In search of the intuitive notion of logical consequence

Catarina Dutilh Novaes
Faculty of Philosophy – Leiden University

I. Introduction

After decades of predominant focus on the notion of logical truth, the debate on the concept of logical consequence among philosophers and logicians was re-ignited by J. Etchemendy’s book *The Concept of Logical Consequence* (1990). His main tenet was that the model-theoretic notion of logical consequence did not capture adequately the corresponding intuitive notion. One of Etchemendy’s central claims was that the intuitive notion could be understood essentially from two different perspectives, one representational and one interpretational – and that the model-theoretic notion failed to match either.

Some years ago, S. Shapiro (1998) sought to vindicate the model-theoretic notion of logical consequence; one of his arguments was that the dichotomy representational/interpretational notion of logical consequence was in a certain way infelicitous, since, according to him, a faithful rendering of the intuitive concept would have to have elements of both notions. Clearly, the resolution of issue as to whether the model-theoretic notion correctly captures the intuitive notion presupposes an at least minimally adequate characterization of this intuitive notion. Shapiro claimed that Etchemendy hadn’t really provided such a characterization\(^1\), and attempted to formulate one himself. He further claimed that, thus characterized, the intuitive notion was indeed correctly captured by the model-theoretic notion (albeit with some adjustments).\(^2\)

In this paper, I do not discuss Shapiro’s defense of the model-theoretic notion; rather I examine his contention that the best rendering of the intuitive notion of logical consequence is what he called the ‘conglomeration’ notion, that is, the hybrid notion that combines both the representational and the interpretational view on consequence. More specifically, I claim that such a hybrid view was held by the medieval logician John Buridan (Cf. Hubien 1976 – henceforth TC), and that this fact offers significant historical support to Shapiro’s version of the intuitive concept of (logical) consequence.

II. Four notions of logical consequence

Both in Etchemendy’s book and in Shapiro’s article, four presumably distinct notions of logical consequence are at issue: the (elusive) intuitive, pre-theoretic notion\(^3\), the model-

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\(^1\) Cf. Shapiro 1998, 143.
\(^2\) ‘My claim is that model-theoretic consequence can be made into a good model of this notion [the ‘hybrid’ intuitive notion of logical consequence] and that both the intuitive notion so characterized and its mathematical model are useful tools for shedding light on the normative/modal/semantic notion of correct reasoning in natural language, the target of logic.’ (Shapiro 1998, 148)
\(^3\) This variety of notions of consequence was already present in Tarski’s seminal ‘On the concept of following logically’ (Tarski 2002); Tarski referred to the concepts of *logical* consequence, *formal* consequence, *material* consequence and also to what he called in the Polish version of the text the
theoretic notion, the representational notion and the interpretational notion. The goal of most philosophers and logicians interested in this issue (among whom Etchemendy, Shapiro and Tarski\(^4\)) seems to be that of capturing the intuitive, pre-theoretical notion – the real notion, so to say – by means of suitable theoretical constructions (for example, one of the three other aforementioned notions).

How does one accomplish that? How can one compare different notions of (logical) consequence? The most straightforward way seems to be to focus on their extensions, that is, on the sets of pairs of antecedent and consequent considered to form valid (logical) consequences according to each notion. For this purpose, Etchemendy introduced the very useful concepts of overgeneration and undergeneration, which can be formulated as follows:

Consider two notions of logical consequence, \(A\) and \(B\). That a pair \(<K, X>\) (where \(K\) is a set of sentences and \(X\) an individual sentence\(^5\)) forms a valid consequence according to \(A\), \(B\) is represented, respectively, by \(K \models_A X\) and \(K \models_B X\); that a pair \(<K, X>\) does not form a valid consequence according to \(A\), \(B\) is represented, respectively, by \(K \not\models_A X\) and \(K \not\models_B X\).

- \(A\) overgenerates w.r.t. \(B\) \(\iff\) There is a pair \(<K, X>\) such that \(K \models_A X\) and \(K \not\models_B X\).
- \(A\) undergenerates w.r.t. \(B\) \(\iff\) There is a pair \(<K, X>\) such that \(K \not\models_A X\) and \(K \models_B X\).

In other words, if a notion \(A\) either overgenerates or undergenerates w.r.t. a notion \(B\), their extensions do not coincide. When one of the two notions is the desired one, undergeneration is not as threatening as overgeneration; for example, logical systems that are sound but not complete w.r.t a given semantics undergenerate w.r.t this semantics, but are still considered to be, to some extent, reliable. But if a logical system overgenerates w.r.t the semantics in question, then it is unsound, and that is of course reason enough to dismiss the system as utterly hopeless and flawed.

a. Model-theoretic notion

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\(^4\) 'The concept of following logically belongs to the category of those concepts whose introduction into the domain of exact formal investigations was not only an act of arbitrary decision on the side of this or that researcher: in making precise the content of this concept, efforts were made to conform to the everyday 'pre-existing' way it is used.' (Tarski 2002, 176).

\(^5\) 'everyday concept' of consequence (Stroinska and Hitchcock 2002, 165). Tarski also employed extensively the word 'intuition' connected with the use of the concept of consequence (Stroinska and Hitchcock 2002, 166). Spelling out how these different concepts relate to one another is one of the aims of the present discussion, but for the moment the reader must bear in mind that, even though my main interest is in the concept of logical consequence, I will occasionally refer to the concept of consequence tout court.

\(^5\) The issue of the relata involved in the relation of consequence is systematically overlooked in the tradition under discussion: are they wff’s of an uninterpreted language? Are they asserted sentences, or else (abstract) propositions? I discuss this crucial issue in (Dutilh Novaes 2004), but constraints of space do not allow me to go into further detail here.
The birth of the model-theoretic notion of logical consequence, as that of model-theory in general, is usually traced back to the works of Alfred Tarski. The historical accuracy of this view is not at issue in the present discussion; for the present purposes, it is sufficient to turn to a precise (and uncontroversial) formulation thereof, as in the passage below:

- A sentence $X$ is a model-theoretic logical consequence of a set of sentences $K$ ($K \models_{MT} X$, for short) just in case every set-theoretic structure which is a model of all the sentences in $K$ is also a model of $X$. If an argument $<K, X>$ is such that $K \models_{MT} X$ then any argument $<K', X'>$ with the same form will be such that $K' \models_{MT} X'$, since the model-theoretic notion of logical consequence is intended for languages where any two sentences of the same form have as models exactly the same structures. (Gomez-Torrente 2000, 529)

Notice that it is the notion of equiformity that connects the language in which an argument is expressed to the structures that are models of the sentences of the language, by means of the crucial concept of form: sentences of the same form (the linguistic, syntactic level) have the same structures as models (the semantic level). In this sense, both interpretational and representational approaches to logical consequence are to some extent represented in the model-theoretic notion, as will become clear below: the interpretational notion is reflected in the idea of sentences having the same form, whereas the representational notion is reflected in the all-models criterion (a logical consequence is valid iff all models satisfying the premises also satisfy the conclusion).

But this is only part of the story. ‘It seems that model theory is not a good model of representational semantics nor is it a good model of interpretational semantics.’ (Shapiro 1998, 143). Etchemendy’s conclusion had been that the model-theoretic notion came the closest to the interpretational / Tarskian notion of logical consequence\(^6\), and that the former simply did not capture the representational notion. Shapiro and others have argued that the model-theoretic notion is also not a good model of interpretational semantics: one of the advantages of the model-theoretic notion over the interpretational one is that, in the latter, the domain of discourse is fixed (the entities of the actual world), whereas the former encompasses the idea of different domains, coming thus closer to the (sound) basic intuitions of the representational approach.\(^7\)

In either case, the model-theoretic notion does not seem to capture the intuitive notion of logical consequence: it applies only to language with specific characteristics (mostly artificial, formal languages), and it is dependent on what structures are to be considered as models. One of the arguments offered by Etchemendy against the model-theoretic notion (in this case, of logical truth) is the following: assume that the (finite) universe has exactly $n$ entities. A sentence stating that there are exactly $n$ entities in the universe will come out as a logical truth (true in all models), even though intuitively the size of the universe is not a matter of logical truth (but rather a merely contingent matter). Thus, the model-theoretic notion overgenerates w.r.t. the intuitive notion, and it also

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\(^7\) Cf. Shapiro 1998, 143.
undergenerates if one considers the fact that it only captures valid arguments in languages satisfying certain requirements (in particular the requirement of linguistic equiformity corresponding to sameness of models).

b. Interpretational notion

The interpretational notion is essentially the Tarskian notion presented in (Tarski 2002), although it has been claimed that Etchemendy’s discussion does not entirely do justice to the subtleties of Tarski’s argumentation (cf. Gomez-Torrente 1999, Stroinska and Hitchcock 2002). Briefly put, the interpretational notion can be formulated as follows:

- **X** is an interpretational logical consequence of **K** (\(K \models_{TI} X\), for short\(^8\)) iff for all interpretations of the non-logical terms of **K** and **X**, if **K** is true then **X** is true.

An interpretation of the non-logical terms of **K** and **X** is the assignment of a sequence of objects of the universe to the sequence of these non-logical terms, such that each object is the denotation of one of the non-logical terms in question. A sequence of objects satisfies a sentence or set of sentences if the latter come out true under the given interpretation. If all sequences that satisfy **K** also satisfy **X**, then **K** \(\models_{TI} X\) (and the converse).

So far, so good. But there are serious problems with the interpretational notion: a sharp distinction between logical and non-logical terms is required (and that is notoriously a difficult task\(^9\)); the interpretational notion fails to capture necessary connections belonging to the level of the meaning of non-logical terms; the domain remains fixed, which means that the modal intuition behind the intuitive notion of logical consequence is at least partially lost. As a result, the interpretational notion undergenerates and overgenerates w.r.t. the intuitive notion.

Take the following example (cf. Beall & Restall 2000, 479): ‘a is red, consequently a is coloured’. Intuitively, this is a valid consequence – in fact, many of us would call it an analytic consequence, since there is a connection between the meanings of ‘red’ and ‘coloured’. However, under the interpretational notion, when these terms are taken to be non-logical terms, this is not a valid consequence. Hence, the interpretational notion undergenerates w.r.t. the intuitive notion.\(^{10}\)

Consider another example: ‘Andrew Jackson was President, thus Andrew Jackson was male.’\(^{11}\) (cf. Shapiro 1998, 144). If only ‘Andrew Jackson’ is taken to be a non-logical term, and different interpretations are assigned to it, then it will turn out that every

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\(^8\) ‘TI’ for ‘Tarskian Interpretational’.

\(^9\) Indeed, Tarski considered this very issue as perhaps the gravest ‘open question’ of his analysis (cf. Tarski 2002, 188).

\(^{10}\) Notice that this argument is also valid according to the representational notion (it is impossible for something to be red without being coloured), thus the interpretational notion also undergenerates w.r.t. the representational notion.

\(^{11}\) It is presupposed, of course that ‘President’ here means ‘President of the USA’; it might have been more polite towards other republics if this presupposition had been made explicit.
interpretation that satisfies the premise also satisfies the conclusion. Thus, according to the interpretational notion, this would be a valid consequence. But intuitively, this is obviously not a valid consequence; it is a matter of pure contingency (or in any case of the history of humanity) that, thus far, all presidents of the USA have been men. The problem here is that of the fixed domain: only entities existing or having existed in the actual world are taken into account, and this is not sufficient to establish the logical validity of an argument. Hence, the interpretational notion also overgenerates w.r.t. the intuitive notion.

c. Representational notion

The representational notion of logical consequence, or variants thereof, has been in operation for a long time in the history of logic\textsuperscript{12}, and can be traced back to Aristotle.\textsuperscript{13} It can also be described as the incompatibility notion of consequence, insofar as it holds that the truth of the premises is incompatible with the falsity of the conclusion. It is most general formulation, it can be phrased as follows:

- \(X\) is a representational logical consequence of \(K\) (\(K \models_{RE} X\), for short) iff it is impossible for \(K\) to be true and \(X\) not to be true.

As much as in interpretational semantics, an account of what it means for a sentence \(X\) to be true must be provided. But the biggest challenge of representational semantics is to give an account of what it means to be impossible for \(K\) to be true and \(X\) not to be true. The representational notion of logical consequence is rightly seen as essentially modal in nature. There is a variety of ways to spell out this modal character, and currently the most popular one employs the idea of possible worlds. While interpretational semantics is based on the rather manageable notions of interpretation, sequences of objects and satisfiability, representational semantics (according to many) is inexorably tangled in all kinds of metaphysical webs, owing to the problematic modal notions.

However, I do not wish to discuss philosophical objections to modal notions and to possible-world semantics.\textsuperscript{14} What matters now is how the representational notion of logical consequence fares when confronted to the intuitive notion. At first sight, the representational notion seems a strong candidate for the office of being an accurate model of the intuitive notion, since it appears to capture successfully its modal nature. But there are difficulties that must be faced.

Consider the following example: ‘\(a\) is a man, consequently \(a\) is an animal’. Under the representational view, this comes out as a valid consequence, since (assuming a reasonable dose of essentialism) it is impossible for whichever entity to be a man without being an animal. But is this an intuitively valid logical consequence? It seems to differ

\textsuperscript{12} Cf. Martin 1986, p. 567.
\textsuperscript{13} ‘A deduction is a discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so.’ Prior Analytics 24\textsuperscript{b}19-20.
\textsuperscript{14} Notice though that the apparent manageability of interpretational semantics when contrasted to representational semantics partially explains the appeal of interpretational semantics. Cf. (Read 1994, 252).
from the analytic consequence involving the terms ‘red’ and ‘coloured’ mentioned above, since the connection between the terms ‘man’ and ‘animal’ does not seem to pertain to the level of their meanings. It is rather a metaphysical connection: animality pertains to the very nature of manhood. I, in any case, would not be prepared to call it an analytic, or logical, consequence.15 Thus, if this is not an intuitively valid logical consequence, but it does come out valid according to the representational notion, then the latter seems to overgenerate w.r.t. the intuitive notion.

Another difficulty concerning the representational notion is related to the problem of how many ‘possible situations’ are to be taken into account, just as in the case of the model-theoretic notion. If the meta-theory is not broad enough and does not consider all possible situations, then an argument may be held valid only because no counterexample has been found among the available situations, while a counterexample (a situation in which the premises are true and the conclusion false) would exist in a meta-theory with a wider range of situations. Again, if the meta-theory is not broad enough, then the representational notion may overgenerate w.r.t. the intuitive notion.

At any rate, it is clear that the representational notion is conceptually and extensionally different from the other two notions considered thus far. Etchemendy seems to be right in insisting on their fundamental differences, and in saying that it is mere contingency if they occasionally intersect.16

d. Intuitive (pre-theoretic) notion

How far have we got in our quest for the intuitive notion of logical consequence? None of the three candidates seems to be a good model of it, as they all seem to undergenerate or, more importantly, overgenerate (or both) w.r.t the intuitive notion.

Another aspect that deserves further investigation is the extent to which the requirement of formality -- which is present in the model-theoretic and in the interpretational notions, but not in the representational notion -- is a part of the intuitive notion of logical consequence. While some people have very strong intuitions according to which validity is mainly determined by the form of arguments, others may feel that what is really at stake is preservation of truth. In other words, it is very well possible that a single intuitive notion of logical consequence does not exist, and that there are as many ‘intuitive notions’ as there are people pondering on the matter. But a less desperate view on the issue may be that there are perhaps two basic intuitive notions of consequence in operation, one focusing on the formal, logical character of validity and the other on truth preservation.

15 Obviously, it can be turned into a logical / formal consequence by the addition of a premise, namely ‘Every man is an animal’. The missing premise is a popular approach to the issue of logical and formal validity: a valid argument is one that can be turned into a formally valid one by the addition of a (necessarily true) premise. But this approach is not without difficulties (cf. Read 1994, pp. 254-259); Buridan, for example, says that the addition of a premise makes the validity of an argument evident, but it does not turn an invalid argument into a valid one (cf. TC, p. 23). So this strategy has at most an epistemological value.

16 Cf. Etchemendy 1990, 63-64.
Clearly, a deeper analysis of the matter is welcome. Where can one turn to for further insight? I propose to turn to the history of logic, and more specifically to the theory of *consequentia* of the medieval (14th century) logician John Buridan.

III. Buridan’s notion of *consequentia*

a. History

By now, the rough lines of the development of the medieval notion of consequence have been established (cf. Stump 1982, Boh 1982), but not all details are known. For example, it is still unclear to which extent, and how, the development of ‘propositional logic’ in the later medieval period is related to Stoic logic, even though the conceptual connection appears quite tight (cf. Boh 1982, 303). What follows is a modest sketch of this development, insofar as it is related to the present discussion.

The story must begin, as always in logic, with Aristotle. For Aristotle and the Aristotelian tradition, the model of logical validity was that of syllogistic validity, according to the patterns described in the *Prior Analytics*. The syllogistic system is a clear case of undergeneration w.r.t. the intuitive notion of logical validity: all valid syllogistic patterns are indeed intuitively valid, but the group of valid arguments described by syllogistics is but a very small subset of all intuitively valid logical arguments.

Besides syllogisms, Aristotle also produced what came to be known as a theory of Topics. Topics are general rules that can serve as tools for argumentation, but their logical status is very much inferior to that of syllogisms; they could at most generate reasonable dialectical arguments. The theory presented in the *Topics* was considered to be at most an art, not having the scientific status attributed to syllogisms. Yet in the post-Aristotelian tradition, in particular among the Latin rhetoricians, the Topics remained an important object of study.

An addition to the Aristotelian framework introduced by Boethius, which was to be influential later in the Middle Ages, was the idea of hypothetical syllogisms (the original Aristotelian system only dealt with categorical syllogisms). Indeed, the first medieval discussions related to the notions of consequence and entailment were prompted by reflections concerning hypothetical syllogisms, and, to a minor extent, the Topics. Abelard (early 12th century) is probably the first medieval logician to have developed an early version of what we could call ‘propositional logic’ (cf. Tweedale 1982), and he also introduced an interesting distinction between perfect and imperfect

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17 Although this did not appear to be Aristotle’s opinion; he seemed to think that every valid argument could be reduced to one of the valid syllogistic forms. Cf. (Smith 1995, 30).
18 Arguably under the influence of Stoic logic, cf. (Boh 1982, 303).
19 This is the case in particular of Garlandus Compotista and Peter Abelard. ‘Garlandus is interested in the Topics because he thinks they are useful in the study of hypothetical syllogisms, which appear to be his main interest in the *Dialectica.*’ (Stump 1982, 276)
Two aspects of this distinction are worth noting: the foundational priority of syllogisms and the idea that there are two ‘sorts’ of inferences. The first aspect would no longer be unanimously accepted in later developments, but the division of consequences between two kinds is one of the most important traits of the evolution of this concept.

In the 13th century, logicians were still very much attached to the idea that syllogistic validity was the only real form of logical validity, and there were no independent treatises on consequences yet. The dominating paradigm was still very much that of Aristotelianism, in particular the logical primacy given to syllogisms. The change occurred in the early 14th century can be seen precisely as the undermining of the Aristotelian view on logical validity, in particular with the ‘growing inclination to see all syllogisms as dependent on Topics’ (Stump 1982, 287). Indeed, the early stages of theories of consequentiae can be described as the realization that the syllogistic system chronically undergenerates; logicians then turned to the Topics to expand the range of consequences accepted as logically valid.

This can be seen in the first two important treatises on consequentiae: the chapters dedicated to the topic in Walter Burley’s Tractatus brevior and Tractatus longior (Burley 2000) and part III-3 of Ockham’s Summa Logicae (all three written in the 1320’s – cf. Stump 1982, 294). In those texts the validity of consequences was still seen as fundamentally dependent on the Topics, but this situation did not last for long: just a few decades later, in Buridan’s treatise (written in the 1330’s) and in Pseudo-Scotus’s commentary on the Prior Analytics (written around 1350 – Pseudo-Scotus 2001), the validity of consequentiae was no longer based on Topics (cf. Boh 1982, pp. 307-310); it finally acquired a status of its own, independent from the Aristotelian notions of syllogisms and Topics.

One important distinction introduced by Ockham is that between formal and material consequences. The criterion used by him to draw the distinction was still very much based on the Topical framework, and was soon discarded by authors such as Buridan and Pseudo-Scotus. But the idea that there are basically two kinds of consequences, and that what distinguishes them is something related to their form, was to remain influential throughout the Middle Ages, and is arguably still current (cf. Read 1994).

Indeed, for most medieval authors, the primitive notion of consequence was that of a material consequence, usually expressed in terms of the ‘modal’, or incompatibility criterion: a consequence holds iff it is impossible for the antecedent to be the case while the consequent is not the case. Moreover, a sub-group of the (materially) valid consequences was thought to satisfy not only the modal criterion, but also another, stricter criterion, such that its member were said to be formally valid consequences. The

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20 ‘He distinguished inferences into those that are perfect and those that are imperfect. The former he identifies with syllogisms, whose form alone guarantees their validity. Imperfect inferences he identifies with Topical arguments, which need to be confirmed by a Differentia and maximal proposition [Topical notions].’ (Stump 1982, 280). See also (Martin 1986, 566).

21 Cf. (Stump 1982, 281); (Boh 1982, 306).

22 Cf. Ockham, Summa Logicae, III-3, cap. 1, 4-9 (p. 587)

different authors did diverge, however, with respect to the criterion differentiating a formal from a material consequence. Some defined formal consequences on the basis of semantic criteria\textsuperscript{24}, others in terms of epistemic notions.\textsuperscript{25}

The notion of formal consequence that is of interest for the present discussion is that advocated by Buridan, and is what we could call a substitutional notion: a consequence is formal iff all substitu
tional instances of its categor
tematic (i.e. non-logical) terms yield material consequences. Therefore, let us now turn to Buridan’s theory of consequence.

b. Buridan: material and formal consequence

Buridan’s theory of consequence is exposed in his \textit{Treatise on Consequences} (Hubien 1976) and, in a less systematic way, in the eighth chapter of his \textit{Summulae de Dialectica} (Buridan 2001), dedicated to sophisms.

Most of Buridan’s discussion on the correct definition of consequence is related to his staunch commitment to proposition-tokens as truth-value bearers, and the amendments to the intuitive definition made necessary by this commitment (cf. Klima 2004; Dutilh Novaes 2004). But this aspect of his theory shall not be dealt with now; I will focus on his distinction between formal and material consequences.

The first formulation of consequence offered by Buridan (which later has to be reformulated to accommodate the behaviour of tokens) is very much in the spirit of the representational notion:

\begin{quote}
[The terms] ‘antecedent’ and ‘consequent’ are predicated correlative
tively; therefore, they need to be described in terms of each other. Many people say of two propositions that one is the antecedent with respect to the other which cannot be true while the other is not true, so that every proposition is antecedent with respect to any other proposition which cannot be true without the other being true. (TC, p.21, 26-32)
\end{quote}

In practice, Buridan adds a few pages later, this definition holds good in all cases in which the very existence of the tokens in question does not interfere with their own modal values, so for the present purposes this is the definition that matters.

Among the consequences that satisfy the modal/representational criterion, some seem to display an interesting feature, namely the fact that their form appears to be connected to their being valid consequences. These are called formal consequences:

\begin{quote}
‘Formal’ consequence means that [the consequence] holds for all terms, retaining the form common to all. Or, if you want to express it according to the proper force of discourse, a formal consequence is that which, for every
\end{quote}

\textsuperscript{24} The early tradition, represented by Ockham and Burley.
\textsuperscript{25} The later, predominantly English, tradition, represented by Ralph Strode and Richard Billingham (cf. Boh 2001, Billingham 2003)
proposition similar in form which might be formed, it would be a good consequence, such as ‘what is A is B; thus what is B is A’. (TC 22/23, 5-9)

The non-logical / logical boundary was less of a problem for medieval logicians such as Buridan because they operated with the distinction between categoric and syncategoric terms; so the form of a proposition is given by its syncategoric terms.

A few things are worth noticing in Buridan’s definition of formal consequence. First, his criterion is clearly what we could call the substitutional / variational criterion of validity, to be rediscovered by Bolzano\textsuperscript{26} in the 19\textsuperscript{th} century and refined by Tarski in the 20\textsuperscript{th} century, yielding what Etchemendy calls the interpretational notion of consequence (cf. Etchemendy 1990, chap. 3). But notice that Buridan is not referring to the set of all putative consequences that satisfy the substitutional criterion; rather, he is interested in the subset of the putative consequences that satisfy the modal / representational criterion, which also satisfy the substitutional criterion. In particular, what defines a formal consequence is the fact that all its substitutional instances are material consequences.

c. Matrices

Buridan’s notion of a valid consequence relies on the idea of impossibility of the concomitant truth of the premises and the falsity of the conclusion. With some anachronism, this idea can be represented by the familiar possible-world semantics. Let \(<W, R, V>\) be the usual possible-world model, where \(W\) is a set of worlds, \(R\) the accessibility relation\textsuperscript{27} and \(V\) a valuation, taking the ordered pair of a proposition and a world into a truth-value T or F.

\[ X \text{ is consequence of } K \text{ iff for all } w_i \text{ such that } V(K, w_i) = T, \ V(X, w_i) = T \]

A convenient way of depicting this definition is by means of truth table-like matrices:

<table>
<thead>
<tr>
<th>Worlds</th>
<th>(w_1)</th>
<th>(w_2)</th>
<th>(w_3)</th>
<th>(w_4)</th>
<th>etc…</th>
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</thead>
<tbody>
<tr>
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<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>(X)</td>
<td>T</td>
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<td>F</td>
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Notice that this matrix is only one-dimensionally modal: only variation of possible worlds occurs, the rest remaining fixed.

By contrast, Buridan’s definition of a formal consequence can be said to be two-dimensionally modal:\textsuperscript{28} it involves variation of worlds and of interpretations.

\textsuperscript{26} Cf. Bolzano 1973, in particular §§ 154-155.

\textsuperscript{27} The accessibility relation plays no role in the present analysis, so for the sake of simplicity just assume it to be S5.

\textsuperscript{28} But if one wants to take into account Buridan’s commitment to tokens, one must also distinguish the context of formation of a proposition from its context of evaluation, adding one extra dimension to the
(substitutional instances) of each proposition. Therefore, each propositional form or set of propositional forms must be represented by a two-dimensional matrix, where each row corresponds to a substitutional instance of the form (assuming that the corresponding rows of each matrix are homogeneous substitutions of the non-logical terms – i.e. in $K_n$ and $X_n$ the non-logical terms are homogeneously substituted).

A

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<th>$w_2$</th>
<th>etc...</th>
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<td>$a_{12}$</td>
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<tr>
<td>$K_2$</td>
<td>$a_{21}$</td>
<td>$a_{22}$</td>
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<td>etc...</td>
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B

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<thead>
<tr>
<th>Worlds Interpretations</th>
<th>$w_1$</th>
<th>$w_2$</th>
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<td>$X_2$</td>
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<td>etc...</td>
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</tbody>
</table>

The elements $a_{ij}$ and $b_{ij}$ of the matrices are the truth-values T and F. Clearly, $X$ is a formal consequence of $K$ iff for all $a_{ij}$ such that $a_{ij} = T$, $b_{ij} = T$ -- or, equivalently, if, for all substitutional instances $K_n$ and $X_n$ of $K$ and $X$, $<K_n, X_n>$ is a material consequence.

IV. The hybrid notion of logical consequence

a. Shapiro and Buridan

Based on his critique of the representational / interpretational dichotomy, Shapiro concludes that the intuitions behind both notions are present in the intuitive notion of logical consequence, but that each of them only partially represents the latter. While in the representational account the language remains fixed and the world of evaluation varies, in the interpretational account the reverse occurs. Shapiro demands that both variations -- of worlds and of language -- occur to qualify a logical consequence. He proposes thus his ‘conglomeration’ notion of logical consequence, which is indeed a blend of the modal and of the semantic characterizations of logical consequence and, according to Shapiro, a characterization as good as any of the intuitive notion.

• $\Phi$ is a logical consequence of $\Gamma$ if $\Phi$ holds in all possibilities under every interpretation of the non-logical terminology in which $\Gamma$ holds. (Shapiro 1998, 148)

In other words, the extension of the intuitive notion of logical consequence seems to be the subset of the representationally valid consequences that are also interpretationally

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definition. In other words, within a token-based semantics, the definition of formal validity must be three-dimensionally modal.
valid. This corresponds precisely to Buridan’s notion of a formal consequence: a formally
valid consequence is a materially valid consequence such that all substitutional instances
of its non-logical terms are (materially) valid consequences.

b. Intersection

In terms of its extension, the intuitive notion of logical consequence simply corresponds
to the intersection between the sets of RE-consequences and of TI-consequences, or, in
Buridan’s framework, to the subset of materially valid consequences that are also
formally valid.

Shapiro’s/ Etchemendy’s terminology:

Representational notion

Intuitive, hybrid notion

Interpretational notion

Buridan’s terminology:

Material consequences

Formal consequences

Purely in terms of extensional adequacy, it is irrelevant whether we consider the
representational notion or the interpretational notion as primitive: the subset of the
representationally valid consequences that are also interpretationally valid is the same as
the subset of the interpretationally valid consequences that are also representationally
valid. But we must bear Etchemendy’s warning in mind: extensional adequacy is not
sufficient, the proper conceptual analysis must also be offered. Etchemendy claims that
that the model-theoretic notion of logical consequence does capture the extension of the
intuitive notion in the case of first-order logic, for example, but that this happens almost
by pure chance (cf. Etchemendy 1990, 8), since the underlying conceptual assumptions
are not correct. In particular, the same (extensional adequacy) does not happen in the case
of second-order logic.

Thus, not only do we seek to capture the extension of the intuitive notion of logical
consequence, we also want an adequate conceptual analysis thereof. For this purpose,
turning to Buridan’s theory seems very fruitful, insofar as, in his account, the
representational notion (corresponding to the notion of material consequence) is clearly
the primitive notion. There is not even a term for the putative consequences that satisfy
the substitutional criterion but not the modal one (see graph above -- and one may even
wonder whether such arguments exist at all).

In sum, in both Shapiro’s hybrid notion of logical consequence and Buridan’s notion of
formal consequence, the modal component seems to be the central feature; now, insofar
as the intuitive concept of validity seems to be essentially that of truth-preservation, the
core of the intuitive concept of logical consequence is indeed the modal intuition. The
interpretational / substitutional criterion is applied only once the representational / modal
criterion has been applied, to capture the set of valid consequences whose validity is not
only a matter of truth-preservation, but also of their form (their logical structures). Hence,
the analysis of Buridan’s theory seems to yield not only extensional adequacy w.r.t. the intuitive notion, but also the adequate conceptual analysis of the issue.

V.  Open questions

The conclusion we can draw from this discussion is that the intuitive notion of logical consequence is indeed what we could call a ‘hybrid’ notion, as Shapiro claims, and that the analysis of Buridan’s theory of consequentia is illuminating for the purpose of attaining a better understanding of this hybrid notion (since Shapiro’s account thereof is rather brief). But some issues remain to be discussed.

1. I have not discussed the crucial distinction between inference and consequence, which is systematically overlooked in the current debates concerning logical consequence (cf. Sundholm 1998). This distinction is however of the utmost importance, and I have discussed elsewhere the implications of this distinction for Buridan’s theory of consequence (cf. Dutilh Novaes 2004).

2. Is formal consequence equivalent to logical consequence? – So far I have been considering the two notions as roughly equivalent; in particular, I claim that Buridan’s notion of formal consequence is a very good model of the intuitive notion of logical consequence. But one may object that there is more to logic than just form – for example, that analytic consequences that are not formal are also in some sense logical. In other words, a deeper discussion of this issue is needed.

3. Consequence and validity are fundamentally modal notions, but logical consequence concerns form. That is precisely the role of the technical device of interpretation \ substitution: it takes care of the formality requirement of logic (isomorphism of models \ variation of interpretation). But in which sense the essential trait of logic is that of being formal is again an issue that deserves a more detailed analysis (cf. MacFarlane 2000)

4. I claim that material consequence must remain the primitive notion (cf. Read 1994). Patterns of logically/formally valid consequences are attempts to model and capture an increasing number of material consequences, but these formal patterns do not offer the very foundations for validity. The issue of logical \ non-logical boundary is precisely a sign of that: the very purpose of logic is to model the logical form of propositions -- but the models remain incomplete, and are always only partial approximations. This however is a general view on the very nature of logic that must be argued for in more detail than what I have offered in the present paper.

References


J. MacFarlane 2000, What does it mean to say that logic is formal?. PhD Dissertation, University of Pittsburgh. Available at http://philosophy.berkeley.edu/macfarlane/Diss.pdf


