Mixed stochastic-deterministic tube MPC for offset-free tracking in the presence of plant-model mismatch

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\section*{ARTICLE INFO}

\textbf{Article history:}
Received 19 October 2017
Received in revised form 27 April 2018
Accepted 29 April 2018
Available online 3 August 2018

\textbf{Keywords:}
Tube-based MPC
Stochastic tubes
Plant-model mismatch
Offset-free tracking

\section*{ABSTRACT}

This paper presents a stochastic-tube model predictive control (MPC) strategy that systematically handles plant-model mismatch, guarantees stability in the presence of changing operating conditions, and ensures offset-free tracking for all reachable operating conditions. The key notion of this work is to separate the system uncertainty into two distinct sources of additive bounded state error. The first source is a non-random uncertainty that represents mismatch between the linear model of the controller and the true plant (as well as other types of persistent disturbances). The second source represents random fluctuations due to either the intrinsic stochastic variability of the system, or exogenous disturbances. Recursive feasibility and stability of the stochastic-tube MPC strategy is guaranteed by defining the terminal invariant set to include the effect of changes in the steady-state operating conditions. Offset-free tracking is achieved with a disturbance estimator that can be tuned independently of the controller. To reduce the conservatism inherent to robust control performance, a new filter model for deterministic disturbances is proposed that predicts uncertainty in the future evolution of the disturbances based on a bound on how quickly the plant-model mismatch can vary. The stochastic-tube MPC strategy is demonstrated on two simulation case studies—a benchmark DC-DC converter and an industrially-motivated fluidized-bed catalytic cracking unit.

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1. Introduction

Model predictive control (MPC) is the most widely used approach for optimal control of multivariable, constrained systems [2,3]. The receding-horizon implementation of MPC, wherein the optimal control problem is repeatedly solved online based on new system measurements, provides some degree of robustness to system uncertainties, which generally stem from mismatch between the plant and model, exogenous disturbances, and measurement noise. However, the marginal robust performance in receding-horizon control may not be adequate in practice to prevent, for example, unstable process operation, or violation of quality constraints for products with stringent quality requirements.

When a probabilistic description of system uncertainties is available, the likelihood of occurrence of different realizations of uncertainties can be explicitly accounted for in the optimal control problem to design control policies that are robust to uncertainties. This notion has led to stochastic MPC (SMPC) whose core component is chance constraints that allow a prespecified level of constraint violation (e.g., see [4,5]). Chance constraints give SMPC the ability to systematically trade-off constraint satisfaction with control performance, thus possibly reducing the conservatism that is often associated with robust control [6].

A comprehensive overview of the many different problem settings and methods for SMPC is given in [5]. One key feature of the different SMPC formulations is their assumed form of the uncertainty description, which can be time varying (additive and/or multiplicative) or time invariant. This work addresses the SMPC problem for linear systems with probabilistic time-variant uncertainties.\textsuperscript{1} To this end, tube-based MPC has emerged as a popular approach (e.g., [4,10,11]). One of the key advantages of tube-based MPC is that the complexity of the MPC

\textsuperscript{1} A preliminary version of this work has been presented at the IFAC 2017 World Congress [1].
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\textsuperscript{1} Interested readers are referred to, for example, [7–9] for SMPC methods that can handle probabilistic time-invariant uncertainties for linear and nonlinear systems.

https://doi.org/10.1016/j.jprocont.2018.04.010
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problem is generally comparable to that of nominal MPC, while also providing guaranteed constraint satisfaction. The concept of tubes, originally developed for set-based uncertainty descriptions [10], was extended to stochastic systems in [12,13]. Recently, Lorenzen et al. [14] proposed a new formulation for stochastic-tube MPC that increases the feasible region of the controller using a specialized first-step constraint.

This paper addresses three open challenges in stochastic-tube MPC for linear systems, which are particularly important for process systems applications. The challenges include: (i) systematically dealing with mismatch between a process model and the true plant, (ii) ensuring recursive feasibility and stability of SMPC when the process operating point is changed, for example, due to economic considerations, and (iii) guaranteeing offset-free tracking for all reachable operating points. The first challenge is addressed by separating the deterministic sources of uncertainty from the stochastic sources, which comprises the key notion of this paper. Plant-model mismatch can be modeled as deterministic system uncertainty, since it is not inherently of random nature. Deterministic uncertainties can also include (e.g., step- or ramp-like) persistent disturbances. On the other hand, stochastic uncertainties arise from the intrinsic random fluctuations of a system and/or exogenous stochastic disturbances. The key advantage of handling deterministic and stochastic sources of system uncertainty separately results from the fact that the dynamics of the deterministic uncertainty due to plant-model mismatch can be directly modeled using a high-fidelity model or process data, which effectively translates to bounding the change in the plant-model mismatch over time. The knowledge of the plant-model mismatch can be readily incorporated into the optimal control problem to mitigate the conservatism inherent to most robust control methods.

An important limitation of stochastic-tube MPC for practical applications arises from the fact that recursive feasibility and stability are ensured with respect to only a single operating point for which the controller is designed. This implies that when the operating point of a SMPC controller is changed, for example, due to a change in the process operation dictated at the real-time optimization level [15], the controller should be redesigned to re-establish feasibility and stability. However, this can pose a computational challenge, particularly when the transitions between the operating points should be accommodated online. In [16,17], an MPC strategy was presented for tracking piece-wise constant changes in the operating point in a deterministic setting by defining the terminal invariant set to include the effect of the steady-state operating points. This approach effectively leads to an enlarged domain of attraction, which is defined as the set of states that can be feasibly steered to the terminal set at the end of the control horizon. Inspired by [17], this paper extends the notion of a terminal invariant set for tracking to stochastic-tube MPC to achieve guaranteed recursive feasibility and stability when the operating point of the controller is changed [18].

When deterministic uncertainties tend to non-zero values (e.g., when the uncertainty results from a deterministic bias in model predictions due to plant-model mismatch), the controlled system cannot realize offset-free tracking of the target operating point. Offset-free MPC strategies have been developed to eliminate the setpoint tracking error (under mild assumptions) by appropriately updating the target setpoint when the disturbances tend to a constant value [19,20]. A similar notion is adopted here for stochastic-tube MPC to enable offset-free tracking in the presence of plant-model mismatch and persistent disturbances. The proposed SMPC strategy hinges on using a disturbance estimator, whose dynamics are independent of the controller and thus can be tuned directly based on the disturbance characteristics [17]. The recursive feasibility and stability of the proposed stochastic-tube MPC strategy can be guaranteed for any possible disturbance estimate, since the terminal invariant set for tracking can accommodate the updated target setpoints.

In what follows, Section 2 defines the SMPC problem of interest. Section 3 presents preliminary results along with the proposed notion of mixed uncertainty tubes, which is an extension of stochastic tubes [12] for handling deterministic disturbances. The proposed stochastic-tube MPC strategy is then presented in Section 4, followed by a discussion on its recursive feasibility and stability properties in Section 5. Section 6 presents an extension of the proposed algorithm for offset-free tracking as well as a method for further enlarging the domain of attraction of the controller by using a particular filter to model the deterministic uncertainties. The stochastic-tube MPC strategy is demonstrated on a benchmark DC–DC converter to illustrate how chance constraints can be guaranteed in the presence of persistent disturbances and model uncertainty. In addition, the control strategy is demonstrated on an industrial fluidized-bed catalytic cracking (FCC) unit to show that setpoints that are unreachable due to plant-model mismatch can be realized when the stochastic-tube–MPC strategy is used in conjunction with the proposed filter for deterministic disturbances (Section 7).

Notation. \( [A]_i \) and \( [a]_i \) denote the \( i \)th row and entry of matrix \( A \) and vector \( a \), respectively. \( A \geq 0 \) and \( A \succeq 0 \) indicate that matrix \( A \) is positive definite and positive semidefinite, respectively; \( \|x\|_2^2 = x^T A x \). \( b \geq a \) denotes \( b \in \{a, a+1, a+2, \ldots \} \) where \( a \) and \( b \) are integers. \( \mathbb{N}^a_b \) denotes the sequence of integers from \( a \) to \( b \). \((x, y) = [x^T, y^T]^T \) denotes vector concatenation. \( \mathbb{P}(A | B) = \mathbb{P}(A | x_0) \) denotes the probability of some event \( A \) occurring conditioned on a given realization of the state \( x_0 \). \( \mathbb{P}(Y | x_0) \) is the conditional expectation of random variable \( Y \) given \( x_0 \). \( x_{k|k} \) denotes the state at time \( k \), while \( x_{k|i} \) denotes the \( i \)th-step ahead predictions of the state from time \( k \). For sets \( A \) and \( B \), \( A \oplus B = \{a + b \mid a \in A, b \in B\} \) denotes the Minkowski sum and \( A \ominus B = \{a \in A \mid a + b \in A, \forall b \in B\} \) denotes the Pontryagin difference.

2. Problem statement

Consider an uncertain system with the following discrete-time linear dynamics

\[
\begin{align}
    x_{k+1} &= Ax_k + Bu_k + d_k + B_u w_{k+1}, \\
    y_k &= Cx_k + Du_k,
\end{align}
\]

(1a)

(1b)

where \( x_k \in \mathbb{R}^n \) is state of the system; \( u_k \in \mathbb{R}^m \) is the manipulated control input; and \( y_k \in \mathbb{R}^p \) is the controlled variable. The “perturbation” signal \( d_k = f_k(x_k, u_k) - Ax_k - Bu_k \) represents mismatch between the true (possibly time-varying and nonlinear) dynamics \( f_k(x_k, u_k) \) and the linear model, which satisfies

\[
d_k \in \mathbb{R}^n.
\]

(2)
The disturbance $w_k \in \mathbb{R}^{nw}$, on the other hand, is assumed to be a realization of a i.i.d. random variables $W_k$ with known distribution $\mathbb{P}_W$ and finite support $\mathbb{W}$ such that

$$w_k \in \mathbb{W} \subset \mathbb{R}^{nw}.$$  \hfill (3)

We briefly highlight the difference between $d_k$ and $w_k$. The signal $d_k$ represents all the deterministic forms of uncertainty including structural plant-model mismatch and unmeasured persistent disturbances, which cannot be directly modeled as random variables since they do not necessarily have a given distribution. Hence, $d_k$ is an unmeasured disturbance that can only be estimated (within some bound) from data. The signal $w_k$, however, represents random fluctuations that either come from intrinsic system variability (e.g., differences in copy number in a collection of biological cells) or from exogenous inputs (e.g., temperature fluctuations outside of a heat exchanger). For clarity, $d_k$ is referred to as the deterministic disturbance while $w_k$ is referred to as the stochastic disturbance.

The main aim of robust MPC methods is to minimize a predicted cost, which is designed specifically to ensure closed-loop stability (in a specific sense), while also satisfying constraints on the state and control input in the presence of disturbances. Since the evolution of (1) is stochastic, it is desired that the state satisfies chance (or probabilistic) constraints

$$\mathbb{P}(H_{j,k} x_k \leq [h_j], \ j \in \mathbb{N}_1, \ k \geq 1, \ \epsilon $$

while the input is required to satisfy hard constraints

$$Gu_k \leq g, \ \ k \geq 0,$$  \hfill (5)

where $H \in \mathbb{R}^{nx \times h}, h \in \mathbb{R}^h, G \in \mathbb{R}^{nx \times m}, g \in \mathbb{R}^g$, and $\epsilon \in [0, 1]^j$ is a vector containing the allowed probability of violation for each state constraint. The values for $[\epsilon]$ may come directly from application requirements, but also can be viewed as a tuning parameter that specifies a trade-off curve between constraint violation and control performance when constraints are active, as discussed in detail in [6]. The control input is required to meet constraints for all admissible disturbance sequences since actuators have finite limitations in practice.

Let $X = \{ x \in \mathbb{R}^n : Hx \leq h \}$ and $U = \{ u \in \mathbb{R}^m : Gu \leq g \}$ be the collection of state and control input constraints, respectively. The following assumptions are made throughout the paper:

**Assumption 1:**

(i) the pair $(A, B)$ is controllable;
(ii) sets $X, U, D$, and $W$ contain the origin as an interior point;
(iii) $U, D$, and $W$ are compact and convex sets;
(iv) $W_k$ for $k = 0, 1, \ldots$ are independent and identically distributed, zero-mean random variables;
(v) the state of the system is measured, and hence $x_k$ is known at each sampling time.

The control objective is to design, via model predictive control, a nonlinear feedback control law that drives the system (1) “as close as possible” to a desired target $y^* k$ while satisfying state chance constraints (4) and hard input constraints (5) in the presence of plant–model mismatch and random disturbances. The target $y^* k$ may vary with time and stability and feasibility should be maintained whenever a switch in $y^* k$ takes place. A further objective is that the feedback controller ensures that offset is removed in the mean of the controlled variables for any $d_k \neq 0$, that is $\lim_{k \to \infty} \mathbb{E} \{ |y_k - y_k^* | \} = 0$ whenever $y_k^*$ is reachable and asymptotically convergent. Lastly, the controller should be simple enough to implement online.

**Remark 1.** No assumption is required on the number of inputs $m$ or number of outputs $p$, which allows for thin $(p > m)$, square $(p=m)$, and fat $(p < m)$ plants to all be considered in the same framework. In addition, $(A, B, C, D)$ is not required to be minimal, which allows for the state-space model to be obtained directly from input–output models with the states represented by past inputs and outputs. These type of representations avoid the necessity of an observer, which can be advantageous.

**Remark 2.** State chance constraints (4) are assumed to be in the form of individual chance constraints since the violation probability is handled separately for each constraint. A more general formulation of chance constraints can be posed in terms of joint chance constraints, which are much more difficult to deal with computationally. Joint chance constraints are often handled by approximating the original joint constraint in terms of a collection of individual chance constraints. We note, however, that the SMPC algorithm presented in this work can handle joint state chance constraints using the concept of risk allocation [21].

**Remark 3.** State chance constraints (4) exclude the initial state at time $k = 0$ due to the fact that its value cannot be influenced by the control inputs. Robust MPC (RMPC) strategies, on the other hand, explicitly enforce $x_0 \in X$. This is an important conceptual difference between RMPC and SMPC, which implies that $x_0 \notin X$ is allowed in SMPC. Hence, chance constraints can be thought of as a systematic way to allow for constraint violation based on a probabilistic description of the system when constraint violation is acceptable. Readers are referred to [22] for more information regarding the various types of feasibility in SMPC and how these concepts relate to RMPC.

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2 The stochastic disturbance is indexed starting from $k + 1$ in (1) as the signal will be assumed to be measured at every time $k \geq 0$ in the final part of the paper for estimation purposes. Even when $w_k$ is measured, its future evolution is unknown and random. Furthermore, since the disturbance distribution is i.i.d., this index change results in no loss in generality.
3. Predictions and mixed uncertainty tubes

Robust and stochastic MPC methods look to enforce constraints (4) and (5) during closed-loop operation by constraining the predictions of the future system evolution based on current measurements. Given the measured state $x_k$ at any time $k$, the predictions can be modeled as

$$x_{i+1|k} = Ax_{i|k} + Bu_{i|k} + d_{i|k} + W_{i+1|k}, \quad x_{0|k} = x_k,$$

where $x_{i|k}$ is the uncertain state prediction, $\overset{\Delta}{=}^{a.s}$ denotes almost sure convergence, $u_{i|k} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are measurable state feedback functions, $d_{i|k}$ is the predicted plant-model mismatch $i$ steps ahead of time $k$, and $W_{i+1|k} = W_{k+i+1}$.

The goal of this section is to develop a set of tighter constraints on nominal system predictions (in the absence of all disturbances) that are sufficient to guarantee (4) and (5) are satisfied in closed-loop operation. These constraints are related to the concept of bounding the states around a “tube” of trajectories, which represent a simple way to deal with uncertainty offline without increasing complexity of the controller [10,17,12]. When $W_{i+1|k} = 0$ standard robust tubes can be used to bound the state trajectory [10]. On the other hand, the case that $d_{i|k} = 0$ has been treated in [12] using so-called “probabilistic tubes,” which is a generalization of robust tubes to handle stochastic uncertainty. This section extends “probabilistic tubes” to also account for other deterministic sources of uncertainty, which we refer to as mixed uncertainty tubes.

3.1. Characterization of steady states

One of the control objectives is to allow the desired target to change, which commonly occurs in applications due to changing economic considerations or disturbances. These changes should be directly accounted for in the restricted constraints. One way to do this is to parametrize the set of all possible steady states for the nominal system. This can be done by noticing that every steady state and input $\tilde{z}^* = (\tilde{x}^*, \tilde{u}^*)$, that is a fixed point of nominal system, must be a solution of the following set of linear equations [23]:

$$[A - I_n \quad B] [\begin{bmatrix} \tilde{x}^* \\ \tilde{u}^* \end{bmatrix}] = 0,$$

and hence is an element of the null space of matrix $[A - I_n \ B]$. Since $(A, B)$ is controllable by Assumption 1, the dimension of this null space is equal to $m$. Therefore, there exists a matrix $M_\theta \in \mathbb{R}^{(n+m)xm}$ such that every nominal steady state and input can be written as

$$\tilde{z}^* = M_\theta \tilde{\theta},$$

for any $\tilde{\theta} \in \mathbb{R}^m$. The nominal steady state outputs are then

$$\tilde{y}^* = N_\theta \tilde{\theta},$$

where $N_\theta = [CD]M_\theta$. The main rationale behind this parametrization, proposed in [23], is to enlarge the terminal invariant set for tracking when compared to standard regulation for a fixed target. This added flexibility thus directly leads to a larger domain of attraction for the controller and ensures feasibility for switches in the target.

3.2. Dual mode prediction method

The key concept in tube methods is to split the predicted states into nominal and uncertain components [4]. Since there are two sources of uncertainty considered, the augmented state decomposition takes the form

$$x_{i|k} = \bar{x}_{i|k} + e_{i|k} + \epsilon_{i|k},$$

where $\bar{x}_{i|k}$ is the nominal state prediction, $e_{i|k}$ is a zero-mean stochastic error, and $\epsilon_{i|k}$ represents the deterministic source of uncertainty. The goal is to define the nominal predictions such that the errors evolve as a stationary processes and, hence, their effect on the constraints can be computed offline.

This can be done based on the “dual mode prediction” paradigm, which splits the predictions of the input sequence into two separate intervals. Mode 1 refers to the predicted input sequence over the first $N$ steps for some user-specified finite horizon $N$ (which are not pre-specified), while Mode 2 denotes the predicted inputs over the remaining part of the horizon (which are fixed according to some terminal feedback law). The terminal feedback controller should regulate the system to the desired target, i.e., $u_{i|k} = \tilde{u}_{i|k} + K(x_{i|k} - \tilde{x}_i)$. Using the steady state characterization (8), we can then specify the predicted dual mode inputs as

$$u_{i|k} = K\epsilon_{i|k} + L\epsilon_{0|k} + c_{i|k},$$

where $c_{i|k}$ are perturbations to the pre-specified terminal controller and $L = [-K\ I_m]M_\theta$ is a known matrix. As mentioned above, to ensure a finite number of decision variables in the problem, only $c_{i|k} = [c_{i|k}^T, \epsilon^T_{i|k}, \ldots, \epsilon_{N-1|k}^T]$ are left free while $c_{i|k} = 0$ for all $i \geq N$. The closed-loop prediction paradigm then must satisfy the following dynamic equations:

$$\ddot{x}_{i+1|k} = A\dot{x}_{i|k} + Bu_{i|k}, \quad \dot{x}_{0|k} = x_k,$$

$$\ddot{c}_{i+1|k} = \Phi \ddot{c}_{i|k} + \Phi^* \epsilon_{i|k}, \quad \epsilon_{0|k} = 0,$$

$$\ddot{\epsilon}_{i+1|k} = \Phi \ddot{\epsilon}_{i|k} + d_{i|k}, \quad \epsilon_{0|k} = 0,$$

where $\Phi = A + BK$. Note that the structure of the predicted input $u_{i|k} = K(e_{i|k} + \epsilon_{i|k}) + \tilde{u}_{i|k}$ is purposely chosen to counteract the effect of disturbances by forcing the true state to lie as close to the nominal trajectory as possible.
It is useful to define the set of error values that are reachable in \( i \) steps from the origin according to (12), i.e.,

\[
R_i^e = \bigoplus_{j=0}^{i-1} \Phi^j B_w W, \quad R_i^c = \bigoplus_{j=0}^{i-1} \Phi^j \mathbb{D}.
\]

The sequence of sets \( R_i^e \) and \( R_i^c \) as \( i \to \infty \) converges to a limit \( R_\infty^e \) and \( R_\infty^c \) as long as \( \Phi \) is strictly stable. Sets \( R_\infty^c \) and \( R_\infty^c \) are commonly referred to as the minimal robust positively invariant (mRPI) sets for their respective dynamics. RPI sets are an important concept in “robust” control that are an intrinsic property of a given system dynamic and admissible disturbance set, and can be defined as follows [24]:

**Definition 1.** A set \( \Omega \) is called robust positively invariant (RPI) for a system \( x_{k+1} = Ax_k + w_k \) with \( w_k \in W \) for all \( k \geq 0 \) if, for any \( x \in \Omega \), \( Ax + w \in \Omega \) for all \( w \in W \). In other words, \( \Omega \) should satisfy \( A\Omega \oplus W \subseteq \Omega \).

The mRPI set can then be thought of as the only RPI set contained in every other RPI set.

### 3.3. Recursively feasible tubes for mixed uncertainty

Here, we will focus on developing the tightened constraints on the nominal states \( \bar{x}_{ik} \) and inputs \( \bar{u}_{ik} \) that ensure satisfaction of (4) and (5). Instead of working with (4) directly, we will work with the following sufficient condition

\[
P[|H|\bar{x}_{i+1} \leq |h| |\bar{x}_i| \geq 1 - |\epsilon|, \quad j \in N_1^i, \quad k \geq 0.
\]

These constraints are clearly sufficient to guarantee (4) as (14) must hold for all reachable states \( x_k \), i.e.,

\[
P[|H|\bar{x}_{i+1} \leq |h| |\bar{x}_i| = \int P[|H|\bar{x}_{i+1} \leq |h| |\bar{x}_i|] dF_{\bar{x}_i}(\bar{x}_i),
\]

where \( dF_{\bar{x}_i}(\bar{x}_i) \) is the probability measure for \( \bar{x}_i \), which integrates to one based on the axioms of probability.

Constraints (14) are the underpinning of the probabilistic tube framework, however, it is important to carefully consider how to guarantee they are satisfied based on predictions of the states. It is necessary that the predictions satisfy

\[
P[\bar{x}_{i+1} \leq |h| |\bar{x}_i| \geq 1 - |\epsilon|, \quad j \in N_1^i, \quad i \geq 0.
\]

The main issue with (15) is that they do not ensure recursive feasibility, that is, whenever there exist a \( c_k \) satisfying (15) at time \( k \), there does not necessarily exist at least one set of control actions \( c_{k+1} \) that satisfies (15) at time \( k+1 \). This is mainly due to the realization of disturbances in the future that have only been accounted for in a probabilistic fashion at time \( k \).

A simple way to way to ensure recursive feasibility is to replace (15) with the stronger condition [22]:

\[
P[\bar{x}_{i+1} \leq |h| |\bar{x}_i| \geq 1 - |\epsilon|, \quad j \in N_1^i, \quad i \geq 0.
\]

These constraints have an interesting structure as conditioning on \( x_{ik} \) implies that the distribution \( P_W \) only impacts the constraints one-step ahead. This simplifies the numerical calculation of the tightened constraints as we can neglect the convolution of the disturbances through the system dynamics.

We can replace (16) with constraints on the nominal state by making the following observation based on (12)

\[
x_{i+1,k} = \xi_{i+1,k} + e_{i+1,k} + e_{i+1,k},
\]

where conditioning on \( x_{ik} \) implies that (16) must hold for all reachable \( e_{ik} \) while \( w_{i+1,k} \) is random as it has not yet been realized. It is therefore sufficient to impose

\[
\bar{x}_{i+1} \in X_i \triangleq X_1 \ominus \Phi \mathcal{R}_i^c \ominus R_i^c,
\]

where \( X_1 = \{ x \in \mathbb{R}^n : Hx \leq h - \eta \} \) and the elements of \( \eta \in \mathbb{R}^r \) are defined as the solution to

\[
[\eta]_j = \min \left( \eta \right) \quad \text{s.t.} \quad P[|H|B_w W_k \leq \eta] = 1 - |\epsilon|.
\]

Similarly, we can replace hard input constraints (5) with constraints on the nominal input based on the fact that \( u_{ik} = \bar{u}_{ik} + K e_{ik} + K e_{ik} \) so that it is sufficient to impose

\[
\bar{u}_{ik} \in \bar{U} \ominus U \ominus KR_c^c \ominus KR_c^c,
\]

The constraints (18) and (20) define the proposed tubes for mixed uncertainty. Note that the distribution of the disturbance is only needed to define \( \eta \) that appears in the set \( X_1 \), which is independent of time due to the fact that the stochastic disturbances are i.i.d. by assumption. The vector \( \eta \) can be determined by solving \( r \) chance constrained optimization problems (19) directly offline (using numerical integration or sampling-based methods) or, whenever the inverse cumulative distribution function \( F_{|H|B_w W_k}(\cdot) \) of \( |H|B_w W_k \) is easily derivable, setting \( [\eta]_j = F_{|H|B_w W_k}^{-1}(1 - |\epsilon|) \). This avoids the need for computing the distribution function \( e_{ij} \) at future times, which requires a multivariate convolution integral.

Using properties of the reachable sets (13), we can define the sets in (18) and (20) recursively as

\[
\tilde{X}_{j+1} = \tilde{X}_j \ominus \Phi B_w W \ominus \Phi \mathbb{D}, \quad j \geq 1,
\]

\[
\tilde{U}_{j+1} = \tilde{U}_j \ominus K \Phi B_w W \ominus K \Phi \mathbb{D}, \quad j \geq 0.
\]
where $\tilde{X}_1 = X_1 \ominus B$ and $\tilde{U}_0 = U$. This definition only requires the set multiply and Pontryagin difference operations to be performed. These recursive operations are computationally cheaper than Minkowski sums, which is an interesting advantage of this approach. As pointed out in [25], Pontryagin differences can be efficiently computed for polytopes by solving a sequence of Linear Programming (LP) problems which scale linearly with the number of half-spaces. On the other hand, Minkowski sums are computationally expensive operations which require either vertex enumeration and convex hull computations in $n$-dimensions or a projection from $2n$ down to $n$ dimensions [25]. Also note that the sets $\tilde{X}_{i+1}$ and $\tilde{U}_i$ for $i \geq 0$ are guaranteed to be convex polytopes.

**Remark 4.** The presented approach has a certain degree of conservatism since satisfaction of (14) are only sufficient (and not necessary) for (4). However, (14) represents a tractable and straightforwardly implementable condition in line with the receding horizon implementation of the controller, in contrast to (4), which generally cannot exactly be accommodated in an optimization procedure. In addition, (14) is still less conservative than a robust implementation of the constraint $x_k \in X$ since the distribution of the uncertainty is taken into account.

### 4. Stochastic-tube MPC for tracking

SMPC strategies almost always neglect possible deterministic disturbances, that is setting $d_k = 0$ for all $k \geq 0$ in (1), which is equivalent to assuming the linear model accurately describes the process dynamics. Additionally, these algorithms consider only the regulation problem, which involves steering the system to a fixed steady state point. Rigorous theoretical results for stability, feasibility, and optimality are thus only valid about that single fixed point indicating that these properties can be lost in the case of a target change [17].

Whenever feasibility is lost due to a target change, the controller would have to be re-designed, which would require a number of expensive online calculations to be performed. In this section, rigorous MPC for tracking [23] is extended to the stochastic case (with plant-model mismatch) using the concept of “mixed uncertainty tubes” presented in Section 3 in order to avoid this online re-design of the controller.

#### 4.1. Cost function and optimal control problem

The proposed cost function directly penalizes the perturbations $c_k$ from the nominal controller. In order to ensure feasibility for any (possibly time-varying) target $y_k^*$, the steady state conditions (parametrized by $\dot{\theta}_k$) are included as decision variables in the optimal control problem (referred to as the artificial reference). To ensure convergence toward $y_k^*$ an offset cost $V_0(\tilde{y}_k^* - y_k^*)$ is added to the cost function, which penalizes deviations between the artificial reference and the desired target. Therefore, given the current state $x_k$ and target $y_k^*$, the cost function is defined as

$$V_N(c_k, \tilde{y}_k; x_k, y_k^*) = \sum_{i=0}^{N-1} \|c_{i+1}\|_Q^2 + V_0(\tilde{y}_k - y_k^*),$$

and $(\tilde{x}_k^*, \tilde{u}_k^*) = M \dot{\theta}_k, \tilde{y}_k^* = N \dot{\theta}_k$, and $\Psi = \Psi^T > 0$ is a suitably chosen weight matrix. The optimal control problem $\mathcal{P}_N(x_k, y_k^*)$ to be solved at every time $k \geq 0$ is now given by:

$$\min_{c_k, \dot{\theta}_k} V_N(c_k, \tilde{y}_k; x_k, y_k^*)$$

s.t. $x_{0:k} = x_k$,

$\tilde{x}_{i+1:k} = A \tilde{x}_{i:k} + B \tilde{u}_{i:k}, \quad i = 0, \ldots, N - 1$,  

$\tilde{u}_{i:k} = K \tilde{x}_{i:k} + \tilde{c}_{i:k}, \quad i = 0, \ldots, N - 1$,

$\tilde{x}_{i+1:k} \in \tilde{X}_{i+1}, \quad \tilde{u}_{i:k} \in \tilde{U}_i, \quad i = 0, \ldots, N - 1$,

$(\tilde{x}_{N:k}, \tilde{y}_k^*) \in \Omega_k^*$,

where $\Omega_k^*$ is a suitably defined terminal set for the proposed stochastic-tube MPC problem for tracking.

Let $V_N^*(x_k, y_k^*)$ denote the optimal cost and let $c_k^*(x_k, y_k^*)$ and $\dot{\theta}_k^*(x_k, y_k^*)$ denote the optimal values of the decision variables that minimize the cost $V_N(c_k, \tilde{y}_k; x_k, y_k^*)$. Based on the receding horizon implementation of the controller, the proposed control law is given by $u_k = \kappa_N(x_k, y_k^*)$

$$\kappa_N(x_k, y_k^*) = \kappa x_k + \tilde{L}^* c_k^*(x_k, y_k^*) + c_{i:k}^*(x_k, y_k^*)$$

where $c_{0:k}^*(x_k, y_k^*)$ is the first (vector) element of $c_k^*(x_k, y_k^*)$. The domain of attraction of the controller is defined as the set of initial states that can be steered into the terminal region within $N$ steps while fulfilling constraints for all admissible disturbances, that is,

$$\mathcal{A}_N = \{x_k \in \mathbb{R}^n : \exists \tilde{\theta}_k \ s.t. \ \mathcal{P}_N(x_k, y_k^*) \text{ feasible} \}.$$

Note that since the set of constraints of $\mathcal{P}_N(x_k, y_k^*)$ are independent of the target $y_k^*$, the domain of attraction $\mathcal{A}_N$ does not depend on the target $y_k^*$.

**Remark 5.** The cost of control perturbations $\sum_{i=0}^{N-1} \|c_{i+1}\|_Q^2$, representing the first term in the proposed cost function (22), can be directly related to the infinite horizon nominal cost for a particular choice of $\Psi$ and $K$. Define

$$V(c_k, x_k) = \sum_{i=0}^{\infty} \|\tilde{x}_{i:k} - \tilde{x}_k^*\|_Q^2 + \|\tilde{u}_{i:k} - \tilde{u}_k^*\|_R^2,$$
where \( Q \geq 0 \) and \( R > 0 \). In the case that \( K = K_{LQR} \) is chosen to be the LQR solution, which exists as long as the pair \( (A, Q^{1/2}) \) is detectable, then

\[
V(c_k, x_k) = \sum_{i=0}^{N-1} \| c_{0k} \|^2_{A_k + B_k P_k} + x_k^T P_k x_k, \tag{26}
\]

where \( P \) solves the Lyapunov equation \( P - \Phi^T P \Phi = Q + K^T K R \). Since the last term is a constant, it is easy to see that minimizing the cost \( \sum_{i=0}^{N-1} \| c_{0k} \|^2_{A_k + B_k P_k} \) is equivalent to minimizing the infinite horizon quadratic cost (25) whenever \( \Psi = R + B^T P B \).

**Remark 6.** Since the proposed cost (22) includes an offset cost that can be non-zero, the proposed stochastic-tube MPC strategy for offset-free tracking might not ensure the “local optimality” property, i.e., that the finite horizon controller converges to the controller with optimal closed-loop performance in a neighborhood of the steady state. As shown in [26], local optimality can be recovered whenever the offset cost function \( V(o, c) \) satisfies

\[
\alpha_1 \| \tilde{y}_k' - y_k' \|_1 \leq V_o(\tilde{y}_k', y_k') \leq \alpha_2 \| \tilde{y}_k' - y_k' \|_1,
\]

for any \( \alpha_1 \geq \alpha^* \) where \( \alpha_1, \alpha_2 \) are some positive constants and \( \alpha^* \) is the maximum Lagrange multiplier for equality constraint \( \| y_k' - y_k' \|_1 = 0 \) over the set of all optimization parameters that lead to a reachable target. This condition directly follows from the exact penalty function method applied to the offset cost.

4.2. **Calculating the terminal set for tracking**

The choice of the terminal set \( \Omega^T_k \) is important for ensuring recursive feasibility of the optimization problem (23), that is, ensuring \( x_{k+1} \in \mathcal{X} \) for all \( x_k \in \mathcal{X} \), \( d_k \in \mathbb{D} \), \( w_{k+1} \in \mathbb{W} \), and \( y_k' \). A necessary condition for this requirement is

\[
(x_{k+1}, d_k) \in \Omega^T_k \Rightarrow (x_{k+1}, d_k) \in \Omega^T_k, \quad \forall d_k \in \mathbb{D}, w_{k+1} \in \mathbb{W}.
\]

As shown in Appendix A, the nominal predictions at time \( k+1 \) are related to those at time \( k \) according to

\[
\begin{pmatrix}
\tilde{x}_{N+k+1} \\
\tilde{d}_{k+1}
\end{pmatrix} =
\begin{pmatrix}
\Phi & BL \\
0 & I
\end{pmatrix}
\begin{pmatrix}
\tilde{x}_N \\
\tilde{d}_k
\end{pmatrix} +
\begin{pmatrix}
\Phi^N \\
0
\end{pmatrix} d_k +
\begin{pmatrix}
\Phi^N B_w \\
0
\end{pmatrix} w_{k+1}.
\]

(27)

Hence, \( \Omega^T_k \) should be chosen as any admissible RPI set for this system that satisfies tightened state \( \tilde{x}_{N+k} \in \mathcal{X} \) and input constraints \( K\tilde{x}_{N+k} + L\tilde{d}_k \in \mathcal{U} \). More precisely, we assume

**Assumption 2.** The terminal set satisfies \( \Omega^T_k \subseteq \{(x, \theta) : x \in \mathcal{X}, Kx + L\theta \in \mathcal{U}\} \) and

\[
\begin{pmatrix}
\Phi & BL \\
0 & I
\end{pmatrix} \Omega^T_k + \begin{pmatrix}
\Phi^N \\
0
\end{pmatrix} \mathbb{D} + \begin{pmatrix}
\Phi^N B_w \\
0
\end{pmatrix} \mathbb{W} \subseteq \Omega^T_{k+1}.
\]

Many different sets can satisfy this assumption. The largest one, termed the maximal RPI (MRPI) set, is often the most convenient to select as it provides the largest possible \( \mathcal{X} \).

The MRPI \( \Omega^q_{\infty} \) is defined as the infinite set of constraints corresponding to the system evolution throughout Mode 2, that is,

\[
(\tilde{x}_{N+k}, \tilde{u}_{N+k}) \in \tilde{X}_{N+k} \times \tilde{U}_{N+k} \quad \forall l \geq 0.
\]

Since the nominal system evolves autonomously in Mode 2, this collection of constraints can be written as

\[
\Omega^q_{\infty} = \{ x : A_k x_k \in \tilde{X}^q_{N+1}, \quad \forall i \geq 0 \},
\]

(28)

where \( A_k = \begin{pmatrix}
\Phi & BL \\
0 & I
\end{pmatrix} \) and

\[
\tilde{X}^q_{N+1} = \{ x = (x, \theta) : x \in \tilde{X}_{N+i}, \quad Kx + L\theta \in \tilde{U}_{N+i+1} \}.
\]

(29)

Algorithms for constructing MRPI sets have been explored in [27]. In order for \( \Omega^q_{\infty} \) to be represented exactly, it must be finitely determined, i.e., a finite integer \( i_0 \) exists such that \( \Omega^q_{i_0} = \{ A_k^i x_k \in \tilde{X}^q_{N+i}, \quad i = 0, \ldots, i_0 \} = \Omega^q_{\infty} \). Since \( A_k \) has unitary eigenvalues, this condition may not be satisfied and \( \Omega^q_{i_0} \) may not be finitely determined. By modifying the constraints

\[
\tilde{X}^q_{N+i} \cap \{ z = (x, \theta) : M_k \theta \in \lambda (\tilde{X}_N \times \tilde{U}_N) \},
\]

the new MRPI can be guaranteed to be finitely determined and arbitrarily close to the \( \Omega^q_{\infty} \) as \( \lambda \rightarrow 1 \) [17].

As noted in [28], the MRPI set (or tight polyhedral approximations) can only be computed exactly for small systems of approximately 6–7 dimensions. Ellipsoidal approximations represent the better if not only choice in higher dimensions even though they may be conservative in this range, as they only require a simple linear program (LP) to be solved [28]. A key advantage of ellipsoidal invariant sets, however, is the fact that the number of constraints introduced is fixed, whereas in the polytopic case, the calculated polytopes may add a large number of constraints, leading to slower computation times and excessive memory requirements. It is also important to note that the crucial property of recursive feasibility is not affected under these approximations in this work since the terminal set does not have to equal the MRPI set. From the terminal set, we can characterize the set of all nominal targets that are reachable by the proposed controller

\[
\tilde{y}_k = (y = N_k \tilde{\theta} : (M_k \tilde{\theta}, \tilde{\theta}) \in \Omega^T_k),
\]

(30)

where \( M_k = [I_n, 0]M_k \).
5. Stability and convergence

In this section, the recursive feasibility, stability, and convergence properties of the proposed stochastic-tube MPC strategy for offset-free tracking (24) are rigorously presented. The controller is based on the solution of the optimization problem $P_N(x_k, y_k^1)$, which has a number of ingredients including weight matrix $\Psi$, horizon $N$, offset cost $V_\infty$, tightened constraints $X_{k+1}^1$, $\bar{U}$, terminal controller gain $K$, and terminal constraint set $\mathcal{C}_k^\infty$. These ingredients should satisfy the following conditions in order to give suitable properties to the controller: 

**Assumption 3:**

(i) $\Psi > 0$ is a symmetric positive definite matrix.
(ii) The eigenvalues of $\Phi = A + BK$ are inside the unit disc.
(iii) The sets $X_{k+1}, \bar{U}_{k+1}$ and $\mathcal{C}_k^\infty$ are non-empty.
(iv) The offset cost $V_\infty : \mathcal{R}^p \rightarrow \mathcal{R}$ is a convex, positive definite, and sub-differential function with $V_\infty(0) = 0$.

These conditions are an extension of standard stabilizing conditions in linear MPC [29]. Whenever these conditions are satisfied, we can formulate the following theorem.

**Theorem 1.** Consider that Assumptions 1–3 hold and the target $y_k^1$ is asymptotically constant. Then, the system (1) under control law (24), i.e.,

$$x_{k+1} = Ax_k + B_{k,N}(x_k, y_k^1) + d_k + B_ww_{k+1},$$

is, for all initial conditions $x_0 \in X_0$ and for every target $y_k^1$, the evolution of the system is robustly feasible, i.e., $x_k \in X_k, \forall k \geq 1$ and satisfies state change constraints (4) and hard input constraints (5) for all admissible disturbances.

(i) For all initial conditions $x_0 \in X_0$ and for every target $y_k^1$, the evolution of the system is robustly feasible, i.e., $x_k \in X_k, \forall k \geq 1$ and satisfies state change constraints (4) and hard input constraints (5) for all admissible disturbances.

(ii) $\lim_{k \rightarrow \infty} \mathbb{E} \|x_k\|_Q = 0$.

(iii) If the asymptotic target is reachable $y_k^* \in Y_k$, then the controlled variable converges (with probability one) to the set $(y_k^*) \oplus (C + DK)\mathcal{R}_\infty$ where $\mathcal{R}_\infty = R_\infty \oplus R_\infty$.

(iv) If $y_k^* \notin Y_k$, then the controlled variable converges (with probability one) to the set $(y_k^*) \oplus (C + DK)\mathcal{R}_\infty$ where

$$\hat{y}_k = \arg\min_{y \in Y_k} V_\infty(y^1 - y_k^1),$$

is the reachable nominal steady state controlled variable that minimizes the offset cost.

**Proof.** See Appendix A for proof. □

We emphasize that Theorem 1 holds for any choice of $K$ that ensures $\Phi = A + BK$ is strictly stable. Due to the fact that $\lim_{k \rightarrow \infty} c_k = 0$, the proposed controller always converges to the terminal control law, which should be chosen to “optimize” the behavior of the process in the absence of constraints. For example, it may be beneficial to choose $K$ to reduce the size of $R_\infty$. Thus, these results help guide the user on how to best tune the controller for any specific application.

There are a number of important properties that should be highlighted regarding $P_N(x_k, y_k^1)$ in addition to the controller (24) retaining feasibility and convergence under target switches. Under Assumptions 1–3, the optimization (23) is guaranteed to be convex, implying that readily available algorithms can be used to solve $P_N(x_k, y_k^1)$ to its global optimum. In addition, the complexity of (23) is nearly identical to the corresponding nominal MPC problem as only $m$ additional decision variables are included. Whenever $V_\infty$ is defined in terms of a 1-, 2-, or $\infty$-norm, then $P_N(x_k, y_k^1)$ is a convex quadratic program (QP). As such, well-known multiparametric QP methods can be applied in order to compute an explicit solution to $P_N(x_k, y_k^1)$ completely offline [30]. Explicit MPC methods are known to be useful for systems with fast sampling times as a way to avoid the online cost of the optimization [31].

**Remark 7.** A common stability condition of interest in stochastic systems is mean-square boundedness of the closed-loop system, that is, $\sup_{k \geq 0} \mathbb{E}_0 \|x_k\|^2 < \infty$. The system is mean-square bounded as a direct result of Theorem 1. Furthermore, the following quadratic stability condition

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{k=0}^{\infty} \mathbb{E} \left\{ \|x_k - \bar{x}_k\|^2_Q + \|u_k - \bar{u}_k\|^2_R \right\} \leq \mathbb{E} \left\{ \|W_k\|^2_P \right\},$$

can be derived by following the same steps as in [14], where $P$ is the solution to the Lyapunov equation $P - \Phi^T P \Phi = Q + K^T R K$ for any $Q \geq 0$ and $R > 0$.

6. Offset-free tracking and enlarging domain of attraction of controller

In this section, we present two extensions to the proposed stochastic-tube MPC strategy for tracking, which are particularly important for practical applications. The first represents an algorithm for minimizing the tracking offset when the deterministic disturbances tend to a constant value. The second extension represents a novel method to further reduce conservatism of the controller by bounding the variability of the deterministic disturbance. This concept, based on an updated model for the disturbance, can easily be applied to any tube-based MPC method.
6.1. Offset removal via disturbance estimation

If the deterministic disturbances asymptotically converge to a constant value, i.e., \( \lim d_k = d_\infty \), which is commonly the case as \( d_k \) represents plant-model mismatch then the output of the closed-loop system will be offset from the desired asymptotic value of the target \( y_c^\infty \). This is a direct result of the fact that the setpoint is designed based on the nominal system (7).

Since the stochastic disturbances persistently excite the system, we are unable to ensure the system converges to any single point. However, we can remove offset with respect to the mean value of the process, that is, the outputs of the closed-loop system “oscillate” around a neighborhood of \( y_c^\infty \) (with bounded variance) for any reachable target. The basic idea is to modify the target, before passing it to the controller (24), using a current best estimate of the deterministic disturbance. These statements are made more precise in the following corollary of Theorem 1.

**Corollary 1.** Let the conditions of Theorem 1 hold, \( \hat{y}_k = \mathbb{E}_0[y_k] \) be the mean value of the controlled variables, \( d_k \) be asymptotically constant, and \( \hat{d}_k \) be the current estimate of the true disturbance \( d_k \). Assume that \( \hat{d}_k \) converges to \( d_\infty \). Then, the closed-loop system

\[
\begin{align*}
x_{k+1} &= Ax_k + Bu_k + d_k + B_ww_{k+1}, \\
u_k &= \kappa_N(x_k, \hat{y}_k^f), \\
\hat{y}_k^n &= y_k^n - H_d\hat{d}_k,
\end{align*}
\]

with \( H_d = (C + DK)(I - \Phi)^{-1} \) will necessarily satisfy:

(i) The output converges to \( \{\hat{y}_\infty\} \subset R^L \) where \( \hat{y}_\infty \) satisfies
\[
\hat{y}_\infty = \lim_{k \to \infty} \mathbb{E}_0[y_k] = \arg\min_{\hat{y}} \mathbb{E}_0 V_U(\hat{y}^f - y_k^n + H_d d_\infty),
\]

meaning that the mean of the controlled variables asymptotically converges to the desired target without offset whenever the limiting value of the modified target is reachable, i.e., \( \hat{y}_\infty = y_c^\infty - H_d d_\infty \in \hat{S}_i \).

(ii) The output covariance converges to
\[
\lim_{k \to \infty} \mathbb{E}_0[(y_k - \hat{y}_k)(y_k - \hat{y}_k)^\top] = (C + DK)\Sigma(C + DK)^\top,
\]

where \( \Sigma \) satisfies \( \Sigma = \Phi \Sigma \Phi^\top + B_w \mathbb{E}_0[W_k W_k^\top] B_w^\top \) and represents the limiting value of the state covariance.

**Proof.** See Appendix B for proof. □

The disturbance estimate \( \hat{d}_k \) can be determined in a number of ways including a Luenberger observer or Kalman filter. Since the measured state includes information about two separate uncertainty sources, we need to make the following assumption to ensure convergence of the disturbance estimate:

**Assumption 4.** \( w_k \) is measured at all \( k \geq 0 \).

This assumption is reasonable because the distribution of \( w_k \) is needed for the proposed stochastic-tube MPC strategy, which is only straightforward to obtain whenever \( w_k \) is measured. Thus, \( w_k \) is treated as a measured disturbance whose future evolution is unknown and random. We note that this assumption is not restrictive as any unmeasured disturbances can always be lumped into the description of \( d_k \), which is a benefit of the proposed approach of including two distinct uncertainty sources.

When both the state and stochastic disturbance are measured, a simple filter of the form
\[
\hat{d}_k = \lambda_x \hat{d}_{k-1} + (1 - \lambda_x)(x_k - Ax_{k-1} - Bu_{k-1} - w_k),
\]

will converge to the true constant disturbance value for any filter constant \( \lambda_x \in (0, 1) \). The estimator dynamic essentially creates an outer feedback loop to the controller. Since the proposed controller (24) is feasible for all targets, the variability in \( \hat{y}_k^f \) will not affect the feasibility of the optimization. In addition, the estimator can be designed independently of the MPC controller to ensure \( \hat{d}_k \to d_\infty \) which, in effect, guarantees the entire system is stable according to Corollary 1.

Note that, due to the removal of the deterministic disturbance, the output converges to a smaller neighborhood around the target that is a function of \( w \) only. Also, note that we provide an expression for the covariance of the output (34) that is an explicit function of the terminal controller \( K \) and the stochastic disturbance covariance \( \mathbb{E}_0[W_k W_k^\top] \).

6.2. Enlarging domain of attraction by bounding the change in plant-model mismatch

In the previous sections, we bounded the predictions of the deterministic disturbances as \( d_{ik} \in D \) for all \( i \geq 0 \). Clearly, we need this type of assumption in order to guarantee robust constraint satisfaction. However, allowing \( d_{ik} \) to vary arbitrarily quickly in \( D \) might be unrealistic (and overly conservative) as the true disturbance will have some associated dynamic in many applications. This is especially true whenever \( d_k \) represents model uncertainty.

Using measurements of \( x_k \) and \( w_k \), we are able to obtain information about the disturbance one step in the past, i.e.,
\[
d_{k-1} = x_k - Ax_{k-1} - Bu_{k-1} - B_w w_k,
\]
is known at any time \( k \geq 1 \). This can be combined with a supplementary model of the disturbance to reduce error in the predictions \( d_{ik} \). We propose that the following simple/flexible model be used for this task
\[
d_k = A_d d_{k-1} + \delta_d,
\]
where $A_d \in \mathbb{R}^{n \times n}$ is a fixed matrix and $\delta d_{k+1} \in \Delta_d \subset \mathbb{R}^n$ is some bounded perturbation. There will always exist a bounded signal $\{\delta d_k\}_{k=0}^\infty$ satisfying this relationship for any stable $A_d$ matrix.

For the case that $\Delta_d = \{0\}$, $d_k$ is a feedforward disturbance to the controller. Whenever $A_d = 0$ and $\Delta_d = \mathbb{D}$, we recover the case that $d_k$ is uncorrelated in time and can vary arbitrarily in $\mathbb{D}$. Hence, this model essentially bridges the gap between perfect knowledge and no knowledge of $d_k$’s variability in time. This added flexibility is important since, in reality, the real situation will often lie in between these two limiting cases.

The matrix $A_d$ represents a tuning parameter for the proposed disturbance model that should be chosen based on experience with the real plant. The main constraint is that $A_d$ should have eigenvalues inside the unit circle to ensure that the predictions are bounded at all times. Effectively, the eigenvalues of $A_d$ specify the “memory” of the disturbance, with larger eigenvalues implying more memory/correlation. For any chosen $A_d$, the bound $\Delta_d$ can be directly obtained from data or a high-fidelity model. The key operations are as follows:

(i) calculate $d_0$ according to (36) where the successor state $x_{k+1}$ and current state $x_k$ are determined from data or a high-fidelity model;
(ii) calculate $\delta d_k = d_k - A_d d_{k-1}$;
(iii) increment $k \leftarrow k + 1$ and repeat steps (i) and (ii) until all data has been processed;
(iv) construct $\Delta_d$ from the min and max values of $\{\delta d_k\}_{k=0}^\infty$.

Clearly, not all choices of $A_d$ will yield an improved model of the disturbance. We can define an additional constraint that the pair $(A_d, \Delta_d)$ must satisfy to guarantee a reduction in conservatism based on the limiting behavior of (37). The limiting behavior is defined in terms of the mRPI set for (37), which is denoted by $R_d^\infty$. Therefore, $(A_d, \Delta_d)$ should also satisfy

$$d_0 \in R_d^\infty \subseteq \mathbb{D},$$

(38)

to ensure a reduction in the prediction error of the deterministic disturbance. Assuming that a pair $(A_d, \Delta_d)$ can be selected a priori to satisfy these conditions is reasonable in the sense that, if these cannot be estimated for a given model, then the model is significantly wrong and it is of little practical use. In this particular case, a new identification procedure should be performed to update the model description.

The new system description, based on (1) and (37), can be written in terms of an augmented state space of the form:

$$\begin{bmatrix} x_{k+1} \\ d_k \\ \end{bmatrix} = \begin{bmatrix} A & A_d \\ 0 & A_d \\ \end{bmatrix} \begin{bmatrix} x_k \\ d_{k-1} \\ \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ \end{bmatrix} u_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \delta d_k + \begin{bmatrix} B_w \\ 0 \end{bmatrix} w_{k+1},$$

(39a)

$$y_k = \begin{bmatrix} C \\ 0 \end{bmatrix} \begin{bmatrix} x_k \\ d_{k-1} \end{bmatrix} + Du_k.$$  

(39b)

Based on this observation, the proposed stochastic-tube MPC strategy (23) can be directly applied to the augmented system (39) by making the appropriate substitutions:

$$X_k \leftarrow \begin{bmatrix} x_k \\ d_{k-1} \end{bmatrix}, \quad A \leftarrow \begin{bmatrix} A & A_d \\ 0 & A_d \end{bmatrix}, \quad B \leftarrow \begin{bmatrix} B \\ 0 \end{bmatrix},$$

$$B_w \leftarrow \begin{bmatrix} B_w \\ 0 \end{bmatrix}, \quad C \leftarrow \begin{bmatrix} C \\ 0 \end{bmatrix}, \quad D \leftarrow \begin{bmatrix} I \\ I \end{bmatrix} \Delta_d, \quad H \leftarrow \begin{bmatrix} H \\ 0 \end{bmatrix}. $$

It is interesting to note that the gain $K$ in this case can be designed with respect to the augmented system, which is based on the chosen disturbance model and includes feedback from $d_{k-1}$. In addition, the complexity of (23) does not increase since the disturbance states are not affected by the control input due to the structure of $B$ so that they do not need to be included as decision variables in the optimization. Note that, in this case, the controllability assumption on the updated $(A, B)$ matrices will not be satisfied as the input does not affect the deterministic disturbance dynamic. However, since $A_d$ is stable, the augmented system will necessarily be stabilizable. As a result, this does not change any of the results shown in the paper.

We demonstrate the advantages of this proposed method for handling the disturbance in simulations in Section 7. We also illustrate how this type of disturbance model could be obtained from data on an industrially-relevant process.

### 7. Case studies

#### 7.1. Benchmark problem: DC-DC converter

In the following, the advantages of the proposed stochastic-tube MPC strategy (including an enlarged domain of attraction, offset free tracking, and non-conservative handling of chance constraints) are demonstrated. The chosen example is based on the DC–DC converter, which is a commonly used problem in the SMPC literature [13,32]. The system can be described as a linear state space model of the form (1) with:

$$A = \begin{bmatrix} 1.0 & 0.0075 \\ -0.143 & 0.996 \end{bmatrix}, \quad B = \begin{bmatrix} 4.798 \\ 0.115 \end{bmatrix}, \quad B_w = I, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0.$$  

Two different examples of $d_k$ are explored below. First, the case of an asymptotically constant disturbance that persists throughout the simulation. Second, the case that $d_k$ represents plant-model mismatch in the form of model uncertainty in the $A$ and $B$ matrices. In both of these cases, the true expression for $d_k$ is unknown to the controller; however, it can be bounded in the set $\mathbb{D} = \{d \in \mathbb{R}^2 : \|d\|_\infty \leq 0.1\}$.
offline through comparisons of the model to data. The distribution $P_W$ of the stochastic disturbance $w_k$ is assumed to follow a Gaussian distribution with zero mean and covariance matrix $0.04^2 I$ with values truncated at $W = \{w \in \mathbb{R}^2 : \|w\|_\infty \leq 0.1\}$.

Four individual chance constraints of the form (4) are considered, i.e.,

$$P[|x_{k,1}| \leq 3] \geq 0.8, \quad P[|x_{k,2}| \leq 3] \geq 0.8, \quad P[-|x_{k,1}| \leq 3] \geq 0.8, \quad P[-|x_{k,2}| \leq 3] \geq 0.8,$$

while the hard input constraints are given by

$$U = \{u \in \mathbb{R} : \|U\|_\infty \leq 0.4\}.$$

The parameters in the stochastic-tube MPC problem were chosen to be $\Psi = 39.85$, $N = 5$, $K = [-0.286 0.491]$, and offset cost $V_0(\tilde{y}^d - y^f) = \|\tilde{y}^d - y^f\|_1^2$. Note that this choice of $\Psi$ and $K$ correspond to the infinite horizon quadratic cost with weight matrices $Q = \text{diag}(1, 10)$ and $R = 1$. The initial state in all simulations was chosen to be $x_0 = (-4.75, -3.6)$. All set calculations were performed using the MPT toolbox [33]. The time for computing the input using quadprog with a standard active set algorithm in Matlab 2008a was approximately 2 ms on a computer running Ubuntu with an Intel Core i7 processor, 8 GB of RAM, and 2.4 GHz with 4 cores.

**Persistence disturbance:** First, we consider the case that $d_k = (-0.1, -0.1)$ is a constant to demonstrate a few key properties of the proposed controller. Fig. 1 compares the closed-loop response of the proposed stochastic-tube MPC strategy to that of SMPC with recursively feasible probabilistic tubes [12]. As expected, the controller from [12] results in much more than 20% violation (as other disturbances have been neglected) and results in a substantial offset from the desired target of $y^f_k = 0$. By robustly accounting for other deterministic sources of uncertainty, the 20% allowed constraint violation can be guaranteed to be respected by the proposed controller. Additionally, the offset in the mean can be removed by estimating the true value of the persistent disturbance as highlighted in Corollary 1.

Moreover, the proposed controller can smoothly handle setpoint changes while ensuring constraint satisfaction and feasibility of the optimization $P_W(x_0, y^f_k)$ as shown in Fig. 2. On top of that, we can clearly see the system behavior converges to the mRPI set $R^e_k$ after every setpoint change, while the closed-loop mean (shown in blue), always converges exactly to the desired target. We also remark that the domain of attraction is enlarged with respect to [12] due to the larger terminal set for tracking (as opposed to regulation) that allows for piece-wise constant changes in the target value.

**Model uncertainty:** Now, the deterministic disturbance takes the form

$$d_k = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.02 \end{bmatrix} x_k + \begin{bmatrix} 0.01 \\ 0 \end{bmatrix} u_k,$$

which represents error in estimates of the A and B matrices in the model. Fig. 3a shows the closed-loop response of the proposed stochastic-tube MPC strategy compared to that of [12]. We again see substantially more violation is observed with [12]. This violation can be quantified using the empirical cdf of the process under these two controllers at $k = 1$ (Fig. 3b). When only model uncertainty is considered, we see that [12] results in $\geq 90\%$ violation while the proposed algorithm yields only $19\%$ violation, which is tight near the desired bound of $20\%$. The latter result is expected as the disturbance respects the a priori selected bounds as shown in Fig. 3c.

We can clearly see from Fig. 3c that $d_k$ has an associated dynamic, so that the signal cannot vary arbitrarily in $\mathbb{D}$. Motivated by this observation, we can apply the idea of introducing a new model for the disturbance as presented in Section 6. A comparison of the proposed controller with and without the improved disturbance model is shown in Fig. 4, which shows the probability that $|x_{k,1}|$ is greater than the active constraint, i.e., $P[|x_{k,1}| > 3]$ versus time. The goal is to be as close to the desired probability bound of $80\%$ as possible. As seen in Fig. 4, including the new disturbance model allows for the controller to push closer to the constraint by lowering the uncertainty in the predictions and, thus, directly reduces conservatism.

### 7.2. Industrial fluidized-bed catalytic cracking (FCC) unit

Fluidized-bed Catalytic Cracking (FCC) units are commonly explored in the robust MPC literature [34,35] due to its practical relevance in chemical process industry. This problem has already been treated in the context of additive disturbances [35] and multi-model description...
Fig. 2. Illustration of the key features of the proposed stochastic-tube MPC strategy for asymptotically constant disturbances. The large light gray set is the domain of attraction $\mathcal{X}_0$ for the proposed approach, the dashed black lines denote the state constraints, the large dark gray set is the nominal terminal tracking set $\mathcal{X}_0^T$, and the solid green line is the set of reachable nominal targets $\mathcal{S}_0$. The evolution of the mean is shown with a blue line that departs from an initial condition in the bottom left corner of $\mathcal{X}_0$ while the red lines show 10 example realizations of the closed-loop process. Three setpoint changes are shown with solid black lines with the system moving to 2, −2, and then 0. For each setpoint change, the system converges to the mRPI set $\mathcal{R}_0^T$ shown in dark gray, which are necessarily inside the combined mRPI for both disturbances $R_0$, shown in dark gray. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

[34]. In this case study, the additive disturbances are composed of plant-model mismatch, unmeasured constant disturbances and stochastic noise which naturally appears in real control problems.

The nominal continuous-time transfer function model is provided in [35] as follows:

$$G_0(s) = \begin{bmatrix} 0.2033 \\ 1.7187s + 1 \\ 0.1886s - 3.8087 \\ 17.7347s^2 + 10.8348s + 1 \end{bmatrix}.$$  (40)

In order to analyze the effect of structural uncertainty, it is assumed that the system is actually described by the following transfer function matrix provided in [34]:

$$G_r(s) = \begin{bmatrix} 0.135 \\ 2.77s + 1 \\ 0.1886s - 2.8 \\ 19.7347s^2 + 10.8348s + 1 \end{bmatrix}.$$  (41)

The following discrete-time realization with a sampling time of $T_s = 1$ is considered for simulation purposes

$$A_r = \begin{bmatrix} 0.6970 & 0 & 0 \\ 0 & 1.5387 & -0.5775 \\ 0 & 1.0000 & 0 \end{bmatrix}, \quad B_r = \begin{bmatrix} 0.2500 \\ 0.2500 \\ 0 \end{bmatrix}.$$  (42)

and

$$C_r = \begin{bmatrix} 0.1636 & 0 & 0 \\ 0 & -0.2081 & -0.2268 \end{bmatrix}.$$  (43)

Since the state vector of the minimal description is not measurable, a non-minimal state-space realization was used with $x_k = (y_k, y_{k-1}, u_{k-1})$. Further details regarding the derivation of the nominal matrices can be found in [36,37]. Moreover, $d_k = [d_{1,k}, d_{2,k}]^T$ due to the nature of the description. The tuning parameters were selected to be $N = 3, Q = 10I_{5,5},R = 1, V_0(\bar{y}_{2,k} - y_{2,t}) = 1000(\bar{y}_{2,k} - y_{2,t})^2$. Disturbances are assumed to be bounded by $||d_k||_\infty < 0.5$ and $||w_k||_\infty < 0.1$ where $w_k$ is assumed to be a zero mean Gaussian with covariance matrix $0.04I$ that is truncated at 0.1. The disturbance dynamics is defined by $[d_{1,k+1} = \lambda d_{1,k} + \lambda d_{2,k}, i = 1, 2$ with $\lambda = 0.4$. The input constraints are given by $|u|\leq 4$ while the following chance constraints are imposed on the outputs

$$P(|y_{k,1} \leq 5| \geq 0.8, \quad P(|y_{k,2} \leq 5| \geq 0.8, \quad P(|y_{k,3} \leq 5| \geq 0.8.$$  (44)

An unmeasured unit step disturbance is also added to the control input at time $140T_s$. 

Fig. 5 shows the setpoint tracking response for different targets with and without the augmented disturbance. The target 1.5 is not admissible with the standard approach, but this setpoint can be tracked if the augmented strategy is used. The same result is verified for the target −1.5 without the constant input disturbance. This result illustrates the benefits of the proposed approach whenever the target operating condition is near the operational constraints. As verified in Fig. 6, the control signal respects the constraint despite multiple sources of uncertainty. Also, note that the control signal gets closer to its upper limit after the setpoint change due to the fact that the proposed approach (with the augmented disturbance model) reduces the conservatism of the tightened mixed tube constraints.
Fig. 3. Comparison of closed-loop response under 1000 stochastic disturbance realizations for the proposed stochastic-tube MPC strategy (red) and SMPC with probabilistic tubes [12] (green) in the presence of model uncertainty: (a) phase plot where the two active state constraints are shown with black dashed lines; (b) empirical cdf at the first sampling time in the simulation, which shows the proposed approach tightly meets the maximum allowed constraint violation of 20% while [12] results in 90% violation; (c) actual value of $d_i$ in the simulations, which clearly respects the bounds shown with black dashed lines. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The disturbance evolution is depicted in Fig. 7, which shows that all of the assumed bounds are respected in the simulation. As previously discussed, the additive disturbances do vary arbitrarily due to the fact that the plant-model mismatch is associated with a smooth dynamic. This kind of smooth disturbance evolution is expected in several types of real problems.

Fig. 8 further illustrates the key advantage of the proposed controller by plotting the one step ahead disturbance bound. The worst-case interval is depicted in red while the interval predicted based on the improved disturbance model is shown in blue. As expected, the blue
Fig. 4. Comparison of chance constraint violation versus time for the proposed stochastic-tube MPC strategy (in the case of model uncertainty) with and without the improved disturbance model, respectively, in blue and red. Including the disturbance model reduces the conservatism in handling the chance constraint as the proposed algorithm can move closer to the maximum allowed 20% violation. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 5. Comparison of the set-point tracking response for different targets with 100 disturbance realizations – stochastic-tube MPC strategy without augmented dynamic (red), stochastic-tube MPC strategy with augmented dynamic (blue), and target (green). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 6. Comparison of the control signal evolution with 100 disturbance realizations – stochastic-tube MPC strategy without augmented dynamic (red), and stochastic-tube MPC strategy with augmented dynamic (blue). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

region is significantly smaller than the red region indicating that the new disturbance model enlarges the feasible region of the controller and directly leads to improved control performance whenever constraints are active.
8. Conclusions and future work

This paper presents a stochastic-tube MPC strategy for offset-free tracking that systematically handles plant-model mismatch through simultaneous consideration of bounded deterministic and stochastic uncertainties. The deterministic uncertainty accounts for structural and/or parameter mismatch between the linear model of the controller and the true plant, while the stochastic uncertainty represents the intrinsic random variability of the system or exogenous disturbances that vary randomly with some known distribution. Constraint tightening is performed offline using the notion of mixed uncertainty tubes, which is a direct extension of probabilistic tubes to handle deterministic uncertainty. Recursive feasibility and stability is guaranteed using the concept of a terminal invariant set for tracking, while offset-free behavior (in the mean) is ensured using a disturbance estimator. Motivated by the fact that the plant-model mismatch error cannot vary arbitrarily quickly in most practical applications, a new filter model is presented for the deterministic disturbances in order to reduce the conservatism that is inherent to robust control. The basic notion of the filter model is to bound how quickly the plant-model mismatch can change, which can be estimated from data or a high-fidelity model. The simulation results indicate that the proposed stochastic-tube MPC strategy can fully guarantee chance constraint satisfaction even when the controller is designed with an incorrect model. In addition, it is shown that the filter model for the deterministic disturbances enables tracking operating points that are otherwise unreachable.

Future work will involve exploring new avenues for further enlarging the feasibility domain of the stochastic-tube MPC strategy. In addition, the control performance and constraint satisfaction can be possibly improved by using a closed-loop prediction strategy that is optimized online [38]. Lastly, demonstrating the benefits of SMPC (in terms of the trade-off between the control performance and constraint satisfaction) on real systems is an important area for future research, which critically hinges on accurate estimation of the uncertainty descriptions from system data and/or high-fidelity models.

Acknowledgments

T.L.M. Santos gratefully acknowledges the Brazilian Funding Agency CAPES for the financial support (grant number: 88881.119983/2016-01).
Appendix A. Proof of Theorem 1

The proof of Theorem 1 is presented in this section, which is divided into two parts corresponding to recursive feasibility and stability of the controller.

A.1 Proof of recursive feasibility

Recursive feasibility is demonstrated by the standard procedure of showing that, given a solution at time $k$, an explicit candidate solution will satisfy constraints at time $k + 1$.

Recall that the solution to $\mathcal{P}_N(x_k, y_k)$ at time $k$, corresponding to the optimal values of (23), are denoted by $c^*_k = (c^*_0, c^*_1, \ldots, c^*_{N-1|k})$ and $\tilde{\sigma}^{\star}_k$. The optimal nominal state profile is then given by

$$\tilde{x}^*_{0:k+1} = \Phi \tilde{x}^*_k + BL \tilde{\sigma}^*_k + Bc^*_k, \quad \tilde{x}^*_0 = x_k.$$  

Let $c^*_k = (c^*_0, \ldots, c^*_{N-1|k}, 0)$ and $\tilde{\sigma}^{\star}_k$ denote the candidate solution at time $k + 1$ and let $x^*_0, i \geq 0$ denote the corresponding nominal state predictions with $\tilde{x}^*_0 = x^*_0$.

Now, observe that $x^*_{k+1} = Ax_k + Bu_k + d_k + Bw_k$, and $u_k = \tilde{u}^*_0 = Kx_k + L\tilde{\sigma}^*_k + c^*_0$. Hence, $x^*_{k+1}$ is related to $\tilde{x}^*_1$ by

$$\tilde{x}^*_1 = Ax_k + Bu_k + d_k + Bw_k.$$  

By definition, $\tilde{x}^*_0 = \Phi \tilde{x}^*_k + BL \tilde{\sigma}^*_k + Bc^*_k$. Making the appropriate substitutions into this expression, we can derive

$$\tilde{x}^*_1 = \Phi (\tilde{x}^*_1 + d_k + Bw_k) + BL \tilde{\sigma}^*_k + Bc^*_k,$$

$$\tilde{x}^*_1 = \Phi \tilde{x}^*_k + BL \tilde{\sigma}^*_k + Bc^*_k + \Phi d_k + \Phi Bw_k.$$  

Thus, we can derive the following general relationship between the $i$ step ahead prediction at time $k + 1$, i.e., $\tilde{x}^*_0$ and the $i + 1$ prediction at time $k$, i.e., $\tilde{x}^*_0$ by induction

$$\tilde{x}^*_i = \tilde{x}^*_0 + \Phi^i d_k + \Phi^i Bw_k, \quad \forall i \in \mathbb{N}^i.$$  

Similarly, the candidate input $\tilde{u}^*_0 = Kx_k + L\tilde{\sigma}^*_k + c^*_0$ can then be rewritten in terms of the previous optimal input by

$$\tilde{u}^*_0 = \tilde{u}^*_0 + K\Phi^i d_k + K\Phi^i Bw_k, \quad \forall i \in \mathbb{N}^i.$$  

Based on the feasibility of the optimal sequence at time $k$, we can directly infer that

$$\tilde{x}^*_i \in \tilde{X}_i, \quad \forall i \in \mathbb{N}^i,$$

$$\tilde{u}^*_i \in \tilde{U}_i, \quad \forall i \in \mathbb{N}^i,$$

from the definition of the tightened state constraints in (21).

Next, we look to show that the tightened constraints are met at the end of the horizon. From the definition of the terminal set, we know that $\tilde{x}^*_N \in \tilde{X}_N$, which guarantees $\tilde{x}^*_N = \Phi^{N-1} d_k + \Phi^{N-1} Bw_k, \quad \forall i \in \mathbb{N}^i$ according to (21a). Similarly, the terminal set ensures that $Kx_N + L\tilde{\sigma}^*_k \in \tilde{U}_N$. From the chosen candidate inputs, we have

$$\tilde{u}^*_N = Kx_N + L\tilde{\sigma}^*_k + c^*_0,$$

$$\tilde{u}^*_N = (Kx_N + L\tilde{\sigma}^*_k + c^*_0).$$  

Therefore, $\tilde{u}^*_N \in \tilde{U}_N$ by the definition provided in (21b).

Lastly, we look to show that the terminal constraint is respected at time $k + 1$. To show this, observe that

$$\tilde{x}^*_N = \tilde{A} \tilde{x}^*_N + B \tilde{B} \tilde{C} + \tilde{B} \tilde{\sigma}^{\star}_k,$$

$$\tilde{x}^*_N = (A \tilde{x}^*_N + BL \tilde{\sigma}^*_k + B \tilde{C} + \tilde{B} \tilde{\sigma}^{\star}_k).$$  


From the RPI property of $\Omega_N^2$ in Assumption 2, we know that, since $(\bar{x}_N, \bar{\theta}_N) \in \Omega_N^2$, the successor state must remain in the terminal set, i.e., $(\bar{x}_{N+1}, \bar{\theta}_{N+1}) \in \Omega_N^2$. Since $\bar{\theta}_{N+1} = \bar{\theta}_N$, the candidate solution is necessarily feasible for all constraints in $P_N(x_k, y_k)$. Therefore, $x_k \in \lambda_N \Rightarrow x_{k+1} \in \lambda_N$ for all admissible disturbances, which completes the proof of (i).

A.2 Proof of stability and convergence

To demonstrate stability/convergence, we introduce the optimal cost function $J_k = V_N^*(x_k, y_k) = \sum_{i=0}^{N-1} c_i^* \| \phi \|_\Psi + V_N(\bar{\theta}_k - y_k)$. We also define the cost associated with the candidate control sequence $c_{k+1}$ and candidate artificial target $\tilde{\theta}_{k+1}$, i.e.,

$$V_N^*(x_{k+1}, y_{k+1}) = \sum_{i=0}^{N-1} \| c_{k+1} \|_\Psi + V_N(\bar{\theta}_{k+1} - y_{k+1}),$$

$$= \sum_{i=0}^{N-1} \| c_{k+1} \|_\Psi + V_N(\bar{\theta}_{k+1} - y_{k+1}),$$

$$= J_k - \| c_{k+1} \|_\Psi.$$ 

Since this is only a candidate solution, the optimal solution may only yield a smaller cost, i.e., $J_{k+1} \leq V_N^*(x_{k+1}, y_{k+1})$ according to the optimality principle. Thus, it is clear that the difference in the optimal cost must satisfy

$$J_{k+1} - J_k \leq -\| c_{k+1} \|_\Psi.$$ 

Hence, $(J_k)_{k=0}^\infty$ is a nonnegative monotonically nonincreasing scalar sequence and, as $k \to \infty$, must converge to $J_\infty < \infty$. Summing this difference for $k$ from 0 to $\infty$, we have

$$\infty > J_0 - J_\infty \geq \sum_{k=0}^{\infty} \| c_{k+1} \|_\Psi \geq 0 \Rightarrow \lim_{k \to \infty} c_{k+1} = 0,$$

which proves (ii) since $\Psi > 0$.

Let $(\bar{x}_k^*, \bar{\theta}_k^*) = M_k \tilde{\theta}_k$ be the optimal steady state and input. Then, $u_k = K(x_k - \bar{x}_k^*) + \bar{u}_k^* + c_{0/k}^*$. We also define the deviation variables $\bar{x}_k = x_k - \bar{x}_k^*$ and $\bar{u}_k = u_k - \bar{u}_k^*$. From the system dynamics (1) and steady state condition (8), we can derive

$$\bar{x}_{k+1} = A \bar{x}_k + B \bar{u}_k + d_k + B_w w_{k+1},$$

$$= \Phi \bar{x}_k + B \bar{c}_{0/k} + d_k + B_w w_{k+1}.$$ 

Furthermore, due to the superposition principle, we have

$$\lim_{k \to \infty} \bar{x}_k = \lim_{k \to \infty} \left[ \Phi \bar{x}_0 + \sum_{j=1}^{k} \Phi^{j-1} B \bar{c}_{0/j} + \sum_{j=1}^{k} \Phi^{j-1} (d_{k-j} + B_w w_{k+1-j}) \right]$$

$$= \lim_{k \to \infty} \left[ \sum_{j=1}^{k} \Phi^{j-1} d_{k-j} \right] + \lim_{k \to \infty} \left[ \sum_{j=1}^{k} B_w w_{k+1-j} \right].$$ 

Then, from the definition of the reachable sets, we have

$$\lim_{k \to \infty} x_k \in \lim_{k \to \infty} (\bar{x}_k^*) + R_\infty^c \cup R_\infty^c,$$

$$\lim_{k \to \infty} u_k \in \lim_{k \to \infty} (\bar{u}_k^*) + KR_\infty^c \cup KR_\infty^c.$$ 

The last result of interest is the convergence of the artificial target. As pointed out in [35], $\tilde{\theta}_k \to \tilde{\theta}$ where $\tilde{\theta} \in \tilde{\Theta}$ minimizes the offset cost $V_0(\bar{\theta}_0 - y_{\infty})$. A detailed proof of this is shown in [17], but a simplified argument is made here in light of the chosen cost function.

Since $u_k$ converges to the desired control law inside of the terminal set, the perturbations $c_{0/k} = 0$, $i \in N_{N-1}$ are feasible as $k \to \infty$ for any feasible artificial target. Hence, the limiting behavior of the two parts of the cost function (22) can be chosen independently. Since $c_{0/k} = 0$ is the global optimal for the first part of the cost, we must have $\lim_{k \to \infty} \tilde{\theta} = \tilde{\theta}$. Any other choice would contradict the optimality of the cost function. This, in turn, proves (iv) since $y_k = Cx_k + Du_k$. Then, (iii) follows directly from the fact that the offset cost can be set to zero (i.e., global optimum) by choosing $\bar{y}_k = y_{\infty}^c$ whenever the target is reachable, which concludes the proof.

Appendix B. Proof of Corollary 1

The proof of Corollary 1 is provided in this section, which consists of two main claims. First, offset-free convergence of the mean of the controlled variables to the target under certain conditions. Second, an exact expression for the limiting behavior of the process covariance.
Whenever $d_k$ is asymptotically constant, the converged mean of the closed-loop system must satisfy

$$
E(x_{\infty}) = AE(x_{\infty}) + BE(u_{\infty}) + d_{\infty},
$$

$$
E(y_{\infty}) = CE(x_{\infty}) + DE(u_{\infty}),
$$

since the expectation is a linear operator. From the convergence proved in Theorem 1, the mean of the control law must satisfy

$$
E(u_{\infty}) = u^* + K(E(x_{\infty}) - \hat{x}^*),
$$

where $(\hat{x}^*, \hat{u}^*) = M_0\theta^*$ and $\hat{y}^* = N_0\theta^*$ represent the converged values of the nominal steady state, i.e.,

$$
\hat{x}^* = A\hat{x}^* + B\hat{u}^*,
$$

$$
\hat{y}^* = C\hat{x}^* + D\hat{u}^*.
$$

Subtracting the process mean values, i.e., $E(x_{\infty})$ and $E(y_{\infty})$ from their converged targets, i.e., $\hat{x}^*$ and $\hat{y}^*$, then substituting the expected value of the control law gives

$$
E(x_{\infty}) - \hat{x}^* = \Phi(E(x_{\infty}) - \hat{x}^*) + d_{\infty},
$$

$$
E(y_{\infty}) - \hat{y}^* = (C + DK)(E(x_{\infty}) - \hat{x}^*).
$$

Since $\Phi$ is stable by assumption, $(I - \Phi)^{-1}$ must be invertible, and consequently $E(x_{\infty}) - \hat{x}^* = (I - \Phi)^{-1}d_{\infty}$. Hence, the offset between the mean of the controlled variables and the converged steady state is exactly

$$
E(y_{\infty}) = \hat{y}^* + H_d d_{\infty}.
$$

Hence, (i) follows directly from Theorem 1 and the definition of the target in this case. Note that the offset will be zero whenever $\hat{y}^* = y^* = H_d d_{\infty} \in \bar{Y}_T$ and the system converges to the PPI set $R_k^*$ as the deterministic disturbance is asymptotically constant.

Now, let $\Sigma_k$ denote the covariance of the close-loop states at time $k$. From (1), we can derive an expression for the dynamics of the covariance

$$
\Sigma_{k+1} = A\Sigma_k A^T + AE(\delta x_k \delta u_k^T)B^T + BE(\delta u_k \delta u_k^T)A^T + BE(\delta u_k \delta u_k^T)B^T
$$

$$
+ B_W E(W_{k+1}W_{k+1}^T)B_{u_k}^T
$$

where $\Sigma_k = E(\delta x_k \delta x_k^T)$, $\delta x_k = x_k - E(x_k)$, and $\delta u_k = v_k - E(u_k)$. Hence, taking the limit as $k \to \infty$ and substituting $\delta u_k \to K\delta x_k$ (as proved in Theorem 1) gives

$$
\Sigma_\infty = A\Sigma_\infty A^T + A\Sigma_\infty K^T B^T + B K \Sigma_\infty A^T + B K \Sigma_\infty K^T B^T + B_{u} E(W_{\infty} W_{\infty}^T)B_{u_k}^T
$$

$$
+ \Phi \Sigma_\infty \Phi^T + B_{u_k} E(W_{\infty} W_{\infty}^T)R_{\infty}^T.
$$

that, in turn, proves (ii), which concludes the proof.

References