Stochastic Model Predictive Control with Enlarged Domain of Attraction for Offset-Free Tracking

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Abstract—The domain of attraction is central to stability and recursive feasibility of model predictive control (MPC). For stochastic linear systems, this paper addresses the problem of enlarging the domain of attraction of stochastic MPC (SMPC) for offset-free tracking. The proposed SMPC strategy relies on using an artificial steady-state target to enlarge the domain of attraction while ensuring recursive feasibility. The key advantage of the proposed strategy is that it alleviates the computation of an additional robust control invariant set to greatly simplify the SMPC design. Furthermore, the proposed SMPC strategy for offset-free tracking circumvents feasibility loss due to target changes since the feasible region is independent of the desired steady-state target. The SMPC strategy is demonstrated on a benchmark DC-DC converter case study, where its performance is compared to that of standard SMPC for regulation based on tighter constraints and a recently proposed SMPC strategy with a first-step constraint.

I. INTRODUCTION

Model predictive control (MPC) is the most widely used approach for the optimal control of multivariable systems subject to constraints [16], [19]. A key challenge in MPC arises from dealing with system uncertainties, which has led to the development of robust and stochastic MPC strategies to explicitly account for, respectively, deterministic and probabilistic uncertainties in the MPC formulation [1], [9], [17]. Stochastic MPC (SMPC) can avoid excessively conservative control performance, in light of robustness to uncertainties, through allowing for an acceptable level of state constraint violation using chance constraints [17].

A central concept to stability and recursive feasibility of MPC is the domain of attraction, which consists of the set of states that can be steered to a terminal invariant set at the end of the control horizon while satisfying all system constraints [10]. Domain of attraction defines the feasible workspace of a MPC controller, thus enlarging the domain of attraction will enable the MPC controller to handle changes in the operating window of the system (e.g., changes in the output targets) while avoiding loss of feasibility. A common approach to enlarging the domain of attraction involves increasing the control horizon of the MPC problem; however, this approach increases the number of decision variables in the optimization problem, thus making the MPC problem more expensive to solve. A recently proposed alternative is to define the terminal constraints in terms of an invariant set for tracking (as opposed to regulation) in order to handle feasible target changes. This approach has shown to be effective for directly enlarging the domain of attraction in MPC (without increasing the control horizon) due to the larger size of the invariant terminal set [5], [11], [12].

The problem of enlarging the domain of attraction in SMPC has recently been addressed in [8], [13], [14]. The concept of strong feasibility was introduced in [8] to reduce the conservatism associated with stochastic tube-based MPC [2]. In [13], [14], a first-step constraint strategy was adopted to enlarge the feasibility region of SMPC. This approach relies on defining an additional robust control invariant set to guarantee recursive feasibility. The computation of the robust control invariant set can, however, be expensive for systems with high state dimension [13], which is a practical drawback of this approach relative to stochastic tube-based MPC [2].

The aforementioned SMPC strategies deal with the regulation problem, that is, the terminal invariant set is defined with respect to the origin. Thus, a new terminal invariant set must be computed every time the target is changed. This can drastically increase the complexity of the SMPC design for tracking problems and result in a potentially smaller domain of attraction [11]. For linear systems with mixed deterministic and stochastic uncertainties, a robust MPC strategy for offset-free tracking has recently been presented in [18] using the notion of artificial steady-state targets. However, recursive feasibility of this robust MPC strategy was not established.

This paper presents a SMPC strategy with enlarged domain of attraction and guaranteed recursive feasibility for offset-free tracking. The approach of [11] is adopted for linear systems with bounded stochastic disturbances to enlarge the domain of attraction of SMPC through computing the terminal invariant set in terms of a set of steady-state targets. To ensure recursive feasibility, the probabilistic knowledge of disturbances is used for one-step ahead state constraint satisfaction, whereas state constraints are enforced with respect to the worst-case disturbance realizations over the remainder of the prediction horizon [9]. This not only circumvents the need to compute an additional robust control invariant set for ensuring recursive feasibility and thus significantly reducing the complexity of the SMPC design, but also avoids feasibility loss due to target changes since the feasible region is independent of the desired steady-state target. The performance of the proposed SMPC strategy for offset-free tracking is demonstrated on a benchmark DC-DC converter.
case study.

Notation. \[ A \]_j denotes the \( j \)-th row of matrix \( A \). For a matrix \( M \) or scalar \( \lambda \) and a given set \( X \), \( MX = \{ Mx : x \in X \} \) and \( \lambda X = \{ \lambda x : x \in X \} \). \( \mathbb{N} \) represents the set of non-negative integers and \( \mathbb{N}_+ \) denotes the set of positive integers. A positive definite (semi-definite) matrix \( T \) is denoted by \( T > 0 (T \geq 0) \). The matrix \( I \) denotes the identity matrix with appropriate dimension. The Minkowski sum of the set \( S \) with the set \( T \) is denoted by \( S \oplus T = \{ u + s + t : s \in S, t \in T \} \), and the Pontryagin set difference is denoted by \( S \ominus T = \{ s \in S : s + t \in S, \forall t \in T \} \). The vector \( x_k \) describes the measured state at \( k \), while \( x_{1:k} \) represents the \( k \) step ahead prediction of the state based on the available information at \( k \). \( E[X] \) denotes the expected value of a random variable \( X \). \( \mathbb{P}[s \in S] \) is the probability of the event \( s \in S \). The notation \( \mathbb{P}_k[s \in S] \), also denoted by \( \mathbb{P}[s \in S|x_k] \), represents the conditional probability of the event \( s \in S \) given the realization \( x_k \). For a sequence of random variables \( X_k, X_k \rightarrow X \) denotes L_1-convergence, i.e., \( \lim_{k \rightarrow \infty} E[\|X_k - X\|_1] = 0 \), to random variable \( X \).

II. Problem Statement and Preliminaries

A. System Description

Consider a linear time-invariant (LTI) system subject to additive stochastic disturbances

\[
\begin{align*}
    x_{k+1} &= Ax_k + Bu_k + B_ww_k, \quad (1a) \\
    y_k &= Cx_k + Du_k, \quad (1b)
\end{align*}
\]

where \( x_k \in \mathbb{R}^n \), \( u_k \in \mathbb{R}^m \), and \( y_k \in \mathbb{R}^p \) denote the system states, control inputs, and outputs, respectively. The random variable \( w_k \) is a realization of a stochastic process \( W_k \), which is assumed to be independent and identically distributed (i.i.d.) with a bounded, zero-mean probability distribution. Hence, \( w_k \) is bounded by a convex, compact set \( \mathcal{W} \) that contains the origin, i.e., \( w_k \in \mathcal{W} \).

The input and state in the system (1) are constrained. Define the polytopes

\[
\begin{align*}
    U &= \{ u \mid H_u u \leq f_u, \; u \in \mathbb{R}^m \}, \\
    X &= \{ x \mid H_x x \leq f_x, \; x \in \mathbb{R}^n \},
\end{align*}
\]

where \( H_u \in \mathbb{R}^{m \times m} \), \( f_u \in \mathbb{R}^m \), \( H_x \in \mathbb{R}^{n \times n} \), \( f_x \in \mathbb{R}^n \). It is assumed that \( X \) and \( U \) contain the origin. Input constraints are enforced as hard constraints, that is,

\[
H_u u_k \leq f_u, \quad k \in \mathbb{N}. \tag{2}
\]

Since the states in (1) evolve as stochastic processes, the state constraints are generally enforced as chance constraints

\[
\mathbb{P}[(H_x)_j x_k \leq [f_x]_j] \geq 1 - \epsilon_j, \quad j = 1, \ldots, n_v, k \in \mathbb{N}_+, \tag{3}
\]

where \( \epsilon_j \in [0, 1] \). This paper addresses the receding-horizon control problem for the stochastic system (1) subject to the input constraints (2) and state chance constraints (3). The key objective of the paper is to present a SMPC strategy for offset-free tracking, so that feasibility loss due to changes in the steady-state target of the controller can be circumvented. In addition, it is intended that the controller has an enlarged domain of attraction relative to the state-of-the-art SMPC strategies, and its recursive feasibility can be guaranteed irrespective of the computation of an additional robust control invariant set in order to reduce the complexity of the SMPC design.

B. Closed-Loop Paradigm

The notions of closed-loop paradigm and constraint tightening are briefly revisited [2], [3]. Consider an unconstrained stabilizing state-feedback law \( u_k = K x_k \), where all the eigenvalues of \( \Phi = A + BK \) are strictly inside the unitary circle. For a given \( x_k = x_{0:k} \) with \( c_{0:k} = 0 \), the evolution of the nominal predictions can be described by

\[
\begin{align*}
    x_{i|k} &= z_{i|k} + \epsilon_i k, \quad (4) \\
    y_{i|k} &= K z_{i|k} + c_{i|k}, \quad (5) \\
    z_{i+1|k} &= \Phi z_{i|k} + B c_{i|k}, \quad (6) \\
    c_{i+1|k} &= \Phi e_i k + B w_k y_{i+k}, \quad (7)
\end{align*}
\]

where \( c_{i|k} \) denotes decision variables of a constrained optimization problem, as the above system description hinges on dual mode control [15]. Note that the error \( \epsilon_i k \) is stochastic due to the random disturbances.

The closed-loop paradigm can be used to bound the disturbance evolution throughout the prediction horizon. That is, the original constraints (2) and (3) can be ensured by imposing tighter constraints on the nominal system represented by \( z_{i|k} \in \mathbb{Z}_i, \; i \in \mathbb{N}^+ \) and \( v_{i|k} \in \mathbb{V}_i, \; i \in \mathbb{N} \) [2]. This work exploits the results of [9, Chapter 8] and [14]. The main difference between these results arises from how the constraint sets \( \mathbb{Z}_i \) and \( \mathbb{V}_i \) are defined, as discussed below.

C. Constraint Tightening

This work adopts the notion of “probabilistic tubes”, which involves rewriting the state chance constraints (3) in terms of linear convex sets \( \mathbb{Z}_i \) based on nominal predictions and the distribution of the disturbance [2]. For any future prediction \( i \in \mathbb{N}_+ \), these sets are defined as (e.g., see [2], [9], [14])

\[
\mathbb{Z}_i = \{ z \in \mathbb{R}^n \mid H_z z \leq \eta_i \}, \quad i \in \mathbb{N}_{[1, N]}, \tag{8}
\]

where each element of \( \eta_i \) is given by

\[
[\eta_i]_j = \max_{\eta} \quad \text{s.t.} \quad \mathbb{P}_k[\eta \leq [f_x]_j - [H_x]_j e_i k] \geq 1 - \epsilon_j. \tag{9}
\]

As shown in [2], the constraints \( \mathbb{Z}_i \) do not directly guarantee recursive feasibility of tube-based SMPC. In order to provide this guarantee, the first set \( \mathbb{Z}_1 \) is defined in terms of the distribution of \( W_k \), while \( \mathbb{Z}_2, \ldots, \mathbb{Z}_{N-1}, \mathbb{Z}_f \) are restricted based on the worst-case realizations of the disturbances, that is, \( \eta_i \) in (8) is replaced with \( \tilde{\eta}_i \leq \eta_i \)

\[
\tilde{\eta}_i = \eta_i - \sum_{l=0}^{i-1} H_x \Phi^l B w_l. \tag{10}
\]

Since the inputs are enforced as hard constraints, the sets \( \mathbb{V}_0, \ldots, \mathbb{V}_{N-1} \) in [2] are directly computed as “worst-case” sets, implying that they hold for all possible disturbances.
An alternative approach to achieving recursive feasibility has recently been proposed in [14], in which the constraints (8) are directly used over the entire prediction horizon. A parameter $\varepsilon_u \in [0,1)$ is introduced to represent the probability with which the future predicted inputs are allowed to violate the hard input constraints. Similar to (8), the sets $V_i$ are defined as

$$V_i = \{ v \in \mathbb{R}^m \mid H_u v \leq \mu_i \}, \quad i \in \mathbb{N}_{[1,N]}, \quad (11)$$

where each element of $\mu_i$ is given by

$$[\mu_i]_j = \max_{\mu} \mu_j \quad \text{s.t.} \quad P_k[\mu \leq [f_a]_j - [H_u]_j e_{i|k}] \geq 1 - \varepsilon_u. \quad (12)$$

As opposed to shrinking the sets $\{Z_{i+1}, \ldots, Z_i\} \in \mathbb{N}_{[0,N-1]}$, based on the worst-case disturbance realizations, in [14] recursive feasibility is ensured by including an additional first-step constraint, as detailed below.

### D. Finite-horizon Optimal Control

Given the measured state $x_k$, consider the finite-horizon optimal control problem (OCP)

$$\min_{c_k} J_N(c_k; x_k) \quad (13a)$$

subject to $C_T$ must be computed recursively from $C_{T,x}^0 = C_{T,x}$ as

$$C_{T,x}^{i-1} = \{ x \in C_{T,x}^i \mid \exists u \in \mathbb{R}^m \text{ s.t. } [x^T u^T]^T \in C_T \}, \quad (15)$$

where $C_{T,x}^{i} = \cap_{i=0}^{i-1} C_{T,x}^i$ by definition. This set is computed recursively until $C_{T,x}^i = C_{T,x}^{i+1}$ for some $j \in \mathbb{N}$, which directly implies $C_{T,x}^j = C_{T,x}^\infty$. The set is said to be finitely determined whenever a finite value for $j$ exists, which is guaranteed under some weak assumptions. Hence, the additional constraint $z_{i|k} \in C_{T,x}^\infty \ominus B_w \mathbb{W}$ must be included in the finite-horizon OCP (13) in order to ensure recursive feasibility [14].

The OCP (13) with the additional first step constraint closes the gap between “recursively feasible probabilistic tubes” [2], which can be recovered with $\varepsilon_f = \varepsilon_u = 0$, and “strongly feasible SMPC” [8], which is achieved with $\varepsilon_f = \varepsilon_u = 1$. In fact, these parameters $\varepsilon_f$ and $\varepsilon_u$ can be used as a way to trade-off between conservatism in constraint handling and performance. However, the approach of [14] has two main shortcomings: (i) the computation of the robust control invariant set $C_{T,x}^\infty$ can be challenging for high dimensional systems [13], and (ii) the best choice of $\varepsilon_f$ and $\varepsilon_u$ is an open issue since performance degradation may be significant if future constraints are not accounted for in the OCP.

In this paper, we propose an alternative SMPC method for enlarging the domain of attraction of the recursively feasible probabilistic tubes in [2]. The main advantages of the proposed approach are: (i) no additional first-step constraint is required, (ii) no control invariant set is computed, (iii) the domain of attraction does not depend on additional tuning parameters, and (iv) offset-free performance is achieved with respect to the mean value of the states.

### III. SMPC FOR OFFSET-FREE TRACKING

This section presents the SMPC strategy with enlarged domain of attraction for offset-free tracking. First, we define the constraint sets in the OCP (13) based on (2) and (3) combined with the concept of probabilistic tubes [2]. Define the set $Z_{1}$ as

$$Z_1 = \{ z \mid H_x z \leq f_x - \Gamma, \ z \in \mathbb{R}^n \} \quad (16)$$

with $\Gamma = [\gamma_1^* \gamma_2^* \ldots \gamma_n^*]^T$, where every $\gamma_j^*$ is obtained offline by solving the auxiliary problem [9]

$$\gamma_j^* = \min_{\gamma_j} \gamma_j \quad \text{s.t.} \quad \mathbb{P}[ H_z ]_{\nu} \leq \gamma_j \geq 1 - \gamma_j. \quad (17)$$

The initial constraint for the control inputs is defined by $\mathbb{V}_0 = \mathbb{U}$. The constraint sets over the prediction horizon can now be recursively obtained from $Z_{1}$ and $\mathbb{V}_0$

$$Z_{j+1} = Z_{j} \ominus \Phi_j B_w \mathbb{W}, \quad j \in \mathbb{N}^*, \quad (18)$$

$$\mathbb{V}_{j+1} = \mathbb{V}_j \ominus \Phi_j B_w \mathbb{W}, \quad j \in \mathbb{N}. \quad (19)$$

In the dual mode control paradigm, the constraint sets (18)-(19) pertain to Mode 1, whereas the terminal set in
the OCP should account for the behavior of the system in Mode 2. Consider the stabilizing control law \( u_k = K(x_k - \pi_t) + L\theta_k \), where \((A - I)\pi_t + B\pi_t = 0\) and \(\pi_t^T \pi_t^T\) is the nominal steady-state target. This stabilizing control law can be parametrized as

\[
u_k = K x_k + L \theta_k,
\]

where \( L = [-K I] M_\theta, [\pi_t^T \pi_t^T]^T = M_\theta \theta_k, M_\theta = [M_x^T M_z^T]^T \) is composed of the base of the nullspace of \([(A - I) B]\), and \( \theta_k \) is a free parameter that defines the artificial steady-state target [11]. Note that \( \gamma_t = M_y \theta_k \), where \( M_y = [C D] M_\theta \).

The system evolution in Mode 2 can now be defined in terms of an augmented state

\[
\begin{bmatrix}
z_{N|k+1} \\
\theta_{k+1}
\end{bmatrix} = \begin{bmatrix}
\Phi & BL \\
0 & I
\end{bmatrix} \begin{bmatrix}
z_{N|k} \\
\theta_k
\end{bmatrix} + \begin{bmatrix}
\Phi^N B_c \\
0
\end{bmatrix} w_k.
\]

The main idea is to guarantee that \( z_{N|k+1} \) is a feasible solution despite the disturbance realization. Let the augmented matrices be represented by \( \Phi_0 = \begin{bmatrix} \Phi & 0 \\
BL & I \end{bmatrix} \) and \( B_a = \begin{bmatrix} \Phi^N B_c \\
0 \end{bmatrix} \). The dual mode approach is obtained from an admissible robust invariant set, \( Z^a_f \), that must satisfy the following two conditions: (i) \( \Phi_0 Z^a_f + B_a W \subseteq Z^a_f \) and (ii) \( Z^a_f \subseteq Z^a_N \), where \( Z^a_N \) is given by

\[
Z^a_{N+i} = \left\{ \begin{array}{c}
\frac{z}{\theta} \in \mathbb{R}^{n+m} \\
|Kx + L \theta| \leq V_{N+i} \\
M_x \theta \in \lambda Z_N \\
M_y \theta \in \lambda W_N
\end{array} \right\}
\]

with \( i = 0 \). The parameter \( \lambda \) is included to ensure that a polyhedral terminal set is finitely determined even in the presence of unitary eigenvalues due to a constant setpoint [11]. The polyhedral robust admissible invariant set can be computed with standard set operations [7].

**Problem 1. (SMPC for offset-free tracking)**

Given the measured state \( x_k \) and a desired output target \( y_t \), the proposed SMPC strategy involves solving the optimal control problem

\[
\min_{c_k, \theta_k} J_N(c_k, \theta_k; x_k, y_t)
\]

s.t. \( z_0(k) = x_k \),

\[
[\pi_t^T \pi_t^T]^T = M_\theta \theta_k,
\]

\[
\bar{y}_t = M_y \theta_k,
\]

\[
z_{i+1|k} = \Phi z_{i|k} + BL \theta_k + B c_{i|k}, \quad i \in \mathbb{N}_{[0,N-1]},
\]

\[
u_{i|k} = K z_{i|k} + c_{i|k} + L \theta_k, \quad i \in \mathbb{N}_{[0,N-1]},
\]

\[
z_{i|k} \in Z_t, \quad i \in \mathbb{N}_{[1,N-1]},
\]

\[
u_{i|k} \in V_t, \quad i \in \mathbb{N}_{[0,N-1]},
\]

\[
[z_{N|k}^T \theta_{k}^T]^T \in Z^a_f,
\]

where

\[
J_N(c_k; x_k) = \sum_{i=0}^{N-1} \left( ||z_{i|k} - \pi_t||^2 + ||v_{i|k} - \pi_t||^2 \right)
\]

\[
+ ||z_{N|k} - \pi_t||^2 + V_o(y_t - \bar{y}_t)
\]

with \( V_o(\cdot) \) representing the offset cost that penalizes deviations between the desired target \( y_t \) and the artificial target \( \bar{y}_t \). Note that the parameter \( \theta_k \) is a decision variable in the OCP (23). In the receding-horizon implementation of the OCP (23), the control inputs at every sampling instant \( k \) are given by a nonlinear state feedback law \( u_k = g_N(x_k, y_t) \), i.e.,

\[
u_k = K x_k + c_{0|k} + L \theta_k(x_k, y_t),
\]

where \( * \) denotes the optimal value of the decision variables. The local optimality of (23) can be established approximately by choosing \( V_o(\cdot) \) as a 2-norm penalty, or recovered exactly by selecting \( V_o(\cdot) \) as a suitably weighted \( \infty \)-norm [4]. Let \( X_N \) denote the domain of attraction for Problem 1.

**Assumption 1.** To establish the recursive feasibility and convergence, the following standard assumptions are made:

1. The pair \((A, B)\) is controllable;
2. The pair \((A, Q^{1/2})\) is observable;
3. \( Q \geq 0 \) and \( R > 0 \);
4. \( \bar{K} \) is defined such that \( \tilde{\Phi} = A + \bar{K} B \) has all its eigenvalues strictly inside the unitary circle;
5. \( P > 0 \) is given by \( \Phi^T P \Phi - P = -(Q + \bar{K}^T R \bar{K}) \);
6. \( V_o : \mathbb{R}^p \to \mathbb{R} \) is a convex, positive-definite and subdifferentiable function with \( V_o(0) = 0 \);
7. \( Z^a_f \) is a non-empty set.\(^2\)

The proofs of recursive feasibility and convergence of the proposed SMPC strategy for offset-free tracking are given in Appendix A and Appendix B, respectively. The effect of the additional decision variable \( \theta_k \) in (23) on the domain of attraction of the SMPC strategy readily follows from the terminal constraint (23i). For the OCP (13) to be feasible, the nominal state at the end of the horizon must satisfy \( z_{N|k} \in \text{Proj}_x(Z^a_f) \). In SMPC for regulation (e.g., [2], [14]), however, the terminal set is defined as a slice of \( Z^a_f \) for a given \( \theta_k \), with a value typically being \( \theta_k = 0 \) for regulation to the origin. Hence, the domain of attraction of the proposed SMPC is enlarged due to the slack provided by including \( \theta_k \) as a decision variable. The following proposition establishes the offset-free tracking property of the SMPC strategy in Problem 1 when the desired steady-state target is admissible.

**Proposition 1.** Consider that Assumption 1 holds and \( y_t \) asymptotically converges to a given value. Then, the closed-loop system \( x_{k+1} = A x_k + B g_N(x_k, y_t) + B_w u_k \) satisfies the following conditions:

(i) For all \( x_0 \in X_N \) and every target \( y_t \), the evolution of the system is robustly feasible, i.e., \( x_k \in X_N \) for all \( k \in \mathbb{N}^* \), and satisfies constraints (2)–(3).

\(^1\)The terminal set can be defined as \( Z^a_f = \{ x_a |\Phi_a x_a \in Z^a_{N+i}, \forall i \in \mathbb{N} \} \).

\(^2\)The verification can be performed either \textit{a priori} (e.g., see [9, Lemma 8.2]) or \textit{a posteriori}.
(ii) The system is mean-square bounded and satisfies the following quadratic stability condition
\[
\lim_{t \to \infty} \frac{1}{t} \sum_{k=0}^{t} \mathbb{E}[\|x_k - \bar{x}_k\|_{Q_k}^2] \leq \mathbb{E}[\|Bw_w\|_{L}^2],
\]
where \(\bar{x}_k = \mathbb{E}[x_k]\) denotes the mean of the state, which converges to a fixed point of the nominal system.

(iii) The controlled variable converges to a neighborhood around its mean value, i.e., \(y_k \in \mathbb{E}[y_k] \oplus (C + D K) R_{\infty}\), where \(R_{\infty}\) is the minimal robust positive invariant (mRPI) set for \(x_{k+1} = \Phi x_k + w_k, \forall w_k \in \mathbb{W}\).

(iv) The mean of the controlled variables converges to the target \(\lim_{k \to \infty} \mathbb{E}[y_k] = y^*\) whenever \(y_k\) is reachable. If \(y_k\) is not reachable, then the mean converges to a value \(\lim_{k \to \infty} \mathbb{E}[y_k] = \tilde{y}_t\) that minimizes the offset cost, i.e.,
\[
\tilde{y}_t = \min_{\bar{y}_t \in Y} V_o(\bar{y}_t - y_t),
\]
where \(Y = \{y = M_y \theta \in \mathbb{R}^p \mid (M_y \theta, \theta) \in \mathbb{Z}^p\}\) is the set of reachable targets.

**Proof.** The proof of (i) follows directly from the recursive feasibility property established in Appendix A, while property (ii) can be demonstrated by following the same steps of [14] since the same decreasing cost function behavior is observed under the same candidate solution. Property (iii) follows from the convergence \(\lim_{k \to \infty} \varepsilon_{k}^* = 0\), as shown in Appendix B. The offset-free property in (iv) is proven in the sequel based on the Dominated Convergence Theorem.

As shown in Appendix B, \(\lim_{k \to \infty} \theta^*_k = \overline{\theta}\). Furthermore, \(\pi^*_k = (A + BK)\pi^*_k + BL\overline{\theta}\) or, alternatively, \(\pi^*_k = [I - (A + BK)]^{-1} BL\overline{\theta}\). Observe that the following limit holds
\[
\lim_{k \to \infty} \mathbb{E}[x_{k+1}] = \lim_{k \to \infty} \{\mathbb{E}[(A + BK)x_k] + \mathbb{E}[BL\overline{\theta}^*_k]\}.
\]
Due to the finite convergence of \(\mathbb{E}[x_k]\) and \(\mathbb{E}[\theta^*_k]\), (25) can be expressed as
\[
\lim_{k \to \infty} \mathbb{E}[x_{k+1}] = (A + BK) \lim_{k \to \infty} \mathbb{E}[x_k] + BL \lim_{k \to \infty} \mathbb{E}[\theta^*_k],
\]
with \(\lim_{k \to \infty} \mathbb{E}[x_{k+1}] = \lim_{k \to \infty} \mathbb{E}[x_k]\). Since \(\theta^*_k\) has finite support due to the fact that \(Z_N + \mathbb{V}_N\) are compact sets, an integrable dominating random variable can be defined to show that \(\lim_{k \to \infty} \mathbb{E}[B\theta^*_k] = B\overline{\theta}\) by virtue of the Dominated Convergence Theorem [20]. Thus, \(\lim_{k \to \infty} \mathbb{E}[x_k] = [I - (A + BK)]^{-1} BL\overline{\theta} = \pi^*_k\).

Under 1000 disturbance realizations, approximately 19% of the state profiles violated constraints at the first time step in all three of the SMPC methods, which is nearly equal to the permitted probability of state constraint violation of 20%. This result is expected since the initial condition is almost in the limit of the domain of attraction.

As previously discussed, the domain of attraction of [14] can be enlarged by increasing \(\varepsilon_f\) and \(\varepsilon_u\), but the trade-off between the control performance and the size of the domain of attraction should be taken into account. In the proposed SMPC strategy in Problem 1, no additional tuning is required because the domain of attraction enlargement

\[\text{IV. Numerical Example}\]

The enlarged domain of attraction and the offset-free tracking properties of the proposed SMPC strategy are demonstrated on a benchmark DC-DC converter case study [13]. The system model takes the form of (1) with
\[
A = \begin{bmatrix} 1 & 0.008 \\ -0.143 & 0.996 \end{bmatrix}, B = \begin{bmatrix} 4.798 \\ 0.0115 \end{bmatrix}, C = [0 \ 1], B_w = I.
\]

Stochastic disturbances are assumed to be bounded by \(\|w_k\|_{\infty} \leq 0.1\), where \(w_k\) follows a truncated normal distribution \(N(0, 0.04^2 I)\). Hard input constraints are given by \(\|u\|_{\infty} \leq 0.2\), while the state constraints are defined to be
\[
\mathbb{P}[|x_{k+1}| \leq 2] \geq 0.8, \quad \mathbb{P}[-|x_{k+1}| \leq 2] \leq 0.8, \quad \mathbb{P}[|x_{k+1}| \leq 3] \geq 0.8, \quad \mathbb{P}[-|x_{k+1}| \leq 3] \leq 0.8.
\]

The SMPC parameters are chosen to be \(Q = \text{diag}(1, 10)\), \(R = 1\), \(N = 4\), and \(V_o(y_t - \overline{y}_t) = \|y_t - \overline{y}_t\|_{\infty}^8\). A constant value of \(\lambda = 0.95\) is used for computing the terminal set.

The domain of attraction of the proposed SMPC strategy is compared to that of the SMPC strategies of [9] and [14] for regulation, where a 5% constraint violation was permitted when computing \(\mathbb{V}_f\) and \(\mathbb{V}_y\) in [14] (i.e., \(\varepsilon_u = \varepsilon_f = 0.05\)).

To effectively demonstrate the benefits of the artificial target approach adopted in the proposed strategy, the first-step constraint was not included in the SMPC controller of [14], as \(C_{T,x}\) can reduce the feasibility region. In this case, the recursive feasibility of the controller was verified a posteriori. Fig. 1 shows the domain of attraction and the closed-loop state evolution (under 200 disturbance realizations) of the three SMPC controllers. As can be seen, the domain of attraction of the proposed strategy is the largest (gray), whereas, as expected, the domain of attraction of [9] (blue) is smaller than that of [14] (red).

The main advantages of the proposed SMPC strategy in comparison with [14] include: (i) the ease of implementation since the additional first-step constraint is no longer required, and (ii) the fact that its domain of attraction (i.e., feasibility region) is independent of the desired steady-state target, which makes the proposed strategy suitable for tracking. Under 1000 disturbance realizations, approximately 19% of the state profiles violated constraints at the first time step in all three of the SMPC methods, which is nearly equal to the permitted probability of state constraint violation of 20%.

Let \(s_k\) be a sample obtained at \(k\) from the normal distribution, \(\{w_k\} = [s_k], \text{if } ||s_k||_{L} \leq 0.1\text{, and } [w_k] = 0.1[s_k]/||s_k||_{L}\text{ if } ||s_k||_{L} > 0.1\).

1. A prediction horizon of \(N = 5\) was used in the SMPC controllers of [9] and [14], so that they have the same number of decision variables as the proposed SMPC controller.
results from the slack of the steady-state parametrization of the terminal control. Even though this type of steady-state parametrization could be applied to the terminal set used in [14], the additional first-step constraint should still be included in the OCP, which can be computationally complex.

Fig. 2 illustrates the offset-free tracking property of the proposed SMPC strategy. In this case, the target is initially at \( y_t = 2 \) before it is changed to \( y_t = -2 \) at \( k = 25 \). Fig. 2 shows the domain of attraction of the controller and the evolution of the closed-loop states under 200 disturbance realizations. It is evident that the domain of attraction of the proposed SMPC controller is independent of the steady-state target, as it remains intact when the target is changed. Fig. 2 indicates that the controller can effectively accommodate the target change and enable offset-free tracking. In summary, the proposed SMPC strategy leads to comparable closed-loop performance and constraint violation levels to [9], [14] (Fig. 1), while providing an enlarged domain of attraction and setpoint tracking ability (Fig. 2).

V. CONCLUSIONS

An SMPC strategy for offset-free tracking of stochastic linear systems is presented. The notion of artificial steady-state targets is adopted to enlarge the domain of attraction and ensure recursive feasibility, while achieving offset-free tracking in terms of the expected value of the outputs. The design complexity of the proposed SMPC strategy is comparable to that of tube-based MPC, as it does not require propagation of the stochastic uncertainties over the prediction horizon and computation of robust invariant sets. Future work will address the conservatism of SMPC for offset-free tracking for systems with structural model uncertainty.

REFERENCES


APPENDIX

A. Recursive Feasibility

Recursive feasibility is demonstrated by following standard steps. Initially, consider that a feasible solution obtained at $k$ is given by $c^i(k) = [c_{0|k}^i, c_{1|k}^i, \ldots, c_{N-1|k}^i]^T$ and $\theta_k^i$. The candidate solution for the next step is defined by $\tilde{c}(k+1) = [c_{1|k}^{T} \ldots c_{N-1|k}^{T} 0^T]^T$ and $\tilde{\theta}_{k+1} = \theta_k^i$.

Now observe that $z_{0|k} = x_k, z_{0|k+1} = x_{k+1}, u_k = v_{0|k} = Kx_k + c_{0|k}^i + L\theta_k^i$, and $z_{1|k} = \Phi z_{0|k} + B(c_{0|k}^i + L\theta_k^i)$. Thus, $z_{0|k+1}$ and $z_{1|k}$ are related by

$$z_{0|k+1} = Ax_k + Bu_k + Bw_k$$

(26)

and

$$z_{1|k} = Bw_k.$$  

(27)

Note that (28) can be rewritten by multiplying both sides by $\Phi$ and then by summing $B(c_{0|k}^i + L\theta_k^i)$

$$\Phi z_{0|k+1} + B(c_{0|k}^i + L\theta_k^i) = \Phi[z_{1|k} + Bw_k] + B(c_{0|k}^i + L\theta_k^i),$$

(29)

which is equivalent to

$$z_{1|k+1} = z_{2|k} + \Phi Bw_k.$$ 

(30)

Then, the general relationship is obtained from induction

$$z_{j-1|k+1} = z_{j|k} + \Phi^j Bw_k$$ 

(31)

with $j \in [1, N]$. Also note that the control candidate is given by $\tilde{v}_{j-1|k+1} = K z_{j-1|k+1} + c_{j|k}^i + L\theta_k^i, j \in [1, N-1]$ with $c^i_{j|k} = K z_{j|k} + c_{j|k}^i + L\theta_k^i, j \in [1, N-1]$ or alternatively

$$\tilde{v}_{j-1|k+1} = v_{j|k}^i + K \Phi^{j-1} Bw_k$$ 

(32)

with $j \in [1, N-1]$. The recursive feasibility for Mode 1 can be directly verified from (31)-(32) combined with the definition of the tight constraints (18)-(19)

$$z_{i|k} \in Z_i \Rightarrow z_{i-1|k+1} \in Z_{i-1}, i \in [2, N],$$ 

(33)

$$v_{j|k}^i \in V_j \Rightarrow \tilde{v}_{j-1|k+1} \in V_{j-1}, j \in [1, N-1].$$ 

(34)

Then, feasibility of the proposed strategy should be verified with respect to $\tilde{v}_{N-1|k+1}$ and $z_{N|k+1}$ (Mode 2). Note that $\tilde{v}_{N-1|k+1} = K z_{N-1|k+1} + L\theta_k^i$, which is equivalent to

$$\tilde{v}_{N-1|k+1} = K(z_{N|k} + \Phi^{N-1} Bw_k) + L\theta_k^i.$$ 

(35)

Now observe that $[z_{N|k}^T \theta_k^i]^T \in Z_f$ and $\nu_{N|k} = K z_{N|k} + L\theta_k^i \in V_N$. In this case, $\tilde{v}_{N-1|k+1} = \nu_{N|k}^i + K\Phi^{N-1} Bw_k$, which guarantees that $\nu_{N|k} \in V_N \Rightarrow \nu_{N-1|k+1} \in V_{N-1}$ because $V_N = V_{N-1} \ominus K \Phi^{N-1} Bw$ by definition.

Finally, it is observed from $z_{N-1|k+1} = z_{N|k} + \Phi^{k-1} w_k$ and $\tilde{v}_{N-1|k+1} = K(z_{N|k} + \Phi^{N-1} Bw_k) + L\theta_k^i$ that the terminal prediction candidate is given by

$$z_{N|k+1} = A z_{N-1|k+1} + B \tilde{v}_{N-1|k+1}$$

(36)

$$= A(z_{N|k} + \Phi^{k-1} Bw_k) + B(K(z_{N|k} + \Phi^{N-1} Bw_k) + L\theta_k^i)$$

(37)

Thus, $[z_{N|k+1}^T \theta_k^i]^T \in Z_f$ and $z_{N|k+1} \in Z_N$ from the definition of the output admissible robust invariant set for the augmented description (21). This completes the proof.

B. Convergence

The cost function and the feasible candidates considered here are the same as in [5]. Details of convergence and input-to-state stability can be found in [5], [6]. When $K$ is obtained from the Linear Quadratic Regulator (LQR) solution, a compact proof of convergence is given for completeness.

As shown in [5], the cost function is equivalent to

$$J_N(\bar{c}(k), \theta_k; x(k)) = \sum_{j=0}^{N-1} ||c^j||^2 \Psi + V_0(\gamma_t - y_t).$$ 

(39)

The difference between the cost of the candidate solution at $k+1$ and the optimal cost at $k$ is given by

$$J_N(\bar{c}(k+1), \theta_k^i; x(k+1)) - J_N(c^*(k), \theta_k^i; x(k)) = -||c^*_0||^2 \Psi,$$

(40)

due to the optimality principle. Thus, as $\Psi > 0$, then $\lim_{k \to \infty} c^*_0 = 0$ and $\lim_{k \to \infty} u^*(k) = \frac{Kx_k + L\theta_k^i}{\alpha}$.

As shown in [5], $\theta_k^i \to \tilde{\theta}$, where $\tilde{\theta}$ minimizes $V_0(\gamma_t - y_t)$. The general proof is presented in [6], but a simplified argument is given for completeness. Since $\lim_{k \to \infty} c^*_0 = 0$, then $\lim_{k \to \infty} c^*(k) = [0^T \ 0^T \ \ldots \ 0^T]^T$ is a feasible steady-state candidate. Also note that $J_N(\lim_{k \to \infty} c^*(k), \theta_k^i; x(k)) = V_0(\gamma_t - y_t)$. This ensures the convergence of $\theta_k^i$ since $V_0(\gamma_t - y_t)$ is non-increasing.

Also observe that $\lim_{k \to \infty} u^*(k) = \bar{u}$ where $\bar{u}$ minimizes $V_0(M_y \theta_k - y_t)$ subject to $[(M_y \theta_k)^T \ \bar{u}^T]^T \in Z_f$, otherwise any other solution would contradict either the convergence of the terminal cost or the optimality principle. Finally, note that $\lim_{k \to \infty} \theta_k^i = \tilde{\theta} = \lim u(k) = Kx(k) + \tilde{\theta}$ so that

$$\lim_{k \to \infty} x_k \in \pi_t^\perp \ominus \mathcal{R}_k,$$

(42)

$$\lim_{k \to \infty} u_k \in \pi_t^\perp \ominus \mathcal{K}\mathcal{R}_k,$$

(43)

where $\mathcal{R}_k$ is the minimal robust positively invariant set given by

$$\mathcal{R}_k = \bigoplus_{i=0}^{\infty} \Phi^i Bw$$

(44)

This completes the proof.