Stochastic Nonlinear Model Predictive Control with Joint Chance Constraints

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Abstract: When the stochastic description of system uncertainties is available, a natural approach to predictive control of uncertain systems involves explicitly accounting for the probabilistic occurrence of uncertainties in the optimal control problem. This work presents a stochastic nonlinear model predictive control (SNMPC) approach for nonlinear systems subject to time-invariant uncertainties as well as additive disturbances. The generalized polynomial chaos (gPC) framework is used to derive a deterministic surrogate for the stochastic optimal control problem. The key contribution of this paper lies in extending the gPC-based SMPC approach reported in our earlier work to handle stochastic disturbances. This is done via mapping the stochastic disturbances onto the space of the coefficients of polynomial chaos expansions, which enables efficient propagation of stochastic disturbances. A sample-based approach to joint chance constraint handling is employed to fulfill the state constraints in a probabilistic sense. A gPC-based Bayesian parameter estimator is utilized to update the probability distribution of uncertain system parameters at each sampling time. In a simulation case study, the closed-loop performance of the SNMPC approach is demonstrated on an atmospheric-pressure plasma jet that is developed for biomedical applications.

Keywords: Stochastic optimal control, Chance constraints, Generalized polynomial chaos, Nonthermal atmospheric-pressure plasmas

1. INTRODUCTION

Predictive control strategies are widely used for advanced control of complex systems owing to their ability to deal with multivariable system dynamics, constraints, and competing control objectives (Morari and Lee, 1999). Even though model predictive control (MPC) exhibits a certain degree of robustness to system uncertainties due to its receding-horizon implementation, the deterministic framework of MPC cannot systematically handle uncertainties (Mayne, 2014). Substantial work has been done in the area of robust MPC with the aim of accounting for system uncertainties in the optimal control problem. Robust MPC generally relies on set-membership uncertainty descriptions (i.e., bounded, deterministic descriptions), and requires constraints be satisfied with respect to all uncertainty realizations (Bemporad and Morari, 1999).

In many practical control applications, however, system uncertainties are of stochastic nature. When probabilistic descriptions of system uncertainties can be characterized, a natural approach to MPC involves explicitly considering the stochastic occurrence of system uncertainties in the optimal control problem. Stochastic MPC (SMPC) with chance constraints provides a systematic framework for optimal control of stochastic systems. Chance constraints, which constitute a key component of SMPC, allow for ensuring an admissible level (in a probabilistic sense) of robustness to uncertainties in constraint handling. Effective constraint handling in a stochastic setting is critical to MPC of uncertain systems, in particular when high-performance operation is realized in the vicinity of constraints.

Recent years have witnessed significant advances in SMPC for linear systems (see (Mesbah, 2016) for an overview of various SMPC formulations). However, stochastic optimal control of nonlinear systems has received relatively little attention mainly due to the challenges associated with efficient uncertainty propagation and establishing the closed-loop properties of SMPC in a nonlinear setting. van Hessel and Bosgra (2006) presented a stochastic nonlinear MPC (SNMPC) approach based on optimizing a deterministic feedforward trajectory and a linear time-varying feedback controller for, respectively, constraint handling and minimizing the closed-loop variance around a reference trajectory. Markov chain and sequential Monte Carlo techniques were employed in (Lecchini-Visintini et al., 2006; Kantas et al., 2009) to develop sample-based approaches to solving stochastic nonlinear optimal control problems. The computational complexity of these approaches, however, prevented their receding-horizon implementation.

The generalized polynomial chaos (gPC) framework (Xiu and Karniadakis, 2002) was used in (Fagiano and Khammash, 2012; Mesbah et al., 2014) to develop SNMPC approaches for nonlinear systems subject to time-invariant probabilistic uncertainties in parameters and initial conditions. In the gPC framework, each stochastic system state is approximated by an expansion of orthogonal polynomial basis functions, defined based on the known descriptions of probabilistic uncertainties. The polynomial
chaos expansions of states provide an efficient machinery for uncertainty propagation. The statistical moments of stochastic states can be efficiently computed from the expansion coefficients or, alternatively, the expansions can be used as a surrogate for the nonlinear system model to perform Monte Carlo simulations efficiently. In (Mesbah et al., 2014), the moments of stochastic states are used to replace individual chance constraints with deterministic approximations. These gPC-based SNMPC approaches, however, cannot account for the effect of stochastic disturbances. For nonlinear control-affine systems with additive stochastic disturbances, Buehler et al. (2016) have recently presented a gPC-based SNMPC approach that uses the Fokker-Planck equation for describing the evolution of probability distributions of states.

This paper addresses the MPC problem for stochastic nonlinear systems subject to uncertain initial conditions, parametric uncertainties, and additive white noise processes. The uncertain initial conditions and system parameters can be described by arbitrary probability distributions with a finite variance. The probability distribution of states evolves efficiently under the assumption of full-state feedback, this paper proposes a SNMPC approach with joint chance constraints. The Galerkin projection method (Ghanem and Spanos, 1991) is used to map the space of stochastic system states conditioned on a realization of the white noise disturbance processes to the space of coefficients of the polynomial chaos expansions of states. The effect of white noise processes on system dynamics is then efficiently accounted for in the space of expansions’ coefficients assuming that the coefficients possess a Gaussian distribution. To update the uncertainty description of parameters based on measurements, a gPC-based histogram filter (Bavdekar and Mesbah, 2016) is utilized to estimate the probability distributions of uncertain parameters at each measurement sampling time. The updated probability distributions of parameters are used to adapt the polynomial basis functions in the controller. A sample-based approach is adopted to approximate the joint chance constraints (Alamo et al., 2010). The performance of the proposed SNMPC approach is demonstrated for regulating the thermal effects of an atmospheric-pressure plasma jet and to a polynomial form if \( f \) is analytic and separable with respect to \( x \) and \( \theta \) (Papachristodoulou and Prajna, 2005).

2. PROBLEM FORMULATION

Consider a discrete-time nonlinear system

\[
x(t) = f(x(t-1), \theta, u(t-1)) + w(t-1),
\]

where \( t \) denotes the time index; \( x(t) \in \mathbb{R}^n \) denotes the system states with uncertain initial conditions \( x(0) \sim P_x \); \( u(t) \in \mathbb{R}^m \) denotes the system inputs; \( \theta \) denotes the uncertain system parameters characterized by the pdf \( P_\theta \); \( w(\omega(t)) \sim \mathcal{N}(\omega; 0, Q) \) denotes zero-mean white noise processes with the known covariance matrix \( Q \in \mathbb{R}^{n \times n} \); and \( f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^n \) denotes the vector function of nonlinear system dynamics. The probabilistic uncertainties \( [x_0 ; \theta] \) \( \in \mathbb{R}^{n+p} \) are defined in terms of the standard random variables \( \xi \in \mathbb{R}^q \), which belong to the Hilbert space \( \mathcal{L}^2(\Omega, \mathcal{F}, P) \). The standard random variables \( \{\xi_j\}^q_{j=1} \) are independently distributed with arbitrary, but known pdfs \( P_{\xi_j} \) that have a finite variance. The measurements of states \( x(t) \) are corrupted by zero-mean Gaussian noise with covariance \( R \in \mathbb{R}^{n \times n} \).

**Assumption 1.** In (1), \( f \) consists of polynomial functions in \( x \) and \( \theta \).

The inputs in (1) are subject to hard constraints

\[
u(t) \in U := \{u(t) \in \mathbb{R}^m : H_u u(t) \leq d_u \},
\]

where \( H_u \in \mathbb{R}^{m \times m} ; d_u \in \mathbb{R}^m \); and \( s \in \mathbb{N} \) denotes the number of input constraints. The stochastic system states must satisfy a joint chance constraint (JCC) of the form

\[
\text{Pr} \left[ g_i(x(t)) \leq 0, \quad \forall i = 1, \ldots, l \right] \geq \beta,
\]

where \( g_i : \mathbb{R}^n \rightarrow \mathbb{R} \) denotes a possibly nonlinear vector function of state constraints; \( l \in \mathbb{N} \) denotes the number of state constraints; and \( \beta \in (0, 1) \) is the lower bound for the probability level that the JCC must be satisfied. The JCC ensures that the state constraints are fulfilled with an admissible probability level in the presence of the stochastic uncertainties in (1).

Under the assumption of full-state feedback, this paper presents a SNMPC approach for the stochastic nonlinear system (1) subject to the hard input constraints (2) and the JCC (3). SNMPC involves solving a stochastic optimal control problem (OCP) in a receding-horizon manner. The main challenge in solving the stochastic OCP lies in efficiently predicting the time-evolution of the probability distributions (or statistics) of the stochastic states. This work adopts the gPC framework (Xiu and Karniadakis, 2002) for uncertainty propagation. The gPC-based SNMPC approach in (Mesbah et al., 2014) is extended to enable propagating the system disturbances \( w(t) \) (along with \( [x_0 ; \theta] \) \( \in \mathbb{R}^{n+p} \) ) as well as handling of the JCC (3). A gPC-based Bayesian estimator (Bavdekar and Mesbah, 2016) is solved in tandem with the stochastic OCP to estimate the probability distributions of the uncertain parameters \( \theta \) at each sampling time. The estimated pdfs of parameters and the measured pdfs of states are used to facilitate receding-horizon implementation of the SNMPC approach. The uncertainty propagation method used to arrive at a computationally tractable formulation for the stochastic OCP is introduced in the next section.

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1 Nonpolynomial functions can be converted to a polynomial form if \( f \) is analytic and separable with respect to \( x \) and \( \theta \) (Papachristodoulou and Prajna, 2005).
3. UNCERTAINTY PROPAGATION

Recently, the gPC framework has been widely used for efficient propagation of time-invariant probabilistic uncertainties in the context of nonlinear estimation and control. A major shortcoming of the gPC framework, however, lies in efficiently dealing with uncertainties of time-varying nature (e.g., stochastic disturbances). The shortcoming arises from the rapidly increasing approximation error of polynomial chaos (PC) expansions over time when time-varying uncertainties are described by fixed polynomial basis functions over time (Wan and Karniadakis, 2005). Inspired by (Konda et al., 2011), this paper presents a two-step procedure based on the gPC framework to propagate parametric uncertainties as well as additive stochastic disturbances through the nonlinear system dynamics in (1). In a nutshell, for a given realization of disturbances, PC expansions are used for propagation of the uncertainties in parameters and initial conditions, resulting in an approximation for the pdf of states conditioned on disturbance realizations. The pdf of states is then computed by integrating the conditional pdf of the PC approximated states over the pdf of disturbances.

3.1 Propagation of parametric uncertainties

In the gPC framework, a stochastic variable \( \psi(\xi) \) is approximated by a finite expansion of orthogonal polynomial basis functions

\[
\psi(\xi) \approx \hat{\psi}(\xi) = \sum_{j=0}^{l_\xi} a_j \varphi_j(\xi) = a^\top \varLambda(\xi),
\]

where \( a = [a_0, \ldots, a_{l_\xi}]^\top \) is the vector of the expansion coefficients; \( \varLambda(\xi) = [\varphi_0(\xi), \ldots, \varphi_{l_\xi}(\xi)]^\top \) is the vector of orthogonal polynomial basis functions that have a maximum degree \( m_\xi \) with respect to the standard random variables \( \xi \in \mathbb{R}^{n_r} \); and the number of terms in the PC expansion is given by \( l_\xi + 1 = \frac{m_\xi (m_\xi + 1)}{2} \) (Xiu and Karniadakis, 2002). The orthogonal basis functions, defined on the support space of the standard random variables, belong to the Wiener-Askey scheme of polynomials. This implies that the inner product \( \langle \varphi_i(\xi), \varphi_j(\xi) \rangle = \int_{\Omega} \varphi_i(\xi) \varphi_j(\xi) \varphi(\xi) \varrho(\xi) d\xi = \langle \varphi_i^2(\xi) \rangle \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker delta function.

The direct use of the gPC framework for propagation of the disturbances \( w(t) \) in (1) would entail defining an independent standard random variable \( \xi \) for each disturbance realization at every time point \( t \). As a result, the number of terms in the PC expansion (4) will increase significantly, rendering the gPC framework computationally prohibitive for real-time applications. To address the latter shortcoming in this work, the coefficients of PC expansions of states are first computed by conditioning their propagation on a particular realization of \( w(t) \). For a given realization of disturbances \( w(t) = \bar{w} \), system (1) is written as

\[
x(t) = f(x(t-1), \theta, u(t-1)) + \bar{w}.
\]

The gPC framework can now be used for efficient propagation of the uncertainties associated with the parameters and initial conditions. By approximating the stochastic states \( x \) and uncertain parameters \( \theta \) with their respective PC expansions (4), (5) takes the form

\[
\begin{align*}
\sum_{j=0}^{l_\xi} \tilde{x}_{i,j}(t) \varphi_j(\xi) &= f_i(\tilde{x}_{i}(t-1) \Lambda^\top(\xi), \ldots, \\
\theta_1 \Lambda^\top(\xi), \ldots, u(t-1)) + \bar{w}_i, \quad i = 1, \ldots, n; \\
\end{align*}
\]

where \( \tilde{x}(t) := [\tilde{x}_{i,0}(t), \ldots, \tilde{x}_{i,l_\xi}(t)]^\top \) and \( \theta_i := [\theta_{i,0}, \ldots, \theta_{i,l_\xi}]^\top \). The dynamics of the coefficients \( \tilde{x}(t) \) can be obtained using the Galerkin projection by computing the inner product

\[
\int_{\Omega} \left[ f_i(\tilde{x}_{1} \Lambda^\top(\xi), \ldots, \theta_1 \Lambda^\top(\xi), \ldots, u) + \bar{w}_i \right] \varpi d\xi
\]

over the support space of \( \xi \) (Ghanem and Spanos, 1991). The Galerkin projection ensures that, for a given disturbance realization \( \bar{w}_i \), the difference between each state \( x_i \) and its approximated PC expansion \( \tilde{x}_i \) (i.e., the PC approximation error) is orthogonal to the functional space spanned by the basis functions \( \varphi_j(\xi) \). For the nonlinear polynomial functions \( f \), the orthogonality of the basis functions allows for analytically deriving a set of deterministic equations that describe the dynamics of the PC expansion coefficients

\[
\tilde{x}(t) = F(\tilde{x}(t-1), \bar{\theta}, u(t-1)) + \bar{w},
\]

where \( \tilde{x}(t) := [\tilde{x}_{1}^\top(t), \ldots, \tilde{x}_{n}^\top(t)]^\top \in \mathbb{R}^{n(n+1)} \) is the vector of coefficients of PC expansions of all states; \( \bar{\theta} := [\tilde{\theta}_1^\top, \ldots, \tilde{\theta}_n^\top]^\top \in \mathbb{R}^{n(n+1)} \) is the vector of coefficients of PC expansions of all uncertain parameters; and the vector function \( F \) describes the deterministic dynamics of the expansion coefficients \( \tilde{x}(t) \).

3.2 Propagation of system disturbances

Eq. (6) indicates that the dynamics of coefficients \( \tilde{x}(t) \) are described by a deterministic nonlinear model that is driven by additive stochastic disturbances \( w \). Hence, the coefficients \( \tilde{x}(t) \) are stochastic variables. To obtain the pdf of states, the pdf of the PC approximated states conditioned on \( \bar{w} \) must be integrated with respect to the pdf of disturbances \( \varpi \).

Assumption 2. At any given time \( t \), the stochastic coefficients \( \tilde{x}(t) \) of PC expansions have a Gaussian distribution. Under the above assumption, the stochastic coefficients \( \tilde{x}(t) \) can be described in terms of their mean and covariance only, i.e.,

\[
\begin{align*}
X(t) &= F(X(t-1), \bar{\theta}, u(t-1)) \\
\Sigma_X(t) &= A(t) \Sigma_X(t) \Lambda(t) + \bar{Q},
\end{align*}
\]

where \( X(t) \in \mathbb{R}^{n(n+1)} \) and \( \Sigma_X(t) \in \mathbb{R}^{n(n+1) \times n(n+1)} \) denote the mean and covariance of the stochastic coefficients \( \tilde{x}(t) \); \( A(t) = \frac{\partial F}{\partial X} \mid_{X(t) = \mu(t)} \); and \( \bar{Q} \in \mathbb{R}^{n(n+1) \times n(n+1)} \) is a diagonal matrix whose \((i-1)n_q + 1\) diagonal entries are the entries of the disturbance covariance matrix \( \bar{Q} \). Note that the derivation of (7) relies on the fact that the stochastic disturbances \( w \) in system (1) have a Gaussian distribution.

The pdf of the stochastic states \( x(t) \) can now be approximated by
\[ P_x(t) \approx \int_\Omega P(\hat{x}(t)|w(\omega)) P_\omega d\omega = \int_\Omega P\left(\sum_{j=0}^{\ell_c} \tilde{x}_{i,j}(t) \varphi_j(\xi)\right) d\tilde{\xi}. \]

The quadrature methods can be used to evaluate (8) in order to obtain the moments of the PC approximated states \( \hat{x}(t) \). Exploiting the orthogonality property of the basis functions, the first two moments of each state are given by

\[ E[x_i(t)] \approx \int_\Omega \sum_{j=0}^{\ell_c} \tilde{x}_{i,j}(t, w) \varphi_j(\xi) w(\omega) X_0(x(t), \Sigma_X(t)) d\tilde{x} = \tilde{x}_{i,0}(t), \]

\[ E[x_i^2(t)] \approx \int_\Omega \left( \sum_{j=0}^{\ell_c} \tilde{x}_{i,j}(t, w) \varphi_j(\xi) \right)^2 w(\omega) X_0(x(t), \Sigma_X(t)) d\tilde{x} = \sum_{j=0}^{\ell_c} (X_{i,j}^2 + \Sigma_{i,j,i,j}(t)) \langle \varphi_j^2(\xi) \rangle \]

where \( X_i(t) \) and \( \Sigma_X(t) \) denote the vector of mean values and covariance matrix of coefficients of the PC expansion of the \( i \)-th state, respectively.

4. SAMPLE-BASED JOINT CHANCE CONSTRAINT APPROXIMATION

The nonconvexity and general intractability of chance constraints necessitate an approximation for solving the stochastic OCP (Ben-Tal et al., 2009). In this work, a sample-based approach is employed to provide an empirical approximation of the probability of state constraint violation in the JCC (3).

Define the indicator function

\[ I_G(x(t)) = \begin{cases} 0, & \text{if } x(t) \in G \\ 1, & \text{otherwise} \end{cases} \]

where 

\[ G := \{ x(t) \in \mathbb{R}^n | g(x(t)) \leq 0 \} \]

and \( x(t) \) denotes the gPC-based approximations of states (see Section 3). The indicator function (9) is used to estimate the probability of state constraint violation based on random samples of the stochastic states \( x(t) \). Given the probability of constraint violation can be estimated by

\[ \hat{\Psi}(t) := \sum_{j=1}^{N_s} I_G(x_i(j)(t)) \]

where \( N_s \) denotes the number of samples, \( \rho \) as the estimated probability of constraint violation computed from (10), and \( \delta \) as a user specified confidence level in the chance constraint approximation. The minimum sample size can be determined using Proposition 1.

Proposition 1. (Alamo et al., 2010): Consider \( x(t) \in \mathbb{R}^n \), \( \rho, \beta \) such that \( 0 \leq \rho < (1 - \beta) < 1 \), and \( \delta \in (0, 1) \).

To address the probability of state constraint violation based on a finite number of samples, \( N_s \), the proposed sample-based approach entails solving the deterministic OCP

\[ \pi^* := \arg \min_{\pi} J(\pi, P_x(t_k), P_\theta(t_k)) \]

s.t.: \( X(t+1) = F(X(t), \tilde{\theta}, u(t)), \quad \forall t = [t_k, t_{k+N-1}] \)

\[ \Sigma_X(t+1) = A(t+1) \Sigma_X(t) A^T + Q + \tilde{Q}, \quad \forall t = [t_k, t_{k+N-1}] \]

\[ u(t) \in U, \quad \forall t = [t_{k+1}, t_{k+N}] \]

\[ x(t_k) \sim P_x(t_k), \quad \theta \sim P_\theta(t_k), \quad \forall t = [t_k, t_{k+N-1}] \]

where \( \Sigma_X(t+1) = (A(t+1))^{\Sigma_X(t)} A^T + Q + \tilde{Q}, \) \( \forall t = [t_{k+N-1}, t_{k+N}] \)

\[ x(t_k) \sim P_x(t_k), \quad \theta \sim P_\theta(t_k), \quad \forall t = [t_{k}, t_{k+N}] \]

5. TRACTABLE SNMPC FORMULATION

The gPC-based uncertainty propagation method and the sample-based chance constraint approximation approach are employed to obtain a computationally tractable formulation for the SNMPC approach. At every sampling time \( t_k \), the proposed SNMPC approach entails solving the deterministic OCP

\[ \pi^* := \arg \min_{\pi} J(\pi, P_x(t_k), P_\theta(t_k)) \]

s.t.: \( X(t+1) = F(X(t), \tilde{\theta}, u(t)), \quad \forall t = [t_k, t_{k+N-1}] \)

\[ \Sigma_X(t+1) = A(t+1) \Sigma_X(t) A^T + Q + \tilde{Q}, \quad \forall t = [t_k, t_{k+N-1}] \]

\[ u(t) \in U, \quad \forall t = [t_{k+1}, t_{k+N}] \]

\[ x(t_k) \sim P_x(t_k), \quad \theta \sim P_\theta(t_k), \quad \forall t = [t_{k}, t_{k+N}] \]

where \( N \) denotes the prediction horizon, and \( \pi := [u^T(t_k), \ldots, u^T(t_{k+N-1})] \) is the control policy over the horizon \( N \) with \( \pi^* \) denoting the optimal policy. In (11), the knowledge of uncertain states and parameters at every sampling time (respectively, \( P_x(t_k) \) and \( P_\theta(t_k) \)) is used to adapt the polynomial basis functions utilized in constructing the PC expansions of states. The cost function

\[ J(\pi, P_x(t_k), P_\theta(t_k)) \]

can be defined in terms of either the probability distribution of states (i.e., \( P_x(t) \)) or their statistics (e.g., \( E[x(t)] \) and \( E[x^2(t)] \)).

The OCP (11) is implemented in a full-state feedback manner as \( P_x(t_k) \) is assumed to be known at each sampling time. At every \( t_k \), the OCP is solved over the horizon \([t_k, t_{k+N}]\), and the first set of the optimal inputs \( u^*(t_k) \) is applied to system (1). To adapt the uncertainty description of parameters \( \theta \) at every sampling point using the system measurements, the OCP is implemented in conjunction
Algorithm 1 The gPC-based SNMPC Approach (implementation at each sampling time \( t_k \))

1: At \( t = t_k \), obtain the measurements \( P_x(t_k) \) from (1)
2: At \( t = t_k \), estimate \( P_\theta(t_k) \) using the gPC-based HF
3: Initialize the control policy \( \pi \)
4: Use \( P_x(t), P_\theta(t), \) and \( \pi \) to predict the mean and covariance of coefficients of PC expansions over \([t_k, t_{k+\Delta}]\) using (7)
5: Use the statistics of PC coefficients to predict the evolution of \( P_x(t) \) over \([t_{k+1}, t_{k+\Delta}]\) using (8)
6: Use \( P_x(t) \), or its moments, to evaluate the cost function \( J(\pi, P_x(t_k), P_\theta(t_k)) \)
7: Draw samples \( \{x^{(j)}(t)\}_{j=1}^N \) from \( P_x(t) \) and evaluate \( I_G(x^{(j)}(t)) \) in (9) over \([t_{k+1}, t_{k+\Delta}]\)
8: Use \( \{I_G(x^{(j)}(t))\}_{j=1}^N \) to approximate the probability of state constraint violation using (10)
9: Compute the optimal control policy \( \pi^* \) by minimizing the cost function \( J \) subject to satisfaction of constraints in the OCP (11)
10: Apply \( u^*(t_k) \) to system (1)

with a gPC-based histogram filter (HF) (Bavdekar and Mesbah, 2016). The gPC-based HF uses the Bayes’ rule

\[
P_\theta(t) := P(\theta(t)|x(t)) = \frac{P(x(t)|\theta(t))P(\theta(t))}{P(x(t))} \]

to estimate \( P_\theta(t) \). In the Bayes’ rule, \( P(\theta(t)|\theta(t-1)) := P_\theta(t-1) \) is the prior pdf of the parameters; \( P(x(t)|\theta(t)) \) is the likelihood; \( P(x(t)) \) is the evidence; and \( P(\theta(t)|x(t)) \) is the posterior pdf of the parameters. The gPC-based HF estimates the posterior pdf of the parameters by constructing histograms of the prior and the likelihood pdfs (Bavdekar and Mesbah, 2016). Algorithm 1 describes the steps involved in solving the proposed gPC-based SNMPC approach at every sampling time.

6. SNMPC OF AN ATMOSPHERIC-PRESSURE PLASMA JET

The gPC-based SNMPC approach is demonstrated for regulating the thermal effects of an atmospheric-pressure plasma (APP) jet for biomedical applications. For a hot jet of Ar gas in contact with a target surface, Gidon et al. (2016) reported a reduced-order model for describing the dynamics of surface temperature \( T_s \)

\[
\rho_s c_{ps} d_A \frac{dT_s}{dt} = -d_A c_{ph} \rho_b \omega_b (T_s - T_b) + AU (T_g - T_s),
\]

\( T_s(0) = T_{b0} \),

where \( \rho_s \) and \( c_{ps} \) are the density and specific heat capacity of the surface, respectively; \( c_{ph}, \omega_b, \) and \( T_b \) are the specific heat capacity, perfusion rate, and temperature of the coolant (e.g., blood stream) in the target surface, respectively; \( A \) is the area of cross section of the surface that is in contact with the hot Ar jet; \( d_A \) is the thickness of surface; \( U \) is the heat transfer coefficient between the hot Ar jet and surface; and \( T_g \) is the temperature of the hot Ar jet in contact with the surface

\[
T_g = T_{in,f} + \frac{\eta c_{in}}{2h} \left[ 0.5 \exp\left(-\frac{2h}{vc_p r}d_{sep}\right) + 4 \exp\left(-\frac{2h}{vc_p r}(0.2d_2 + d_{sep})\right) - 4.5 \exp\left(-\frac{2h}{vc_p r}(0.2(d_1 + d_2) + d_{sep})\right) \right].
\]

In (14), \( T_{in,f} \) is the ambient temperature; \( c_{in} \) is the specific heat capacity of the hot Ar gas; \( P_{in} \) is the input power per unit volume to the plasma jet; \( \eta \) is the efficiency of the plasma jet; \( c_{ps} \) is the specific heat of the hot Ar gas; \( R \) is the radius of the plasma tube; \( h \) is the heat-transfer coefficient between the hot Ar gas and the surrounding air; \( d_1 \) is the length of chamber that contains the electrodes that ignite and sustain the plasma; and \( d_2 \) is the length of the chamber beyond the electrodes but before the plasma is exposed to ambient air. A scheme of the APP jet, along with further modeling details and parameter values, can be found in (Gidon et al., 2016).

The input power \( P_{in} \) and Ar inlet velocity \( v \) are the manipulated inputs of the APP jet. The system state in (13) is affected by an additive zero-mean white noise process described by \( w(t) \sim \mathcal{N}(0,0.01) \). In addition, the model parameters \( \eta \) and \( \omega_b \) are uncertain with known initial probability distributions \( \eta \sim \mathcal{N}(0.96, 4 \times 10^{-3}) \) and \( \omega_b \sim \mathcal{N}(1, 2 \times 10^{-3}) \). The surface temperature \( T_s \) is measured at regular sampling intervals of 10 s, and is corrupted with a zero-mean Gaussian noise with variance \( R = 0.001 \).

The control objective is to regulate the surface temperature \( T_s \) around a setpoint \( T_s^* \) in the presence of parametric uncertainties and stochastic disturbances. The cost function in the OCP (11) is defined by

\[
J = \sum_{i=1}^{N} \left( T_s - E[T_s(t_i)] \right)^2, \tag{14}
\]

where \( N \) corresponds to the prediction horizon of 100 s. The system inputs are subject to constraints given by \( 8 \text{ W} \leq P_{in} \leq 35 \text{ W} \) and \( 8 \text{ m/s} \leq v \leq 25 \text{ m/s} \). The surface temperature \( T_s \) is constrained in a probabilistic sense as

\[
\Pr[T_s \leq 319 \text{ K}] \geq 0.9.
\]

Algorithm 1 is used for implementing the proposed gPC-based SNMPC approach, in which \( r = 0.06 \) and \( \delta = 0.05 \) for chance constraint approximation.

Open-loop simulations were first carried out to investigate the effectiveness of the proposed gPC-based uncertainty propagation method. The state pdfs obtained using the proposed method were compared to those obtained through Monte Carlo simulations on the original system model using 1000 samples. The average CPU time for uncertainty propagation between two sampling times using the proposed method was 0.18 s, while the average CPU time required by the Monte Carlo-based approach was approximately 0.70 s. Thus, the gPC-based uncertainty propagation method led to an approximately 75% reduction in the computational time. A comparison of the histograms obtained from the two approaches (not shown) indicated
To evaluate the performance of the SNMPC approach, 70 closed-loop simulation runs were performed based on different uncertainty realizations. The setpoint is initially defined at $T_s = 318$ K and changed to $T_s = 312$ K at $t = 250$ s to avoid thermal damage to the surface. Fig. 1 shows the closed-loop surface temperature profiles for the 70 runs. As can be seen, the SNMPC approach can effectively track the setpoint change, while containing the state constraint violation within the admissible level of 10%. The case study is currently being extended to include a complex model of jet, and comparisons will be made to nominal and Monte Carlo-based NMPC approaches.

7. CONCLUSIONS

This paper presents a stochastic nonlinear model predictive control approach that can deal with parametric uncertainties and additive stochastic disturbances. An uncertainty propagation approach based on generalized polynomial chaos is presented to efficiently propagate system disturbances. This is done through mapping the stochastic disturbances to the space of coefficients of polynomial chaos expansions, which are then used to compute the probability distribution of states along the prediction horizon. The preliminary closed-loop simulation results on an atmospheric-pressure plasma jet indicate the effectiveness of the proposed approach for stochastic predictive control of nonlinear systems with short sampling times. In future, the performance of the proposed approach will be investigated on more complex system models, and its theoretical properties will be established.

REFERENCES


