ALGEBRAIC CONNECTIVE $K$-THEORY OF A SEVERI-BRAUER VARIETY WITH PRESCRIBED REDUCED BEHAVIOR

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Abstract. We show that Chow groups of low dimension cycles are torsion free for a class of sufficiently generic Severi-Brauer varieties. Using a recent result of Karpenko, this allows us to compute the algebraic connective $K$-theory in low degrees for the same class of varieties. Independently of these results, we show that the associated graded ring for the topological filtration on the Grothendieck ring is torsion free in the same degrees for arbitrary Severi-Brauer varieties.

1. Introduction

The goal of this paper is to determine some low dimension algebraic connective $K$-groups for a class of generic Severi-Brauer varieties having arbitrary reduced behavior (see Example 2.4). Using a recent result of Karpenko [Kar19] this problem reduces to checking that the canonical surjection, from the Chow groups to the associated graded groups for the topological filtration on the Grothendieck group of coherent sheaves, is an isomorphism in low dimension for these varieties. This is accomplished in Theorem 4.1 where we show that these Chow groups are torsion free.

Independently of this goal, we determine in Theorem 3.6 the topological filtration on the Grothendieck group of an arbitrary Severi-Brauer variety in low (homological) degrees. As a consequence, in the degrees where Theorem 3.6 applies the associated graded group for the topological filtration turns out to be torsion free. This allows for some pretty elaborate descriptions of the torsion elements appearing in algebraic connective $K$-theory of a Severi-Brauer variety. For example, it follows that the algebraic connective $K$-group $CK_i(X)$ of a Severi-Brauer variety $X$ associated to a central simple algebra of $p$-primary index (for $p$ a prime) contains torsion when $i \leq p - 2$ if and only if there exists a numerically trivial $j$-cycle on $X$ that is not rationally trivial for some $j \leq i$.

Although the results of this paper are new, the techniques that go into their proofs have mostly appeared already in other places. This is especially true for our proof of Theorem 4.1 that employs a number of tools that have been developed in [Kar17b, KM19, Mac19a]. Because of this, we’ve tried writing the proofs here so that they can be used as a complement to other sources. In the case of Theorem 4.1 and the proofs of [Kar17b, KM19, Mac19a]: if a proof is sufficiently elaborated on in one of these sources, then it is mostly omitted here; if a proof has been mostly omitted in one of these sources, then we elaborate on it here.

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The contents of this paper are structured as follows. In Section 2, we recall a number of results on Severi-Brauer varieties (e.g. we recall the construction of a Severi-Brauer variety from its associated central simple algebra). Here we’re just setting up notation to be used throughout the remainder of the paper. An expert could easily skip this part (although it could still be useful to look at Example 2.4).

Section 3 contains our treatment of the Grothendieck group. Here we prove Theorem 3.6 that determines the topological filtration on the Grothendieck group for an arbitrary Severi-Brauer variety. Since Theorem 3.6 is independent of the other sections, the reader only interested in our computation for the algebraic connective $K$-theory and Chow groups of generic Severi-Brauer varieties with prescribed reduced behavior can skip this section.

Section 4 is likely the most nuanced section of this paper. Here we pull a number of results from other recent works of the author and Karpenko to get our Theorem 4.1 on the computation of the Chow groups of a Severi-Brauer variety. We’ve tried to write this section so that it can be read independently of these other works but, for the full proof one will have to look elsewhere.

Finally, we conclude in Section 5 with some observations on the algebraic connective $K$-theory of arbitrary smooth varieties.

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Notation and Conventions. Throughout this note we fix an arbitrary base field $k$. When no confusion should arise, we may also use $k$ as index.

For any field $F$, an $F$-variety, or simply a variety if the field $F$ is clear from context, is a separated scheme of finite type over $F$.

For a central simple $k$-algebra $A$ we denote by $\deg(A)$ the degree of $A$, i.e. the square root of the $k$-dimension of $A$; we denote by $\text{ind}(A)$ the index of $A$, i.e. the reduced dimension of a minimal left (or right) ideal of $A$; we denote by $\exp(A)$ the exponent of $A$, i.e. the order of $A$ in the Brauer group $\text{Br}(k)$ of $k$.

Lastly, if $p$ is a prime then by $v_p$ we mean the usual $p$-adic valuation.

2. Severi-Brauer varieties

In this section we recall some notation and conventions regarding Severi-Brauer varieties. Recall that a Severi-Brauer $k$-variety is a variety $X$ over $k$ such that there exists a separable field extension $L/k$ and an isomorphism

$$X_L \cong \mathbb{P}^n_L$$

between the scalar extension of $X$ to $L$ and projective space of dimension-$n$ over $L$. Any field $F$, not necessarily separable over $k$, that admits an isomorphism between $X_F$ and a projective space over $F$ is called a splitting field for $X$; in this case, $F$ is said to split $X$ and
the choice of such an isomorphism is called a splitting of \( X \) over \( F \).

From any Severi-Brauer variety \( X \), one can canonically associate a central simple \( k \)-algebra \( AZ(X) \) to \( X \) in such a way that
\[
AZ(X) \otimes_k F = AZ(X_F)
\]
for any field extension \( F/k \). Conversely, to any central simple \( k \)-algebra \( A \) one can canonically associate a Severi-Brauer variety \( X = SB(A) \) to \( A \) in such a way that
\[
SB(A)_F = SB(A \otimes_k F)
\]
for every field \( F/k \). These associations are inverse to each other in the sense
\[
SB(AZ(X)) \cong X \quad \text{and} \quad AZ(SB(A)) \cong A
\]
for any Severi-Brauer \( k \)-variety \( X \) and for any central simple \( k \)-algebra \( A \).

Starting with a central simple \( k \)-algebra \( A \) of degree \( n \), the Severi-Brauer variety \( SB(A) \) is defined as the subvariety
\[
SB(A) \subset Gr(n, A)
\]
of the Grassmannian of \( n \)-dimensional planes in \( A \) that represents the functor with \( R \)-points, for any finite type \( k \)-algebra \( R \), the set
\[
SB(A)(R) = \{ I \subset A \otimes_k R : I \text{ is a right ideal} \} \subset Gr(n, A)(R).
\]
In the other direction, one starts with a Severi-Brauer variety \( X \) and considers the (unique up to scaling) nontrivial class \( \eta \in \text{Ext}^1(O_X, \Omega_X) \). The class \( \eta \) corresponds to a short exact sequence
\[
0 \to \Omega_X \to \zeta_X \to O_X \to 0
\]
that becomes isomorphic to the Euler exact sequence on projective space over any splitting field \( F \) of \( X \). The middle term \( \zeta_X \) of this exact sequence is called the tautological sheaf on \( X \). The central simple \( k \)-algebra \( AZ(X) \) is defined as the endomorphism algebra of the dual of \( \zeta_X \), i.e.
\[
AZ(X) = \text{End}(\zeta_X^\vee).
\]

As a consequence of the above, any Severi-Brauer variety \( X \) is isomorphic to a subvariety of the Grassmannian of \( n \)-planes in \( A = AZ(X) \),
\[
X = SB(A) \subset Gr(n, A).
\]
The tautological sheaf \( \zeta_X \) can be identified with the pullback of the universal subsheaf on \( Gr(n, A) \) under this embedding. For any finite type \( k \)-algebra \( R \) and any \( R \)-point \( x \) of \( X \) corresponding to a right ideal \( I \subset A \otimes_k R \), the sheaf \( x^*\zeta_X \) can be canonically identified with \( I \) when considered as an \( R \)-module. In particular, \( \zeta_X \) is a right module under the constant sheaf \( A \) and it makes sense to define sheaves
\[
(\Theta.1) \quad \zeta_X(i) = \zeta_X^\otimes i \otimes_{A^\otimes i} M_i
\]
for every \( i \in \mathbb{Z} \) and for simple left \( A^\otimes i \)-modules \( M_i \).

One can check that there’s an equality
\[
(\Theta.2) \quad \text{rk}(\zeta_X(i)) = \text{ind}(A^\otimes i)
\]
for every \( i \in \mathbb{Z} \).
for any $i \in \mathbb{Z}$. The ranks of the sheaves $\zeta_X(i)$ as $i$ varies forms an invariant of the Severi-Brauer variety $X$ and of the central simple algebra $A$ called the behavior of $X$ or the behavior of $A$. We’ll be interested in a reduced version of this invariant.

**Definition 2.1.** For any prime $p$, we define the reduced $p$-behavior $r\text{Beh}(p, X)$ of $X$ as the sequence

$$r\text{Beh}(p, X) := \left( v_p \left( \text{rk} \left( \frac{\text{ind}(A)}{p^{\text{exp}(A)}} \right) \right) \right)_{i=0}^{v_p(\text{exp}(A))}$$

of $p$-adic valuations of the ranks of these sheaves.

If $A = M_r(k) \otimes \bigotimes_{q \text{ prime}} A_q$ is a decomposition of $A$ into division algebras $A_q$ of $q$-primary index, then we similarly define the reduced $p$-behavior $r\text{Beh}(p, A)$ of $A$ to be the sequence

$$r\text{Beh}(p, A) := \left( v_p \left( \text{ind} \left( A_p^{\otimes i} \right) \right) \right)_{i=0}^{v_p(\text{exp}(A))}$$

of $p$-adic valuations of the indices of these tensor powers of $A_p$.

Because of (??), the two sequences $r\text{Beh}(p, A)$ and $r\text{Beh}(p, X)$ are actually the same and we often refer to either of them as the reduced $p$-behavior. If $A$ is a central simple $k$-algebra with index a power of a prime $p$, then we’ll often omit the prime $p$ and refer to the reduced $p$-behavior of $A$ (or of $X$) as simply the reduced behavior $r\text{Beh}(A)$ of $A$ (or $r\text{Beh}(X)$ of $X$ respectively).

**Remark 2.2.** Let $p$ be a prime, let $n \geq 0$ be an integer, and let $A$ be a central simple $k$-algebra of index $\text{ind}(A) = p^n$. Then the index $\text{ind}(A^\otimes i)$ depends only on the $p$-adic valuation $v_p(i)$:

$$\text{ind}(A^\otimes i) = \text{ind}(A^{\otimes p^{v_p(i)}}).$$

For example, if $i$ is prime-to-$p$, then $\text{ind}(A^\otimes i) = p^n$. In this way one can reconstruct the behavior of $A$ from the reduced behavior.

**Remark 2.3.** For any central simple algebra $A$, the reduced $p$-behavior of $A$ is a strictly decreasing sequence of nonnegative integers starting with $v_p(\text{ind}(A))$ and ending with 0, see [Kar98, Lemma 3.10]. Conversely, the proof of [Mac19b, Theorem 4.11] shows that for any finite collection of primes $p_1, ..., p_r$ and for any $r$ strictly decreasing sequences $S_1, ..., S_r$ of nonnegative integers ending in 0, there is an algebra $B$ with $r\text{Beh}(p_i, B) = S_i$ for each $1 \leq i \leq r$ and with $r\text{Beh}(q, B) = (0)$ for all primes $q$ outside of $p_1, ..., p_r$.

A specific example of the construction from Remark 2.3 that we’ll revisit often is the following:

**Example 2.4.** Here we construct for any sequence of integers $S = (n_i)_{i=0}^m$ with $n = n_0 > n_1 > \cdots > n_m = 0$ a Severi-Brauer variety $X^S$ associated to a division algebra $A^S$ of index $\text{ind}(A^S) = p^n$ with reduced behavior

$$r\text{Beh}(X^S) = (n_0, n_1, ..., n_m).$$

The variety $X^S$ that we construct will be defined over a large field extension $K$ of the base field $k$ and $X^S$ will have the additional property that the Chow ring $\text{CH}(X^S_K)$ of the scalar extension of $X^S$ to the composite field $LK$ is generated by CH-Chern classes for every algebraic extension $L/k$. 

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To construct $X^S$ we start by fixing a power $r = p^n$ of a prime $p$ and we set $G = \text{PGL}_n$. Next we find an embedding $G \to \text{GL}(V)$ of $G$ into the general linear group of a finite dimensional $k$-vector space $V$ and set $E$ to be the generic fiber of the quotient map $\pi : \text{GL}(V) \to \text{GL}(V)/G$. The group $G$ acts on $\mathbb{P}^r$ via conjugation and we choose $P \subset G$ to be the stabilizer of a rational point under this action. The quotient $E/P$ is a Severi-Brauer variety over the field $F = k(\text{GL}(V)/G)$ and the central simple $F$-algebra $\mathcal{A}Z(E/P)$ associated to $E/P$ is a division algebra having index and exponent equal $p^n$, cf. [KM06, Proof of Theorem 6.4 (2), Theorem 6.9, and Section 8.1].

In particular, by [Kar98, Example 3.9] the reduced behavior of $\mathcal{A}Z(E/P)$ is
\[ r\text{Beh}(\mathcal{A}Z(E/P)) = (n, n - 1, \ldots, 1, 0) \]
the sequence whose $i$th term, starting from $i = 0$, is $n - i$. Now for any strictly decreasing sequence like $S = (n_i)_{i=0}^m$ with $n_0 = n$ and $n_m = 0$ we can consider the varieties
\[ Z = \prod_{i=0}^m Z_i \quad \text{and} \quad Z_i = \text{SB}(p^{n_i}, \mathcal{A}Z(E/P)^{\otimes p^i}) \]
with $\text{SB}(p^{n_i}, \mathcal{A}Z(E/P)^{\otimes p^i})$ the generalized Severi-Brauer variety of reduced dimension-$p^{n_i}$ right ideals inside of $\mathcal{A}Z(E/P)^{\otimes p^i}$.

We claim that the pair
\[ X^S = (E/P)_{F(Z)} \quad \text{and} \quad A^S = \mathcal{A}Z(E/P)_{F(Z)} \]
defined over the field $K = F(Z)$ has the specified properties. First, by an application of index reduction, see [Kar98, Lemma 3.10], the $K$-division algebra $A^S$ has reduced behavior
\[ r\text{Beh}(A^S) = (n_0, n_1, \ldots, n_m) \]
as claimed. Next, we observe:

(1) CH($E/P$) is generated by CH-Chern classes by [Kar18, Proposition 6.1] and [Kar17a, Proof of Lemma 2.1]

(2) CH($E/P \times Z$) is generated by CH-Chern classes as $E/P \times Z$ is a chain of Grassmannian bundles over $E/P$ [Kar18, Proposition 6.1] and [Kar17a, Proof of Lemma 2.1]

(3) CH($X^S$) is generated by CH-Chern classes since $X^S$ is the generic fiber of the projection $E/P \times Z \to Z$.

Finally, the proofs of (1), (2), and (3) above go through unchanged for the varieties $E_{LF}/P_L$, $E_{LF}/P_L \times Z_{LF}$, and $X^S_{LK}$ for every algebraic extension $L/k$ showing that $X^S$ and $A^S$ have all of the claimed properties.

The next lemma is standard. We include it here because it’s essential to some of our arguments.

**Lemma 2.5.** Let $A$ be a central simple $k$-algebra with index $\text{ind}(A) = p^n$ for some prime $p$ and some $n > 0$. Then there exist finite field extensions
\[ k \subset L_0 \subset \cdots \subset L_i \subset \cdots \subset L_n \]
with
\[ [L_{i+1} : L_i] = p \quad \text{and} \quad \text{ind}(A_{L_i}) = p^{n-i} \]
for all $0 \leq i \leq n$. Moreover, the degree $[L_0 : k]$ is prime-to-$p$. 
Proof. We explicitly construct fields $L_i$ with the given properties. To start, let $D_A$ be the underlying division algebra of $A$ and let $L \subset D_A$ be a maximal separable subfield of $D_A$. The field $L$ has degree $[L : k] = p^n$ and splits $A$.

Let $F$ be a Galois closure of $L$ and let $G_p \subset \text{Gal}(F/k)$ be a Sylow $p$-subgroup. We set $L_0 = F^{G_p}$ to be the fixed field of $G_p$; note that $[L_0 : k]$ is prime-to-$p$. The composite field $L_0L$ corresponds to a subgroup $H_n \subset G_p$ and has degree $[L_0L : L_0] = p^n$. If $H_n$ is normal in $G_p$ then the $p$-group $G_p/H_n$ has subgroups of every prime power dividing the order $\#(G_p/H_n)$. In this case we can choose a sequence of such groups

$$H_n \subset H_{n-1} \subset \cdots \subset H_{n-i} \subset \cdots \subset H_0 = G_p$$

with $[H_i : H_{i+1}] = p$ corresponding to fields $L_i = F^{H_i}$ with $[L_{i+1} : L_i] = p$. It follows from the relation (see [GS06, Corollary 4.5.11])

$$\text{ind}(A_{L_0}) = [L_i : L_0] \text{ind}(A_{L_i})$$

that the fields $L_i$ have the claimed property.

On the other hand, if $H_n$ is not normal in $G_p$ then we consider the normalizer $N_{G_p}(H_n)$ of $H_n$ in $G_p$. The group $H_n$ is normal in $N_{G_p}(H_n)$ so, if $p^r = \#(N_{G_p}(H_n)/H_n)$ then one can find subgroups $H_n \subset H_{n-1} \subset \cdots \subset H_{n-r}$ contained in $N_{G_p}(H_n)$ and containing $H_n$ with $[H_i : H_{i+1}] = p$ as before. If the normalizer is normal then we can conclude by the previous paragraph. If the normalizer isn’t normal in $G_p$, then we can consider the normalizer $N_{G_p}(N_{G_p}(H_n))$ and so on. Since $G_p$ is finite, we will eventually construct fields as in the lemma statement. 

The next example won’t be used in the remainder of this text but, we include it here because of it’s explicit nature. To elaborate, we’ll often be working with the reduced behavior of some algebras both over a ground field and over fields like the ones constructed in Lemma 2.5. Generally, the reduced behavior over a field extension can behave fairly sporadically and depends on the actual algebra under consideration. The next example is one of the few cases where one can completely describe the way this reduced behavior changes.

Example 2.6. Fix a prime $p$ and let $S = (n_i)_{i=0}^m$ be a sequence of integers with

$$n = n_0 > n_1 > \cdots > n_m = 0.$$

Let $X^S$ and $A^S$ be the Severi-Brauer variety and central simple algebra constructed from $p$ and $S$ as in Example 2.4. Let $L_0 \subset \cdots \subset L_i \subset \cdots \subset L_n$ be a chain of field extensions constructed from $A^S$ as in the proof Lemma 2.5. Then, for any $0 \leq i \leq n$, the reduced behavior of $A^S_{L_i}$ can be determined by index reduction [Bla91, Theorem 5] or [SVdB92, Theorem 2.5] and equals

$$r\text{Beh}(A^S_{L_i}) = (n_0 - i, \min\{n_0 - 1 - i, n_1\}, \ldots, \min\{n_0 - j - i, n_j\}, 0)$$

where $j$ is the largest integer satisfying $\min\{n_0 - j - i, n_j\} > 0$.

3. K-theory

In this section, we recall notations and definitions from K-theory. We also prove Theorem 3.6 that describes some terms of the topological filtration for any Severi-Brauer variety.
associated to a $p$-primary indexed central simple algebra for an arbitrary prime $p$. We start under the assumption that $X$ is an arbitrary $k$-variety and, eventually, we specialize to the case $X$ is a Severi-Brauer variety.

For any $k$-variety $X$ we write $K(X)$ for the Grothendieck ring of locally free sheaves of finite rank on $X$ and we write $G(X)$ for the Grothendieck group of coherent sheaves on $X$. These two objects are comparable by a morphism

$$(K \rightsquigarrow G)_X : K(X) \to G(X)$$

that takes the class of a locally free sheaf $[\mathcal{F}] \in K(X)$ to the element $[\mathcal{F}] \in G(X)$. When $X$ is smooth, the comparison morphism is an isomorphism giving $G(X)$ the structure of a ring.

The group $G(X)$ comes equipped with an ascending $\mathbb{Z}$-indexed filtration $\tau_*(X) \subset G(X)$ called the topological filtration on $G(X)$. By definition, the $i$th piece of this filtration is the subgroup of $G(X)$ with generators coherent sheaves supported in dimension-$i$ or less,

$$(\mathcal{O} .3) \quad \tau_i(X) = [\mathcal{F}] : \dim(\text{Supp}(\mathcal{F})) \leq i \subset G(X).$$

When $X$ is equidimensional of dimension-$n$ the topological filtration can be determined by the codimension of the support of a coherent sheaf and we write

$$(\mathcal{O} .4) \quad \tau^i(X) = \tau_{n-i}(X) = [\mathcal{F}] : \text{codim}_X(\text{Supp}(\mathcal{F})) \geq i \subset G(X)$$

for the $i$th piece of the descending $\mathbb{Z}$-indexed filtration $\tau^*(X) \subset G(X)$.

We’ll often be concerned throughout this text with the graded object associated with the topological filtration on $G(X)$. Our notation for these objects will be

$$\text{gr}_{\tau,i}G(X) = \tau_i(X)/\tau_{i-1}(X) \quad \text{and} \quad \text{gr}_\tau G(X) = \bigoplus_{i \in \mathbb{Z}} \text{gr}_{\tau,i}G(X).$$

When $X$ is equidimensional, we’ll also use the notation

$$\text{gr}^i_\tau G(X) = \tau^i(X)/\tau^{i+1}(X) \quad \text{and} \quad \text{gr}_\tau G(X) = \bigoplus_{i \in \mathbb{Z}} \text{gr}^i_\tau G(X).$$

If $X$ is smooth and connected then the filtration $\tau^*(X)$ is multiplicative and the associated graded object $\text{gr}_\tau G(X)$ is a graded ring.

The ring $K(X)$ and the ring $G(X)$, when $X$ is smooth, come equipped with $K$-Chern classes. For a smooth variety $X$, they can be constructed as follows. Starting with an effective divisor $D \subset X$ with invertible ideal sheaf $\mathcal{O}(-D)$, the first $K$-Chern class of the dual $\mathcal{O}(-D)^\vee = \mathcal{O}(D)$ is defined to be the class of the structure sheaf $[\mathcal{O}_D]$ in $K(X)$ or $G(X)$,

$$(\mathcal{O} .5) \quad c_1^K(\mathcal{O}(D)) = 1 - [\mathcal{O}(-D)] = [\mathcal{O}_D] \in K(X) \text{ or } G(X).$$

The formal group law for $K$-theory is multiplicative, meaning

$$(\mathcal{O} .6) \quad c_1^K(\mathcal{L} \otimes \mathcal{L}') = c_1^K(\mathcal{L}) + c_1^K(\mathcal{L}') - c_1^K(\mathcal{L})c_1^K(\mathcal{L}')$$

for any pair of line bundles $\mathcal{L}, \mathcal{L}'$ on $X$, so one can extend the definition of the first $K$-Chern class to arbitrary line bundles. Higher $K$-Chern classes for arbitrary vector bundles
can then be constructed from the first $K$-Chern class and the projective bundle formula for these groups.

For the remainder of this section, we specialize to the case $X = \text{SB}(A)$ is the Severi-Brauer variety associated to a central simple $k$-algebra $A$. After recalling various structure theorems for $K(X)$, we turn to proving the main result of this section, Theorem 3.6, that determines the low degree terms of the filtration $\tau_*(X)$ in the case $A$ is a $p$-primary indexed central simple $k$-algebra. As an immediate corollary to this computation, we also get a description of the low degree terms of the filtration $\tau_*(X)$ for arbitrary central simple $k$-algebras.

Our main tool in the remainder of this section is the simple description of $K(X)$ that’s been given by Quillen:

**Theorem 3.1** ([Qui73, §8, Theorem 4.1]). The homomorphism of $K$-groups

\[ \bigoplus_{i=0}^{\text{deg}(A)-1} K(A^\otimes i) \to K(X), \]

sending the class of a left $A^\otimes i$-module $M$ to the class of $\xi^i_X \otimes A^\otimes i M$, is an isomorphism. □

Quillen’s theorem shows that $K(X)$ (and hence $G(X)$ since $X$ is smooth) is a free $\mathbb{Z}$-module of rank $\text{deg}(A)$ with a basis the collection of classes of sheaves $\xi_X(i)$ (see (3).1 for the definition of these sheaves) where $i$ varies over the interval $0 \leq i < \text{deg}(A)$. It’s important to note that the description of $K(X)$ given by Theorem 3.1 is a group description.

To describe the ring structure of $K(X)$ we can observe that the morphism of Theorem 3.1 commutes with the pullback along a projection from a scalar extension of the base field. When this scalar extension is to an algebraic closure $\overline{k}$ of $k$ (or to any splitting field of $X$) this gives the following description of the rings $K(X)$ and $G(X)$.

**Lemma 3.2.** Let $\overline{X} = X_{\overline{k}}$ be the base change of $X$ along the extension of scalars $\overline{k} \supset k$ to an algebraic closure of $k$. Choose an isomorphism $\phi : \overline{X} \cong \mathbb{P}^{n-1}$ and let $\mathcal{O}(-1)$ be the tautological line bundle on $\overline{X}$ under $\phi$.

Then the class $\xi$ of $\mathcal{O}(-1)$ in $G(\overline{X})$ is independent of the isomorphism $\phi$ and, in this setting, there is a ring isomorphism

\[ \mathbb{Z}[x]/(1 - x)^{\text{deg}(A)} \cong G(\overline{X}) \]

sending $x$ to $\xi$. Under this isomorphism, the pullback along the projection

\[ G(X) \to G(\overline{X}) \]

identifies $G(X)$ with the subring of $\mathbb{Z}[x]/(1 - x)^{\text{deg}(A)}$ that’s additively generated by elements $\text{ind}(A^\otimes i)x^i$ with $0 \leq i \leq \text{deg}(A) - 1$. □

It follows from Lemma 3.2 that $K(X)$ is completely determined, as a group or as a ring, by the degree $\text{deg}(A)$ and the reduced $p$-behaviors of $A$ as $p$ varies over all primes. This is sufficient for the purposes of this section but, in Section 4 we’ll need the even stronger description of the $\lambda$-ring structure of $K(X)$ when $A$ is a central simple $k$-algebra of $p$-primary index. Recall that the $\lambda$-ring structure on $K(X)$ is the collection of operations

\[ \lambda^i : K(X) \to K(X) \]
defined by the condition $\lambda^i([\mathcal{F}]) = [\wedge^i \mathcal{F}]$ for any vector bundle $\mathcal{F}$ on $X$. These operations satisfy a number of functorial properties that we don’t list here (the reference [Man69] is a good introduction to this subject).

In the case $X = \text{SB}(A)$ for a $p$-primary indexed central simple $k$-algebra $A$, our description of the $\lambda$-ring structure on $K(X)$ is facilitated by the following definition.

**Definition 3.3.** Let $p$ be a prime and let $n \geq 0$ be an integer. Suppose $X = \text{SB}(A)$ is the Severi-Brauer variety associated to a central simple $k$-algebra $A$ of index $\text{ind}(A) = p^n$. We define the level set $S_X$ of $X$ as the set

$$S_X = \{1 \leq i \leq v_p(\exp(A)) : v_p(\text{rk}(\xi_X(p^i))) < v_p(\text{rk}(\xi_X(p^{i-1}))) - 1\}$$

$$= \{1 \leq i \leq v_p(\exp(A)) : v_p(\text{ind}(A^{\otimes p^i})) < v_p(\text{ind}(A^{\otimes p^{i-1}})) - 1\}.$$

Informally, the level set of $X$ consists of those places in the reduced behavior where the sequence decreases by more than one from the previous place.

Specifically, we’ll need the next lemma showing that level set of $X$ keeps track of the $\lambda$-ring generators of $K(X)$.

**Lemma 3.4 ([KM19, Lemma A.6]).** Let $p$ be a prime and let $n \geq 0$ be an integer. Suppose $X = \text{SB}(A)$ is the Severi-Brauer variety associated to a central simple $k$-algebra $A$ of index $\text{ind}(A) = p^n$. Then $K(X)$ is generated as a $\lambda$-ring by the vector bundles $\{\xi_X(p^i)\}_i$ with index $i \in S_X \cup \{0\}$. \hspace{1cm} \square

For now, however, we only need to consider the group and ring structures of $K(X)$, and of $G(X)$, when $X = \text{SB}(A)$ is the Severi-Brauer variety of a central simple algebra $A$ of $p$-primary index. Our goal is to describe the topological filtration $\tau_i(X)$ in low degrees $i \leq p - 2$. To do this, we use Lemma 3.2 to compare $\tau_i(X)$ to an explicit description of the topological filtration $\tau_*(\overline{X})$.

**Lemma 3.5.** Identify $G(\overline{X})$ with the ring $\mathbb{Z}[x]/(1-x)^{\deg(A)}$ as in Lemma 3.2. Set $h = 1-x$. Then the topological filtration $\tau_*(\overline{X})$ of $G(\overline{X})$ is determined as

$$\tau_i(\overline{X}) = \bigoplus_{j \leq i} h^{\deg(A) - 1 - j} \mathbb{Z}$$

for any integer $i \in \mathbb{Z}$. \hspace{1cm} \square

The description for $\tau_*(X)$ is then:

**Theorem 3.6.** Suppose that $A$ is a division algebra of index $p^n$ for some prime $p$ and some $n \geq 1$. Identify $G(\overline{X})$ with $\mathbb{Z}[x]/(1-x)^{p^n}$ as in Lemma 3.2. Set $h = 1-x$. Identify $G(X)$ with a subring of $G(\overline{X})$ via the pullback along the projection.

Then the topological filtration $\tau_*(X)$ on $G(X)$ is determined as

$$\tau_i(X) = \bigoplus_{j \leq i} p^n h^{p^n - 1 - j} \mathbb{Z} = p^n \tau_*(\overline{X})$$

for every $i \leq p - 2$. \hspace{1cm} 9
Proof. By Lemma 3.5, there’s an equality
\[ \tau_i(X) = \bigoplus_{j \leq i} h^{p^n - 1 - j} \mathbb{Z} \]
so, in order to prove the result it suffices to show that \( \tau_i(X) = p^n \tau_i(X) \). Note that the containment \( p^n \tau_i(X) \subseteq \tau_i(X) \) is immediate from [Mac19b, Lemma 5.7] so that we only need to show the reverse containment.

For any integer \( 0 \leq i \leq p - 2 \) (the nontrivial cases), an arbitrary element for \( \tau_i(X) \subseteq \tau_i(X) \)
is of the form
\[ \sum_{j=p^n-1-i}^{p^n-1} s_j h^j \]
for some integers \( s_j \). Writing this element as a sum of powers of elements \( a_i(1 - h)^{p^n - 1 - l} \) with \( a_i = \text{ind}(A \otimes^i) \) (cf. Lemma 3.2) gives an equality
\[ (\mathcal{O}.7) \quad \sum_{j=p^n-1-i}^{p^n-1} s_j h^j = \sum_{l=0}^{p^n-1} t_l a_i(1 - h)^{p^n - 1 - l} \]
with \( t_l \) some integers. Since powers of \( h \) form a basis for \( G(X) \) we can expand the right side of \( (\mathcal{O}.7) \) and compare the coefficients of \( h^j \) to find
\[ (\mathcal{O}.8) \quad s_j = \sum_{l \geq j} t_l a_i \left( \frac{p^n - 1 - l}{p^n - 1 - j} \right). \]
Now all of the \( a_i \) that appear in \( (\mathcal{O}.8) \) have \( p^n - 1 \geq l \geq j \geq p^n - p + 1 \). Hence \( l \) is prime-to-\( p \) and \( a_i = \text{ind}(A \otimes^i) = p^n \) by Remark 2.2. It follows that \( \tau_i(X) \) is contained in \( p^n \tau_i(X) \) for every \( 0 \leq i \leq p - 2 \) as claimed. \( \square \)

It follows immediately from Theorem 3.6 that one can describe the associated graded objects \( \text{gr}_{\tau_i}G(X) \) for the topological filtration on \( G(X) \) in the degrees \( 0 \leq i \leq p - 2 \).

**Corollary 3.7.** Suppose that \( A \) is a central simple \( k \)-algebra of index \( p^n \) for some prime \( p \) and some \( n \geq 1 \). Then \( \text{gr}_{\tau_i}G(X) \) is torsion free for all \( 0 \leq i \leq p - 2 \). \( \square \)

A description for the low degree terms of \( \tau_i(X) \) for an arbitrary Severi-Brauer variety \( X \) also follows from Theorem 3.6. We record it here for completeness.

**Corollary 3.8** (cf. [Mac19b, Corollary 5.9] and [Kar95, Corollary 1.3.2]). Suppose that \( A \) is an arbitrary central simple \( k \)-algebra. Identify \( G(X) \) with the ring \( \mathbb{Z}[x]/(1 - x)^{\deg(A)} \) as in Lemma 3.2. Let \( h = 1 - x \). Identify \( G(X) \) with a subring of \( G(X) \) via the extension of scalars map.

Let \( \text{ind}(A) = p_1^{n_1} \cdots p_r^{n_r} \) be a prime factorization of the index labeled so that \( p_1 < p_2 < \cdots < p_r \). Then the topological filtration on \( G(X) \) is determined as
\[ \tau_i(X) = \bigoplus_{j \leq i} \text{ind}(A) h^{\deg(A) - 1 - j} \mathbb{Z} = \text{ind}(A) \tau_i(X) \]
for every \( i \leq p_1 - 2 \). \( \square \)

And the corresponding result for the associated graded objects:
Corollary 3.9. Let $A$ be an arbitrary central simple $k$-algebra and let $\text{ind}(A) = p_1^{n_1} \cdots p_r^{n_r}$ be a prime factorization of the index labeled so that $p_1 < p_2 < \cdots < p_r$. Then $\text{gr}_{\tau,i} G(X)$ is torsion free for all $0 \leq i \leq p_1 - 2$. \hfill $\square$

4. CHOW THEORY

Throughout this section we assume $X = \text{SB}(A)$ is the Severi-Brauer variety associated to a central simple $k$-algebra $A$ having index $\text{ind}(A) = p^n$ for some prime $p$ and for some $n \geq 1$. Applying Lemma 2.5 to $A$, we also fix a chain of finite field extensions of $k$

$$k \subset L_0 \subset L_1 \subset \cdots \subset L_i \subset \cdots \subset L_n$$

such that the degree $[L_0 : k]$ is prime-to-$p$ and

$$[L_{i+1} : L_i] = p \quad \text{and} \quad \text{ind}(A_{L_i}) = p^{n-i}.$$ 

We’re going to be working with the integral Chow ring $\text{CH}(X)$ of $X$ and with the integral Chow groups $\text{CH}_i(X)$ of dimension-$i$ cycles on $X$ modulo rational equivalence. As is common, we often write $\text{CH}^i(X)$ for the group of codimension-$i$ cycles on $X$ mod rational equivalence. The main result of this section is the following theorem.

Theorem 4.1. Let $X = \text{SB}(A)$ and assume $\text{ind}(A) = p^n$ for some prime $p$ and for some $n \geq 1$. Fix an integer $0 \leq i \leq p - 2$ and assume $\text{CH}_i(X)$ is generated by polynomials in CH-Chern classes. Then $\text{CH}_i(X) = \mathbb{Z}$.

Example 4.2. Let $p$ be a prime and let $S$ be a strictly decreasing sequence of nonnegative integers ending in 0. Let $X^S$ be the Severi-Brauer variety constructed from $p$ and $S$ as in Example 2.4. Then $\text{CH}(X^S)$ is generated as a ring by CH-Chern classes so that Theorem 4.1 applies for all $0 \leq i \leq p - 2$.

Since the groups $\text{CH}_i(X)$ can contain only $p$-primary torsion, it follows from a restriction-corestriction argument that $\text{CH}_i(X)$ and $\text{CH}_i(X_{L_0})$ coincide. In order to simplify some of our notation, we’ll assume from now on that $k = L_0$ by possibly making a scalar extension of $k$ if this equality didn’t already hold. We also remark that since torsion in $\text{CH}_i(X)$ is $p$-primary, it suffices to prove the result after localizing at the prime ideal $(p)$ (i.e. we show that $\text{CH}_i(X) \otimes \mathbb{Z}_{(p)}$ is torsion free instead).

The proof of Theorem 4.1 in this setting requires a number of structure results for some subrings $\text{CT}(i; X)$ of $\text{CH}(X)$ that are generated by particular CH-Chern classes and is given at the end of this section. Our starting point will be these structure results.

Definition 4.3. For any $i \in \mathbb{Z}$, we let $\text{CT}(i; X)$ be the subring of $\text{CH}(X)$ generated by Chern class of $\zeta_X(i)$. The ring $\text{CT}(i; X)$ is canonically graded, and we write $\text{CT}^j(i; X) \subset \text{CH}^j(X)$ for its $j$th graded summand.

We recall that the groups $\text{CT}^j(i; X)$ are torsion free and of rank one for any $0 \leq j \leq \dim(X)$. Further, the localization of these groups $\text{CT}^j(i; X) \otimes \mathbb{Z}_{(p)}$ have a canonical generator that we denote by $\tau_i(j)$:

Proposition 4.4 ([KM19, Proposition A.8]). For any $i > 0$, the ring $\text{CT}(i; X) \otimes \mathbb{Z}_{(p)}$ is a free $\mathbb{Z}_{(p)}$-module. Moreover, for $0 \leq j < \deg(A)$ the degree $j$ summand $\text{CT}^j(i; X) \otimes \mathbb{Z}_{(p)}$ is generated by the element

$$\tau_i(j) := c_p^v(\zeta_X(i))^{a_0} c_{n_1}(\zeta_X(i))$$
where $p^v$ is the largest power of $p$ dividing $\text{ind}(A^{\otimes i})$ and $j = p^v s_0 + s_1$ with $0 \leq s_1 < p^v$. □

When there’s possible ambiguity for where these classes are defined (e.g. if we work over multiple different fields simultaneously) we’ll include a superscript like $\tau_i^X(j)$ to mean these classes are defined inside $\text{CT}(i; X) \otimes \mathbb{Z}_{(p)}$. We collect here a number of results on the rings $\text{CT}(i; X)$.

**Lemma 4.5** ([Mac19a, Lemma 3.4]). Let $F/k$ be a finite field extension and $X_{F/k} : X_F \to X$ the projection. Then the composition

$$\text{CT}^j(i; X) \subset \text{CH}^j(X) \xrightarrow{\pi_{\text{F/k}}^*} \text{CH}^j(X_F)$$

of the inclusion and flat pullback $\pi_{\text{F/k}}^*$ has image contained in $\text{CT}^j(i; X_F)$. Moreover, if the composition (resp. this composition with $\mathbb{Z}_{(p)}$-coefficients)

$$\text{CT}^j(i; X_F) \subset \text{CH}^j(X_F) \xrightarrow{\pi_{\text{F/k},*}} \text{CH}^j(X)$$

of the inclusion and proper pushforward $\pi_{\text{F/k},*}$ has image contained in $\text{CT}^j(i; X)$ (resp. has image contained in $\text{CT}^j(i; X) \otimes \mathbb{Z}_{(p)}$) then the projection formula holds for $\pi_{\text{F/k},*}, \pi_{\text{F/k}}^*$ and the compositions

$$\pi_{\text{F/k}}^* \circ \pi_{\text{F/k},*} \quad \text{and} \quad \pi_{\text{F/k},*} \circ \pi_{\text{F/k}}^*$$

are both multiplication by $[F : k]$. □

**Lemma 4.6** ([Kar17b, Proposition 3.5] and [Mac19a, Lemma 3.5]). Let $F/k$ be a finite extension splitting $A$ and $X_{F/k} : X_F \to X$ the projection. Then the composition

$$\text{CT}^j(i; X_F) \subset \text{CH}^j(X_F) \xrightarrow{\pi_{\text{F/k},*}} \text{CH}^j(X)$$

of the inclusion and the proper pushforward $\pi_{\text{F/k},*}$ has image contained in $\text{CT}^j(i; X)$ for any $j \geq 0$ and for any $i \geq 1$. □

**Lemma 4.7** ([Kar17b, Proposition 3.5]). Let $j \geq 0$ be an integer and, in the notation above, let $F = L_{n-v_p(i)}$. Then the composition

$$\text{CT}^j(1; X_F) = \text{CH}^j(X_F) \xrightarrow{\pi_{\text{L_F/k},*}} \text{CH}^j(X)$$

has image contained in $\text{CT}^j(1; X)$. Further, this composition is surjective onto $\text{CT}^j(1; X)$. □

The proof of Theorem 4.1 follows a technique which was developed in [Kar17b] to compute the Chow ring of a generic Severi-Brauer variety and which was used in [KM19] to compute the Chow ring of some Severi-Brauer varieties with prescribed reduced behavior. Most of the tools that we use here have actually already appeared in these references. Essentially, we’re going to use the projection formula to show that any element of $\text{CH}_i(X) \otimes \mathbb{Z}_{(p)}$ that can be realized as a polynomial of $\text{CH}$-Chern classes is a multiple of $\tau_i(\dim(X) - i)$ for any $0 \leq i \leq p - 2$. The next lemma gives a useful description of any such polynomial of $\text{CH}$-Chern classes.

**Lemma 4.8.** Let $S_X = \{i_1 < \cdots < i_k\}$ be the level set of $X$. Let $j \geq 0$ be an integer and let $\alpha$ be an element of $\text{CH}^j(X) \otimes \mathbb{Z}_{(p)}$ that can be realized as a polynomial in $\text{CH}$-Chern classes. Then $\alpha$ is contained in the subgroup of $\text{CH}^j(X) \otimes \mathbb{Z}_{(p)}$ generated by elements

$$\tau_1(a_0)\tau_p^1(a_1)\cdots\tau_p^k(a_k)$$

where $a_0 + \cdots + a_k = j$ for some integers $a_0, \ldots, a_k \geq 0$. □
Proof. Because of Lemma 3.4, every polynomial in CH-Chern classes is a polynomial in the Chern classes of \( \{ \zeta_X(i) \} \), where \( i \in S_X \cup \{ 0 \} \) (cf. [KM19, Proposition A.5]). Grouping terms using Proposition 4.4 gives exactly these generators. □

Now we separate the proof of Theorem 4.1 into two cases. We want to show that each generator from Lemma 4.8 is contained in \( \text{CT}(1; X) \otimes \mathbb{Z}(p) \). To do this we note that either \( \tau_1(a_0) \) appears in such a generator with \( a_0 > 0 \) or this term doesn’t appear at all. The latter case is easy to handle.

**Lemma 4.9.** Let \( S = \{ i_1 < \cdots < i_k \} \) be an arbitrary collection of integers. Let \( a_1, \ldots, a_k \geq 0 \) be integers such that \( p^n - p + 1 \leq a_1 + \cdots + a_k \). Then

\[
\tau_{p^1}(a_1) \cdots \tau_{p^k}(a_k)
\]

is contained in \( \text{CT}(1; X) \otimes \mathbb{Z}(p) \).

**Proof.** Since \( a_1 + \cdots + a_k \geq p^n - p + 1 \) by assumption, there must exist an index \( r \) with \( a_r \geq p^n - i_r \). Indeed, if this was false then for every \( 1 \leq r \leq k \) there would be an inequality \( a_r < p^n - i_r \). Since \( p^n - i_r < p^n - r \) this would imply the inequality

\[
a_1 + \cdots + a_k < \sum_{l=1}^{k} p^{n-l} < \sum_{l=1}^{n} p^{n-l} = \frac{p^n - 1}{p - 1}.
\]

But since one also has

\[
\frac{p^n - 1}{p - 1} < p^n - p + 1
\]

for all \( p > 2 \) and for all \( n > 1 \) this can’t happen.

Assume then that \( a_r \geq p^n - i_r \) for some \( 1 \leq r \leq k \). It follows from the proof of [KM19, Corollary A.13] there is an element \( x \) in \( CH(X_{L_n}) \) with

\[
\pi_{L_n/k, *}(x) = \tau_{p^r}(a_r).
\]

Applying the projection formula to the product \( \tau_{p^1}(a_1) \cdots \tau_{p^k}(a_k) \) and using Lemma 4.6 gives the result. □

On the other hand, if \( \tau_1(a_0) \) appears in the product of such a generator then the same proof becomes a little more complicated. We’ll need the following lemma.

**Lemma 4.10.** Let \( 0 \leq j \leq n \) be an integer. Define

\[
\nu_j = v_p(\text{rk}(\zeta_X(p^j))) \quad \text{and} \quad \alpha_{i,j} = v_p(\text{rk}(\zeta_{X_{L_i}}(p^j))).
\]

Let \( \pi_{L_i/k} : X_{L_i} \to X \) be the projection along \( L_i/k \). Then there’s an equality

\[
\pi_{L_i/k}^*(\tau_{p^l}^X(k)) = \beta_{i,k}^j \tau_{p^l}(k)
\]

for some \( \beta_{i,k}^j \in \mathbb{Z}(p) \) with

\[
v_p(\beta_{i,k}^j) = \begin{cases} 
 n_j - \alpha_{i,j} - v_p(k/p^{\alpha_{i,j}}) & \text{if } k \equiv 0 \pmod{p^{\alpha_{i,j}}} \\
 n_j - \alpha_{i,j} & \text{if } k \not\equiv 0 \pmod{p^{\alpha_{i,j}}}.
\end{cases}
\]
Proof. The proof follows the lines of [KM19, Lemma A.12]. Pulling back the total CH-Chern polynomial of $\zeta_X(p^j)$ to $L_i$ we find

$$\pi_{L_i/k}^* c_t(\zeta_X(p^j)) = c_t(\zeta_{X_{L_i}}(p^j)) p^{n_j-a_{i,j}}$$

$$= \left(1 + \tau_{p^j}^{X_{L_i}}(1)t + \cdots + \tau_{p^j}^{X_{L_i}}(p^{a_{i,j}})t^{p^{a_{i,j}}}\right)^{p^{n_j-a_{i,j}}}$$

$$= 1 + \beta_{i,1}^j \tau_{p^j}^{X_{L_i}}(1)t + \cdots + \beta_{i,p^{n_j}}^j \tau_{p^j}^{X_{L_i}}(p^{n_j})t^{p^{n_j}}.$$

The $p$-adic valuations of the $\beta_{i,k}^j$ can be determined by the multinomial formula. In fact, this has already been done explicitly in [KM19, Lemma B.4]; set $r = a_{i,j}$ and $s = n_j$. \qed

Proof of Theorem 4.1. We proceed by showing that every generator as in Lemma 4.8 is contained in $\text{CT}(1;X) \otimes \mathbb{Z}(p)$. If in such a generator

$$\tau_1(a_0) \tau_{p^1}(a_1) \cdots \tau_{p^k}(a_k)$$

one has $a_0 = 0$ then we can apply Lemma 4.9. We’re left in the case $a_0 > 0$.

Set $v = v_p(a_0)$ and let $F = L_{n-v}$ (if $v \geq n$ then the claim follows from the proof of [KM19, Corollary A.13]). By Lemma 4.7, there is an element $x \in \text{CH}^{a_0}(X_F) \otimes \mathbb{Z}(p)$ such that the pushforward of $x$ along the projection $\pi_{F/k} : X_F \to X$ equals

$$\pi_F/x_{\ast}(x) = \tau_1^X(a_0).$$

Since one has $\pi_{F/k} \circ \pi_{F/k}^x = [F : k] = p^{a-v}$ by the construction of $F$ and, one has

$$\pi_{F/k}^x(\tau_1^X(a_0)) = \beta_{n-v,a_0}^0 \tau_1^X(a_0)$$

with $v_p(\beta_{n-v,a_0}^0) = n - v$ by Lemma 4.10, it follows that $x = \alpha \tau_1^X(a_0)$ for some $\alpha \in \mathbb{Z}(p)$ not divisible by $p$.

Applying the projection formula one finds

$$\tau_1^X(a_0) \tau_{p^1}^X(a_1) \cdots \tau_{p^k}^X(a_k) = \pi_{F/k, \ast}(\alpha \tau_1^X(a_0)) \tau_{p^1}^X(a_1) \cdots \tau_{p^k}^X(a_k)$$

$$= \pi_{F/k, \ast} \left(\alpha \tau_1^X(a_0) \cdot \pi_{F/k}^* \left(\tau_{p^1}^X(a_1) \cdots \tau_{p^k}^X(a_k)\right)\right)$$

$$= \pi_{F/k, \ast} \left(\beta \alpha \tau_1^X(a_0) \tau_{p^1}^X(a_1) \cdots \tau_{p^k}^X(a_k)\right).$$

We won’t use it but, the coefficient $\beta$ can also be explicitly determined by Lemma 4.10. We claim that

$$\tau_{p^1}^X(a_1) \cdots \tau_{p^k}^X(a_k)$$

is contained in $\pi_{L_n,F, \ast}(\text{CH}(X_{L_n}) \otimes \mathbb{Z}(p)) \subset \text{CT}(1;X_F) \otimes \mathbb{Z}(p)$. But, this is true by Lemma 4.9 since there is an integer $1 \leq r \leq k$ with $a_r \geq p^{v-i}$. In particular, there is an element

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y ∈ CH(X_{L_n}) ⊗ \mathbb{Z}_{(p)} with

\[ \tau_i^X (a_0) \tau_{p_1}^X (a_1) \cdots \tau_{p_k}^X (a_k) = \pi_{F/k,*} \left( \beta \alpha \tau_i^X (a_0) \tau_{p_1}^X (a_1) \cdots \tau_{p_k}^X (a_k) \right) \]

\[ = \pi_{F/k,*} \left( \beta \alpha \tau_i^X (a_0) \pi_{L_n/F,*}(y) \right) \]

\[ = \pi_{F/k,*} \left( \pi_{L_n/F,*} (y) \cdot \pi_{L_n/F,*} (\beta \alpha \tau_i^X (a_0)) \right) \]

\[ = \pi_{L_n/k,*} \left( \pi_{L_n/F,*} (\beta \alpha \tau_i^X (a_0)) \right) \]

and this last element is contained in CT(1; X) ⊗ \mathbb{Z}_{(p)} by Lemma 4.6. □

5. Connective K-theory

Throughout this section we let X be a k-variety. We write \( \mathcal{M}(X) \) for the category of coherent sheaves on X. The category \( \mathcal{M}(X) \) is abelian and admits a filtration

\[ 0 = \mathcal{M}_{-1}(X) \subset \mathcal{M}_0(X) \subset \cdots \subset \mathcal{M}_n(X) = \mathcal{M}(X) \]

by Serre subcategories \( \mathcal{M}_i(X) \) defined as the category of coherent sheaves on X supported in dimension-i or less. Each category \( \mathcal{M}_i(X) \) is also abelian and it makes sense to consider the Grothendieck group \( K(\mathcal{M}_i(X)) \).

We refer to [Cai08] for our treatment of the algebraic connective K-theory \( \text{CK}(X) \) of X. By definition \( \text{CK}(X) \) is the sum of groups \( \text{CK}_i(X) \) that can be realized as the image of the induced map on \( K \)-groups under the exact inclusion \( \mathcal{M}_i(X) \subset \mathcal{M}_{i+1}(X) \):

\[ \text{CK}(X) = \bigoplus_{i \in \mathbb{Z}} \text{CK}_i(X) \quad \text{and} \quad \text{CK}_i(X) = \text{Im} \left( K(\mathcal{M}_i(X)) \to K(\mathcal{M}_{i+1}(X)) \right) . \]

When X is smooth, connected, and of dimension-n the group \( \text{CK}(X) \) has the structure of a commutative and graded ring. In this case, we often write \( \text{CK}^i(X) \) for the group summand \( \text{CK}_{n-i}(X) \) and the multiplication is a collection of maps

\[ \text{CK}^i(X) \otimes \text{CK}^j(X) \to \text{CK}^{i+j}(X) \]

for any \( i, j \in \mathbb{Z} \).

For any integer \( i \), the group \( \text{CK}^i(X) \) has the structure of left \( K(X) \)-module induced by the tensor product of sheaves. Indeed, tensoring by a locally free sheaf is exact on \( \mathcal{M}_{n-i}(X) \) and this descends to a morphism

\[ K(X) \to \text{End}(\text{CK}^i(X)) \]

sending a class \([\mathcal{F}]\) to the endomorphism sending \([\mathcal{G}]\) to \([\mathcal{F} \otimes \mathcal{G}]\).

In some instances, it’s known that \( \text{CK}^i(X) \) can be naturally associated with a subgroup of the ring \( K(X) \). More precisely, for any \( i \in \mathbb{Z} \) one can consider the morphism

\[ \psi_X^i : \text{CK}^i(X) \to \tau^i(X) \]

to the \( i \)th piece of the topological filtration (O.4) defined via the map

\[ \text{Im} \left( K(\mathcal{M}_{n-i}(X)) \to K(\mathcal{M}_{n-i+1}(X)) \right) \to \text{Im} \left( K(\mathcal{M}_{n-i}(X)) \to K(\mathcal{M}(X)) \right) \]
induced by the inclusion $\mathcal{M}_{n-i+1}(X) \subset \mathcal{M}(X)$; for $i \leq 2$ the map $\psi^i_X$ is an isomorphism. In general, the $K(X)$-module structure on $\text{CK}^i(X)$ is related to the ring structure of $\text{CK}(X)$ via $\psi^i_X$ and the following lemma.

**Lemma 5.1.** Let $\beta$ be the Bott element, i.e. the element of $\text{CK}^{-1}(X)$ represented by the class of $\mathcal{O}_X$. Then the diagram below, with top horizontal arrow induced by the $K(X)$-module structure morphism on $\text{CK}^i(X)$ and bottom horizontal arrow the ring structure map on $\text{CK}(X)$, is commutative for every $i, j \in \mathbb{Z}$.

$$
\begin{align*}
\tau^j(X) \otimes \text{CK}^i(X) & \longrightarrow \text{CK}^i(X) \\
\psi^i_X \otimes 1 & \quad \beta \quad 1
\end{align*}
$$

$$
\text{CK}^j(X) \otimes \text{CK}^i(X) \longrightarrow \text{CK}^{i+j}(X)
$$

In particular, the composition

$$
\tau^j(X) \otimes \text{CK}^i(X) \to K(X) \otimes \text{CK}^i(X) \to \text{CK}^i(X)
$$

has image contained in $\beta^i \text{CK}^{i+j}(X)$.

**Proof.** This follows from the fact that the multiplication by $\beta$ map and the map induced by the inclusion $\mathcal{M}_{n-i}(X) \to \mathcal{M}_{n-i+2}(X)$ are identical on algebraic connective $K$-theory, cf. [Cai08, Proof of Theorem 7.1]. In particular, one has

$$
\psi^j(\mathcal{F}) \cdot \mathcal{G} = (\beta^j \cdot \mathcal{F}) \cdot \mathcal{G} = \beta^j \cdot (\mathcal{F} \cdot \mathcal{G}) = \psi^j(\mathcal{F} \cdot \mathcal{G})
$$

for all coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ supported in codimension-$j$ and codimension-$i$ respectively. \qed

Alternatively, one can relate the $K(X)$-module structure on $\text{CK}(X)$ to the ring structure in the following way. Give the tensor product $\text{CK}^i(X) \otimes \text{CK}^j(X)$ a $K(X)$-module structure for any $i, j \in \mathbb{Z}$ by acting on the left of either $\text{CK}^i(X)$ or $\text{CK}^j(X)$. The multiplication map is then a morphism of $K(X)$-modules:

**Lemma 5.2.** For any $i, j \in \mathbb{Z}$, the multiplication map

$$
\text{CK}^i(X) \otimes \text{CK}^j(X) \to \text{CK}^{i+j}(X)
$$

is a morphism of $K(X)$-modules where $\text{CK}^i(X) \otimes \text{CK}^j(X)$ is considered as a $K(X)$-module by the action of $K(X)$ on either $\text{CK}^i(X)$ or $\text{CK}^j(X)$.

**Proof.** This is a consequence of the proof of Lemma 5.1. \qed

To define CK-Chern classes for a smooth variety $X$, we start with any effective divisor $D \subset X$ having invertible ideal sheaf $\mathcal{O}(-D)$. The first CK-Chern class in $\text{CK}^1(X)$ is then defined as

$$
c_1^{\text{CK}}(\mathcal{O}(D)) = [\mathcal{O}_D] \in \text{CK}^1(X).
$$

It’s easy to see from (0.5) that under the isomorphism

$$
\psi^1_X : \text{CK}^1(X) \cong \tau^1(X)
$$

the first CK-Chern class is mapped to the first $K$-Chern class. The formal group law for algebraic connective $K$-theory is also multiplicative, meaning that similar to (0.6) one has

$$
c_1^{\text{CK}}(\mathcal{L} \otimes \mathcal{L}') = c_1^{\text{CK}}(\mathcal{L}) + c_1^{\text{CK}}(\mathcal{L}') - \beta c_1^{\text{CK}}(\mathcal{L})c_1^{\text{CK}}(\mathcal{L}')
$$

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for any pair of line bundles $\mathcal{L}, \mathcal{L}'$ on $X$ allowing one to extend the definition of the first CK-Chern class to arbitrary line bundles. Higher CK-Chern classes for an arbitrary vector bundle can then be defined using this definition of the first CK-Chern class and the projective bundle formula [Cai08, Theorem 6.3].

The following lemma shows the relationship between $K$-Chern classes, CK-Chern classes, and the $K(X)$-module structure on $CK(X)$.

**Lemma 5.3.** Let $\mathcal{F}$ and $\mathcal{G}$ be two vector bundles on $X$. Then

$$c^K_i(\mathcal{F}) \cdot c^{CK}_j(\mathcal{G}) = \beta^i c^{CK}_i(\mathcal{F}) c^{CK}_j(\mathcal{G})$$

for any pair of integers $i, j \geq 0$.

**Proof.** The multiplication map

$$CK^i(X) \otimes CK^j(X) \to CK^{i+j}(X)$$

is a morphism of left $K(X)$-modules when the tensor product is given its left $K(X)$-module structure by Lemma 5.2. This means that, by possibly moving to a successive chain of projective bundles over $X$ and applying the projective bundle formula [Cai08, Theorem 6.3], it suffices to check the formula when $i = j = 1$ and when $\mathcal{F}, \mathcal{G}$ are both line bundles; here the claim is obvious. \qed

**Theorem 5.4.** Consider the following statements:

1. $CH(X)$ is generated as a ring by $CH^1(X)$
2. $CK(X)$ is generated as a $K(X)$-algebra by $CK^1(X)$ and $\beta$
3. $CK^i(X)$ is generated as a $K(X)$-module by polynomials of $CK$-Chern classes for every $i \in \mathbb{Z}$
4. $CK^i(X)$ is generated as a $K(X)$-module by polynomials of $CK$-Chern classes for some $i \in \mathbb{Z}$
5. $CK^i(X)$ is generated additively by polynomials of $CK$-Chern classes and $\beta CK^{i+1}(X)$ for some $i \in \mathbb{Z}$
6. $CH^i(X)$ is generated additively by polynomials of $CH$-Chern classes for some $i \in \mathbb{Z}$.

Then (1) $\iff$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5) $\iff$ (6).

**Proof.** (1) $\implies$ (2): For any $i \in \mathbb{Z}$ the exact sequence of [Cai08, Theorem 7.1]

$$0 \to \beta CK^{i+1}(X) \to CK^i(X) \to CH^i(X) \to 0$$

shows that $CK^i(X)$ is generated additively by polynomials of first CK-Chern classes (lifts of the generators of $CH^i(X)$) and $\beta CK^{i+1}(X)$. Now Lemma 5.3 shows that $\beta CK^{i+1}(X)$ is generated as a $K(X)$-module by polynomials of first CK-Chern classes as well.

(2) $\implies$ (1): The canonical surjection $CK(X) \to CH(X)$ takes CK-Chern classes to CH-Chern classes and has kernel $\beta CK(X)$. Normally, this would imply $CH(X)$ is generated as a $K(X)$-algebra by $CH^1(X)$ but, in this case any element of $K(X) = \mathbb{Z} \oplus \tau^1(X)$ acts on $CH(X)$ only via by its rank because of Lemma 5.1.

(2) $\implies$ (3) $\implies$ (4): This is obvious.
(4) $\implies$ (5): Let $\{p_i\}_{i \in I}$ be a set of polynomials in CK-Chern classes that generate $\text{CK}^i(X)$ as a $K(X)$-module. Then every element $x$ of $K(X)$ can be written as $x = \text{rk}(x) + (x - \text{rk}(x))$ where $\text{rk} : K(X) \to \mathbb{Z}$ is the rank map. The element $x - \text{rk}(x)$ is contained in $\tau^1(X)$ so for any index $i$ one finds
\[
x p_i = \text{rk}(x)p_i + (x - \text{rk}(x))p_i = \text{rk}(x)p_i + \beta z
\]
for some element $z$ of $\text{CK}^{i+1}(X)$ by Lemma 5.1.

(5) $\iff$ (6): One can use again that the surjection $\text{CK}^i(X) \to \text{CH}^i(X)$ takes CK-Chern classes to CH-Chern classes and, by [Cai08, Theorem 7.1], has kernel $\beta \text{CK}^{i+1}(X)$. □

**Remark 5.5.** In general, the implications (5) $\implies$ (4) $\implies$ (3) $\implies$ (2) between the properties of Theorem 5.4 are false.

**Remark 5.6.** Fix an integer $\ell$. Suppose that $X$ is a smooth and connected variety satisfying one of either (1)-(2), or (5)-(6) for all $i \geq \ell$, of Theorem 5.4. Then every projective bundle over $X$ and every localization of $X$ satisfies the same properties. To see this, one can use the projective bundle formula and the localization exact sequence for the Chow ring.

Theorem 5.4 can be combined with [Kar19, Theorem 2.2] to give the following corollary.

**Corollary 5.7.** Denote by
\[
\varphi^i_X : \text{CH}^i(X) \to \tau^i(X) \quad \text{and} \quad \varphi^i_X : \text{CH}^i(X) \to \text{gr}^i_G(X)
\]
defined by taking the class of an irreducible variety $V$ to the class of the structure sheaf $[\mathcal{O}_V]$.

Then (1) $\iff$ (2) $\implies$ (3) $\implies$ (4) $\iff$ (5).

**Proof.** (1) $\iff$ (2): By [Kar19, Theorem 2.2] the assumption $\varphi^i_X$ is an isomorphism for all $i \geq 0$ implies that $\psi^i_X$ is an isomorphism for every $i \in \mathbb{Z}$ hence this is a consequence of Theorem 5.4.

(2) $\implies$ (3) $\implies$ (4) $\iff$ (5): By [Kar19, Theorem 2.2 and Remark 2.4] the assumption $\varphi^i_X$ is an isomorphism for all $i \geq \ell$ implies that $\psi^i_X$ is an isomorphism for every $i \geq \ell$ hence this is a consequence of Theorem 5.4. □
Finally, the following result on the algebraic connective $K$-theory of a Severi-Brauer variety follows from Theorem 4.1 and Corollary 5.7.

**Corollary 5.8.** Let $X = \text{SB}(A)$ be the Severi-Brauer variety associated to a central simple $k$-algebra $A$ with $\text{ind}(A) = p^n$ for some prime $p$ and some integer $n \geq 0$. Assume that $\text{CH}_i(X)$ is generated by polynomials of CH-Chern classes for all $0 \leq i \leq p-2$. Then the canonical morphism

$$\psi^\dim(X)-i_X : \text{CK}_i(X) \to \tau_i(X)$$

is an isomorphism for all $0 \leq i \leq p-2$. \qed

**References**


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