THE ARITHMETIC GENUS OF A COMPLETE INTERSECTION CURVE

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Abstract. The purpose of this short note is to relate two formulas for the genus of a curve that can be realized as a complete intersection in some projective space.

Fix a field $k$. Without any loss of generality, one can suppose that $k$ is algebraically closed throughout this note. Let $X$ be a projective $k$-variety and choose an embedding $X \subset \mathbb{P}^n = \text{Proj}(k[x_0, \ldots, x_n])$.

We say that $X$ is a complete intersection (with respect to this embedding) if $X$ is the vanishing locus $X = V_+(f_1, \ldots, f_c)$ of $c = \text{codim}(X, \mathbb{P}^n)$ homogeneous equations $f_1, \ldots, f_c$ of the coordinate ring $k[x_0, \ldots, x_n]$ that form a regular sequence for this ring.

When $X$ is a complete intersection curve (i.e. $\dim(X) = 1$), the arithmetic genus of $X$ has been calculated in [AS98, Corollary 2].

Theorem 0.1. Suppose that $X = V_+(f_1, \ldots, f_{n-1}) \subset \mathbb{P}^n$ is a complete intersection curve. Then the arithmetic genus $g(X)$ of $X$ equals

\begin{equation}
(\text{no.1}) \quad g(X) = \sum_{i=1}^{n-1} (-1)^{i+n-1} \left( \sum_{1 \leq a_1 < \cdots < a_i \leq n-1} \frac{d_{a_1} + \cdots + d_{a_i} - 1}{d_{a_1} + \cdots + d_{a_i} - n - 1} \right)
\end{equation}

where for each $1 \leq i \leq n-1$ we write $d_i = \text{deg}(f_i)$. \hfill \Box

Briefly, the proof of Theorem 0.1 utilizes the fact that the Koszul complex gives a resolution for the structure sheaf of $X$ by sums of twists of the tautological bundle on $\mathbb{P}^n$; the Euler characteristic of $X$ (and hence the arithmetic genus) can then be determined explicitly from the computation [Sta20, Tag 01XT] of the cohomology of these twists.

The purpose of this note is to prove the following simplification of formula (no.1).

Theorem 0.2. Suppose that $X = V_+(f_1, \ldots, f_{n-1}) \subset \mathbb{P}^n$ is a complete intersection curve. Then the arithmetic genus $g(X)$ of $X$ equals

\begin{equation}
(\text{no.2}) \quad g(X) = 1 + \frac{1}{2} (d_1 + \cdots + d_{n-1} - n - 1) d_1 \cdots d_{n-1}
\end{equation}

where for each $1 \leq i \leq n-1$ we write $d_i = \text{deg}(f_i)$.

Remark 0.3. If $X = H_1 \cap \cdots \cap H_{n-1}$ is the intersection of hypersurfaces $H_i \subset \mathbb{P}^n$ such that the sequence

$H_1, \ H_1 \cap H_2, \ H_1 \cap H_2 \cap H_3, \ \ldots, \ H_1 \cap \cdots \cap H_{n-1}$

consists of smooth schemes, then Theorem 0.2 can be proved using the adjunction formula and induction; note that $X$ is not assumed smooth, or even reduced, in Theorem 0.2.
Before giving the proof, we make some initial observations. Consider the following set of points $S_{>0}^{n-1} \subset \mathbb{A}_Q^{n-1}(\mathbb{Z})$ consisting of tuples of integers with positive coordinates
\[(S0) \quad S_{>0}^{n-1} = \{(d_1, \ldots, d_{n-1}) : d_i \in \mathbb{Z}, \ d_1, \ldots, d_{n-1} > 0\}.
\]
The arithmetic genus $g(X)$ from (no.2) agrees with the polynomial of $\mathbb{Q}[X_1, \ldots, X_{n-1}]$
\[(gn) \quad g_n(X_1, \ldots, X_{n-1}) := \sum_{i=1}^{n-1} (-1)^{i+n-1} \left( \frac{1}{n!} \sum_{a_i < \cdots < a_i} \prod_{j=1}^{n-1} (X_{a_i} + \cdots + X_{a_i} - j) \right)\]
evaluated at the corresponding point of $S_{>0}^{n-1}$. Because of the following lemma, we’ll often work with the latter description of the arithmetic genus.

**Lemma 0.4.** Fix an integer $n \geq 2$. Let $V \subset \mathbb{A}_Q^{n-1}$ be an arbitrary closed subvariety. Then there is a containment $S_{>0}^{n-1} \subset V$ if and only if $V = \mathbb{A}_Q^{n-1}$. In particular, if a polynomial $f(X_1, \ldots, X_{n-1}) \in \mathbb{Q}[X_1, \ldots, X_{n-1}]$ vanishes on $S_{>0}^{n-1}$, then $f(X_1, \ldots, X_{n-1}) = 0$.

**Proof.** Let $V = V(f_1, \ldots, f_m)$ be the affine variety defined as the vanishing locus of some nonconstant polynomials $f_1, \ldots, f_m \in \mathbb{Q}[X_1, \ldots, X_n]$. We’ll show that there is a point of $S_{>0}^{n-1}$ not contained in $V$; to do this it suffices to work with any of the hypersurfaces $V(f_i)$, and without loss of any generality, we’ll assume $V = V(f)$. Since $\mathbb{Q}$ is infinite, there is a point $p \in \mathbb{A}_Q^{n-1}(\mathbb{Q})$ outside of $V$; we can also assume that $p$ has all positive coordinates. Let $\ell$ be the line connecting $p$ and the origin. Then the restriction of $f$ to $\ell$ has finitely many zeros and $\ell$ intersects $S_{>0}^{n-1}$ infinitely often.

**Lemma 0.5.** Let $n \geq 3$ be an integer. Then $g_n(1, X_2, \ldots, X_{n-1}) = g_{n-1}(X_2, \ldots, X_{n-1})$.

**Proof.** Identify $S_{>0}^{n-1}$ with the intersection $S_{>0}^n \cap V(X_1 - 1) \subset \mathbb{A}_Q^n$, i.e. with the restriction of $S_{>0}^n$ to the hyperplane where $X_1 = 1$. In this case, $g_n(X_1, \ldots, X_{n-1}) - g_{n-1}(X_2, \ldots, X_{n-1})$ vanishes on every point of $S_{>0}^{n-1}$, as they both compute the arithmetic genus. Applying lemma 0.4 gives the result.

**Lemma 0.6.** Keep notation as in Lemma 0.7. Then there is an equality
\[g_n(X_1 + 1, X_2, \ldots, X_{n-1}) = g_n(X_1, X_2, \ldots, X_{n-1}) + \sum_{i=1}^{n-1} (-1)^{i+n-1} \left( \frac{1}{(n-1)!} \sum_{1 < a_2 < \cdots < a_i} \prod_{j=1}^{n-1} (X_1 + \cdots + X_{a_i} - j) \right)\]
as elements of $\mathbb{Q}[X_1, \ldots, X_{n-1}]$.

**Proof.** Restricted to the set $S_{>0}^{n-1}$ of (S0), the polynomial $g_n(X_1, \ldots, X_{n-1})$ agrees with the function
\[g_n^*(X_1, \ldots, X_{n-1}) := \sum_{i=1}^{n-1} (-1)^{i+n-1} \left( \sum_{1 < a_1 < \cdots < a_i \leq n-1} \left( \frac{X_{a_1} + \cdots + X_{a_i} - 1}{X_{a_1} + \cdots + X_{a_i} - n - 1} \right) \right).
\]
Because of the recursive formula for binomial coefficients,
\[
\binom{m}{k} = \binom{m-1}{k-1} + \binom{m-1}{k}
\]
for some polynomial $h$ we work with the individual summands $\leq$ for each 1

$$g_n(x_1 + 1, x_2, \ldots, x_n) - g_n(x_1, x_2, \ldots, x_n - 1)$$

vanishes restricted to $S_{>0}^{n-1}$; the claim follows from Lemma 0.4.

The proof of Theorem 0.2 is dependent on the following lemma.

**Lemma 0.7.** For any $n \geq 2$, there’s an equality

$$g_n(x_1, \ldots, x_n) = 1 + x_1 \cdots x_{n-1}h_n(x_1, \ldots, x_{n-1})$$

for some polynomial $h_n(x_1, \ldots, x_{n-1}) \in \mathbb{Q}[x_1, \ldots, x_{n-1}]$ with

$$h_n(x_1, \ldots, x_{n-1}) = a_1 x_1 + \cdots + a_{n-1} x_{n-1} + c$$

for some $a_1, \ldots, a_{n-1}, c \in \mathbb{Q}$.

**Proof.** The claim is clear when $n = 2$ so assume $n \geq 3$. We’ll use the recursive formula

$$g_n(x_1 + 1, x_2, \ldots, x_n) = g_n(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{n-1} (-1)^{i+n-1} \left( \frac{1}{(n-1)!} \sum_{1<a_2<\cdots<a_i} \prod_{j=1}^{n-1} (X_1 + \cdots + X_{a_i} - j) \right).$$

After setting $X_1 = 0$ in the above recursion one gets the equality

$$g_n(1, x_2, \ldots, x_n) = g_n(0, x_2, \ldots, x_n) - 1 + g_{n-1}(x_2, \ldots, x_{n-1}).$$

Since there’s also an equality $g_n(1, x_2, \ldots, x_n) = g_{n-1}(x_2, \ldots, x_{n-1})$ by Lemma 0.5, it follows that

$$g_n(0, x_2, \ldots, x_{n-1}) - 1 = 0.$$

As $g_n(x_1, \ldots, x_{n-1})$ is symmetric in the variables $X_i$, it follows $X_i$ divides $g_n(x_1, \ldots, x_{n-1}) - 1$ for each $1 \leq i \leq n - 1$, which proves the first part of the lemma that there’s an equality

$$g_n(x_1, \ldots, x_{n-1}) = 1 + X_1 \cdots x_{n-1}h_n(x_1, \ldots, x_{n-1})$$

for some polynomial $h_n(x_1, \ldots, x_{n-1}) \in \mathbb{Q}[x_1, \ldots, x_{n-1}]$.

Now we show that $h_n(d_1, \ldots, d_{n-1})$ as defined above is linear of the given form. To do this, we work with the individual summands

$$(FF) \quad \frac{1}{n!} \prod_{j=1}^{n} (X_{a_1} + \cdots + X_{a_i} - j).$$

Subtracting 1 from $g_n(x_1, \ldots, x_{n-1})$ and dividing the result by $X_1 \cdots X_{n-1}$ is a polynomial in $\mathbb{Q}[x_1, \ldots, x_{n-1}]$ so, after expanding any of the summands (FF) and dividing by $X_1 \cdots X_{n-1}$, all monomials with nontrivial denominator must vanish after summing over all other terms.
with the same denominator. This leaves just the last term of the sum from \((gn)\), when \(i = n - 1\), as a contributing factor to \(h_n(X_1, ..., X_{n-1})\). Expanding this term shows
\[
\frac{1}{n!} \prod_{j=1}^{n}(X_1 + \cdots + X_{n-1} - j) = \\
\frac{1}{n!} \left( (X_1 + \cdots + X_{n-1})^n + (-1)^{n-1}n(n+1) \frac{(X_1 + \cdots + X_{n-1})^{n-1} + L(X_1, ..., X_{n-1})}{2} \right)
\]
where the summand \(L(X_1, ..., X_{n-1})\) is comprised of terms of degree smaller than \(n - 1\), and doesn’t contribute to the polynomial \(h_n(X_1, ..., X_{n-1})\). After expanding \((X_1 + \cdots + X_{n-1})^n\), the monomial summands divisible by \(X_1 \cdots X_{n-1}\) are multiples of \(X_1 \cdots X_{n-1}\). As \(h_n(X_1, ..., X_{n-1})\) is the polynomial that one gets after dividing the sum of these summands by \(X_1 \cdots X_{n-1}\), this shows \(h_n(X_1, ..., X_{n-1})\) is linear of the given form, as claimed.

**Proof of Theorem 0.2.** By Lemma 0.7, we have that
\[
g_n(d_1, ..., d_{n-1}) = 1 + d_1 \cdots d_{n-1} h_n(d_1, ..., d_{n-1})
\]
for a linear polynomial
\[
h_n(d_1, ..., d_{n-1}) = a_1d_1 + \cdots + a_{n-1}d_{n-1} + c.
\]
Note that, when \(n = 2\), the equation (no.1) becomes
\[
g_2(d_1) = \frac{(d_1 - 1)}{d_1 - 3} = \frac{(d_1 - 1)(d_1 - 2)}{2} = 1 + \frac{1}{2}(d_1 - 3)d_1.
\]
Hence, when \(n \geq 3\), one finds
\[
g_2(d_i) = g_n(1, ..., d_i, ..., 1) = 1 + d_i h_n(1, ..., d_i, ..., 1)
\]
by setting \(d_j = 1\) for all \(j \neq i\). It follows that \(a_i = 1/2\) for all \(1 \leq i \leq n - 1\). Finally, one can solve for \(c\) using the relation \(0 = g_n(1, ..., 1)\) where \(d_i = 1\) for all \(1 \leq i \leq n - 1\). \(\square\)

**References**


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