An Acoustic Variational Principle and Component Mode Synthesis Applied to the Analysis of Acoustic Radiation from a Concentrically Stiffened Plate

J. H. Ginsberg
J. G. McDaniel

Sc. J1 of Mechanical Engineering,
Georgia Institute of Technology,
Atlanta, GA 30338

This study concerns the effects of circumferential ring stiffeners on the vibration and acoustic radiation of a submerged circular elastic plate supported by an infinite rigid baffle. The analysis employs variational principles, in which surface pressure and velocity are represented by assumed modal functions. It is shown that the use of a Fourier series expansion of the Green's function, along with Bessel function pressure modes, greatly simplifies computation of the fluid-structure interaction. Two structural formulations are addressed, differing by whether the connection between the plate and stiffeners is implicitly or explicitly enforced. The results for surface response are used in conjunction with the Kirchhoff-Helmholtz integral theorem to evaluate near- and farfield pressures. An example of a single stiffener illustrates the influence of stiffener location on radiated power.

Introduction

Analysis of steady-state radiation from smooth, closed, finite surfaces has proceeded along several branches. Exact analytical solutions for the field variables are possible for radiating surfaces that conform to coordinate systems in which the wave equation can be separated. If the shape does not conform to a separable coordinate system, the most widely employed approach seems to be one of several boundary element formulations derived from the Kirchhoff-Helmholtz integral theorem, some of which are summarized by Chien et al. [1]. Problems in the initial formulations arose due to nonuniqueness at the eigenfrequencies of the interior Dirichlet or Neumann problem, but improvements have been made by combining a surface Helmholz integral formulation either with an interior Helmholtz integral [2], or with the normal derivative of the surface Helmholtz integral formulation [3].

Formulations based on finite element representations, often in conjunction with infinite elements, avoid difficulties at the characteristic frequencies of the interior domain, but this comes at the penalty of raising the number of state variables required for a simulation. A variational principle offers the prospect of developing a simulation that is, in a sense, optimal for a specified number of independent variables [4]. Many variational principles have been derived from Kirchhoff-Helmholtz integral corollaries of the wave equation, as summarized by Wu et al. [5], whose variational principle is the basis of this work. This variational principle, which considers variations in surface pressure for a given velocity, has been successfully applied by Ginsberg, Chen, and Pierce [6] to the axisymmetric vibration of a thin flexible disk (plate or membrane). In their model, the plate support was either an infinite baffle with fluid on one side or a rigid annulus with fluid on both sides. Their study used the method of assumed modes [7]; surface pressure and displacement were expanded in series of assumed modal functions with unknown amplitudes. The coefficients were found by inserting these series into the acoustic variational principle and into Hamilton's Principle, which led to a set of linear equations that are coupled in the pressure and displacement amplitudes. The results achieved by this analysis agreed well with those found by Alper and Magrab [8], who used spheroidal wave functions for pressure.

The present study is concerned with the extension of this approach to the analysis of bodies with discrete stiffeners, which shall be demonstrated by analyzing the problem of a thin circular plate in a stationary baffle to which concentric ring stiffeners are attached at various radii. The plate will be excited by a harmonically time-varying central point force, which results in an axisymmetric problem with no stiffener bending. For simplicity, numerical results will be given for a clamped plate in an infinite baffle with one stiffener, and compared to the unstiffened case.

The connection of an elastic body to stiffeners usually involves three effects: (1) additional mass of the connected stiffener, (2) a force exerted on the body by the stiffener, and (3)
a moment exerted on the body by the stiffener. The effect of a "concentric reinforcing ring" on a clamped circular plate was analyzed as early as 1948 by Nash [9], who found in the static case that maximum deflection of a clamped circular plate with a uniform surface pressure is reduced by the presence of the stiffener. Additional work has been done recently by Demes et al. [10] on the optimal design of stiffeners, again for the static case.

Seybert and Tsui [11] in their analysis of mass-loaded, slender, baffled beams concluded that adding a mass to a uniform body lowers the natural frequency and changes the mode shape. However, the quantitative effect of a stiffener on the radiation of a plate is not clear from the outset. We will see that no general rule applies to a vibrating plate with a stiffener. Indeed, adding a stiffener might increase the amplitude of vibration.

In formulating the structural equations of motion, the stiffeners act as potential and kinetic energy storage devices. The analysis can proceed along two different paths and arrive at the same result. The stiffeners can be treated as separate bodies using component-mode synthesis (CMS), according to which the transmitted forces and moments are unknown. Correspondingly, compatibility of displacement and rotation at each stiffener location must be enforced explicitly. Alternately, the plate and stiffener can be treated as a single structure, in what could be termed a unified structure (US) analysis. The energies of the plate must be altered to include the energy storage effects of the stiffeners, but the compatibility conditions are implicitly satisfied in such a formulation.

Although the problem will be formulated for an arbitrary number of stiffeners, the example will treat the case of a single stiffener in order to focus on the quantitative effects of stiffener positioning. It will be shown that placement of a stiffener of constant mass may increase or decrease radiation. Plots of surface pressure and velocity will show the effect of stiffeners whose location minimize or maximize radiated power. For comparison, the stiffener mass will be uniformly "smearred" over the surface of the plate, resulting in an increased plate thickness. In all cases, the thickness ratio (plate radius/plate thickness) will not be below 25, which Alper and Magrab [8] consider to be a thin plate. It must be noted, however, that the technique presented here could be easily altered to address vibration of a thick plate by using a Timoshenko-Mindlin shear correction [12].

1 Variational Principle for Acoustic Radiation

A thorough derivation of a variational formulation describing the fluid-structure interaction is presented by Ginsberg et al. [13]. Briefly, it involves regularizing the normal derivative of the surface Helmholtz integral according to the procedure of Maucc [14] and Stallybrass [15]. Multiplying the result by a virtual increment of surface pressure and integrating over all of the wetted surface leads to

\[ \delta J[p] = 0, \]

where the functional \( J \) is given by

\[ J[p] = \int_{\partial S} \left[ p(\xi) \left. U_\nu(\xi) \right| \right] d\Gamma \]

\[ + \int_{\Omega} \left[ \frac{k^2}{2} \int_{\partial S} \left[ \left. \left( n(\xi) \times \nabla p(\xi) \right) \cdot \nabla \left( p(\xi) \right) \right| \cdot \delta \left( n(\xi) \right) \right] d\Gamma \cdot d\Omega + \frac{1}{2} \int_{\partial S} \left[ \left. \left( n(\xi) \times \nabla \left( p(\xi) \right) \right) \cdot \nabla \left( p(\xi) \right) \right| \right] d\Gamma \cdot d\Omega \]

\[ = 0 \]

and \( G(\xi,\ell) \) is the free-space Green's function,

\[ G(\xi,\ell) = \frac{1}{R}, \quad R = |\xi - \ell|. \]

The term \( U_\nu(\xi) \) is associated with the surface velocity distribution. Its evaluation involves a limiting process which brings a field point \( x = \xi + e_n(\xi) \) to the surface.

\[ U_\nu(\xi) = \lim_{\epsilon \to 0} \frac{1}{4\pi} \int_{\partial S} u_\nu(\xi) \cdot \nabla G(x|\xi) d\Gamma. \]

The evaluation of \( U_\nu(\xi) \) was treated in the derivation of the variational principle [6]. The integral was decomposed there into the contribution of the singular domain, for which the smallness of \( R \) is crucial, and the nonsingular Cauchy principle part, which accounts for the region far away from \( \xi \). The latter vanishes for a plate because \( \mathbf{v}_\nu G(x|\xi) \) is tangent to the surface when \( x \) is situated on the surface. The contribution of the singular part is found to be \(-1/(2\pi u_\nu(\xi))\), corresponding to a \( 2\pi \) solid angle that is illuminated by a point on the surface. Hence, for a disk in an infinite planar baffle, one obtains

\[ U_\nu(\xi) = \frac{1}{2} \left. u_\nu(\xi) \right| \]

Since the plate is circular, a polar coordinate system, centered at the plate's center, is chosen for the above integration. It is convenient to nondimensionalize the system equations by using the plate radius \( a \), sound speed \( c \), and \( \rho c^2 \) to scale distance, velocity, and pressure, respectively. With \( \nu^* \) used to denote the nondimensional quantity, we define

\[ \eta = \frac{r}{a} = \text{radial distance from plate center to } \xi, \]

\[ \nu = \frac{R}{a} = \text{radial distance from plate center to } \xi, \]

\[ \hat{R} = \frac{R}{a} = (\rho^2 + \eta^2 - 2\rho \eta \cos \theta)^{1/2} = \text{distance between } \eta_{\xi} \text{ and } \eta_{\xi}, \]

\[ \hat{b} = \frac{b}{a}, \quad \hat{\beta}(\hat{\rho}) = \frac{\beta(\rho)}{\rho c^2}, \quad \hat{\nu}_\nu(\xi) = \frac{\nu_\nu(\xi)}{c}. \]

where \( b \) is the baffle radius. Substitution of these definitions in equation (3) gives the following nondimensional Green's function:

\[ G(\xi,\ell) = (\omega) G(\xi,\ell) = \exp(ik\hat{R})/\hat{R}. \]

Gradients of the surface pressure are given in polar coordinates as

\[ \hat{\nabla} p(\xi) = \rho c^2 \frac{d}{d\theta} [\hat{\beta}(\hat{\rho})] e_{\ell}, \quad \hat{\nabla} \nu(\xi) = \rho c^2 \frac{d}{d\theta} [\hat{\beta}(\hat{\rho})] e_{\ell}. \]

Since the surface is flat, \( e_{\ell} \) is constant, therefore

\[ [n(\xi) \times \hat{\nabla} p(\xi)] - [n(\xi) \times \hat{\nabla} \nu(\xi)] = \left[ \frac{\rho c^2}{a} \right] \frac{d}{d\theta} [\hat{\beta}(\hat{\rho})] \frac{d}{d\theta} [\hat{\beta}(\hat{\rho})] \cos \theta. \]

Since all variables have been nondimensionalized, we will drop the \( \nu^* \) notation. Introducing equations (5)-(9) into equation (2) gives the nondimensional form of the variational principle for a fluid loaded flat plate in an infinite baffle,

\[ J = \frac{1}{8\pi} \int_0^{2\pi} \left[ (ka)^2 C_1(r\rho') d\rho' \right] \left[ \frac{d}{d\theta'} [p(\rho')] d\theta' \right] d\theta \]

\[ - C_2(r\rho') \left[ \frac{d}{d\theta'} [p(\rho')] d\theta' \right] d\theta' \]

\[ \frac{1}{4\pi} \int_0^{2\pi} p(\rho) \nu_\nu(\rho) d\theta, \]

where \( C_1 \) and \( C_2 \) are the integrated Green's functions, given by

\[ C_1(r\rho') = \int_0^{2\pi} [G(\xi,\ell)] d\theta \]

402 / Vol. 113, JULY 1991

Transactions of the ASME
and
\[ C_2(r|r') = r^2 \int^{2\pi}_0 [G(\xi_1|\xi_2) \cos \theta] d\theta. \]

Both terms have a singularity at \( r = r' \). Their evaluation was described in detail in the earlier analysis of an unstimmed plate [6]. However, the evaluation of these coefficients may be simplified substantially through the use of a Fourier series expansion of the Green's function. This simplification is described in Section VI.

2 Fluid-Structure Coupling: Analysis of the Surface

The variational principle addresses the surface pressure that arises from a specified motion of the wetted surface. In order to analyze this coupling, surface pressure and normal velocity will be represented by series expansions in a set of basis functions in the method of assumed modes. Surface pressure is written as
\[ p(r) = \sum_{j=1}^{N} p_j \phi_j(r), \]
where \( p_j \) will be referred to as a complex pressure amplitude and \( \phi_j(r) \) is a nondimensional basis function.

Each basis function is an admissible function that need only satisfy geometric (imposed) boundary conditions. Since the infinite baffle is unbounded in the \( r \) direction, the surface pressure must decay with increasing \( r \). This condition is satisfied in the present formulation by selecting a distance \( b = a \) beyond which the pressure is considered to vanish. (Selecting \( a \) such that \( b = a \) is one acoustic wavelength is generally adequate, in the sense that the solution obtained from the variational principle with larger \( a \) indicates that the pressure at this distance is small.) Therefore, the condition imposed on the pressure functions is
\[ \phi_j(r) = 0; \ r \geq a. \]

Substitution of the pressure expansion into equation (10) yields the following quadratic sum:
\[ J = \frac{1}{2} \sum_{j=1}^{N} \sum_{j=1}^{N} A_{ij} p_j p_i - \sum_{j=1}^{N} B_j p_j, \]
where
\[ A_{ij} = A_{ij} = \int_{a}^{b} \left\{ \frac{1}{2} \left[ (k\omega)^2 C_1(r|r') \phi_j(r) \phi_j(r') \right. \right. \]
\[ - C_2(r|r') \left. \frac{d}{dr} \phi_j(r) \right]^{r'}_{r} \left. \left. \frac{d}{dr} \phi_j(r'). \right\} \right\} r' dr' dr \]
\[ B_j = 2\pi ika \sum_{j=1}^{N} \phi_j(r) v_n(r) \rangle dr. \]

In order to impart an infinitesimal variation to \( J \), the pressure amplitudes \( p_j \) are given a virtual increment, while the normal velocity remain constant. Employing equation (1), we obtain
\[ \frac{\partial J}{\partial p_j} = 0; \ j = 1, 2, \ldots, N \]
from which the following matrix equation for the complex pressure amplitudes is obtained:
\[ [A] \left\{ \begin{array}{c} p \\ \end{array} \right\} = [B]. \]  \( \quad \) (18)

The transverse displacement of the plate, whose dimensional value is \( \omega w \), will also be expressed in an assumed mode series,
\[ w(r) = \sum_{j=1}^{M} q_j \phi_j(r), \]
where \( q_j \) is a generalized coordinate, \( \phi_j(r) \) is an assumed plate displacement mode, and \( M \) is the number of plate displacement modes. Each \( \phi_j(r) \) is an admissible function which must satisfy the imposed boundary conditions,
\[ \phi_j(r) = 0; \ r \geq 1. \]  \( \quad \) (20)
For a plate with a clamped edge, the boundary conditions imposed on \( \phi_j \) are
\[ \phi_j(1) = 0 \quad \text{and} \quad \frac{\partial \phi_j(r)}{\partial r} \bigg|_{r=1} = 0. \]  \( \quad \) (21)
Due to the \( \exp(-i\omega t) \) time dependence, the normal velocity derived from equation (19) is
\[ v_n = (-ika) \sum_{j=1}^{M} q_j \phi_j(r). \]  \( \quad \) (22)

Substitution of this velocity distribution into equation (16) leads to a representation of each coefficient \( B_j \) as a linear combination of the \( q_j \). The corresponding partitioned matrix form of equation (18) is
\[ \left[ \begin{array}{c} \{A\} - 2\pi(k\omega)^2 [H]^T \{P\} \\ \{Q\} \end{array} \right] = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right]. \]  \( \quad \) (23)
where \( \gamma_j \) is identified as the modal coupling factor
\[ \gamma_j = \int_0^1 \phi_j(r) \phi_j(r) r dr \]
\( \quad \) (24)

3 Vibrational Energies of Plate and Stiffeners

The equations of motion for the stiffened plate are derived by using Hamilton's Principle. First, it is necessary to formulate the kinetic and potential energies of the plate stiffeners. Additionally, the virtual work done by the surface pressure and the central harmonic forcing function must be calculated. Figure 1 shows displacement and rotation variables for the plate and a stiffener. According to classical plate theory, which neglects the effects of transverse shear and rotatory inertia, the kinetic and potential energies of the plate are given by
\[ T_p = \frac{1}{2} \int_0^1 \rho_p u^2 r dr \]
and
\[ V_p = \frac{1}{2} (2\pi k) \int_0^1 \left( \frac{\partial w}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial r} \right)^2 r dr. \]  \( \quad \) (25)

where \( \rho_p \) is the density of the plate, \( h \) is the plate thickness, and \( K \) is the flexural rigidity of the plate.

We consider the cross-sectional dimensions of the stiffener to be comparable to the plate thickness. The deformation of the stiffener is considered to depend on the displacement and rotation of the centroid. Figure 1 shows the movement of stiffener \( n \) as its cross-section rotates through an angle \( \theta_n \). The
The virtual work done by the central harmonic forcing function is given by

\[ \delta W_f = aF(t) \sum_{j=1}^{M} \delta q_j \phi_j(0). \]  

(35)

The general form for the virtual work due to the central harmonic forcing function is given by

\[ \delta W_f = 2\pi a^3 p_c \sum_{j=1}^{M} (Q)_j \delta q_j, \]  

(36)

where \((Q)_j\) is obtained by matching equations (36) and (35),

\[ (Q)_j = \phi_j(0) \left( \frac{F_j}{2\pi a^3 p_c} \right) \]  

(37)

For numerical results, we will set the quantity \(F_j/2\pi a^3 p_c\) equal to unity.

4 Unified Structure Approach

The foregoing has not addressed the plate-stiffener interaction. The US approach treats stiffeners and the plate as a single structural entity. This requires that stiffener displacement and rotation be implicitly compatible with the plate motion, which is achieved by evaluating the displacement \(w\) in equation (19) and its derivative at a stifferner location in order to describe the displacement and rotation of that stifferner. Specifically, this means that for stifferner \(n\) at \(r_n\), the displacement \(w_n(t)\) and rotation \(\theta_n\) are given by

\[ w_n = \sum_{j=1}^{M} q_j \phi_j(r_n) \]  

and

\[ \theta_n(t) = \frac{\partial w(r_n, t)}{\partial r} \]  

(38)

The kinetic energy of the system (stiffeners and plate) has the form of a quadratic sum,

\[ T = T_s + T_p = \pi \rho_a h^2 \sum_{i=1}^{M} \mu_{ij} \phi_i \phi_j, \]  

(39)

where \(\mu_{ij}\) is identified from equations (25), (30), and (39) to be

\[ \mu_{ij} = \int_{0}^{L} \left[ \eta \phi_i(r) \phi_j(r) \right] dr + \sum_{m=1}^{L} \bar{M}_m \phi_i(r_m) \phi_j(r_m) \]  

(40)

where \(L\) is the total number of stiffeners attached to the plate. Similarly, the potential energy of the system is given as a quadratic sum of the form

\[ V = V_p + V_s = \pi \rho_a h^2 E \sum_{i=1}^{M} \sum_{j=1}^{M} \kappa_{ij} q_i q_j, \]  

(41)

where \(\kappa_{ij}\) is found from equations (25), (30), and (41) to be

\[ \kappa_{ij} = \int_{0}^{L} \left[ \eta \phi_i(r) + \frac{1}{r} \phi_i'(r) \right] \left[ \phi_j(r) + \frac{1}{r} \phi_j'(r) \right] dr \]  

(42)

Substitution of kinetic energy, potential energy, and equations (37) and (34) for the generalized forces into Lagrange’s equations gives the following equation for structural motion:
The combination of these equations with equation (24) may be written in partitioned matrix form as

\[
\begin{bmatrix}
\{A\} & -2\pi(k_a)^2 \{I\} \{p\}
\end{bmatrix} \begin{bmatrix}
\{q\}
\end{bmatrix} = \begin{bmatrix}
\{0\}
\end{bmatrix},
\]

with [D] identified as the dynamic stiffness matrix, given by

\[
D = \frac{E}{\rho c^2 \kappa} \begin{bmatrix}
\frac{\partial (k_a)^2}{\partial \rho} & \frac{\partial (k_a)^2}{\partial \rho}
\end{bmatrix},
\]

(45)

It is necessary to solve the above set of simultaneous equations for the unknown pressure and displacement modal amplitudes. Substitution of these amplitudes into equations (12) and (19) will yield expressions for the surface pressure and displacement at the surface of the plate. Stiffener rotation and displacement are then evaluated using equation (38). Observe that this set of equations is the same form as those obtained previously for an unstiffened plate [6], the only difference being the dynamic matrix [D].

5 Component Mode Synthesis Approach

An alternative approach is CMS, in which the movement of the stiffeners is treated as independent of the plate. As such, the displacement \(w_\rho\), rotation \(\theta_\rho\) of each stiffener \(n\) is treated as independent. Continuity of stiffener and plate displacement and rotation is enforced explicitly through geometric constraint equations. In addition, constraint forces and moments exerted between the stiffeners and the plate must be accounted for in the virtual work term.

The general form for the kinetic energy is given by

\[
T = \frac{\pi}{2} \rho c^2 \left[ \sum_{\rho} \sum_{\alpha=1}^{M} (\mu_\rho) \frac{\partial^2 w_\alpha}{\partial \rho^2} \right] + \frac{\pi}{2} \rho c^2 \left[ \sum_{\alpha} \sum_{\rho=1}^{M} (\mu_\rho) \frac{\partial^2 w_\alpha}{\partial \rho^2} \right],
\]

where equations (25), (30), and (46) lead to

\[
\begin{align*}
\mu_\rho & = \int \phi (r) \phi (r) r dr, \\
(\mu_\rho)_n & = M_n, \\
(\mu_\rho)_n & = M_n, (\kappa_\rho)_n = (\kappa_\rho)_n, \\
(\mu_\rho)_n & = M_n, (\kappa_\rho)_n = (\kappa_\rho)_n.
\end{align*}
\]

The general form for the strain energy of the system is given by

\[
V = \frac{\pi}{2} \rho c^2 \left[ \sum_{\rho} \sum_{\alpha=1}^{M} (\kappa_\rho) \frac{\partial^2 \delta_\rho}{\partial \rho^2} \right] + \frac{\pi}{2} \rho c^2 \left[ \sum_{\alpha} \sum_{\rho=1}^{M} (\kappa_\rho) \frac{\partial^2 \delta_\rho}{\partial \rho^2} \right],
\]

where the coefficients are found from equations (25), (30), and (48) to be

\[
\kappa_\rho = \frac{K}{E h c^2} \int \left[ \phi (r) + \frac{1}{r} \frac{\partial \phi (r)}{\partial r} \right] \left[ \phi (r) + \frac{1}{r} \frac{\partial \phi (r)}{\partial r} \right] dr,
\]

(49)

and

\[
(\kappa_\rho)_n = (\kappa_\rho)_n.
\]

In addition to the virtual work due to the surface pressure and central force, there is work done on the plate by the stiffener and on the stiffener by the plate. Recall that both forces and moments exist between the plate and the stiffeners. Therefore, a general form for the total virtual work done on the plate and stiffeners is

\[
\delta W = \delta W_f + \delta W_p + \delta W_c,
\]

(32)

where \(\delta W_f\) is given by equation (35), \(\delta W_p\) is given by equation (32), and \(\delta W_c\) is the work of the constraint forces. The latter is given by

\[
\delta W_c = \sum_{\rho} \sum_{\alpha=1}^{M} (\delta \beta_\rho) \frac{\partial^2 \delta_\rho}{\partial \rho^2} + \frac{\pi}{2} \rho c^2 \left[ \sum_{\alpha} \sum_{\rho=1}^{M} (\kappa_\rho) \frac{\partial^2 \delta_\rho}{\partial \rho^2} \right],
\]

(44)

where the generalized forces are

\[
(\delta q_\rho) = -2\pi \rho c^2 \left[ \begin{array}{c}
\delta \beta_\rho \\
\delta \beta_\rho \\
\end{array} \right] \frac{\partial^2 \delta_\rho}{\partial \rho^2},
\]

(51)

Substitution into Lagrange's equations of the kinetic energy, potential energy, and generalized forces gives a matrix equation for structural vibration. Combination of these equations with equation (18) gives the overall partitioned matrix equation

\[
\begin{bmatrix}
\{ [A] \} \{ [B] \} \{ [C] \} \\
\{ [D] \} \{ [E] \} \{ [F] \}
\end{bmatrix} \begin{bmatrix}
\{ p \}
\{ q \}
\end{bmatrix} = \begin{bmatrix}
\{ 0 \}
\{ 0 \}
\end{bmatrix},
\]

(53)

The matrices forming the partitions are

\[
\begin{bmatrix}
\{ [A] \} \{ [B] \} \{ [C] \} \\
\{ [D] \} \{ [E] \} \{ [F] \}
\end{bmatrix} = \begin{bmatrix}
\{ [A] \} \{ [B] \} \\
\{ [D] \} \{ [E] \}
\end{bmatrix},
\]

(54)

In these definitions, \([D]_x\) and \([D]_y\) are dynamic stiffness matrices for the collection of stiffeners

\[
[D]_x = \frac{\partial^2}{\partial \alpha^2} (\mu_\rho) \frac{\partial^2 \delta_\rho}{\partial \rho^2},
\]

(55)

while \([F]_x\) and \([F]_y\) represent the plate displacement and rotation at stiffener locations,

\[
[F]_x = \phi (r) \frac{\partial^2 \delta_\rho}{\partial \rho^2},
\]

(56)

In addition, \([I]\) denotes a unit matrix, \([A]\) is given by equation (15), \([I]\) is given by equation (24), and \([D]\) is given by equation (45).

As mentioned previously, the US analysis is mathematically equivalent to CMS. This can be shown by solving the last partition row of equation (53) for \([c]\) and the second partition row for \([s]\) and then using the results to simplify the first partition row. This will yield equation (44).

6 A Simplification for Bessel Function Modes

It was noted earlier that the evaluation of the integrated Green's functions given in equation (11) can be greatly simplified through the use of a Fourier series expansion of the Green's function. Consider two points \(x'\) and \(x\) whose cylindrical coordinates are \((\alpha', \theta', z')\) and \((\alpha, \theta + \theta', z)\), corresponding to separation distance \(dR\). The expansion given by Morse and Ingard [17] may be written in nondimensional form as
\[ G(\xi | \zeta) = \frac{1}{R} \exp(iR) = \sum_{m=0}^{\infty} i(2 - \delta_m) \cos(m\theta) \]
\[ \times \int_0^{\pi} [J_m(\mu k_r')]I_m(\mu k) \exp[i\alpha_k l(\zeta - \zeta')] \frac{d\mu}{\sigma}, \quad (57) \]

where
\[ \sigma = \begin{cases} (1 - \mu^2)^{1/4}, & 0 \leq \mu \leq 1, \\ (\mu^2 - 1)^{1/4}, & \mu > 1. \end{cases} \quad (58) \]

The integral can be split into two parts extending over the ranges given in equation (58). Accordingly, define
\[ \lambda = \eta (\mu^2 - 1)^{1/4}; \quad \mu > 1, \]
which enables one to write the two parts of the integral as
\[ I_m = \int_0^{\pi} [J_m(\mu k_r')I_m(\mu k) \exp[i\alpha_k l(\zeta - \zeta')] \frac{d\mu}{\sigma}, \quad \text{and} \]
\[ II_m = \int_0^{\pi} [J_m(\mu k_r')I_m(\mu k) \exp[-\lambda k_l(\zeta - \zeta')] \frac{d\mu}{\lambda}. \quad (59) \]

Further, the singularities in the denominators of these integrals may be removed by changing variables such that \( \mu = \sin \psi \) for \( I_m \) and \( \mu = \sinh \psi \) for \( II_m \). When equation (57) is used to form the integrated Green's function in equation (11), one finds that only the \( m = 0 \) azimuthal harmonic contributes to \( C_1 \), and only the \( m = 1 \) harmonic contributes to \( C_2 \). Then the integrated Green's functions for a flat plate are reduced to
\[ C_1(r') = 2\pi(ka)r' \quad (60) \]
\[ C_2(r') = 2\pi(ka)r' \quad (61) \]

The virtue of describing \( C_1 \) and \( C_2 \) in the foregoing manner is that it makes it possible to reduce the evaluation of the coefficients \( A_{ij} \) from a double to a single integration if Bessel functions are selected for surface pressure modes. Specifically, if
\[ \psi_j(r) = J_0(\alpha_j r/\sigma); \quad J_0(\alpha_j) = 0, \quad (62) \]
then the double integrals for \( A_{ij} \) are reduced to two single integrals, given by
\[ A_{ij} = A_{ij} = (ka)^2 A_{00} - A_{ij}, \quad (63) \]

where
\[ A_{ij} = \left[ \begin{array}{l} 2\pi(ka) \int_0^{\pi} F_0(\sigma \sin \psi, \alpha_j, \sigma) F_1(\sigma \cos \psi, \alpha_j, \sigma) \sin \psi \, d\psi \\ + 2\pi(ka) \int_0^{\pi} F_0(\sigma \cos \psi, \alpha_j, \sigma) F_1(\sigma \sin \psi, \alpha_j, \sigma) \cos \psi \, d\psi \end{array} \right] \]

and \( F_0 \) is given by
\[ F_0(\sigma \alpha_k, \alpha_j, \sigma) = \begin{cases} \sigma \alpha_k^2, & \sigma \alpha_k \neq \alpha_j, \\ \frac{\sigma \alpha_k^2 - (\sigma \alpha_k)^2}{2\sigma \alpha_k^2 F_1(\sigma \alpha_k)^2}, & \sigma \alpha_k = \alpha_j, \end{cases} \quad (64) \]

Note that simplification is analogous to the manner in which the Rayleigh integral for radiation from a circular projector in an infinite baffle is converted to the King integral. Both transform a double integral to a single integral through a Fourier series expansion of the Green's function.

7 Numerical Results

Since CMS yields explicit information about the plate-stiffener interaction, it provides a useful tool for evaluating a stiffener's effect. Also note that the most computationally intensive partition above is \( I^1 \), which does not depend on the stiffener configuration. Therefore, it is possible to consider alternative stiffener configurations at a relatively low cost in computer time by saving \( I^1 \).

There are many parameter studies that could be performed, since this analysis has provided for an arbitrary number of stiffeners, each with four parameters: translational inertia, rotational inertia, stiffness, and radius. The study presented here involves placing a physically reasonable stiffener at various radii and calculating the effects on radiated power. In designing a plate and stiffener system, one may wish either to decrease or to increase radiated power. It will be shown that, for the stiffener criteria chosen, there are two identifiable radii which correspond to minimum and maximum radiated power.

In order to have a physically reasonable stiffener, we choose the following criteria for a stiffener:

- The stiffener mass must be equal to half of the plate mass.
- The cross-sectional area of the stiffener must be geometrically reasonable.
- The natural frequency of an isolated stiffener in axisymmetric rotation, \( \theta_0 \) constant (see Fig. 1), must equal the natural frequency of the plate.

For a plate thickness \( a/h = 37.5 \), these criteria yield the following numerical values for the stiffener:
\[ M_{\text{th}} = 0.25, \quad (\beta_h) = 0.00002963, \quad \text{and} \quad (\beta_h) = 6.350 \times 10^{-7}. \quad (65) \]

Radiated power may be calculated by integrating over the surface of the disk according to the following expression:
\[ \tilde{P}_{\text{rad}} = \frac{1}{2} \text{Re} \left[ \int \left[ \nabla \cdot (\sigma \tau \nabla \phi(r)) \right] dA \right], \quad (66) \]

The CMS approach was programmed and used along with equation (66) to give radiated power as a function of stiffener placement \( r \). The results are plotted in Fig. 2. It was found that 35 fluid modes \( (N = 35) \) and 15 displacement modes \( (M = 15) \) gave a convergent solution for a \( ka = 3.35 \).

It should be mentioned that, due to the small stiffener cross-section, the stiffener was found to store negligible strain energy—the only effect of a stiffener with the above properties was its capacity to store kinetic energy. However that one capacity enabled the stiffener to raise and lower the radiated power when compared to the smeared plate, in which stiffener mass is redistributed uniformly over the surface of the plate.

Figures 3, 4, and 5 show surface velocity and pressure for

![Fig. 2 Amplitude of time-averaged radiated power as a function of stiffener radius \( r \), for \( ka = 3.35 \). The stiffener has \( M_{\text{th}} = 0.25 \), \( (\beta_h) = 0.00002963 \), and \( (\beta_h) = 6.350 \times 10^{-7} \). The power radiated from the smeared plate was 149.](attachment:image.png)
Fig. 3 Complex surface pressure and velocity amplitudes as a function of radial distance for Case A: stiffener mass smeared ($ka = 3.35$ and $a/h = 25$). ---: surface pressure, - - - : surface velocity

Fig. 4 Complex surface pressure and velocity amplitudes as a function of radial distance for Case B: maximum power radiation corresponding to stiffener placement at $r_1 = 0.244$ ($ka = 3.35$ and $a/h = 37.5$). ---: surface pressure, - - - : surface velocity

Fig. 5 Complex surface pressure and velocity amplitudes as a function of radial distance for Case C: minimum power radiation corresponding to stiffener placement at $r_1 = 0.002$ ($ka = 3.35$ and $a/h = 37.5$). ---: surface pressure, - - - : surface velocity

Note that the largest internal force and moment correspond to the case of maximum power radiation. Additionally, the nondimensional in-vacuo natural frequencies below 4.0 were calculated ($k_1, a = \omega_d / c$) for comparison with the nondimensional driving frequency ($ka = 3.35$). The in-vacuo natural frequencies were found to be:

<table>
<thead>
<tr>
<th>Mode</th>
<th>$ka$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case A</td>
<td>Case B</td>
</tr>
<tr>
<td>1</td>
<td>0.48</td>
</tr>
<tr>
<td>2</td>
<td>1.8</td>
</tr>
<tr>
<td>3</td>
<td>4.1</td>
</tr>
<tr>
<td>4</td>
<td>7.4</td>
</tr>
</tbody>
</table>

Recall that fluid loading lowers the frequency for peak response. In view of this, it seems likely that the enhanced radiation at $ka = 3.35$ in Case B is attributable to the proximity of that frequency to a “fluid-loaded resonance.”

8 Conclusions

Among the modern techniques for studying fluid-structure interaction, variational principles hold a special position. One advantage is the ability to benefit from a priori knowledge of the physics of the problem. This study has shown how theory can be applied to systems with discrete stiffeners. The combination of a variational principle for surface pressure and Hamilton’s principle for structural motion was well suited to the inclusion of stiffeners into the model. In fact, many possible configurations could be analyzed with very little computation time (solving linear equations). While the Unified Structure approach showed how to treat a plate and its attached stiffeners as a single system, Component Mode Synthesis was preferred for the purpose of examining stiffener-plate interaction.

Numerical results were presented which show that it is possible to position a stiffener to maximize or minimize radiation...
for a given wavenumber. Finally, the Kirchhoff-Helmholtz integral was evaluated numerically to give the near- and farfield radiation patterns. Future work could use the present development to address the possibility of using stiffeners to shape radiation patterns.

Acknowledgments

This work was supported by the Office of Naval Research, Code 1132-SM.

References


16 Carlson, R. M., Combined Bending and Torsion in Circular Rings of Thin-Walled Open Section, University Microfilms, Ann Arbor, Michigan, 1969, pp. 79-80.