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Directionality of flexural intensity in orthotropic plates

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This work investigates composite plates and their ability to direct flexural intensity, which has important implications for noise and vibration control. It is well known that a composite plate supports a flexural wave whose wavenumber depends strongly on its angle of propagation. This suggests that a composite plate will direct more flexural intensity in some directions than others. The present work considers a thin multi-layered plate in which each layer is constructed from an orthotropic material and has a chosen orientation relative to the other layers. Such an approach may be used to design highly directive structures. An analysis is presented in which a two-dimensional Fourier transform is analytically applied to the equation of motion, yielding algebraic expressions for displacements and stress resultants. Next, a two-dimensional discrete inverse Fourier transform is applied to compute displacements and stress resultants at discrete locations. Flexural intensity is computed at these locations.

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I. INTRODUCTION

The intent of this work is to examine flexural intensity associated with local excitations applied to composite plates with orthotropic layers. This research examines how intensity depends on the orientations of orthotropic layers relative to each other. Much of the published literature on forced vibration of plates deals with isotropic cases. Exceptions include Bhat and Sinha, who expanded upon the classical problem of an isotropic plate subject to a point force by investigating cases of “sandwich plates,” and Chen who examined two joined semi-infinite plates, one of which is isotropic and the other anisotropic. Magliula and McDaniel examined orthotropic plates that support flexural waves whose wavenumbers depend on their angle of propagation and investigated the effect of fluid loading on this angular dependence.

With regards to vibrational intensity, Piaud and Nicolas investigated the relationship between vibrational and acoustical energy of infinite plates using geometrical optics theory. Structural intensity was examined in nonhomogeneous beams and plates by Pierce. Daley and Hambrick predicted the structural intensity patterns in flat isotropic plates excited by spatially random pressure fields. This previously published work prompts one to consider the effects of anisotropy on the spatial pattern of flexural intensity generated by localized excitations.

The present work, which utilizes classical laminated plate theory, examines the case of an infinite plate with one point force. The infinite plate affords the clarity of examining the direct field from the force, without the complicating effects of reflections that are always present in finite plates. This effectively removes the influence of boundary conditions on our observations. Results have captured the direct field with the goal of illuminating the fundamental physics of single- and multi-layered plates comprised of orthotropic materials whose fibers may be oriented with the intent of designing directive structures. Solutions found in infinite domains may be added together to yield solutions with boundaries. In this way, infinite plate results can be applied to finite plates using the method of images.

The response to a point excitation is found in the wave-number domain and is transformed into the spatial domain by a discrete inverse Fourier transform method. The stress resultants for the multi-layered plate are evaluated in the wave-number domain and are transformed to the spatial domain by a discrete inverse Fourier transform. Stress resultants and the associated velocities are used in calculating intensity in the spatial domain. Numerical examples using conventional orthotropic materials indicate relatively high intensity in certain directions. For a two-layer orthotropic plate, intensity was found to vary by a factor of ten depending on the direction from the excitation. Portions of this work were presented at the ASME International Mechanical Engineering Congress and Exposition, Boston, November 2008.

II. ANALYSIS OF LAYER ROTATION

For convenience of referral, the present authors summarize the derivation of material properties in a rotated coordinate system. The complete analysis is given by Whitney. Consider an $x-y$ coordinate system rotation by an angle $\theta$, where the rotated axes are denoted by $x'$ and $y'$. Refer to Fig. 1 for standard fiber orientation angle conventions and notation corresponding to this angle rotation. The transformed stresses and strains are used in conjunction with generalized anisotropic Hooke’s law to determine a rotated stiffness matrix.

More generally, it is possible to determine the elastic properties with respect to an $x_1$, $x_2$, and $x_3$ coordinate...
system when the elastic stiffnesses are known relative to a rotated \( y'x'_1, x'_2, \) and \( x'_3 \) coordinate system, i.e., in matrix notation,

\[
\sigma' = C' \epsilon',
\]

where \( C' \) is a known orthotropic stiffness matrix. It is necessary to determine a relationship for the transformed system of the form,

\[
\sigma = C \epsilon,
\]

where \( C \) is to be determined from \( C' \), \( \sin(\theta) \), and \( \cos(\theta) \). Substituting equations for transformed stress and strain into Eq. (1), an expression relating stresses and strains in the transformed coordinate system is obtained using the known stiffness matrix \( C' \),

\[
T_\sigma \sigma = C'T_\epsilon \epsilon,
\]

where \( T_\sigma \) and \( T_\epsilon \) are stress and strain transformation matrices. Premultiplying Eq. (3) by the inverse of \( T_\sigma \) yields a relationship that enables the determination of a transformed stiffness matrix in terms of known elasticity parameters,

\[
C = T_\sigma^{-1} C'T_\epsilon.
\]

### III. NUMERICAL SOLUTION FOR POINT EXCITATION OF AN INFINITE COMPOSITE PLATE

Consider a plate comprised of orthotropic layers, excited by a time-harmonic point force at \((x_0, y_0)\). In all that follows, a time dependence of \( \exp(i\omega t) \) has been assumed. Classical laminated plate theory results in the following equation of motion,

\[
D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} - \rho \omega^2 w = F_0 \delta(x - x_0) \delta(y - y_0),
\]

where \( w \) is the complex amplitude of the normal displacement of the plate. This equation assumes that the plate thickness is small relative to a flexural wavelength. As frequency increases, the wavelengths get smaller; therefore this assumption places an upper limit of validity on the frequency range.

The variables \( D_{ij} \) represent the flexural stiffnesses of the plate, given by

\[
D_{ij} = \int_{-h/2}^{h/2} Q_{ij}^{(k)} z^2 dz,
\]

where the superscript refers to the \( k \)th layer of the plate. Reduced stiffness terms \( Q_{ij}^{(k)} \) that assume an approximate state of plane stress for a thin shell are given by

\[
Q_{ij} = C_{ij} - C_{i3}C_{j3} C_{33}.
\]

Each layer of the multi-layered plate has its own mass density, therefore an equivalent density must be determined that takes into account the mass per unit area and thickness of each layer \( k \),

\[
\rho = \int_{-h/2}^{h/2} \rho_0^{(k)} dz,
\]

where \( \rho_0^{(k)} \) is the mass per unit volume of the \( k \)th layer and \( h \) is the thickness of the entire plate.

The equation for the complex amplitude of the normal displacement of the plate can be transformed from the spatial \((x, y)\) domain into the wavenumber \((k_x, k_y)\) domain as follows:

\[
\tilde{w}(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(x, y) \exp[-i(k_xx + k_yy)] dx dy.
\]

The point force is placed at \((x_0, y_0)\) as shown on the right hand side of Eq. (5) and was transformed into the wavenumber domain using the same definition of the wavenumber transform. Analytic transformation of the Dirac delta function was performed to arrive at the final expression for the point force in the wavenumber domain,

\[
\tilde{f}(k_x, k_y) = F_0 \exp[-i(k_xx_0 + k_yy_0)].
\]

Substitution of Eqs. (9) and (10) into Eq. (5) yields an analytically derived equation which may be solved numerically using a method of inverse Fourier transforms.
\[
\frac{\dot{w}}{F_0} = \frac{\exp[-i(k_x x_0 + k_y y_0)]}{D_{11} k_x^2 + 4D_{16} k_x^3 k_y + 2(D_{12} + 2D_{66})k_x^2 k_y^2 + 4D_{26} k_x k_y^3 + D_{22} k_y^4 - \rho \omega^2}.
\]

### IV. ANALYSIS OF FLEXURAL INTENSITY IN THE WAVENUMBER DOMAIN

For linear flexural vibration in thin flat plates, the total active structural intensity vectors consist of components due to transverse shear resultants (\(Q_x \) and \(Q_y \)), bending moments (\(M_x \) and \(M_y \)), and twisting moment (\(M_{xy} \)). Following the conventions of Daley and Hambric, the orthogonal components of intensity in W/m² may be written as

\[
I_x = -\frac{\langle Q_x(i\omega) \rangle}{h} + \langle M_x(i\omega \theta_y) \rangle + \langle M_{xy}(i\omega \theta_x) \rangle, \tag{12}
\]

\[
I_y = -\frac{\langle Q_y(i\omega) \rangle}{h} + \langle M_y(i\omega \theta_x) \rangle + \langle M_{xy}(i\omega \theta_y) \rangle. \tag{13}
\]

Rotations in the \(x \) and \(y \) directions are defined in the spatial and wavenumber domains as follows, where the presence of a “tilde” represents conversion into the wavenumber domain,

\[
\theta_x = \frac{\partial w}{\partial y}, \quad \tilde{\theta}_x = i k_x \tilde{w}, \quad \theta_y = \frac{\partial w}{\partial x}, \quad \tilde{\theta}_y = i k_y \tilde{w}. \tag{14}
\]

The bending moments along \(x \) and \(y \) are defined in Eqs. (15) and (16), respectively, the twisting moment is defined in Eq. (17), and the transverse shear resultants along \(x \) and \(y \) are defined in Eqs. (18) and (19), respectively, as

\[
M_x = -D_{11} \frac{\partial^2 w}{\partial x^2} - 2D_{16} \frac{\partial^2 w}{\partial x \partial y} - D_{12} \frac{\partial^2 w}{\partial y^2}, \tag{15}
\]

\[
M_y = -D_{12} \frac{\partial^2 w}{\partial x^2} - 2D_{26} \frac{\partial^2 w}{\partial x \partial y} - D_{22} \frac{\partial^2 w}{\partial y^2}, \tag{16}
\]

\[
M_{xy} = -D_{16} \frac{\partial^2 w}{\partial x^2} - 2D_{66} \frac{\partial^2 w}{\partial x \partial y} - D_{26} \frac{\partial^2 w}{\partial y^2}, \tag{17}
\]

\[
Q_x = -D_{11} \frac{\partial^3 w}{\partial x^3} - 3D_{16} \frac{\partial^3 w}{\partial x^2 \partial y} - (D_{12} + 2D_{66}) \frac{\partial^3 w}{\partial x \partial y^2} - D_{26} \frac{\partial^3 w}{\partial y^3}, \tag{18}
\]

\[
Q_y = -D_{16} \frac{\partial^3 w}{\partial x^3} - (D_{12} + 2D_{66}) \frac{\partial^3 w}{\partial x^2 \partial y} - 3D_{26} \frac{\partial^3 w}{\partial x \partial y^2} - D_{22} \frac{\partial^3 w}{\partial y^3}. \tag{19}
\]

Sign conventions employed in the above expressions for moments and transverse shear resultants can be found in Whitney, and the appropriate orthotropic bending rigidity values for each layer are determined using Eq. (6). While some of Eqs. (12)–(19) differ in sign from the corresponding set of equations found in Daley and Hambric, the final intensity relations agree.

The stress resultants for the multi-layered plate are found in the spatial domain by a discrete inverse Fourier transform and are used in calculating intensity in the spatial domain. When transformed into the wavenumber domain, Eqs. (15)–(19) are rewritten as

\[
\tilde{M}_x = -D_{11}(ik_x)^2 \tilde{w} - 2D_{16}(ik_x)(ik_y) \tilde{w} - D_{12}(ik_y)^2 \tilde{w}, \tag{20}
\]

\[
\tilde{M}_y = -D_{12}(ik_y)^2 \tilde{w} - 2D_{26}(ik_y)(ik_x) \tilde{w} - D_{22}(ik_x)^2 \tilde{w}, \tag{21}
\]

\[
\tilde{M}_{xy} = -D_{16}(ik_x)(ik_y) \tilde{w} - (D_{12} + 2D_{66})(ik_x)(ik_y) \tilde{w} - D_{26}(ik_y)^2 \tilde{w}, \tag{22}
\]

\[
\tilde{Q}_x = -D_{11}(ik_x)^3 \tilde{w} - 3D_{16}(ik_x)^2 (ik_y) \tilde{w} - (D_{12} + 2D_{66})(ik_x)(ik_y)^2 \tilde{w} - D_{26}(ik_y)^3 \tilde{w}, \tag{23}
\]

\[
\tilde{Q}_y = -D_{16}(ik_x)(ik_y)^2 \tilde{w} - (D_{12} + 2D_{66})(ik_x)^2 (ik_y) \tilde{w} - 3D_{26}(ik_x)(ik_y)^3 \tilde{w} - D_{22}(ik_x)^3 \tilde{w}. \tag{24}
\]

### V. ANALYSIS BY DISCRETE INVERSE FOURIER TRANSFORMATION

Plate response, given in the wavenumber domain by Eq. (11), is discretely sampled in the wavenumber domain and inverse Fourier transformed to the spatial domain via a fast Fourier transform. The first step in this process is the definition of discrete wavenumbers, which has the standard form

\[
k_n = \frac{2\pi n}{\Delta}, \tag{25}
\]

where \(\Delta\) represents spacing in either the \(x\) or \(y\) directions, \(\Delta\) defines the window length, and \(n\) ranges from \((-N/2)\) \(\leq n \leq \{N/2\} - 1\). In the numerical results below \(\Delta = 0.06\), \(\Delta\) was chosen to be much less than relevant wavelengths, and \(\Delta\) was chosen to be sufficiently large to observe wavenumber directivity.

Use of the discrete Fourier transform implies a spatial periodicity in both the \(x\) and \(y\) directions with a spatial period of \(\Delta\) in each direction. Physically, this periodicity implies an infinite two-dimensional array of point forces separated by a distance \(2\Delta\) in both \(x\) and \(y\). Since the intent is to examine the response due to a single point force on a plate without boundaries, the window length in each direction is chosen to be large enough in our calculations so that responses from other point forces are negligible within our windows due to spreading and decay.

### VI. NUMERICAL RESULTS

Results for normal displacement \(w\) and magnitude of intensity, \(\sqrt{P_x^2 + P_y^2}\), are obtained for two separate composite plates. All results are obtained at a fixed frequency of 1 kHz. Both plates are subject to a time-harmonic point force of
1 N, applied to the center of the plate. The first plate under investigation is comprised of a single layer of graphite–epoxy, $h = 0.005$ m in thickness, with fibers oriented at an angle of $\theta = 0^\circ$. The second plate investigated in this work is comprised of two layers, each made of graphite–epoxy, and each of thickness $h = 0.005$ m. The first layer has a fiber orientation angle of $\theta = 0^\circ$, while the second layer has a fiber orientation angle of $\theta = 90^\circ$. The third plate is comprised of three layers, each of thickness $h = 0.005$ m. The first and third layers are comprised of glass–epoxy and the second (middle) layer is made of graphite–epoxy. The three layers had fiber orientation angles of $\theta = 45^\circ$, $\theta = 0^\circ$, and $\theta = 145^\circ$, respectively. Material properties for graphite–epoxy and glass–epoxy can be found in Table I.

In the following numerical results, damping was omitted from the model. If damping were added, each of the variables $D_{ij}$ would be multiplied by a factor of $(1 + i\eta)$, where $\eta$ represents the applicable damping coefficient for the material. The addition of this damping factor would influence Eq. (5), which would in turn alter the expression for $\tilde{w}$ in Eq. (11). In this manner, the addition of damping would propagate through to the equations used in calculating intensity.

The focus of this work was an examination of wave propagation, not wave decay, therefore damping was omitted from the model so that propagation directivity could be clearly observed.

Figures 2–4 show the displacement patterns for the single-, double-, and triple-layer plates, respectively. In each case, it is seen that the displacement field is most pronounced in the direction corresponding to the fiber orientation. Each of the figures shows significant beaming in the directions corresponding to the fiber orientations of each layer. The dramatic effect of orthotropy on the displacement field is clearly shown in these figures.

Since plates often connect vibrating equipment to other radiating structural elements, their transmission of intensity in a given direction is what often matters most. The displacement patterns for each plate serve as inspiration to examine

<table>
<thead>
<tr>
<th></th>
<th>Graphite–epoxy$^a$</th>
<th>Glass–epoxy$^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_x$</td>
<td>138 GPa</td>
<td>48.3 GPa</td>
</tr>
<tr>
<td>$E_y$</td>
<td>8.9 GPa</td>
<td>19.8 GPa</td>
</tr>
<tr>
<td>$E_z$</td>
<td>8.9 GPa</td>
<td>19.8 GPa</td>
</tr>
<tr>
<td>$G_{xy}$</td>
<td>5.176 GPa</td>
<td>8.96 GPa</td>
</tr>
<tr>
<td>$G_{xz}$</td>
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<td>8.96 GPa</td>
</tr>
<tr>
<td>$G_{yz}$</td>
<td>2.89 GPa</td>
<td>6.19 GPa</td>
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<tr>
<td>$\nu_{xy}$</td>
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</tr>
<tr>
<td>$\nu_{yz}$</td>
<td>0.54</td>
<td>0.60</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$1600$ kg/m$^3$</td>
<td>$1800$ kg/m$^3$</td>
</tr>
</tbody>
</table>

$^a$Material properties taken from Ref. 13.
associated intensity patterns. These intensity results are depicted in Figs. 5–7 for the single-, double-, and triple-layer configurations, respectively. It can be seen from these figures that the strongest intensity corresponds with the direction of the fibers. Significant beaming can be seen in the case of the double-layer plate, where intensity can vary by a factor of ten depending upon the direction. This was determined by examining the variations in intensity observed with varying angle at a fixed radius from the center of the plate.

In order to arrive at a local approximation to the dispersions of waves seen in the two-dimensional case, let a coordinate $x'$ be defined in the propagation direction of a flexural wave in the orthotropic plate. Coordinate $y'$ is defined in the direction perpendicular to the wavefront. Therefore, $x'$ is aligned with the wavefront vector and the displacement is independent of all other orthogonal coordinates. The equation of motion of the plate assumes the form of Eq. (5) with derivatives with respect to $y$ neglected. Considering Eq. (5) for the case of a plane wave with no $y'$ dependence, the wavenumber $k_p$ is given analytically by

$$k_p = \sqrt{\frac{\rho \omega^2}{D_{11}}},$$

where $D_{11}$ varies as the $x$ axis is rotated, thus producing an angular dependence of $k_p$. For an isotropic plate, there is no dependence of wavelength on propagation direction. Figures 8–10 examine the wavelength of a plane wave propagating on a composite plate as a function of propagation direction specified by an angle $\alpha$ from the $x$ axis. For example, at $\alpha = 0^\circ$ there is a plane wave propagating in a strictly horizontal direction. The limiting case of plane wave propagation helps to confirm the wavelength dependence on direction that is exhibited in the displacement plots for the two-dimensional case.

![FIG. 5. Magnitude of intensity (W/m²) for single-layer graphite–epoxy plate, $h = 0.005$ m and fiber angle $= 0^\circ$.](image)

![FIG. 6. Magnitude of intensity (W/m²) for double-layer graphite–epoxy plate, $h_1 = 0.005$ m, $h_2 = 0.005$ m, and fiber angles $= 0/90^\circ$.](image)

![FIG. 7. Magnitude of intensity (W/m²) for triple-layer (glass–epoxy/graphite–epoxy/glass–epoxy) plate, $h_1 = 0.005$ m, $h_2 = 0.005$ m, $h_3 = 0.005$ m, and fiber angles $= 45/0/135^\circ$.](image)

![FIG. 8. Wavelength (m) versus position on single-layer plate.](image)

![FIG. 8. Wavelength (m) versus position on single-layer plate.](image)
VII. Conclusions

An analysis has been presented for calculating the flexural response and intensity due to a local excitation on a composite plate. This study was based on a wavenumber analysis of the plate variables followed by an inverse Fourier transform to the spatial domain, which was implemented in a discrete fashion. Analysis of the bending moments, twisting moment, and transverse shear resultants in the wavenumber domain allowed for analytic differentiation of the response field. The numerical results obtained in this work show that the directionality of both the displacement field and flexural intensity can be significantly influenced by the choice of material, the fiber orientation angle of each layer, and the number of orthotropic layers. The most significant beaming is experienced in directions corresponding to the fiber directions.

The difference between the single-, double-, and triple-layer plates, and in particular the dramatic difference in intensity, suggests an avenue for designing plate-like structures that preferentially direct vibratory intensity in certain directions. This avenue would not involve substantial addition of material or material development, as the benefits derive intrinsically from the orientation of the material stiffness directions during construction. Future work will seek to extend this avenue to curved structures and to structures with damping layers, in an attempt to both direct and dissipate intensity.

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