Structural wave reflection coefficients of cylindrical shell terminations: Numerical extraction and reciprocity constraints
P. W. Smith, J. Gregory McDaniel, Kevin D. LePage, and Robert Barile

Citation: The Journal of the Acoustical Society of America 101, 900 (1997); doi: 10.1121/1.418109
View online: https://doi.org/10.1121/1.418109
View Table of Contents: https://asa.scitation.org/toc/jas/101/2
Published by the Acoustical Society of America

ARTICLES YOU MAY BE INTERESTED IN

Sound, Structures, and Their Interaction, 2nd edition by Miguel C. Junger and David Feit
The Journal of the Acoustical Society of America 82, 1466 (1987); https://doi.org/10.1121/1.395243

Acoustics of fluid-filled boreholes with pipe: Guided propagation and radiation
The Journal of the Acoustical Society of America 105, 3057 (1999); https://doi.org/10.1121/1.424635
Structural wave reflection coefficients of cylindrical shell terminations: Numerical extraction and reciprocity constraints

P. W. Smith, Jr., J. Gregory McDaniel, Kevin D. LePage, and Robert Barile

BBN Corporation, 70 Fawcett Street, Cambridge, Massachusetts 02138

(Received 1 July 1996; accepted for publication 12 September 1996)

A fluid-loaded cylindrical shell guides vibratory energy along its axial direction in the form of a finite set of distinct traveling wave types. On a finite shell, the energy incident upon a termination is redirected into reflected waves of all types and into sound radiated into the farfield. (The total field near the termination also includes various evanescent components.) Reflection coefficients relate the amplitudes of individual reflected wave types to the amplitudes of individual incident wave types. In this paper, values for the reflection coefficients of several terminations have been extracted from results of finite element analysis for the forced response of a finite structure excited by a variety of different sources. The numerical results exhibit good agreement with the analytical constraints consequent to the principle of reciprocity. The constraints are developed here by an analysis adaptable to other multimodal waveguides by appropriate modification of detail. © 1997 Acoustical Society of America. [S0001-4966(97)02901-9]

PACS numbers: 43.40.Ey, 43.20.Gp [CBB]

INTRODUCTION

We are concerned with the dynamics of a finite fluid-loaded structure that consists of a long, homogeneous thin cylindrical shell, bounded by terminations at each end and reinforced by a few localized ring discontinuities spaced at large intervals along the shell. In this paper, the shell is forced into vibration by internal mechanical sources. It is plausible to conjecture (subject to later verification) that vibratory energy in such a structure is carried between adjacent terminations or ring discontinuities in the form of natural shell waves, indistinguishable from similar waves on an infinite shell having the same dynamical properties.

It is sometimes argued from asymptotic analysis that grazing sound propagation, with associated forced motion of the shell, must eventually be a more important coupling mechanism than natural shell waves, because the sound amplitude decays only as a power of distance while the shell waves decay exponentially. But our interest lies in their relative strength at a finite distance. Suffice it to say that our numerical results at typical distances and frequencies of interest show no reason to include grazing sound as a coupling mechanism.

A. Multiple reflection model

We are led by this conjecture to a multiple reflection model of the total vibratory field on the structure. Except near the discontinuities, the vibratory field consists of a superposition of natural waves propagating in both directions. When one such wave strikes a discontinuity, energy is sent out into many distinct paths: reflected natural waves of all types, transmitted waves of all types (when the discontinuity is not a termination), and sound radiated into the fluid.

If the discontinuities are widely enough spaced, one expects that the reflection process at any one of them is sensibly independent of the others, except insofar as their presence may affect the amplitudes of the incident waves. The reflection process at one discontinuity is then indistinguishable from reflection at the same discontinuity located on an infinite shell (or at the end of a semi-infinite shell in the case of a termination) and irradiated by the same incident wave field.

B. Reflection coefficients

The principal goal of this analysis is to evaluate the complex reflection coefficients, functions of frequency, which relate the amplitudes of the reflected fields to the amplitudes of all of the incident wave types. Each different discontinuity is characterized by a different set of reflection coefficients, but the values for any one discontinuity are, for widely spaced discontinuities, independent of the presence or the location of other discontinuities. This independence is the pre-eminent advantage of the multiple reflection model.

In general, stringent analytical difficulties arise in trying to evaluate these coefficients. We know of only one class of discontinuity that is amenable to the procedures of classical analysis: a thin interior ring discontinuity that contacts the shell on a single cross section. In this case the interaction of shell and discontinuity can be described by collocated ring forces and moments. A classical formulation expresses the total vibratory field on the infinite shell as the sum of an incident field and a scattered field, and imposes appropriate continuity conditions at the connection between shell and discontinuity. The analysis can be carried out formally in terms of the solutions for two sub-problems: (i) the shell’s response to localized ring forcing, and (ii) the discontinuity’s response to forcing on its outer perimeter. The first problem is easily formulated classically, using Fourier analysis, although numerical calculations will often be necessary to evaluate the resulting integral forms. The second problem may also have a simple solution if the design of the discontinuity is sufficiently simple.

In the case of terminations and ring discontinuities that have significant axial extent, two serious complications arise:
(i) elastic deformation occurs within the discontinuity, with the possibility for multiple resonances, and (ii) acoustic coupling arises among the elements of the discontinuity and adjacent shell surfaces. No simple approach is then evident. The solution to this problem is the topic of this paper.

C. This study

This study illustrates a technique by which the reflection coefficients of a discontinuity, specifically a termination, can be extracted from results of finite element analysis (FEA) of a particular finite structure containing that discontinuity. The finite structure used here is a homogeneous fluid-loaded thin shell with a termination on each end. For simplicity in the exposition, both shell and termination are taken as axisymmetric. The analysis, which is clearly based on the multiple reflection model, has several parts.

First, the properties of the natural wave types propagating on the shell are determined by classical analysis for an infinite fluid-loaded shell. The equations for pure-tone vibrations in the absence of external forcing are solved for the complex wave numbers and for the associated wave shapes or “polarizations” that specify the relative amplitudes of the three components of vector displacement in the natural wave.

Second, the velocity pattern associated with forced response of the finite structure is found by FEA. It is useful to think of these calculations as a numerical experiment, in which the target termination is located at one end of a long unobstructed segment of shell and a mechanical source and closing termination are located at the other end. The whole structure thus consists of three sections: a source end, a transmission and measurement section, and the target end. The total velocity pattern in the transmission section is then extracted from results of finite element analysis of any single wave type. Then the displacement vector, $d$, has the general form

$$d = A \begin{bmatrix} \cos n(\phi - \phi_0) \\ 0 \\ 0 \end{bmatrix} \times S e^{i(kz - \omega t)},$$

where $A$ is an arbitrary complex constant, $n$ is an integer, $\phi_0$ is an arbitrary reference angle, $S$ is a unit “polarization” vector, $k$ is the complex axial wave number, and $\omega$ is the real angular frequency. (The diagonal matrix should be replaced by the identity matrix in the case $n = 0$.) Different wave types have distinct characteristic values of $k$ and $S$, which vary with frequency.

These characteristic values are found in the customary manner, with numerical algorithms similar to those proposed by Scott. The differential equations of unforced motion with fluid loading reduce to a matrix equation,

$$Zd = 0,$$

which has nontrivial solutions only if the determinant of the impedance matrix $Z$ vanishes. That determinant is well known to be a function of $k^2$. Therefore, if $k$ is a root corresponding to a wave propagating in the positive $z$ direction there is a root $-k$ corresponding to an identical wave propagating in the opposite direction. In the complex $k$-plane, both upper and lower half-planes have four distinct principal roots of the determinantal equation, $|Z(k)| = 0$, for each $n$ and every frequency. Once these are found, the polarization vector is evaluated from the cofactors of $Z$. In the results reported here, the Donnell–Mushtari thin shell theory has been used.

In contrast to most prior studies, we have included shell dissipation by means of a non-vanishing loss factor in the elastic moduli of the shell material. This enhances both the generality and the realism of the results, and even improves the performance of computer algorithms (the wave number as a function of frequency is not only continuous but has a continuous derivative if the damping is nonzero); there is no possibility of a strictly real root.

A. Natural wave types

Prior studies, omitting shell dissipation, have elucidated the general properties of the roots of the determinantal equation for a fluid-loaded thin shell. In general, the presence of fluid loading or of small dissipation in the shell material leads only to small changes in the wave numbers from the idealized case most often studied.

There are four principal roots in each half of the $k$ plane. [Scott (Ref. 5, p. 254 ff. and Appendix D) delineates other, highly attenuated roots called “Stokes’ roots” that exist for large values of $n$. These are intimately related to “creeping waves” in the fluid surrounding the shell.] The principal roots are:

(i) quasi-longitudinal: The principal displacement is axial although radial motion is nonvanishing. In the idealized case, the wave can be resolved into a pair of helical traveling waves whose phase speed, at frequencies well above cutoff, is nearly the speed of longitudinal waves in a thin flat plate.

I. NATURAL WAVES

Consider the natural, pure-tone waves that can exist on an infinite fluid-loaded thin shell. We take a cylindrical coordinate system where $z$ is the axial coordinate, $\phi$ is the circumferential angle, and $r$ is the radial coordinate; the shell’s midsurface lies at $r = a$. Then the displacement vector

$$d = [u, v, w]^T$$

of any single wave type has the general form

$$d = A \begin{bmatrix} \cos n(\phi - \phi_0) \\ 0 \\ 0 \end{bmatrix} \times S e^{i(kz - \omega t)},$$
(ii) quasi-shear: The principal displacement is circumferential although radial motion is nonvanishing except for \( n = 0 \). In the idealized case, the wave can be resolved into a pair of helical travelling waves whose phase speed, at frequencies well above cutoff, is nearly the speed of shear waves in a thin flat plate.

(iii) quasi-flexural: The principal displacement is radial. In the idealized case, the wave can be resolved into a pair of helical traveling waves whose phase speed, at frequencies well above both cutoff and ring frequencies, is nearly the speed of bending waves in a thin flat plate.

(iv) evanescent: The principal displacement is radial. This wave is highly attenuated in the axial coordinate and is significant only near inhomogeneities. We dismiss it as a means of coupling between discontinuities on the shell. For simplicity, we shall omit the “quasi” prefix in future discussions.

Thus, there are three natural wave types for each value of \( n \) that can propagate between the discontinuities, if the frequency is above their cutoff frequencies. (Cutoff frequencies can be roughly estimated from analytical results for a shell in vacuo without dissipation.) When the frequency moves below the cutoff frequency of one of these wave types, the wave number acquires an increasingly large imaginary part. At some point, it is necessary for numerical reasons to omit such a wave type from consideration; for example, if \( \text{Im} \, ka > 2 \), the amplitude attenuates by more than 17 dB in a distance of one radius.

B. Typical results

Some typical numerical results are given in Fig. 1 for \( n = 0 \) waves on a steel shell in water. The nondimensional parameters of the shell are: thickness/radius, \( h/a = 0.01 \); specific gravity, \( \rho / \rho_0 = 7.8 \); loss factor, \( \eta = 0.02 \); relative Young’s modulus, \( E / \rho_0 c_0^2 = 88(1 - i \eta) \); Poisson’s ratio, \( \nu = 0.3 \); where \( \rho_0 \) and \( c_0 \) are the density and sound speed of the ambient fluid. The graphs show the real and imaginary parts of the natural wave number as functions of frequency expressed by the value of \( k_\alpha a \). Figure 1(a) is for the flexural wave. As shown in prior studies, this subsonic wave gains a large phase speed below the ring frequency (about \( k_\alpha a = 3.6 \)); the dispersion leads to a small group speed and an associated peak in the attenuation rate. Figure 1(b) is for the longitudinal wave. This supersonic wave is only slightly dispersive, and the attenuation is primarily due to radiation. We omit the shear wave which for \( n = 0 \) is purely torsional and unaffected by the ambient fluid.

II. FINITE ELEMENT EXPERIMENTS

A. The model

The so-called experiments run by FEA determine the response velocity of a simple fluid-loaded shell with two terminations, the shell being excited by mechanical ring sources located near one of the terminations (see Fig. 3). The mechanical ring sources, all of which were distributed about the circumference with specified circumferential harmonic

![FIG. 3. Water-loaded cylindrical shell with terminations and mechanical ring excitations, concentrated in axial location but distributed circumferentially.](image)
n, consisted of bending moments about the circumferential coordinate, radial forces, and axial forces. These sources have usually been located at a distance of 0.9L from the target termination.

Since the shell and terminations are taken as axisymmetric, a 2-D code suffices; calculations are carried out for each circumferential harmonic in isolation. The raw output data consist of the three directional components of complex shell velocity at each axial location along the shell. We have used the 2-D SARA code,8 which represents fluid loading by a layer of fluid elements bounded by essentially nonreflective infinite elements.

The shell modeled in the experiments is identical to that used in the analysis of wave numbers. Elastic parameters are appropriate to steel and water and the ratio of shell thickness h to radius a is h/a = 0.01. The length of the shell is somewhat arbitrary, determined principally by the two requirements that, at all frequencies of interest, (i) evanescent elastic fields near the terminations shall be negligible in the center of the shell, and (ii) the propagating wave types shall not be so highly attenuated in the length of the shell as to give rise to numerical inaccuracies. The ratio of cylindrical length L to radius a in our studies ranged from 13 < L/a < 18.5.

Two termination geometries have been tested, as shown in Fig. 4. For modeling convenience, the same design is used for the termination at each end. In order to assess the effects of termination dynamics, two studies of the hemispherical termination were conducted. In one study, it was modeled as steel while in the other study it was “rigidized” with a Young’s modulus many times larger than steel, thus highlighting the effects of interior dynamics on the reflection process.

III. EXTRACTION OF REFLECTION COEFFICIENTS

A. Traveling wave amplitudes

The first task is to extract estimates of the traveling wave amplitudes from the FEA data for velocity on the shell in the nth circumferential order. We search for the best fit to the data of an analytical model of the velocity pattern, expressed as a sum of traveling waves having the previously determined natural wave numbers. Let the axial coordinates of points on the shell extend from z = 0 at the source termination to z = L at the target termination. Then the model for, say, total radial velocity is

\[ v(z, \phi) = \sum_n \cos(n\phi) \sum_q \left[ V_q^{inc} e^{ikq(z-L)} + V_q^{ref} e^{ikq(L-z)} \right], \]

where the unknown complex amplitudes \( V_q \) of the qth wave type have superscripts “inc” for the waves incident on the target termination and “ref” for the reflected waves. (Subscripts \( n \) on the complex amplitudes and natural wave numbers have been suppressed.)

The values of the \( V_q \) are determined by a least-mean-square matching to the data from, typically, 0.1 < z/L < 0.9, the source region and the ends of the shell being truncated in order to exclude regions where the field is dominated by evanescent terms, including the response to grazing sound. Note that the amplitudes are normalized in magnitude and phase to the values at the discontinuity, z = L.

The same procedure can be applied to the axial or circumferential velocity patterns, and the data for all three vector components could be combined in a single extraction process by using the polarization vector \( \mathbf{S} \) [see Eq. (1)] determined from analysis of waves on the infinite shell. One must however be careful to distinguish between the polarization vectors of incident and reflected waves. The whole procedure, which must be repeated for all frequencies and all order numbers \( n \) of interest, is greatly facilitated by flexible software for manipulating complex matrices.9

Figure 5 shows some results of this extraction process. Since the residual after extracting the natural waves has a level more than 50 dB less than the level of the total shell velocity, except near the ends of the shell, it is clear that the natural waves are the dominant mechanism coupling the two ends of the shell. The conjecture in the Introduction is thus confirmed by the data.
B. Reflection coefficients defined

At this point the reflection coefficients can be introduced formally as a square matrix \( \mathbf{R} \) relating the reflected amplitudes to the incident amplitudes:

\[
\mathbf{V}_k^{\text{ref}} = \mathbf{R} \mathbf{V}_k^{\text{inc}},
\]

(4)

where the wave amplitudes for the different wave types have been assembled into vectors:

\[
\mathbf{V}_k^{\text{ref}} = \begin{bmatrix}
V_{1,k}^{\text{ref}} \\
V_{2,k}^{\text{ref}} \\
V_{3,k}^{\text{ref}}
\end{bmatrix} \quad \text{and} \quad \mathbf{V}_k^{\text{inc}} = \begin{bmatrix}
V_{1,k}^{\text{inc}} \\
V_{2,k}^{\text{inc}} \\
V_{3,k}^{\text{inc}}
\end{bmatrix}.
\]

(5)

The subscript \( k \) has been added to these variables as a placeholder for an index number identifying the source, for use in the subsequent discussion of multiple sources. (In the axisymmetric case, the shear wave is often omitted and the order of the matrices reduced to two.)

C. Multiple sources; Solution for \( \mathbf{R} \)

The wave amplitudes determined for a single source do not determine the desired reflection coefficients. Because each incident wave type can reflect into every wave type, one has in Eq. (4) three scalar equations with nine unknown coefficients. Our procedure for resolving this problem is to use multiple sources, exploiting the fact that the reflection matrix is independent of the values of the incident wave amplitudes.

Suppose that we have wave amplitudes for \( K \) different source arrangements on the structure. The result for all the sources are combined into the one matrix equation

\[
\mathbf{V}_k^{\text{ref}} = \mathbf{R} \mathbf{V}_k^{\text{inc}},
\]

(6)

where

\[
\mathbf{V}_k^{\text{ref}} = [V_1^{\text{ref}}, \ldots, V_K^{\text{ref}}] \quad \text{and} \quad \mathbf{V}_k^{\text{inc}} = [V_1^{\text{inc}}, \ldots, V_K^{\text{inc}}].
\]

(7)

When the number of sources \( K \) is equal to 3 (or 2 for \( n = 0 \)), the matrix \( \mathbf{V}_k^{\text{inc}} \) can be inverted, if it is not singular or ill-conditioned, and the desired solution found:

\[
\mathbf{R} = \mathbf{V}_k^{\text{ref}} \mathbf{V}_k^{\text{inc}}^{-1}.
\]

(8)

For larger \( K \), a matrix pseudo-inverse is used.

These mathematical statements have interesting physical interpretations. First, the matrix \( \mathbf{V}_k^{\text{inc}} \) is singular if its columns are not linearly independent. This would imply that the mixes of incident wave types achieved in the different experiments are not sufficiently distinct; an examination of the mixes achieved may suggest a more desirable source. Second, postmultiplying \( \mathbf{V}_k^{\text{inc}} \) by any matrix is equivalent to synthesizing new compound sources by superposing the original sources (and their responses) in proportion to the columns of that matrix. If the multiplier is the inverse of \( \mathbf{V}_k^{\text{inc}} \), so that the product is the identity matrix, each of those synthetic sources generates a new incident field in which only a single wave type is nonvanishing. We have then achieved those ideal sources which were described in the Introduction.

D. Results

Figure 6 shows the amplitudes of reflection coefficients for the \( n = 0 \) flexural and longitudinal waves for a rigid hemispherical termination, a steel hemispherical termination, and a steel ‘‘conical-spherical’’ termination. Note the dramatic effects of termination shape as well as flexibility in the direct reflection coefficients (flexural-to-flexural and longitudinal-to-longitudinal). Energy considerations restrict these direct reflection amplitudes to be equal to or less than unity.

IV. RECIPROCITY CONSTRAINTS

One is aware from many studies that the principle of reciprocity constrains the results of any two transmission measurements in which the directions of transmission are reversed (Ref. 10, p. 550 ff.). The objective of this section is to develop analytical expressions of those constraints as applied to the reflection coefficients. The classical statement of reciprocity (Ref. 11, Volume 1, p. 150 ff.) involves transmission between two transducers, each of which can be used either as a source or as a receiver. We shall define a canonical reflection problem of that type, using ring transducers on a semi-infinite shell. However, both incident and reflected fields then involve all the propagating wave types. Further analysis is necessary to find the constraints on individual reflection coefficients, each of which corresponds to a single incident wave type and another single reflected wave type.

A. Transducers

We conceptualize the ideal structural transducer as a ring located at some axial coordinate \( z = z_0 \) that, as a source, exerts a localized normal traction in the outward radial direction and, as a receiver, measures the outward radial velocity. Transducers having different circumferential orders \( n \) are required.

As a source, the transducer of order \( n \) applies a generalized force \( F_n \) at an axial coordinate \( z_0 \) by generating a normal stress or pressure

\[
p(z, \phi) = F_n \delta(z - z_0) \Phi_n(\phi),
\]

(9)
having the ‘‘mode shape’’

$$\Phi_n(\phi) = \frac{e_n}{2\pi a} \cos(n \phi), \quad (10)$$

where $e_n$ is the Neumann factor, equal to 1 for $n = 0$ and 2 for $n > 0$. (There is no consensus on the normalization of terms in a Fourier series expansion; the one adopted here may differ from that used in any particular finite element program.)

As a receiver excited by an arbitrary radial velocity field $v(z, \phi)$, this transducer’s output is the generalized velocity

$$V_n = \int_0^{2\pi} v(z_0, \phi) \Phi_n(\phi) a \ 2 \pi a \ d\phi, \quad (11)$$

corresponding to the series expansion for a general velocity field having symmetry in $\phi=0$ in the form

$$v(z_0, \phi) = \sum_{n=0} \ V_n \cos(n \phi). \quad (12)$$

We pause to verify that the normalization of the generalized variables demanded by the principle of reciprocity has been achieved. [See Rayleigh (Ref. 11, p. 157). Rayleigh expresses this constraint on normalization in the equivalent terms of work done rather than complex power.] The surface integral for time-averaged complex power is readily found to satisfy

$$\frac{1}{2} \int \int p(z, \phi) \superscript{*} w(\phi) a \ 2 \pi a \ d\phi \ dz = \frac{1}{2} F_n V_n^* \quad (13)$$

where the superscript asterisk denotes complex conjugation. The form on the right side is the desired standard expression for complex power in terms of the generalized variables.

For complete generality, one should simultaneously require transducers with different angular symmetry [$\sin(n \phi)$ as well as $\cos(n \phi)$] and with different orders $n$, since reflection from a general, non-axisymmetric termination could involve both symmetries and all orders. The added complexity, easily achieved, is unnecessary in the case at hand where the structure is axisymmetric. One could also generalize the analysis by considering transducers that apply different tractions (e.g., axial) to the shell. The reciprocity constraints would be no different, so long as the proper normalization of generalized variables is maintained [cf. Eq. (13)]. In the present case, nothing is gained by that procedure since all wave types except the $n = 0$ shear wave have nonvanishing radial motion. Explicit denotation of the circumferential order by means of the subscript $n$ will be dropped in the subsequent discussion.

### B. Canonical reflection problem

The canonical reflection problem for coupling between different structural wave types consists of a semi-infinite shell closed by the target termination, with two ring transducers on the shell at a distance from the termination (see Fig. 7). The transducers numbered 1 and 2 are located at large distances $d_1$ and $d_2$ from the termination, so that the evanescent nearfields of the transducers do not couple significantly with the termination. Without loss of generality, we assume that both transducers are of the same type (radial force, axial force, etc.).

Now consider two reciprocal experiments, $a$ and $b$, in which first one then the other transducer is active as a source, exerting generalized forces $F_1$ and $F_2$. The inactive transducer acts as a receiver, measuring generalized velocities $V_2$ and $V_1$, respectively. The mathematical expression of reciprocity is

$$[V_2/F_1]_a = [V_1/F_2]_b. \quad (14)$$

In the absence of trust in Lord Rayleigh, one may readily derive Eq. (14) from Eqs. (9)–(12) with the condition of point-to-point reciprocity in the Green’s function.

The measured velocities in each of these two experiments is the sum of two parts: direct transmission from the source (superscript ‘‘dir’’ below) and the field reflected from the termination (superscript ‘‘ref’’ below). This decomposition is expressed in the equations

$$[V_2]_a = [V_2^{\text{dir}} + V_2^{\text{ref}}]_a \quad \text{and} \quad [V_1]_b = [V_1^{\text{dir}} + V_1^{\text{ref}}]_b. \quad (15)$$

The direct transmission is defined as the transmission for the same source and receiver on an infinite shell of the same design; the reflected transmission is the difference between the results on the semi-infinite and infinite shells.

But, measurements on the infinite shell are also governed by reciprocity, so that the direct components alone must satisfy

$$[V_2^{\text{dir}}/F_1]_a = [V_2^{\text{dir}}/F_2]_b, \quad (16)$$

whence the reflected components alone satisfy a similar equation:

$$[V_2^{\text{ref}}/F_1]_a = [V_2^{\text{ref}}/F_2]_b. \quad (17)$$

This is not the desired final form, mainly because the reflected fields include contributions from all propagating wave types, each of which may have been generated by all propagating wave types in the incident field. The next section unravels that complication.

### C. Multiple wave types

In both experiments, $a$ and $b$, the measured reflected field is a sum over components, each of which represents a signal propagating from the source to the termination in one wave type and propagating from the termination to the re-
ceiver in another wave type (or the same). Hence, the total reflected field is a double sum with exponential phase factors dependent on the distances $d_1$ and $d_2$; the exponential factors contain the complete dependence on those distances.

The result for experiment $a$ can be written as

$$\left[\frac{V_r}{F_1}\right]_{a} = \sum_{q} \sum_{r} Y_{q} R_{qr} e^{i(k_q d_2 + k_r d_1)},$$

(18)

where the admittance $Y_{r}$ is defined by the following form for the amplitude of the $r$-th component of the incident field that is generated by $F_1$ at a distance $d_1$:

$$V_{r}^a = F_1 Y_{r} e^{ik_r d_1}.$$  

(19)

Note that $Y_{r}$, a “source strength” as we shall call it, is independent of the distance from the source although dependent on the type of source and the wave type $r$. (Sometimes the response to the source on an infinite shell does not have left–right symmetry; a ring moment is a good example. Then the value of $Y_{r}$ also depends on the orientation of the source. Care is required to avoid mistakes.) Its evaluation is examined below.

The corresponding result for experiment $b$ is identical except that the distances $d_1$ and $d_2$ must be interchanged. For purposes of the argument it is desirable also to interchange the subscripts $r$ and $q$ inside the summations so that the exponential factor is unchanged; the result is

$$\left[\frac{V_b}{F_2}\right]_{b} = \sum_{q} \sum_{r} Y_{q} R_{rq} e^{i(k_q d_2 + k_r d_1)}.$$ 

(20)

Now comes the crux of the argument. Equation (17) indicates that the right sides of Eqs. (18) and (20) must be equal—their difference must vanish—at all values of $d_1$ and $d_2$. Since the natural wave numbers are distinct, it can then be argued (see the Appendix) that the constant coefficients of the variable exponentials must be equal term by term:

$$Y_{r} R_{qr} = Y_{q} R_{rq}.$$  

(21)

which can be written in the nondimensional form

$$R_{qr} Y_{r} Y_{q} = R_{rq} Y_{q} Y_{r}.$$  

(22)

D. Source strength

We turn to the evaluation of the wave type source strengths $Y_{r}$ defined by Eq. (19). For purposes of illustration, consider a radial ring source, as defined in Eq. (9), located at $z_0 = 0$ on an infinite fluid-loaded shell. We must evaluate the strength of the $r$th wave type at a large distance $d_1$ from the transducer when it exerts a generalized force $F_1$. For convenience, we evaluate the strength of the radial component of shell velocity, other components being related to it by the polarizations determined in Sec. II.

The analysis proceeds by standard means, using a Fourier transform in the $z$-coordinate. (For examples, see Refs. 1–3.) The transforms of the applied pressure and of the response radial velocity on the shell are related by a spectral impedance determined by the fluid-loaded shell equations:

$$\tilde{Z}(k) \tilde{v}(k) = \tilde{p}(k),$$  

(23)

where $k$ is the transform wave number. The transform of pressure defined by

$$\tilde{p}(k) = \int_{-\infty}^{\infty} p(z) e^{-ikz} dz$$  

(24)

is easily evaluated from Eq. (9), with the result

$$\tilde{v}(k) = F_1 \Phi_n(\phi) \tilde{Z}(k)^{-1}.$$  

The total velocity at a distance $d_1$ is given by the inverse transform of $\tilde{v}(k)$:

$$v(d_1, \phi) = F_1 \Phi_n(\phi) \frac{1}{2\pi} \int \frac{e^{ikd_1}}{\tilde{Z}(k)} dk.$$  

(25)

That part of the total velocity represented by the $r$th wave type is the pole contribution of the integral (closed in the upper half-plane) at the natural wave number $k_r$, a simple pole of the integrand where $\tilde{Z}(k)$ vanishes by definition. The pole contribution is readily found and written [using the notation of Eq. (19)] as

$$v(d_1, \phi) = V_{r} \cos(n \phi)$$

$$= i F_1 \Phi_n(\phi) e^{ik_r d_1} \left[ \frac{\partial \tilde{Z}(k)}{\partial k} \right]_{k=k_r}^{-1}.$$  

(26)

so that the reciprocal of the desired source strength is given by

$$\frac{1}{Y_r} = \frac{-i2\pi a}{e_n} \left[ \frac{\partial \tilde{Z}}{\partial k} \right]_{Z=0; k=k_r},$$  

(27)

where $e_n$ is the previously defined Neumann factor.

Surprisingly, an algebraic form can be developed for the derivative of the spectral impedance as a function of wave number, frequency, and $n$, so that the source strength is readily calculated. The spectral impedance is the sum of a shell impedance, algebraic in the variables, and a transcendental fluid impedance. The derivative of the shell part is easily expressed algebraically. The derivative of the fluid impedance can be shown, by using the differential equation governing Bessel functions, to involve only algebraic factors and the fluid impedance itself. But, at the wave number where the total impedance vanishes, the fluid impedance equals the negative of the shell impedance, so that the desired derivative of the total spectral impedance can be expressed algebraically.1

In the case of natural waves that are not attenuated by radiation or internal dissipation, it has been shown that the derivative with respect to wave number of the spectral impedance is the dominant factor in the characteristic wave impedance, which relates wave power to the square of response velocity. (See Ref. 12, Section 11.7, and Ref. 13 for further discussion.) Then, the square of Eq. (22) corresponds to requiring equality of the reflection coefficients for power. A similar interpretation for waves with nonvanishing attenuation has not been developed.

E. Results

The values of the reflection coefficients derived in the manner described in Sec. IV have been tested for their con-
sonance with the principle of reciprocity, as expressed in Eqs. (22) and (27). Specifically, Fig. 8 shows the absolute value of the ratio of reflection coefficients (longitudinal-to-flexural divided by flexural-to-longitudinal) predicted both by our numerical experiments and by reciprocity considerations. The agreement is remarkable in that no knowledge of the termination is needed for the reciprocity result.

V. CONCLUDING REMARKS

The literature of acoustics is replete with problems involving the reflections of waves on a waveguide by an isolated inhomogeneity, those results being needed as building blocks for the analysis of some realistic situation involving several discontinuities. The problem is usually selected so as to be amenable to classical methods of analysis. However, in the case at hand the existence of multiple traveling waves, significant fluid loading, and resonant behavior internal to the discontinuity all combine to impede a classical approach to the reflection problem.

We have demonstrated that the essentially analytic multiple-reflection model of response can be retained in such complicated problems, by developing a methodology for evaluating its parameters. Appropriate reflection coefficients are evaluated from an analysis of finite element results for a set of particular problems which share the same discontinuities. The results are tested for agreement with the analytical constraints consequent to the principle of reciprocity.

ACKNOWLEDGMENTS

This paper reports a part of a broader study of the dynamics of fluid-loaded structures supported by the Office of Naval Research, under the supervision and encouragement of Dr. Geoffrey L. Main. The insight of Dr. R. Preuss was critical with respect to the Appendix.

APPENDIX

When the results of the two reciprocal experiments, Eqs. (18) and (20), are subtracted, one obtains a function of the distances \( d_1 \) and \( d_2 \) which must vanish for any values of \( d_1 \) and \( d_2 \); it has the form of a double summation over the wave types:

\[
g(d_1,d_2) = \sum_{q,r=1}^{N} A_{qr} e^{ikqd_2} e^{ikrd_1} = 0, \quad (A1)
\]

where \( A_{qr} = Y_q R_{qr} - Y_r R_{rq} \) and \( N \) is the number of wave types. It is to be proven that \( A_{qr} \) must vanish identically for all values of \( q \) and \( r \).

For all pairs of integers \((n,m)\) ranging from 0 to \( N-1\), consider the derivatives

\[
G_{nm} = \frac{n^m m^m}{i^{n+m} \partial d_2^m \partial d_1^n}, \quad (A2)
\]

which have the form

\[
G_{nm} = \sum_{n,m} k_q^n q^m A_{qr} e^{i(kqd_2 + krd_1)}. \quad (A3)
\]

Since \( g \) is constant with value zero, these derivatives all vanish. The result can be written as the matrix equation

\[
G = V^T B V = 0, \quad (A4)
\]

where \( B_{qr} = [A_{qr} e^{i(kqd_2 + krd_1)}] \) and \( V \) is the Vandermonde matrix:

\[
V = \begin{bmatrix}
1 & k_1 & k_1^2 & \cdots & k_1^{N-1} \\
1 & k_2 & k_2^2 & \cdots & k_2^{N-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & k_N & k_N^2 & \cdots & k_N^{N-1}
\end{bmatrix} \quad (A5)
\]

A Vandermonde matrix in the form of \( V \) is known to be nonsingular, and thus to have an inverse, if \( k_q \neq k_r \) for every \( q \neq r \). Since the natural wave numbers are known to be distinct, it follows from Eq. \( (A4) \) that \( B \) equals the null matrix, \( A_{qr} = 0 \), and \( Y_q R_{qr} = Y_r R_{rq} \) for every \( r \) and \( q \), as was to be demonstrated.

5Lord Rayleigh (John William Strutt), Theory of Sound (Dover, New York, 1945), 2nd ed.