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Applications of the causality condition to one-dimensional acoustic reflection problems

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The causality condition is examined as a means of determining frequency-domain information about a submerged object from a partial knowledge of its acoustic reflection characteristics. A one-dimensional problem is considered in which an acoustic wave reflects from an object that is described by the impedance it presents to the fluid. Two new applications of the causality condition to the frequency-domain analysis of this problem are investigated and illustrated by numerical examples. In each application, the causality condition is used to find the object’s complex impedance from a knowledge of the reflected wave’s magnitude. The first application is to experimental studies where one desires a knowledge of an object’s complex impedance but practical limitations only allow a measurement of the reflected wave amplitude. Analysis shows that the causality condition may be used to determine the phase of the reflected wave, and hence the object’s impedance, if the reflection coefficient is minimum phase. When this is true, examples suggest that the phase of the reflection coefficient may be accurately determined from the causality condition even in the presence of noise and band-limited data. The second application is to design situations, where one wishes to create an object that reflects sound with a specified frequency-dependent magnitude. The causality condition may aid the designer by providing a knowledge of all causal object impedances that produce the same reflection coefficient magnitude. A numerical example is presented in which a variety of causal object impedances produce the same reflection coefficient magnitude over an infinite frequency range. © 1999 Acoustical Society of America.

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INTRODUCTION

This paper presents applications of the causality condition to problems in which acoustic waves reflect from submerged objects. For simplicity we shall restrict our attention to the one-dimensional system shown in Fig. 1, however, extensions of the concepts to multi-dimensional scattering problems are discussed in Sec. IV. The system consists of an object that presents an impedance to the fluid through a rigid massless piston. The impedance defines the ratio of force to velocity such that \( Z = F/V \) where \( F \) is the force applied to the piston in the direction of the velocity \( V \) indicated in the figure. An acoustic wave reflects from the object when there is any difference between the fluid impedance \( \rho_f c_f \) and the object impedance \( Z \), where \( \rho_f \) and \( c_f \) are the fluid’s mass density and sound speed, respectively, and \( A \) is the area of the piston. If the pressure due to the incident wave is zero for \( t < 0 \), then one would expect that the piston velocity and the pressure due to the reflected wave would also be zero for \( t < 0 \). This expectation is known as the causality condition, which in more general terms states that a response cannot precede its cause. In time-domain analysis and in experimental data, the condition is implicit in the solution. However, in the frequency domain the condition is much more subtle and powerful. In particular, it allows one to relate the real and imaginary parts and, in some cases, the magnitude and phase of the Fourier transform of a causal response.

We shall present two applications of the causality condition in the frequency domain. In both applications, the frequency dependence of the reflected wave’s magnitude is known and one desires a knowledge of the object’s complex impedance. A key issue in both applications is the non-uniqueness of the reflected wave’s phase in the absence of other knowledge about the system. In particular, the causality condition allows one to compute the minimum phase from the magnitude of the response’s Fourier transform. However, the actual phase may not be the minimum phase and there is no definitive way of knowing this from only a knowledge of the magnitude. Nonetheless, the minimum phase is useful as a starting point for constructing other phases that satisfy the causality condition.

Some of the earliest applications of the causality condition were made by Kronig in 1926 and Kramers in 1927 and involved the dispersion of X-rays. Their work led to integral relations between the index of refraction and the atomic absorption coefficient. Since then, the causality condition has been applied to many engineering problems, perhaps the most notable being the design of feedback amplifiers by Bode. The concept was so critical to amplifier designs that it appeared in a design patent (U.S. Patent Number 2,123,178), and a corrected form of his causality equations appeared in Terman’s Radio Engineer’s Handbook. This and other work in circuit analysis and design is discussed by Guillemin. In addition, there is a large body of work which involves applications of the causality condition to electromagnetic scattering matrices. An introduction is...
given by Holbrow and Davidson,6 more detailed treatment is found in Refs. 7 and 8. The present work differs in that it uses the causality condition to find properties of the scatterer in the frequency domain, such as its impedance, that directly relate to its construction.

Before stating the mathematical implications of the causality condition, let us define \( f(t) \) as the response of a linear system to an excitation \( g(t) \) that is zero for \( t < 0 \). Furthermore, let us define the complex Fourier transform \( \tilde{f}(\omega) \) and its inverse transform as:

\[
\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt
\]

and

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t} d\omega.
\]

The causality condition in the time domain leads to Hilbert transform relations between the real and imaginary parts of the complex Fourier transform of the response. Discussions and proofs of these relations are found in various texts,9–13 and in an article by MacDonald and Brachman.14 Writing \( \tilde{f}(\omega) = \tilde{f}_r(\omega) - i\tilde{f}_i(\omega) \), the Hilbert transform relations are

\[
\tilde{f}_r(\omega) = -\frac{i}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\tilde{f}_i(x)}{\omega - x} dx,
\]

and

\[
\tilde{f}_i(\omega) = \frac{i}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\tilde{f}_r(x)}{\omega - x} dx,
\]

where \( \mathcal{P} \) indicates the principal value of the integral. These may be derived by either invoking the convolution theorem in the time domain or by contour integration in the frequency domain (see Ref. 10 with \( j \) replaced by \( -i \)).

The relationships between magnitude and phase of \( \tilde{f}(\omega) \) may be derived from Eqs. (3) to (4) if \( \tilde{f}(\omega) \) is analytic and has no zeros in the upper half of the complex plane (henceforth referred to as the causal half-plane). The assumption of analyticity guarantees causality while the assumption of no zeros in the causal half-plane is known as the minimum phase condition. This condition will be discussed in more detail below. Magnitude and phase relations are derived by writing \( \tilde{f}(\omega) = \exp[-\alpha(\omega) + i\theta(\omega)] \) and taking the natural logarithm, which yields

\[
\theta(\omega) = \frac{\omega}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\alpha(x)}{\omega - x} dx,
\]

and

\[
\alpha(\omega) = \alpha(0) - \frac{\omega^2}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\theta(x)}{\omega^2 - x^2} dx.
\]

The phase \( \theta \) found from Eq. (5), which is the phase of \( \tilde{f} \) that satisfies the minimum phase condition, is known as the minimum phase.

Direct numerical evaluation of either Eq. (6) or (5) is complicated by the singularities at \( x = \pm \omega \). A more straightforward and robust numerical procedure is made possible by the Wiener–Lee transform15–16 defined as \( \omega = -\tan(\delta/2) \) and illustrated in Fig. 2. From Eqs. (5) and (6), we find that the transformed magnitude and phase, \( \tilde{\alpha}(\delta) \) and \( \tilde{\theta}(\delta) \), may be expressed in terms of the following Fourier series

\[
\tilde{\alpha}(\delta) = \sum_{n=0}^{\infty} d_n \cos(n \delta)
\]

and

\[
\tilde{\theta}(\delta) = \sum_{n=1}^{\infty} e_n \sin(n \delta),
\]

where

\[
d_n = -e_n.
\]

This transform will be used in numerical examples to determine the minimum phase \( \tilde{\theta} \) by expanding the magnitude \( \tilde{\alpha} \) in the cosine series in Eq. (7), finding \( e_n \) from Eq. (9), and then evaluating the sine series in Eq. (8). This approach has been applied to a related problem involving phase reconstruction from uniform fiber Bragg gratings,17–18 in which the amplitude was measured from power measurements and the phase was reconstructed by the Wiener–Lee transform.

The minimum phase condition is subtle and, as pointed out by Victor,19 has been missed by previous researchers. For both of the applications considered here, an understand-
ing of the condition and its implications is critical. Let us briefly review a key analytical result presented by Victor, which states that the Fourier transform \( \tilde{f}(\omega) \) of a causal response may be decomposed as follows:

\[
\tilde{f}(\omega) = e^{i\omega D} Q(\omega) \prod_{q=1}^{Q} P(\omega, \omega_q)
\]

(10)

where \( e^{i\omega D} \) is a phase shift which accounts for delays between excitation and response. The delay \( D \) is the largest value for which the time-domain response between excitation and response. If the delay \( D \) is causally causal transforms \( f(t) \) will be explored in Sec. II. If one is interested in finding all causal transforms of causal responses by choosing any number of causal transforms of causal responses by choosing any number of

If the magnitude \( |\tilde{f}(\omega)| \) is known over an infinite frequency range, then its minimum phase may be found from Eq. (5). Whether or not this corresponds to the actual phase cannot be known without further assumptions on \( \tilde{f}(\omega) \). In this regard, certain assumptions relevant to the system in Fig. 1 will be explored in Sec. II. If one is interested in finding all causal transforms \( \tilde{f}(\omega) \) given \( |\tilde{f}(\omega)| \) over an infinite frequency range, then the minimum phase may be found from Eq. (5). Equation (10) may then be used to find all possible transforms of causal responses by choosing any number of values for the \( \omega_q \) under the condition that each \( \omega_q \) lie in the causal half-plane.

I. APPLICATION OF THE CAUSALITY CONDITION TO THE ACOUSTIC REFLECTION PROBLEM

In addition to placing requirements on the Fourier transform of the response, the causality condition may be used to infer dynamic properties of a system such as the one shown in Fig. 1. In the frequency domain, linear systems obey the algebraic relationship \( \tilde{f}(\omega) = \tilde{g}(\omega) H(\omega) \), where \( \tilde{g}(\omega) \) is the Fourier transform of the excitation and \( H(\omega) \) is generally known as the system or transfer function and obeys the Hilbert transform relations given in Eqs. (3)–(6). To see this, consider an impulsive excitation described by \( g(t) = \delta(t) \), so that \( \tilde{g}(\omega) = 1 \) and \( \tilde{f}(\omega) = H(\omega) \). From this equality, we see that the Hilbert transform relations in Eqs. (3)–(6) also apply to the system function \( H(\omega) \). This recognition, which will be used extensively in the present work, is significant in that the system function is independent of excitation and often leads directly to the design of the system.

It shall be expedient to derive the system function from the steady-state complex amplitudes of a time-harmonic response and excitation. If the steady-state response and excitation are written as \( f_{ss}(t) = \mathfrak{R}\{F(\omega)e^{-i\omega t}\} \) and \( g_{ss}(t) = \mathfrak{R}\{G(\omega)e^{-i\omega t}\} \), respectively, then the system function is \( H(\omega) = F(\omega)/G(\omega) \). The application of the causality condition to a system function derived from steady-state response is a source of confusion in that steady-state responses are assumed to exist for all time. We must keep in mind only the mathematical equivalence.

Turning to the particular problem illustrated in Fig. 1, we identify the incident and reflected pressures at \( x = 0 \) as the excitation and response, respectively. Other response variables, such as the velocity of the piston, may be chosen but our interest here is in the reflected pressure because it is most often experimentally measured or specified in a design problem. The incident and reflected pressures satisfy the acoustic wave equation and have the form of forward and backward propagating waves, respectively, given by

\[
p_{\text{inc}}(x, t) = P_{\text{inc}}(\omega)e^{-i(\omega t - kx)}
\]

(12)

and

\[
p_{\text{ref}}(x, t) = P_{\text{ref}}(\omega)e^{-i(\omega t + kx)}.
\]

(13)

The acoustic wave number is given by \( k = \omega/c_f \).

The complex reflection coefficient, defined as \( R(\omega) = P_{\text{ref}}(\omega)/P_{\text{inc}}(\omega) \), may be interpreted as the system function because it is the ratio of the response to the excitation. The problems considered here involve the determination of the object’s impedance \( Z \), which represents the ratio of force on the piston to its velocity, from a specification of the magnitude of the reflection coefficient over an infinite frequency range. In the frequency domain, the impedance and reflection coefficient are algebraically related by

\[
z = \frac{1 + R}{1 - R} \quad \text{or} \quad R = \frac{z - 1}{z + 1},
\]

(14)

where \( z = Z/(\rho_f c_f A) \) is the specific acoustic impedance of the object. The applications described in the following two sections proceed by applying the causality condition to determine the phase of \( R \) from a knowledge of \( |R| \), and then using Eq. (14) to find the complex-valued impedance \( z \).

II. IMPEDANCES THAT CREATE MINIMUM PHASE REFLECTION COEFFICIENTS

The reflection coefficient is minimum phase when none of its zeros are in the causal half-plane. If this condition is satisfied and the magnitude of \( R \) is known over an infinite frequency range, then the phase may be uniquely determined from Eq. (5). In this section, we shall illustrate the implications of this condition by examining the system shown in Fig. 3, in which the object impedance \( Z \) shown in Fig. 1 has been replaced by a mass-spring-dashpot system. The param-
Parameters $K$, $C$, and $M$ represent the usual spring, mass, and dashpot elements. The specific impedance of this system is

$$z = \frac{Z}{\rho_j c_j A} = c + i\beta \left( \frac{1}{\Omega - \Omega_1} \right), \quad (15)$$

where $\Omega = \omega/\sqrt{KM}$, $c = C/(\rho_j c_j A)$, and $\beta = \sqrt{KM}/(\rho_j c_j A)$.

For this impedance, the zeros of $R$ are given by

$$\Omega_{1,2} = -i \left( \frac{(c-1) \pm \sqrt{(c-1)^2 - 4\beta^2}}{2\beta} \right). \quad (16)$$

Neither of these zeros are in the causal half-plane when $c > 1$, but both zeros are in the causal half-plane when $c < 1$. Therefore, we conclude that $R$ will be minimum phase when $c > 1$ or $\Omega > \rho_j c_j A$. Physically, this means that the reflection coefficient is minimum phase when the damping of the object, which is represented by the dashpot constant $C$, is larger than the impedance $\rho_j c_j A$ of the fluid.

Let us further illustrate this concept and the computation of the minimum phase from Eqs. (7) to (9) by way of two numerical examples. In both examples, it is assumed that the magnitude of the reflection coefficient is known and one is interested in finding its phase. The impedance in Eq. (15) was computed at frequencies defined by 315 values of the transform variable $\delta$ that ranged from $-\pi$ to $-0.001$ in equal steps of 0.01. The sine series in Eq. (7) was truncated at 315 terms and the $d_n$ were found by a collocation technique that required that equation to hold at each value of $\delta$.

In the first example, the parameters of the object impedance were taken as $c = 2$ and $\beta = 10$ so that $R$ is minimum phase. The magnitude of the reflection coefficient for this system is shown in Fig. 4 and is labeled “actual.” In order to simulate what one might measure in an experiment, noise was added to the reflection coefficient magnitude to produce the curve labeled “noisy” in the same plot. The noise was taken to be normally distributed about zero with a maximum amplitude of 0.01. The noisy data was used to reconstruct the phase from Eqs. (7) to (9). This phase approximates the actual phase very well as shown in Fig. 4 because the system is minimum phase. In Fig. 5, the impedances computed from the noisy reflection coefficient are compared to the actual impedance of the structure. The agreement is good except at low frequencies where the impedance approaches infinity and its phase is very sensitive to errors in the reflection coefficient, as may be seen by Eq. (14).

An important practical concern is the bandwidth over which the reflection coefficient magnitude must be measured in order to determine its phase by causality. The results shown in Figs. 4 and 5 required a knowledge of the reflection coefficient magnitude at 315 frequencies with the maximum frequency of $\Omega = 3,375$. Is such a large bandwidth required when the magnitude and phase of $R$ appear to vary only slightly for $\Omega > 2$? To investigate this question, the calculation was performed with the noisy data truncated above $\Omega = 1$ and $\Omega = 2$. Above each upper limit, the magnitude was extrapolated to the constant value given at the upper limit. From Fig. 4, we see that this is a reasonable approximation to the actual reflection coefficient when the upper limit is $\Omega = 2$ but not when it is $\Omega = 1$. Using the same sampled frequencies which were used in the first example, the causality condition was used to reconstruct the phases.

The phases are plotted in Fig. 6 and, as expected, the $\Omega = 2$ truncation agrees very well with the actual phase, which is replotted for reference, but the $\Omega = 1$ truncation does not approximate the actual phase. Considering only the $\Omega = 2$ truncation, this plot indicates that if a reflection coefficient is measured over a finite frequency band in the presence of noise, its phase may be accurately determined by the causality condition if the magnitude is accurately extrapolated outside of the measurement band. We expect this to be true in general when the measurement band contains the resonances of the system, as shown here for measurements up to $\Omega = 2$.

In the second example, the parameters $c = 1/2$ and $\beta = 10$ were chosen, so that the system was not minimum phase. The magnitude of the reflection coefficient of this system is shown in Fig. 7. The minimum phase of the reflec-
tion coefficient, determined from the magnitude and Eqs. (9), is plotted along with the actual phase. The difference between the two phases is due to the fact that the actual phase is not the minimum phase. The actual reflection coefficient \( R_a \) is related to the minimum phase reflection coefficient \( R_{mp} \) by the following form of Eq. (10):

\[
R_a = e^{i\omega D} R_{mp}(\omega) \prod_{q=1}^{Q} P(\omega, \omega_q).
\] (17)

These observations can be generalized to an object with an arbitrary impedance, as shown in Fig. 1, by writing the object’s specific impedance as \( z(\omega) = z_r(\omega) + i z_i(\omega) \). From Eq. (14), the zeros of \( R \) satisfy

\[
z_r(\omega_n) - 1 + i z_i(\omega_n) = 0, \text{ for } n = 1,2,\ldots, N.
\] (18)

Whenever all of the \( \omega_n \) lie in the lower half of the complex plane, the reflection coefficient will be minimum phase and the causality condition may be used to uniquely recover the phase of \( R \) from its magnitude.

III. CAUSAL IMPEDANCES FROM A REFLECTION COEFFICIENT MAGNITUDE

In this section, the nonuniqueness of the reflection coefficient’s phase is exploited to find all causal object impedances that reflect waves with the same magnitude over an infinite frequency band. The analysis is expected to aid design studies by identifying very different conceptual designs that have the same acoustic properties. As an example, we shall consider the system in Fig. 1 and assume that one desires the magnitude of the reflection coefficient to be

\[
|R| = 0.3 \tan^{-1}[5(\Omega - 1)] + 0.5.
\] (19)

The magnitude is shown in the upper plot of Fig. 8. This assumed form is meant to represent a structure that is essentially a “high-pass filter,” reflecting very little sound at low frequencies and giving almost perfect reflection at higher frequencies. The minimum phase of \( R \) found from Eqs. (7) to (9) is shown in the lower plot.

Other causal phases are indicated by a form of Eq. (10):

\[
R = e^{i\omega D} R_{mp}(\omega) \prod_{n=1}^{N} P(\Omega, \Omega_n),
\] (20)

where \( R_{mp} \) is the minimum phase reflection coefficient, whose magnitude and phase are shown in Fig. 8. To illustrate the various object impedances produced by choices of \( \Omega_n \), let us limit the product in Eq. (20) to one term \( (N=1) \) and compute causal reflection coefficients for the cases \( \Omega = i, 10i, \) and \( 1+i \). The unwrapped phases of the reflection coefficient for these cases are shown in Fig. 9, where the minimum phase is indeed always lower than the nonminimum phases. The corresponding impedances are shown in Fig. 10.

Every impedance shown in Fig. 10 represents a physically realizable object that obeys the causality condition. In design situations one typically faces constraints and objectives in choosing an impedance. For example, limiting the total mass of the object may be expressed as a constraint on

FIG. 6. A plot of the phase of the reflection coefficient predicted by causality and a knowledge of the magnitude of the reflection coefficient over a finite frequency range.

FIG. 7. Plots of the magnitude and phase of a nonminimum phase reflection coefficient. The minimum phase in the lower plot was computed by Eqs. (7)–(9).

FIG. 8. A plot showing the desired reflection coefficient, given in Eq. (19), and the minimum phase computed by Eqs. (7)–(9).
impedance. Eq. (20), as illustrated in Fig. 10, offers the possibility of considering many impedances that give the same magnitude of reflected sound. Given a set of design constraints and objectives, one could proceed by applying optimization procedures that vary the number and values of $\Omega$, to achieve a desired impedance, as calculated from Eqs. (14) and (20).

IV. CONCLUSIONS

The analyses presented here indicate the conditions under which the causality condition allows one to determine an object’s impedance from a knowledge of the reflected wave’s magnitude. When the reflection coefficient is minimum phase, the causality condition may be useful in experimental studies in which the impedance of an object is to be determined by measuring only the magnitude of the reflected wave. When this is true, numerical examples suggest that calculations of the object’s impedance from a knowledge of the reflection coefficient magnitude may be accurate even when noise is present and when the magnitude is only specified over a finite frequency band. In design problems, the causality condition may be used to find a class of causal object impedances that produce the same reflection coefficient magnitude. The designer may then choose the most desirable object based on other considerations, such as ease of construction. The Wiener–Lee transform has suggested a robust and efficient computational algorithm for implementing the causality condition in the examples presented here.

In principle, these applications may be extended to multi-dimensional scattering. For example, consider a time-harmonic scattering problem in which the incident pressure amplitude $\{P_{\text{inc}}\}$ at locations on an object is related to the scattered pressure amplitude $\{P_{\text{sc}}\}$ at a set of locations in the acoustic medium. Writing this relationship as $\{P_{\text{sc}}\} = [S] \times \{P_{\text{inc}}\}$, we note that each element of the scattering matrix $[S]$ may be interpreted as a system function in the same way as the reflection coefficient $R$ in the one-dimensional problem. In particular, the $(i,j)$ element of $[S]$ represents the scattered pressure at the $i$th field point caused by a unit incident pressure at the $j$th location on the body with no incident pressure at the other points. Therefore, if only the magnitudes of the scattered pressures were measured at the field points, the causality condition could be used to reconstruct the minimum phases of every element of $[S]$. It may be possible to develop conditions to insure that the actual phases are the minimum phases by placing assumptions on the impedance of the scattering body, as was done for the one-dimensional case in Sec. II.

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