conductive heat transfer across the ice-water interface, \( Ste \) characterizes phase change through the ratio of latent to sensible heat, and \( St \) measures the importance of convective heat loss with respect to water heat capacity. Reynolds number, here written for open-channel flows, depends on \( Nu \) through the relationship \( Re = 4 \kappa (Bu)^{-1} Nu \).

Further nondimensional parameters featuring in the present problem are

\[ v = \frac{D_0}{\rho_0 U_0^2}, \quad (2a) \]

\[ G = \frac{\rho_0 D_0}{\Delta}, \quad (2b) \]

\[ r_n = \frac{h_F}{h_i}, \quad (2c) \]

\[ C_{f0} = \frac{g}{D_0^{1/2}}, \quad (2d) \]

where \( v \) is the ratio between convective and morphological time scales, and \( G \) is the nondimensional temperature gradient within ice. The ratio between heat transfer coefficients, \( r_n \), quantifies the importance of heat exchange between film and atmosphere (\( h_F \)) with respect to heat exchange between film and ice wall (\( h_i \)). Finally, since the flow is fully turbulent, Chezy’s parameterization is adopted to describe shear stresses at the ice surface, \( C_{f0} \) being the dimensionless friction factor, and \( n \) the Manning coefficient.

### 3.1. Governing Equations

Nondimensional mass, momentum, and heat conservation equations for the film flow are obtained by averaging the corresponding three-dimensional equations over the flow depth (\( \eta \leq z \leq \eta + D \)), with kinematic, no-slip, and impermeability boundary conditions set during the integration procedure by means of the Leibniz formula [Izumi and Parker, 1995; Liggett, 1994, p. 267]. They read

\[ \chi \frac{\partial D}{\partial t} + \nabla \cdot (\mathbf{UD}) = - \chi \frac{\rho_s}{\rho_0} \frac{\partial \eta}{\partial t}, \quad (3) \]

\[ \chi \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} = - \Gamma \nabla (\eta + D) - \frac{C_{f0} \mathbf{U} |\mathbf{U}|}{D} + \nabla \cdot (\eta \nabla \mathbf{U}) + \mathbf{F}, \quad (4) \]

\[ \chi \frac{\partial \Theta^i}{\partial t} + \mathbf{U} \cdot \nabla \Theta^i = \nabla \cdot (D_i \nabla \Theta^i) - St \frac{|\Theta^i|}{D} + r_n St \frac{(\Theta_{eq} - \Theta^i)}{D}, \quad (5) \]

where \( \nabla = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\} \); \( C_f = C_{f0} D^{-1/3} \); \( \Gamma = \cos \varphi / Fr^2 \); and \( \mathbf{F} = \{ \Gamma \tan \varphi, 0 \} \) (nondimensional variables are listed in Table 1). Dispersive terms accounting for the deviation of velocity and temperature profiles from the local integrated values have been neglected, as they are overall small for turbulent flows.

Mass conservation is stated in equation (3), where the source term on the r.h.s. accounts for the water influx due to melting at the interface. Equation (4) expresses momentum balance in the \( x \) and \( y \) directions, with the r.h.s. featuring bed and flow depth-induced hydrostatic pressure gradient, wall friction, Reynolds stresses, and gravity terms. Following Izumi [1993] and Izumi and Parker [1995], Reynolds stresses are modeled according to the
Table 1. Parameters and Nondimensional Variables

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi$</td>
<td>Glacier slope (°)</td>
<td>[5, 30]</td>
</tr>
<tr>
<td>$D_0$</td>
<td>Scale flow depth (m)</td>
<td>[0.003, 0.03]</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>Temperature difference</td>
<td>[0.01, 0.1]</td>
</tr>
<tr>
<td>$G$</td>
<td>Temperature gradient ice (K/m)</td>
<td>[0.2, 1.5]</td>
</tr>
<tr>
<td>$T_{aw}$</td>
<td>Air temperature (°C)</td>
<td>[3, 10]</td>
</tr>
<tr>
<td>$C_R$</td>
<td>Friction factor</td>
<td>[0.003, 0.007]</td>
</tr>
<tr>
<td>$Fr$</td>
<td>Froude number</td>
<td>[3, 12]</td>
</tr>
<tr>
<td>$Nu$</td>
<td>Nusselt number</td>
<td>[20, 900]</td>
</tr>
<tr>
<td>$Re$</td>
<td>Reynolds number</td>
<td>$[5 \times 10^3, 5 \times 10^5]$</td>
</tr>
<tr>
<td>$Ste$</td>
<td>Stefan number</td>
<td>$[1.5 \times 10^3, 1.5 \times 10^4]$</td>
</tr>
<tr>
<td>$St$</td>
<td>Stanton number</td>
<td>$6.4 \times 10^{-4}$</td>
</tr>
<tr>
<td>$G$</td>
<td>Nondim. temperature</td>
<td>[0.005, 5]</td>
</tr>
<tr>
<td>$\Theta_{eq}$</td>
<td>Equilibrium water temperature</td>
<td>[30, 900]</td>
</tr>
<tr>
<td>$\chi$</td>
<td>Time scales ratio</td>
<td>$[10^{-3}, 10^{-2}]$</td>
</tr>
<tr>
<td>$r_t$</td>
<td>Heat transfer coeff. ratio</td>
<td>$[10^{-3}, 10^{-2}]$</td>
</tr>
<tr>
<td>$D$</td>
<td>Flow depth</td>
<td></td>
</tr>
<tr>
<td>$\eta$</td>
<td>Ice surface</td>
<td></td>
</tr>
<tr>
<td>$\Theta$, $\Theta^S$</td>
<td>Liquid and solid temperature</td>
<td></td>
</tr>
</tbody>
</table>

Boussinesq assumption, with the dimensionless depth-averaged transversal eddy viscosity defined as $\nu_t = a_tC_{t/2}^2D(U)$.

Experimental results [Fischer, 1979] show that the dimensionless coefficient $a_t$ is in the range $[0.1, 0.3]$ for open-channel flows.

Heat conservation is stated in equation (5). Assuming the validity of the Reynolds analogy between momentum and heat transfer, the dimensionless eddy diffusivity, $D_t$, is taken equal to the momentum diffusion coefficient, $r_t$, i.e., the turbulent Prandtl number is $O(1)$ [Bejan, 2004]. The effect of molecular heat diffusion is negligible if compared to its turbulent counterpart, and will be ignored. The last two r.h.s. terms of equation (5) describe heat exchange with the ice wall and the atmosphere, respectively. The former accounts for convective heat exchange [Lock, 1990], while the latter for radiative, conductive, evaporative, and convective fluxes exchanged between the film and the atmosphere (for details, see Edinger et al. [1968]; Gulliver and Stefan [1986]; Gu and Li [2002]). The equilibrium water temperature, $\Theta_{eq}$, is defined as the water temperature at which the net rate of heat exchange between film and air would be zero [Edinger et al., 1968]. The parameterization for the dimensional equilibrium water temperature and the heat transfer coefficient between water and atmosphere is displayed in Appendix B.

As for the solid domain, the three-dimensional heat diffusion equation reads

$$r_t Nu \frac{\partial \Theta^S}{\partial t} - \nabla^2 \Theta^S - \frac{\partial^2 \Theta^S}{\partial z^2} = 0 \quad \text{for} \quad -\infty \leq z < \eta,$$

with boundary conditions

$$\Theta^S = 0 \quad \text{on} \quad z = \eta,$$

$$\frac{\partial \Theta^S}{\partial z} \to 0 \quad \text{as} \quad z \to -\infty.$$

$r_c = k_I / k_S$ being the ratio of thermal conductivities. Equation (7a) states that the ice surface is at the melting point, while equation (7b) requires the temperature gradient to match the one observed within the surface ice layer. If the ice-water interface is flat, such boundary condition produces a linear temperature profile with slope equal to $G$. In perturbed conditions, equation (7b) requires the gradient of perturbed temperature to fade out far from the interface, so that the unperturbed temperature gradient is recovered in the far field.

Finally, governing equations for the liquid and solid domains are coupled through the Stefan equation

$$\frac{\partial \eta}{\partial t} = \frac{1}{r_t Nu} \left\{ \nabla \Theta^S \cdot \frac{\partial \Theta^S}{\partial z} \right\} \cdot \mathbf{n}_t - \Theta^S |U| \quad \text{on} \quad z = \eta,$$

which describes the thermal energy balance of the ice-water interface [Worster, 2000]. The rate of melting or solidification (l.h.s.) depends on the difference between the diffusive heat flux removed by ice and the convective heat flux released from the film (r.h.s.), with $\mathbf{n}_t$ the unit vector normal to the interface and pointing upward.

Equations (3)–(8) can be partly simplified according to the order of magnitude analysis reported in Table 1. The independent physical parameters are surface slope, $\varphi$, flow depth, $D_0$, ice temperature gradient, $G$, and air...
The governing equations are then forced by a perturbation of the ice-water interface such that temperature $T_{air}$, whereas the Manning coefficient, $n$, is assumed as constant and equal to 0.01 m$^{-1/3}$ s [Fountain and Walker, 1998; Isenko et al., 2005] due to the scarcity of data. The order of magnitude of flow depth has been estimated in section 2, while the minimum value of slope is chosen so that turbulent conditions are ensured. $G$ has been determined according to measured temperature profiles; in this sense, relevant field data are reported, among the others, by Blatter and Haeberli [1984], Sobota [2009], Eisen et al. [2009], Gusmeroli et al. [2010], Gilbert et al. [2012], Du et al. [2013], and Ryser et al. [2013].

The characterization of heat transfer between meltwater and atmosphere (Appendix B) requires to set air temperature, and parameters related to atmospheric conditions like net solar radiation, relative humidity, water surface temperature, and wind speed. The limited availability of field data to compare our results with provides a criterion to constrain the parameter regime, along with the sensitivity of $h_v$ and $\Theta_{ai}$ to the above-mentioned atmospheric parameters. Our choice is to allow variations of air temperature, whereas the remaining parameters are kept constant and matching typical conditions on midlatitude alpine glaciers, for which field data are available [Karlstrom et al., 2014, and our field data presented in section 7]. The interval of $T_{air}$ values is reported in Table 1, with the lower bound chosen so that water temperature is above the freezing point, while parameters describing atmospheric conditions are chosen according to the selected setting and listed in Table 2.

We now consider the nondimensional parameters. Since $\chi \ll 1$ and $Ste \gg r_F Nu$, it follows that all the unsteady terms can be neglected except the temporal derivative featuring in the Stefan equation, which is $O(1)$ as a consequence of the chosen time scale. From a physical point of view, this means that the flow and temperature fields develop much faster than the ice-water interface, so that it is legitimate to assume that they instantly adjust to the slowly time-dependent configuration of the ice wall. This kind of “quasi steady approximation” has been numerically assessed and validated by Camporeale and Ridolfi [2012] for a laminar water flow over an inclined ice surface and adopted in a similar context also by Parker [1975] and Karlstrom et al. [2013]. A second remark concerns the Stefan equation (8), where the diffusive heat flux, $\propto Nu^{-1}$, plays a secondary role with respect to the convective flux, which is instead $O(1)$. Last, the Stanton number is a constant $\ll 1$, which underlines that heat fluxes through the liquid domain boundaries are negligible if compared to convective and turbulent heat fluxes within the film. However, fluxes across the boundaries are relevant at leading order and are therefore retained.

### 4. Linear Stability Analysis

The transformation of variables

$$\zeta = z - \eta(x, y, t),$$

is adopted in order to map the solid domain $(-\infty, \eta]$ into the rectangular domain $(-\infty, 0]$. The governing equations are then forced by a perturbation of the ice-water interface such that

$$\eta = \eta_0(t) + \eta'(x, y, t),$$

where $\eta_0(t)$ accounts for the lowering of the ice-water interface due to the excess heat flux released at the wall, and $\eta'$ is the imposed perturbation. Here we investigate the effect of small-amplitude harmonic perturbations about the slowly time-dependent configuration $\eta_0$ the following normal mode ansatz is adopted.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
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<tbody>
<tr>
<td>$g$</td>
<td>Gravitational constant (m/s$^2$)</td>
<td>9.81</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>Water density (kg/m$^3$)</td>
<td>999</td>
</tr>
<tr>
<td>$\rho_i$</td>
<td>Ice density (kg/m$^3$)</td>
<td>916.2</td>
</tr>
<tr>
<td>$k_s$</td>
<td>Water thermal conductivity (W/mK)</td>
<td>0.56</td>
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<tr>
<td>$k_i$</td>
<td>Ice thermal conductivity (W/mK)</td>
<td>2.1</td>
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<tr>
<td>$\gamma_s$</td>
<td>Water thermal diffusivity (m$^2$/s)</td>
<td>$1.34 \times 10^{-7}$</td>
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<tr>
<td>$\gamma_i$</td>
<td>Ice thermal diffusivity (m$^2$/s)</td>
<td>$1.1 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>Latent heat of solidification (kJ/kg)</td>
<td>334</td>
</tr>
<tr>
<td>$\alpha_s$</td>
<td>Eddy viscosity coefficient</td>
<td>0.2</td>
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<tr>
<td>$\nu$</td>
<td>Water kinematic viscosity (m$^2$/s)</td>
<td>$1.7 \times 10^{-6}$</td>
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<tr>
<td>$n$</td>
<td>Manning coefficient (s/m$^{1/3}$)</td>
<td>0.01</td>
</tr>
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**Heat Transfer Parameterization**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w$</td>
<td>Relative humidity</td>
<td>0.5</td>
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<tr>
<td>$R$</td>
<td>Net solar radiation (W/m$^2$)</td>
<td>143</td>
</tr>
<tr>
<td>$U_w$</td>
<td>Wind speed (m/s)</td>
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<tr>
<td>$T_{surf}$</td>
<td>Water surface temperature (°C)</td>
<td>0.1</td>
</tr>
</tbody>
</table>

*The value of $\alpha_s$ is provided by Izumi and Parker [1995], and net solar radiation by Hock [2005] (Vernagtferner Glacier, Austria, 2970 m a.s.l.).

Table 2. Physical Constants for Water and Ice at Standard Conditions and Parameters Required by the Heat Transfer Parameterization Described in Appendix B.
\[ \eta' = \varepsilon (e^{ix + i\gamma} \cos (fy) + \text{c.c.}) + O(\varepsilon^2) \quad \varepsilon \ll 1, \]  

where \( \varepsilon \) is the small amplitude of bed corrugations; \( \alpha \) and \( \beta \) are the real streamwise and spanwise wave numbers, respectively; \( \omega = \omega_r + i\omega_i \) is the complex temporal growth rate, with \( \omega_r \) determining whether the perturbation grows (\( > 0 \)) or decays (\( < 0 \)) in time, and \( \omega_i \) setting the propagation direction through the phase velocity \( c_p = -\omega_i / \alpha \). c.c. refers to the complex conjugate, and subscripts \( r \) and \( i \) denote real and imaginary part of any complex quantity hereafter.

The response of governing equations to the forcing (10-11) is found by posing the following expansion about the steady solution:

\[
\begin{pmatrix}
U \\ V \\ \Theta^i \\ \Theta^s(\zeta) \\ D
\end{pmatrix} = \begin{pmatrix}
1 \\ 0 \\ 1 \\ 1
\end{pmatrix} + \varepsilon \begin{pmatrix}
u \\ -i\nu \tan (fy) \\ \beta^2(\zeta) \\ \beta^2(\zeta)
\end{pmatrix} \cos (fy)e^{ix + i\alpha \delta + \text{c.c.}} + O(\varepsilon^2). \tag{12}
\]

The peculiar structure of transversal velocity perturbation is chosen in order to simplify computations, thus allowing the complex conjugate equations to be disregarded.

By introducing the expansions (10)–(12) into equations (3)–(8) and linearizing about the steady state, at leading order we obtain

\[
F_r^2 = \frac{\sin (\sigma)}{C_{f0}}, \tag{13a}
\]

\[
\Theta_{eq} = 1 + \frac{1}{r_h}, \tag{13b}
\]

\[
\Theta_{eq}^s = C_{eq} \approx \zeta < 0, \tag{14a}
\]

\[
\frac{d\eta_0}{dt} = \frac{G}{r_cNu} - 1. \tag{14b}
\]

Integration of equation (14b) yields a uniform lowering of the ice-water interface

\[
\eta_0(t) = \left( \frac{G}{r_cNu} - 1 \right) t, \tag{15}
\]

due to the mismatch between diffusive heat flux toward ice (first term in brackets) and convective heat flux from meltwater (second term in brackets). The leading order control on melting of ice is therefore the local energy balance, where atmospheric conditions feature indirectly through meltwater temperature. Notice that the resulting meltwater production does not affect film hydrodynamics, as it occurs on a time scale much longer than the convective one.

Under the quasi steady approximation, at \( O(\varepsilon) \) the governing equations for the liquid domain (3)–(5) are transformed into the following algebraic system:

\[
x(\alpha + \beta \nu) - \beta \nu = 0, \tag{16}
\]

\[
2C_{f0} + i\alpha + a_x \sqrt{C_{f0} k^2} u - \frac{4}{3} C_{f0} d + i\sigma(1 + d) = 0, \tag{17}
\]

\[
C_{f0} + i\alpha + a_x \sqrt{C_{f0} k^2} \nu - i\nu(1 + d) = 0, \tag{18}
\]

\[
St(1 + r_h) + i\alpha + a_x \sqrt{C_{f0} k^2} \beta^2 + St \beta u = 0, \tag{19}
\]

\( k = (x^2 + \beta^2)^{1/2} \) being the modulus of the wave vector. The linearized heat diffusion equation for ice reads
with boundary conditions

\[
\frac{d\theta^\delta}{dz} = 0 \quad \text{on } \zeta = 0, \tag{21a}
\]

\[
\frac{d\theta^\delta}{dz} \to 0 \quad \text{as } \zeta \to -\infty, \tag{21b}
\]

which ensure the melting point to be attained at the ice-water interface, and bed-induced perturbations of the base state temperature gradient to vanish in the far field. Integration of equation (20) with boundary conditions (21) yields

\[
\theta^\delta = G(1 - \theta^\delta) \quad -\infty < \zeta < 0, \tag{22}
\]

while the unknowns \( u, v, d, \) and \( \theta^\delta \) are obtained analytically from equations (16)–(19). Their solution is displayed in Appendix C.

Finally, the linearized Stefan equation

\[
\omega = -\left(\theta^\delta + u\right) + \frac{1}{r_x Nu} \frac{d\theta^\delta}{dz} \bigg|_{\zeta=0} = -(\theta^\delta + u) - \frac{kG}{r_x Nu} \tag{23}
\]

serves as dispersion relation and allows the temporal growth rate \( \omega \) to be computed analytically.

The key processes driving the instability are embodied in equation (23) and their investigation will be the primary focus of the next two sections. Here we only recall that the terms on the r.h.s. of equation (23) represent the perturbation of the convective heat flux released from water to the ice wall \( (q^L = \theta^\delta + u) \) and the perturbation of the conductive heat flux absorbed from ice \( (q^S = -kG/r_x Nu) \), respectively. Convective flux perturbation represents the major source of complexity in the dispersion relation, as it encompasses the effects of the whole set of hydrodynamic variables through the coupling of \( u \) to \( v \) and \( d \) via equation (16)–(18).

5. Channel Formation

Results are presented for the case of ice and pure water at standard conditions, and values of employed constants are listed in Table 2. Control parameters entering the dispersion relation (23) are the surface slope \( \varphi \), the friction factor \( C_f \), and the thermal parameters \( G \) and \( r_x \). Notice that \( Nu \) is inversely proportional to \( C_f \) through the relationship \( Nu = C_f^{-\frac{5}{2}} \sin \theta \sinh^2 \beta \eta_2 / \eta_0 \).

The distinctive features of the instability are depicted in Figure 3, where a typical solution of the dispersion relation is plotted in the \( \alpha - \beta \) plane. The unstable domain is made up of two separated regions labeled with C and R, respectively. The maximum growth rate \( \nu_{\alpha,\beta,\text{max}} \), denoting the most amplified waveform, always lies in region C, which will be the focus of our study. \( \nu_{\alpha,\beta,\text{max}} \) is localized in correspondence of nonzero \( \alpha \) and \( \beta \), meaning that the selected waveform is three-dimensional, with along-flow wavelength about one order of magnitude longer than the cross-flow one. As a
result, the predicted pattern features evenly spaced channels superimposed on a long-wavelength bed undulation transversal to the flow.

The less unstable R-region is instead associated to a streamwise instability, and the corresponding pattern features bed undulations perpendicular to the base flow. This follows from the observation that the maximum growth rate relative to the R-region lies on the \( \alpha \) axis in the majority of parametric conditions. Since our major interest is channelization, which is also the most unstable pattern, the analysis of the R-instability is not developed further. However, investigations not reported here have shown that the R-instability migrates upstream, is mostly driven by free-surface dynamics and exhibits centimeter-scale typical wavelengths. This suggests the R-instability to be the turbulent counterpart of laminar ice-ripples investigated by Camporeale and Ridolfi [2012].

5.1. Processes

In this section we provide a phenomenological picture of processes leading to channel formation. To this aim, the spatial behavior of relevant variables in correspondence of \( x = x_{\text{max}} \) is shown in Figure 4, while a conceptual scheme of the processes that are to be illustrated is presented in Figure 5. We start our analysis from the solid domain, represented by the conductive heat flux \( q_S \), which exhibits a maximum in correspondence of troughs. This behavior can be explained (left part of Figure 5) considering that a perturbation of the ice-water interface ends up in a compression (stretching) of the unperturbed temperature profile below troughs (crests), which in turn results in an increase (decrease) of the vertical temperature gradient. The spatial structure of the perturbed interface therefore drives a lateral diffusive heat flux from troughs to crests which tends to restore a flat interface, and is thus stabilizing.

Concerning the film flow, a first observation ensuing from Figure 4 is that the convective heat flux \( q_L \) almost coincides with the perturbed velocity \( u \). We hence infer that \( \Theta \) plays a marginal role in the determination of the maximum growth rate and focus on the effect of bed perturbations on film hydrodynamics only (right part of Figure 5, thick arrows and solid boxes). We will further discuss the role of \( \Theta \) in section 6. Given that the vertical pressure distribution under the shallow-water approximation is hydrostatic, the direct effect of a bed perturbation \( g(x, y, t) \) is to alter the base state pressure field through the term \( Q \) in equation (4). The system responds to this forcing via the whole set of hydrodynamic variables and attains a perturbed configuration resulting from the \( O(\varepsilon) \) balance of mass and momentum conservation equations.

The core of the perturbed dynamics is produced by the flow depth response to bed perturbations, which on the one hand counteracts bed-induced pressure gradients, and on the other hand forces \( u \) through a perturbation of wall shear stress. In fact, Figure 4 shows that \( d \) is \( 180^\circ \) out-of-phase with respect to the bed, so that \( \nabla g \sim -\nabla D \) and the pressure gradient tends to vanish. Additionally, the perturbed equilibrium of \( U \) is such to offset the flow depth-induced perturbation of wall friction. This is achieved with the velocity perturbation \( u \) displayed in Figure 4, which exhibits a maximum in correspondence of troughs and, hence, fosters the instability by enhancing \( q_L \).

The above-presented feedbacks constitute the backbone of the instability dynamics. Indeed, they account for the existence of the instability and provide the conceptual framework to investigate the effect of control parameter variations (see section 5.2). Nevertheless, when wavelength selection is under investigation, further processes must be accounted for, which are enclosed within dashed boxes in Figure 5. To this concern, it’s worth mentioning that the small mismatch between bed and flow depth perturbation amplitudes, and

![Figure 4. Spatial behavior of bed perturbation (bottom box), and perturbed variables (top box) \( d \) (dashed), \( u \) (dashed-dotted), \( q_L \) (solid, thick), \( v \) (dotted, 10X) and \( q_S \) (solid thin, 10X) in the y direction, in correspondence of the maximum growth rate of Figure 3 (\( \alpha = 0.0023, \beta = 0.0575 \)). Parameters as in Figure 3.](image-url)
the nonzero transversal velocity $v$ observed in Figure 4 are the signature of downstream advection. Its effects, along with the competition with Reynolds stresses, will be explored from a conceptual and mathematical point of view in section 6.

5.2. Parameter Sensitivity

We now address the effect of control parameters on the instability. The role of hydrodynamic parameters, $C_{f0}$ and $\varphi$, is investigated first, with our analysis starting from marginal stability curves, i.e., the growth rate isoline with $\omega_r = 0$, which are reported in Figure 6 as a function of $C_{f0}$ and $\varphi$. The instability is observed in the full range of parametric conditions, although with modifications of both the extent and the position of the instability region. Variations of the friction factor mostly affect the band of unstable transversal wave numbers, with its width inversely proportional to $C_{f0}$. Conversely, the position of the maximum growth rate in the $\alpha$-$\beta$ plane remains almost unaltered. On the other hand, changes in surface slope produce purely a shift of the unstable region and of the maximum growth rate.

The behavior of the maximum growth rate is depicted in Figure 6c, whence $\omega_{r,max}$ results to be inversely proportional to the slope, while exhibiting a nonmonotonic behavior with respect to $C_{f0}$. Such a scenario can be accounted for by considering how these parameters alter the fundamental dynamics described in section 5.1. Variations of the slope angle affect the ingredient mostly relevant to flow field perturbations, i.e., the bed-induced pressure gradient. In fact, such pressure gradient (represented by the last term on the r.h.s of equation (4)) is modulated by the hydrostatic pressure distribution at the base state through the coefficient $\Gamma = C_{f0}\cot\varphi$, which is a decreasing function of $\varphi$. Therefore, increments in slope reduce the magnitude of pressure perturbation and, consequently, the amplitude of $u$. The ice-water interface is thus supplied with less convective heating, which results in a reduction of the growth rate.

Figure 5. Conceptual illustration of the physical processes controlling the onset of channelization. The diamond block marks the starting point. Rectangular blocks enclose processes controlling the instability; those occurring in the liquid are bordered in blue (on the right of the start), whereas gray rectangular blocks (on the left) refer to ice. The fundamental dynamics described in section 5 is labeled by thick blocks, whereas thin blocks are relevant to wavelength selection only. Oval blocks in the lower part of the schematic assess the stabilizing/destabilizing effect of each process. Yellow-bounded ovals mark the two competing processes ultimately responsible for the fate of bed perturbations (section 5.1), whereas competition between the processes enclosed in the dashed box controls wavelength selection (section 6).
As for the friction factor, the key point is that shear stresses at the ice-water interface are proportional to $C_f^0$. Two main consequences thus follow from an increase of the friction factor: on the one hand $Nu$ is reduced, hence producing a reduction of the base state convective heat flux. This yields an increase in $q_S$ via the ratio between unperturbed heat fluxes, $G = \frac{r_k Nu}{C_5^0}$ (see equation (23)), and is thus stabilizing. On the other hand, the first-order $x$ momentum balance is supplied with a stronger wall friction forcing, which requires $u$ to grow in order to be balanced. This can be easily proved in the case of purely spanwise perturbations (i.e., $\alpha = 0$), where equations (16)–(18) yield

$$ u = \frac{q^2}{4/3 + a_1 C_1^{1/2} \beta^2}, $$

whence $|u|$ is demonstrated to depend on $C_1^{1/2}$. As a consequence, the first-order convective heat exchange is promoted and the instability favored.

The interplay between the two above presented feedbacks is responsible for the nonmonotonic behavior of $\chi_{r,\text{max}}$, with the destabilizing one ($\propto C_1^{1/2}$) dominant at small values of the friction factor, and the stabilizing one ($\propto C_5^0$) dominant in the upper limit. Furthermore, the stabilizing feedback also explains the reduction of the instability region with increasing friction factor, revealing that the wave number cutoff is primarily influenced by $q^2$.

From a physical perspective, the extensive range of unstable parametric conditions accounts for supraglacial channelization being widespread. Moreover, the model predicts that low slope angles are the most prone to develop the instability, whereas intermediate values of the flow depth (friction factor) are those expected to shape the network at the onset.

Moving to thermal parameters, Figure 7 shows marginal stability curves as a function of $G$ (Figure 7a) and $r_b$ (Figure 7b), where $G$ is related to ice thermal structure and $r_b$ to the base state thermal conditions of the meltwater film (equation (13b)). The system turns out to have a certain sensitivity to variations of $G$, whereas it exhibits an extremely robust behavior with respect to variations of $r_b$. This latter point is relevant in two respects: on the one hand, it ensures that meltwater thermal conditions, which are a major source of uncertainty when comparing model predictions to field observations, do not affect wavelength selection. On the other hand, it confirms that neglecting the time-dependent nature of solar radiation is not a limiting

**Figure 6.** Effects of variations in the hydrodynamic parameters on the instability. (a, b) Marginal stability curves ($\alpha_j = 0$) in the $\alpha - \beta$ plane for different values of the friction factor $C_f^0$ (Figure 6a, constant $\varphi = 10^7$) and of the slope angle $\varphi$ (Figure 6b, constant $C_f^0 = 0.005$), with $C_f^0 \in [0.003, 0.007]$ and $\varphi \in [5, 30^\circ]$. Curves are evenly spaced, with blue (min) to red (max) color scale. For each case, the maximum growth rate is labeled with a cross. (c) Maximum growth rate against $\varphi$ for different values of $C_f^0$. Color scale as in Figure 6a. $G$ and $r_b$ are kept constant in all figures, and equal to 1 and 0.005, respectively.
assumption when the onset of channelization is considered, as it has no effect on channel spacing in the linear framework.

As for ice thermal structure, changes in $G$ modify both the extent of the unstable region and the maximum growth rate, in a manner similar to variations of $C_f$. The position of the maximum growth rate is unchanged alike. This follows from the fact that $q_S/G$, which means that a steeper base state temperature distribution enhances the perturbed diffusive heat flux and damps the instability. Finally, we remark that taking into consideration the vertical temperature profile within ice is fundamental to obtain a finite-width unstable wave numbers band and separate instabilities of type $C$ from those of type $R$.

6. Wavelength Selection

In order to shed light on the physics underlying the spatial structure of supraglacial drainage networks, the behavior of perturbed variables is now investigated as a function of $\beta$ (Figure 8) for three different cases: $\alpha = 0$, $\alpha = \alpha_{\text{max}}$, and $\alpha > \alpha_{\text{max}}$, where $\alpha_{\text{max}}$ corresponds to the maximum growth rate of Figure 3. In the top row of Figure 8 the real part of variables involved in the dispersion relation is displayed, while modulus and argument of complex variables are plotted in middle and bottom rows.

Starting from the top row of Figure 8, we note that the maximum of the growth rate always originates from a minimum in $q_L$, whereas the stabilizing action of $q_S$ becomes effective only in the limit of large $b$. The increase of $|q|\beta$ with wave number is the signature of a diffusive process; indeed, smaller wavelengths produce steeper lateral temperature gradients, thus enhancing heat diffusion from troughs to crests ($q_S/k$). This mechanism is responsible for the cutoff in the band of unstable wave numbers, with its magnitude modulated by the ratio of leading order heat fluxes.

In order to account for wavelength selection, the behavior of $q^2$ has to be analyzed. We first focus on the paradigmatic case of purely spanwise perturbations ($\alpha = 0$), where mass conservation (16) reduces to $\nu = 0$, which forces the spanwise momentum conservation (18) to yield $d = -1$. This means that the bed-induced $y$ pressure gradient $i\beta I$ is fully balanced by flow depth response independently of $\beta$. As for the $x$ direction, the resulting perturbation of wall friction becomes the forcing term in momentum conservation (17), reading now...
Since $d$ is real and negative, $u$ features a 180° phase shift with respect to the bed (Figure 8c), meaning that it is maximum in correspondence of troughs (as in Figure 4), thus playing a destabilizing role. Conversely, the magnitude of flow field response strongly depends on $b$, with two asymptotic regimes clearly identified in Figure 8b. For very long wavelengths ($\beta \to 0$), the forcing term of equation (25) is balanced by the first l.h.s. term; accordingly, the perturbation of wall friction vanishes and $u$ attains the constant value $-2/3$.

\begin{equation}
(2C_{00} + a_1 \sqrt{C_{00} \beta^2}) u = \frac{4}{3} C_{00} d.
\end{equation}

**Figure 8.** Behavior of perturbed variables against $\beta$ for (a–c) $\alpha = 0$, (d–f) $\alpha = \alpha_{\text{max}} = 0.0023$, and (g–i) $\alpha = 0.005$, where $\alpha_{\text{max}}$ corresponds to the maximum growth rate in the $\alpha - \beta$ plane. Parameters as in Figure 3. First row: growth rate and real part of variables involved in the dispersion relation. Second and third rows: absolute value and argument of complex variables.
Notice that, despite the absence of a streamwise perturbation, this scenario corresponds to the fundamental dynamics presented in section 5, and produces the strongest response of $u$. By contrast, at short wavelengths ($\beta \to \infty$) the Reynolds stress term is dominant, and $|u|$ tends to zero as $\beta^{-2}$. The transition between these two regimes is expected when the flow field-induced perturbation of the wall shear stress in equation (25) balances the Reynolds stress term, with the solution given by equation (24).

Moving to the temperature field, heat conservation yields

$$\frac{\partial \theta}{\partial t} = -\frac{St \ u}{St(1+n_s)+a_tC_{10}^2 \beta^2}, \quad (26)$$

showing that $\partial \theta$ is locked to $u$ due to the structure of the convective heat flux term. On the one hand, this results in a similarity in the shape of the amplitude response, even though for large $\beta$ $|\partial \theta| \sim \beta^{-4}$ as a consequence of the combined effect of turbulent heat diffusion and flow field forcing. On the other hand, heat conservation forces a $180^\circ$ phase shift between $u$ and $\theta$. This means that the temperature perturbation cools the troughs, thus counteracting the destabilizing effect of $u$. The competition between $u$ and $\theta$ results in a minimum of the convective heat flux $q_u$, and consequently in a maximum of the growth rate (Figure 8a).

Despite the above presented dynamics providing a wavelength selection mechanism, Figure 8 proves that largest growth rates are obtained when a streamwise perturbation is also allowed, and that an optimum value of $\alpha$ exists. We will now describe the processes leading to such a behavior, primarily referring to the case of $x = x_{\text{max}}$ (Figures 8d–8f). The origin of the optimum $x$ value will be elucidated by comparison with the case where $x > x_{\text{max}}$ (Figures 8g–8i).

Inspection of Figures 8d–8f suggests the existence of three different regimes, clearly emerging in the behavior of $q_u$: indeed amplitude exhibits a strong response in the region of intermediate $\beta$, while it vanishes at lower and upper $\beta$ bounds, with the transitions corresponding to sharp jumps in the argument. Such a behavior is to be ascribed to different perturbed stress balances in the film, resulting from the initial perturbation of the ice-water interface. We will now describe them in more detail, as the transition between these regimes is at the origin of wavelength selection.

1. Regime (I). If $\beta \ll x$ the continuity equation reduces to $u = -d$, while the $x$ momentum balance reads

$$\left[\frac{10}{3} C_{10} + i\alpha(1-\Gamma) + a_t C_{10}^2\right] u = -ix\Gamma. \quad (27)$$

Since the whole unstable region is located in correspondence of $x \ll 1$, advection and Reynolds stresses in equation (25) are negligible with respect to wall friction. Therefore, we have that the bed-induced pressure perturbation on the r.h.s. is balanced by the wall shear stress perturbation (first term in brackets), yielding

$$u \approx -\frac{3}{10} i\alpha \cot (\varphi). \quad (28)$$

With a similar argument, from the balance between streamwise heat advection and convective heat loss we obtain

$$\partial \theta \approx \frac{3}{10} St \cot (\varphi). \quad (29)$$

while $y$ momentum conservation yields

$$\nu \approx i\beta \cot (\varphi)(1+d) = i\beta \cot (\varphi) \left[ \frac{2}{10} \alpha \cot (\varphi) + i \right]. \quad (30)$$

Therefore we understand that, if $x \neq 0$, streamwise advection prevents the flow depth perturbation from cancelling out the bed-induced lateral gradient of pressure $(i/\beta)$. The $y$ component of the velocity is thus activated, and the corresponding component of wall friction enters the $y$ momentum balance.

The inset of Figure 8e allows the validity of the above-presented approximation to be assessed, the asymptotic solutions from equations (28) and (29) being $u = -0.0039i$ and $\partial \theta = 0.0011$; notice that phase shifts (Figure 8f) are equally consistent. From a physical point of view, we have demonstrated that, if $\beta \ll x$, $u$ is responsible for bed perturbation migration, but does not contribute to the growth rate. Conversely, $\partial \theta$ exerts...
The optimum exceedingly shift convective heating from the destabilizing effect. In fact, the main difference between the cases produces a peak in the growth rate; regime (III) is dominated by transversal negative growth rate; regime (II), established when \( q^2 \) jumps from positive to negative values, thus resulting in a strong destabilizing effect. As for \( \theta^2 \), locking to \( u \) is now evident both in magnitude and argument, with the \( 180^\circ \) phase lag that allowed \( \theta^2 \) to counteract the effect of \( u \) here strongly reduced as a consequence of streamwise advection.

2. Regime (II). As shown by equation (30), if \( \beta \ll \alpha \) then \( v \) is linear in \( \beta \), so that the second term of mass conservation (16) increases in magnitude proportionally to the wave number. When \( \beta \sim \alpha \), the two terms of equation (16) are equally important, and the former dominant balance is broken. This results in an abrupt response of \( d \) (Figure 8e), which on the one hand contributes to balance the increased \( y \) pressure gradient and, on the other hand, perturbs the streamwise component of wall shear stress, triggering the fundamental dynamics described in Figure 5. The fact that \( \alpha = 0 \) is the signature of this process. Also, the argument of \( u \) jumps to negative values, thus resulting in a strong destabilizing effect. As for \( \theta^2 \), locking to \( u \) is now evident.

3. Regime (III). Finally, if \( \beta \gg \alpha \) the second term of mass conservation (16) is dominant and the \( x = 0 \) scenario is recovered. In fact, \( v \) is forced to vanish by mass conservation. Accordingly, \( d \) must offset the streamwise pressure gradient, thus attaining the asymptotic value \(-1\). This entails a jump in the argument of \( d \), \( u \), and \( v \), which achieve the same configuration (either in phase or \( 180^\circ \) out of phase) as the \( x = 0 \) case.

To sum up (see Figure 5), regular channel spacing originates from the competition among three asymptotic regimes: regime (I), observed when \( \beta \ll \alpha \), is dominated by a weak \( x \) pressure forcing resulting in a slightly negative growth rate; regime (II), established when \( \beta \sim \alpha \), corresponds to the fundamental dynamics described in section 5, and produces a peak in the growth rate; regime (III) is dominated by transversal Reynolds stresses and provides a stabilizing mechanism in the limit of large \( \beta \).

Therefore, a three-dimensional bed perturbation has manifold effects: first of all it allows regime (I) to exist and produces a peak in \( u \). Additionally, streamwise advection is activated with strength proportional to \( \alpha \). The optimum \( \alpha \) must thus be sufficiently large to allow regime (I) to exist, but sufficiently small not to exceedingly shift convective heating from the \( 180^\circ \) phase configuration, which produces the strongest destabilizing effect. In fact, the main difference between the cases \( \alpha = \alpha_{\text{max}} \) and \( \alpha > \alpha_{\text{max}} \) (Figures 8g–8i) is in the argument of convective heating in correspondence of the maximum growth rate, which jumps from \(-170^\circ \) for the former to \(-145^\circ \) for the latter. Finally, since momentum and heat are advected differently, a 3-D bed perturbation produces a phase lag between \( u \) and \( \theta^2 \) that prevents the latter from counteracting the destabilizing effect of flow field perturbations, as is the case when \( x = 0 \). We can thus conclude that wavelength selection is controlled by the hydrodynamics. Therefore, no difference is expected to occur when \( q^2 \) is suppressed, as in the case of temperate glaciers.

7. Relevance to Field Observations

The ultimate outcome of our analysis is channel spacing, which is displayed in Figure 9 as a function of \( \phi \), \( C_{00} \), and \( r_n \). The dependence on \( G \) does not need to be assessed as \( q^2 \) turned out to be irrelevant to wavelength selection. A few comments are worthwhile: first, spacing is inversely proportional to both slope and friction coefficient. The dependency on slope stems from the reduction of the bed-induced cross-flow pressure gradient \((\propto \cot \phi)\) as the slope grows. Therefore, larger \( \beta \) (smaller \( \lambda \)) are required for the a weak stabilizing effect, so that \(|q^2|\) is small and the growth rate negative.
transversal velocity to overcome the stabilizing effect of the streamwise pressure gradient and enter the asymptotic regime where the growth rate peaks. Variations of $C_Q$ affect instead the $\beta \gg \alpha$ regime. At constant slope, the larger is the friction coefficient, the stronger is the response of the flow field to bed perturbations (equation (24)). Therefore, larger $\beta$ are required for the lateral Reynolds stress to damp the instability. As for $r_h$, it turns out to be irrelevant to wavelength selection, as testified by the proximity of dashed (upper $r_h$ bound) and solid (lower $r_h$ bound) curves in Figure 9. Finally, selected wavelength exhibits a rather robust behavior with respect to parameter variations, with $\lambda$ lying in the interval 0.5–6 m.

Our analysis identifies ice surface slope, meltwater depth and small-scale ice roughness as the primary controls on the spatial structure of supraglacial drainage networks. While surface slope and meltwater depth can be obtained experimentally, high uncertainty surrounds ice roughness, and $n = 0.01 \text{ s m}^{-1/3}$ is to be considered only an estimate of the Manning coefficient. However, in the theoretical framework here developed variations of $n$ are enclosed in the friction coefficient, and in general we expect channel spacing to be inversely proportional to ice roughness.

Regularly spaced supraglacial channels are often observed in the field [e.g., Knighton, 1981, 1985; Kostrzewski and Zwolinski, 1995; Irvine-Fynn et al., 2011]. However, the only available spacing data are those provided by Karlstrom et al. [2014], who present a comprehensive data set characterizing a drainage basin on Llewellyn Glacier, British Columbia. Even though this site is characterized by permeable surface ice, they observe channel inception to occur "when melt production exceeds transport capacity (of surface ice) and ensuing surface flow locally enhances thermal erosion", which appears to be the same process as the one here described and modeled. Also, they report close and regular channel spacing in the upper part of the drainage network, with wavelength spanning the interval 1–10 m.

Even though it has been initially neglected in order to keep the mathematical formulation as simple as possible, the situation reported by Karlstrom et al. [2014] can be accounted for in our formulation. In fact, if melting is sufficient to saturate the subsurface, a meltwater film forms above the porous ice surface. The only modification to the model concerns the Stefan equation, whose l.h.s. should be multiplied by a coefficient accounting for the jump of water mass fraction across the interface between water and partially melt ice, as proposed by Hutter [1983, p. 46]. This means that the morphological time scale is reduced proportionally to the substrate water content, but no effect is exerted on wavelength selection. Then the conclusions drawn above should remain valid also in the case of permeable surface ice saturated by meltwater. It thus follows that the structure measured by Karlstrom et al. can be interpreted as the signature of the here presented linear inception mechanism, thus suggesting that our model is able to properly capture the order of magnitude of channel spacing.

Finally, we performed measurements of supraglacial channel spacing at two sites in the Italian Alps, Ciardoney Glacier (Orco catchment, 2900 m a.s.l.) and Indren Glacier (Lys catchment, 3200 m a.s.l.). Measurements have been taken in the upper part of the ablation region, with both sites featuring fairly impermeable bare ice. Data were collected on 6 September 2006 and 6 September 2013, respectively, and average channel spacing of 3 m (Ciardoney Glacier, slope 10°, Figure 2a) and 1 m (Indren Glacier, slope 20°) were recorded. These data are entirely compatible with model predictions.

As for the along-flow direction, the model predicts streamwise wavelengths of the order of 5–50 m, depending on the friction factor only. To this concern, situations are often reported in literature [Knighton, 1985; Irvine-Fynn et al., 2011] where supraglacial channels exhibit a step-and-pool configuration with comparable wavelength. The same configuration is also observed in the englacial network [Gulley et al., 2009a], which is known to be potentially originated by cut-and-closure of subaerial channels. Depending on ice thickness, wavelengths of the order of tens of meters might be set by subglacial topography. However, despite all the uncertainties related to a linear analysis, our results suggest that a stepped bed profile could also result from a morphological instability. The precise circumstances under which such pattern might develop certainly deserve further investigations.

8. Conclusions

The present study has addressed the inception of supraglacial channelization with a novel approach, which allowed us to provide a possible explanation for the occurrence of regularly spaced channels in
supraglacial drainage networks. We have tackled from a mathematical perspective the idealized problem of a turbulent meltwater film flow down a flat, tilted ice surface, and demonstrated that the ice-water interface is unstable to small perturbations. The resulting morphological instability has a thermal origin, and exists only if the heat flux released from the meltwater film exceeds conduction toward ice. This occurs in a wide range of glaciologically relevant parametric conditions, with the features of the instability depending on slope, friction coefficient, and ice as well as water thermal conditions.

A three-dimensional pattern compatible with supraglacial channelization is always obtained, ensuing from the competition between three asymptotic regimes of water mass conservation. The key to channel spacing is the interplay among lateral pressure gradients, flow depth response, wall shear stress, and Reynolds stresses, whereas ice thermal structure appears to be irrelevant to wavelength selection. Owing to the mathematical similarity with the problem of rill patterning, this physical insight integrates the seminal work by Izumi and Parker [1995] for the alluvial environment.

Model results compare well with field observations of channel spacing. However, further field work aimed at assessing the dependence of channel spacing on glacier slope would certainly be useful to perform a full validation of our model.

Any attempt to relax the simplifications our model is based on would be an improvement to the present theory. One central issue is the effect of unsteadiness in the flow rate driven by solar forcing, which might affect the nonlinear evolution of the network. Further issues concern the presence of sediment. In fact, it is well known that supraglacial streams can sometimes carry sediment load that is particularly prone to absorb solar radiation because of its low albedo. Sediment certainly affects the heat balance of the stream; therefore, it should be considered in order to precisely compute temporal growth rates. However, we do not expect it to alter wavelength selection, as temperature perturbation has been shown to play a marginal role. The interplay between sediment transport and hydrodynamics should be considered as well, as it might cause localized sediment deposition and ice melting, thus altering the evolution of the ice-water interface.

**Appendix A: Scaling**

The dimensional base state solution is obtained in steady conditions with a flat ice-water interface. Governing equations thus reduce to

\[-C_0 \frac{\dot{U}_0^2}{D_0} + g \sin \varphi = 0, \quad (A1)\]

\[\beta \dot{U}_0(T_1 - \dot{T}_0) + h_f(T_{eq} - \dot{T}_0) = 0, \quad (A2)\]

yielding

\[\dot{U}_0 = \frac{1}{n} \sqrt{\frac{g}{\rho D_0}} \dot{T}^{2/3}, \quad (A3)\]

\[\dot{T}_0 = \frac{T_{eq} + h_f T_{eq}}{1 + h_f}, \quad (A4)\]

where \(\dot{U}_0\) is the reference scale for velocity, while \(\dot{T}_0\) is required for the definition of the temperature scale \(\Delta = \dot{T}_0 - \dot{T}_f\).

By substituting these scales in the dimensional Stefan equation, the morphological time scale

\[\tau = \frac{\rho_s \Delta \dot{T}_0}{B \Delta \dot{U}_0} \quad (A5)\]

is eventually obtained.

**Appendix B: Parameterization of Heat Transfer Between Water and Atmosphere**

The equilibrium water temperature (K) is defined as [Gu and Li, 2002]
\[ T_{eq} = (T_d + 273.15) + \frac{R}{h_F} \]  

(B1)

where \( T_d \) is the dew point temperature (°C), \( R \) is the net solar radiation (W/m²), and \( h_F \) is the heat transfer coefficient with the atmosphere (W/m² K). The dew-point temperature definition reads

\[ \tilde{T}_d = \frac{237.3[T^* + \ln(w)]}{[17.27 - \ln(w) - T^*]}, \]

(B2)

where

\[ T^* = 17.27 \left( \frac{T_{air} - 273.15}{237.3 + (T_{air} - 273.15)} \right), \]

(B3)

and \( T_{air} \) is expressed in Kelvin. Finally, the heat transfer coefficient is defined as

\[ h_F = 4.5 + 0.05(\tilde{T}_{air} - 273.15) + \mu f(\dot{U}_w) + 0.47f(\dot{U}_w), \]

(B4)

with the slope of the saturated water pressure versus temperature curve given by

\[ \mu = 0.35 + 0.015 \frac{T_{surf} + \tilde{T}_d}{2} + 0.0012 \left( \frac{T_{surf} + \tilde{T}_d}{2} \right)^2, \]

(B5)

\( T_{surf} \) being water surface temperature. The wind function (wind speed \( \dot{U}_w \) in m/s)

\[ f(\dot{U}_w) = 9.2 + 0.46 \dot{U}_w^2 \]

accounts for wind-driven convective heat exchange.

**Appendix C: Solution of the Linearized Problem**

\[ u = \frac{C_0\cot(\phi)\left[-4\beta^2 C_0 + 3z^2(C_0 + iz + a_1\sqrt{C_0}k^2)\right]}{C}, \]

(C1)

\[ v = \frac{2\beta C_0\cot(\phi)\left(10C_0 + 3iz + 3a_1\sqrt{C_0}k^2\right)}{C}, \]

(C2)

\[ d = \frac{3C_0\cot(\phi)\left(C_0(z^2 + 2\beta^2) + izk^2 + a_1\sqrt{C_0}k^2\right)}{C}, \]

(C3)

\[ \theta^l = -\frac{St u}{St(1 + f_h) + iz + a_1\sqrt{C_0}k^2}, \]

(C4)

\[ C = 3C_0\cot(\phi)\left[(izk^2 + a_1\sqrt{C_0}k^2 + C_0(z^2 + 2\beta^2)) + iz(iz + a_1\sqrt{C_0}k^2 + C_0)(iz + 3a_1\sqrt{C_0}k^2 + 10C_0)\right]. \]

(C5)

**References**


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