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Imprecise Epistemic Values and Imprecise Credences

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ABSTRACT

A number of recent arguments purport to show that imprecise credences are incompatible with accuracy-first epistemology. If correct, this conclusion suggests a conflict between evidential and alethic epistemic norms. In the first part of the paper, I claim that these arguments fail if we understand imprecise credences as indeterminate credences. In the second part, I explore why agents with entirely alethic epistemic values can end up in an indeterminate credal state. Following William James, I argue that there are many distinct alethic values that a rational agent can have. Furthermore, such an agent is rationally permitted not to have settled on one fully precise value function. This indeterminacy in value will sometimes result in indeterminacy in epistemic behaviour—that is, because the agent’s values aren’t settled, what she believes might not be.

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1. Introduction

Here are two hard questions in epistemology:

i. What is the relationship between alethic and evidential norms?

ii. How should an agent’s epistemic values affect her epistemic behaviour?

The first question concerns the relationship between evidence and the truth. Following your evidence is generally a guide to the truth, but truth and evidence can come apart. Can the goal of truth ever conflict with the goal of following one’s evidence?

The second question concerns the relationship between what an epistemic agent cares about and what she should believe. In the practical case, rational agents who like vanilla will behave differently from agents who prefer chocolate. Epistemic agents also have different epistemic values—some care about truth, while others care about explanation or justification. Should such differences lead to differences in what they think, even when they have the same evidence?

I don’t pretend to have full answers to these questions. However, there has recently been an alleged conflict between two major programmes in formal epistemology whose resolution will, I think, shed some light on both.
The first is the Imprecise Credences programme, which is motivated primarily by evidential considerations. According to orthodox Bayesianism, rational agents should assign a precise level of confidence—represented as a single number—to each proposition under consideration. This requirement, as we’ll see below, seems like an undue restriction on doxastic states that prevents agents from responding reasonably to some kinds of evidence.

An alternative model allows for imprecise credences. Instead of using a single probability function to represent a doxastic state, we use a set of them. An agent’s confidence in some proposition can be represented by, say, the interval \([.2, .3]\). Here, she’s between 20% and 30% confident, but there’s no number \(x\) such that she’s exactly \(x\)% confident.

Despite their evidential advantages, imprecise credences appear incompatible with a second programme, called accuracy-first epistemology, which is motivated by alethic considerations. Accuracy-firsters think that the only thing of final epistemic value is accuracy—proximity between doxastic attitude and truth-value. On the precise model, it’s relatively easy to cash out what this means: if \(p\) is true (false), then the closer your credence is to 1 (0), the more accurate it is.

It’s less clear how to understand what accuracy could amount to when we move to an imprecise model. In fact, a number of philosophers have argued that any attempt to measure the accuracy of imprecise credences will render these idle and unmotivated at best from an alethic point of view.

So we’re in a quandary. If you like accuracy-first epistemology, it appears that you have no way of rationalizing an imprecise-credence model. Given the attractiveness of imprecise credences from an evidential perspective, this is a cost. Conversely, if you favour imprecise credences, it appears that you must deny that they have anything to do with pursuit of the truth. The evidential norms that you favour are, by your lights, disconnected from the aim of fitting your doxastic state to the world.

The first goal of this paper is to propose one way to resolve that tension. The second goal is to explore how purely alethic epistemic value can affect epistemic behaviour. The basic picture goes as follows. Even if all that a rational epistemic agent should care about is accuracy, there are different reasonable ways of precisifying and pursuing accuracy. Rationality doesn’t force agents to choose any precise notion of accuracy or any precise strategy for achieving accuracy. Rational agents are therefore permitted to have indeterminate yet entirely alethic epistemic values.

Indeterminate values lead to indeterminate credences, which we can identify with imprecise credences. On this interpretation, when we say that an agent has imprecise credence \([.2, .3]\) toward some proposition \(X\), we don’t mean that her credence is literally the interval \([.2, .3]\). Instead, there’s no fact of the matter as to whether her credence is really .22, .29, or any other element of \([.2, .3]\). This understanding of imprecise credences both escapes the incompatibility arguments and provides a positive account of how they fit with accuracy-first epistemology.

To be clear, the aim isn’t to propose a knock-down argument for imprecise credences or for interpreting them as indeterminate. Instead, the goal is to present a well-motivated picture of epistemic value that renders accuracy-first epistemology and imprecise credences compatible and that allows epistemic value to affect epistemic behaviour.

1 This account is developed in Rinard [2015].
Here’s the plan. **Section 2** presents imprecise credences, accuracy-first epistemology, and the argument for their incompatibility. **Section 3** motivates the indeterminate interpretation and shows why such an interpretation evades the incompatibility arguments. **Section 4** explores the nature of alethic epistemic value and demonstrates that indeterminate values lead to indeterminate credences. **Section 5** contrasts our account of imprecise credences with some alternatives. First, we show that on our view agents with imprecise credences need not update by pointwise conditionalisation. Second, we argue that the imprecise view has some advantages over permissive Bayesianism. **Section 6** wraps up.

### 2. Imprecise Bayesianism and Accuracy-First Epistemology

#### 2.1. Motivating Imprecise Credences

We often have evidence that is incomplete and non-specific. Consider the claims that the person next to you has at least three cans of garbanzo beans in her cupboard, that Greece will leave the Euro by 2030, or that Homer was a woman. Assigning a precise credence to these claims commits you to more, you might think, than the evidence supports.

To bring this out, we’ll use the following tractable example as our paradigm case.

**MYSTERY COIN.** The only evidence you have that is relevant to whether Heads is that the objective chance of Heads is between 0.05 and 0.95.

Imprecise Bayesians argue that, at the least, the nature of the evidence permits the agent not to have a completely precise attitude toward Heads. Her evidence is unspecific and incomplete, and her opinion can be, too.

Joyce [2010: 283] notes that in this case a uniform prior commits you to expecting that, in 100 tosses, the chance the coin will land heads fewer than 17 times is around 0.118, which is just a little more probable than drawing a red Queen, King, or Ace from a deck.2

Joyce’s point is that precise credences always mandate a firm opinion about both the absolute and the comparative likelihoods of every proposition under consideration. You must have your mind made up about whether X is more, less, or equally likely as Y. This requirement for totality sometimes goes beyond your evidence—you might have no basis for your opinion about their likelihoods. Precise credences are thus in some evidential circumstances at best optional and at worst irrational, according to imprecise Bayesians.

For what follows, we’ll look at the former option—namely, making sense of the permissibility of imprecise credences, as follows:

**IMPRECISE.** In the face of certain kinds of evidence, it is sometimes rationally permissible to adopt imprecise credences.

This permissive version of Imprecise seems natural from an evidentialist perspective. That is, imprecise credences seem like a perfectly appropriate response to some kinds of evidence.

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2 The case here is slightly changed.
Our goal will be to motivate IMPRECISE from an alethic perspective. This will require more than merely rendering imprecise credences and accuracy compatible, which, as we’ll see, is itself a challenge. We also aim to show why an agent who cares only about accuracy—and does not value following her evidence, except in so far as doing so promotes accuracy—might end up with imprecise credences.

2.2. Accuracy-First Epistemology (AFE)

According to accuracy-first epistemology, accuracy is the sole source of epistemic value. The higher your credence in truths, and the lower your credence in falsehoods, the better off you are, all epistemic things considered.

Many norms of interest in epistemology—probabilism, conditionalization, the principal principle, and so on—don’t tell us directly to be accurate, however. So, accuracy-firsters must justify these norms through their connection with the rational pursuit of accuracy. Either violating them renders agents unnecessarily inaccurate, or following them is somehow a good means toward the end of accuracy.

2.3. The Epistemic Utility Programme

Accuracy functions as a measure of a credence’s epistemic value at a world, so it’s natural to treat accuracy as epistemic utility akin to practical utility. We can then turn to decision theory to exploit principles of rational choice to derive norms of interest. This idea is best illustrated with an example.

2.3.1. Probabilism

Joyce [1998, 2009] argues that a (precise) agent’s credence function should obey the axioms of probability, as follows: Suppose that $b$ isn’t a probability function. Then, on any legitimate measure of accuracy, there’s some probability function $b'$ that is strictly more accurate at every world than $b$ is. That is, on every legitimate measure, every non-probabilistic credence function $b$ is strictly accuracy-dominated by some coherent function. Furthermore, no probability function is even weakly accuracy-dominated.3

Joyce then argues that only non-dominated options are rational, since accuracy is what you ultimately should value. Therefore, only probability functions are candidates for being rationally permissible credence functions.

So, Joyce identifies epistemic utility with some privileged measures of accuracy and invokes the decision-theoretic norm of dominance-avoidance to establish probabilism. There are two clear ways to derive additional epistemic norms. The first is to change the principle of rational choice. Greaves and Wallace [2006] argue for conditionalization, for example, by appealing to expected utility maximisation instead of to dominance. Similarly, Pettigrew [2013] argues for the Principal Principle by appeal to stochastic dominance. That is, on the same measures of

3 That is, if $c$ is a probability function, then there’s no $c'$ that’s at least as accurate as $c$ at every world and strictly more accurate at some world.
epistemic utility to which Joyce appeals, the updating policy that leads to the most expected epistemic accuracy is conditionalization, and the only credence functions that avoid stochastic dominance are those that obey the Principal Principle.

The second is to change the class of legitimate epistemic utility functions—that is, measures of accuracy. As we’ll see in a moment, however, epistemic utility theorists tend to agree on certain constraints that any good measure must satisfy. Although we will agree to the constraints that accuracy-firsters like, we’ll see that changing one’s epistemic utility function can have important effects on one’s epistemic behaviour.

2.3.2. Measuring Inaccuracy
It will now be useful to get clearer on what accuracy-firsters mean by accuracy.

First, a quick simplification. For technical convenience, it’ll be easier to use measures of inaccuracy or negative accuracy. That is, we seek constraints on acceptable measures of divergence from truth-value, instead of proximity to truth-value. According to AFE, rational agents seek to 
minimize
this divergence.

Now back to the main question. One non-negotiable principle is that credences closer to truth-values are less inaccurate. A credence of .8 in a truth, for example, is less inaccurate than a credence of .7. Likewise, a credence function that’s uniformly closer to the truth than another is overall less inaccurate.

We now cash this out by using possible world semantics. Let \( W \) be a set of worlds, and \( \mathcal{F} \) be a set of propositions over \( W \). Let \( w_X = 1 \) (\( w_X = 0 \)) if \( X \) is true (false) at \( w \). Then \( \text{bel}(\mathcal{F}) \) is the set of credence functions over \( \mathcal{F} \), where a credence function assigns some number in \([0,1]\) to each proposition in \( \mathcal{F} \).

A measure of inaccuracy (that is, a \textit{scoring rule} \( J \)) is a function from \( \text{bel}(\mathcal{F}) \times W \rightarrow \mathbb{R}_{\geq 0} \) that measures how close a credence function is to the truth at a given world. The fundamental constraint on legitimate inaccuracy measures is this:

\textbf{TRUTH-DIRECTEDNESS.} If \( |b_X - w_X| \leq |c_X - w_X| \) for all \( X \), and \( |b_Y - w_Y| < |c_Y - w_Y| \) for some \( Y \), then \( J(b,w) < J(c,w) \).

In addition to \textbf{TRUTH-DIRECTEDNESS}, accuracy-firsters universally require an additional immodesty constraint. The idea is that if you have a credence of \( x \) in some proposition, you should expect \( x \) to be the least inaccurate of the alternatives. Otherwise, you could never rationally hold \( x \) as a credence, since it would be dispreferred to some alternative. Likewise, every probability function should expect itself to be the least inaccurate. More formally:

\textbf{PROPRIETY.} If \( J \) is a legitimate measure of inaccuracy and \( b \) is a probability function, then for all distinct credence functions \( c \), we have:

\[ E_b J(b) < E_b J(c) , \]

where \( E_b \) denotes the expected value function according to \( b \).

So, \textbf{PROPRIETY} says that each probability function assigns itself lowest expected inaccuracy according to any legitimate measure.

For our dialectical purposes, we can just accept this requirement, since we’ll argue that imprecise Bayesians should think that there are \textit{multiple} legitimate measures of accuracy, and \textbf{PROPRIETY} narrows the space of such measures.
2.3.3. An Example
Later, we’ll explore various proper measures in more detail. For now, let’s look at a single example for concreteness. One way to score an agent’s credal state at a world is to identify inaccuracy with mean squared error—namely,

\[
\text{brier score: } BS(b, w) = \frac{1}{|F|} \sum_{X \in F} (w(X) - b(X))^2
\]

The Brier Score simply sums the square of the difference between the agent’s credence in each proposition, and then takes the average.

Sometimes, we’ll want to score individual credences instead of entire credence functions. We can likewise evaluate a particular credence simply by looking at its squared error—namely,

\[
\text{local brier score: } BS(x, i) = (i - x)^2
\]

Here, \(x\) is the agent’s credence, and \(i\) is the truth-value of the proposition in question. Context should make it clear below whether the local or global version is intended.

The Brier Score (along with infinitely many other strictly proper scoring rules) will vindicate Joyce’s argument for probabilism, along with the other accomplishments of epistemic utility theory. That is, if we use the Brier Score as a measure of epistemic disutility, then all and only non-probability functions are dominated, conditionalization is the policy that minimizes expected epistemic disutility, and so on.

2.4. The Incompatibility Argument
Let’s turn to the challenge of reconciling accuracy-first epistemology with imprecise credences.

Recently, multiple authors have published impossibility results that aim to show that accuracy-first epistemology and imprecise credences are incompatible [Seidenfeld, Schervish, and Kadane 2012; Mayo-Wilson and Wheeler 2016; Schoenfield 2017]. Specifically, any way of measuring the inaccuracy of imprecise credal states is sure to yield an unattractive result. The formal arguments are rather nuanced, but I’ll provide a simplified and less general version without bells and whistles.4

Compare the following cases:

MYSTERY COIN. The only evidence you have that is relevant to whether Heads is that the objective chance of Heads is between 0.05 and 0.95.

FAIR COIN. You know the chance of Heads is exactly .5.

Suppose that in MYSTERY COIN you have credence [.05, .95] toward Heads, whereas in FAIR COIN you have credence .5. In each case, there are only two outcomes: one in which Heads is true, and one in which Heads is false.

We assume that, on any good measure of inaccuracy, \(J(.5, 1) = J(.5, 0)\). That is, if you have credence .5, your level of inaccuracy is fixed.5 For instance, on the Brier Score, your inaccuracy is .25, regardless of whether Heads or Tails.

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4 The argument that I give follows Schoenfield [2017] most closely, although the other two arguments cited are similar. Lindley [1982] makes a general case that accuracy-first epistemology is compatible only with precise probability, and rules out imprecise credences in particular.

5 More sophisticated variations of the argument do away with this assumption.
Similarly, it seems, $J([.05, .95], 1)$ should be the same as $J([.05, .95], 0)$, since [.05, .95] doesn’t favour Heads over Tails or vice versa. We then set $J([.05, .95], 1) = J([.05, .95], 0) = m$ and $J(.5, 1) = J(.5, 0) = s$.

Now, $m$ might not be a single number. After all, we might think that imprecise states should get precise scores. But we know that one of the following holds:

i. $m$ is a better score than $s$
ii. $m$ is a worse score than $s$
iii. $m$ is neither better nor worse than $s$.

If (i) is true, it must always be irrational to have credence $.5$ in any proposition. After all, since accuracy depends just on your credal state and truth-values, you would do better in FAIR COIN by having credence [.05, .95] instead of .5. In other words, if $m$ is better than $s$, .5 is accuracy-dominated and therefore always less epistemically valuable than a credence of [.05, .95]. Surely, however, it’s sometimes rational to have a credence of .5 (for instance, if you know that a coin is fair). So, $m$ can’t be better than $s$.

Analogously, if (ii), it’s always irrational to have a credence of [.05, .95]. A generalised version of this argument then rules out imprecise credences from being rationally permissible. This option gives up the game.

If $s$ is neither better nor worse than $m$, then imprecise credences do no alethic work. That is, an accuracy-seeking agent never has reason to prefer an imprecise credence to some precise one. Imprecise credences thus would be unmotivated from an accuracy point of view, since reasons for adopting them would be purely evidential and non-alethic.

However, the situation is in fact worse than that, as Schoenfield [2017] points out. In FAIR COIN, having credence [.05, .95] violates the Principal Principle, which is a rational requirement. However, if (iii) holds, it’s never determinately worse to have credence [.05, .95] than credence .5. So, this option permits violations of the Principal Principle. Since the Principal Principle is better established than either accuracy-first epistemology or imprecise credences, one of the latter two should go.

To respond to this argument, we first look at two subtly different ways of understanding what imprecise credal states amount to.

3. Indeterminacy and Imprecise Bayesianism

3.1. The Formal Representation

As mentioned above, when we’re interested in an imprecise agent’s attitude toward a single proposition, we can represent it with a set or interval. For instance, in MYSTERY COIN, we might represent her with the interval [.05, .95].

We can also use sets of probability functions to represent her entire doxastic state over more than one proposition. Suppose that Alice has an imprecise credence [.2, .3] in $X$, imprecise credence [.3, .4] in $Y$, and precise credence .8 toward $Z$. Assuming that she has no other precise views, Alice’s doxastic state $R$ (called her Representor) is

$$\{c \in \text{Prob} : c(X) \in [.2, .3], c(Y) \in [.3, .4], c(Z) = .8\}$$
Facts about Alice’s opinion correspond to facts that are true of each element of \( \mathcal{R} \). For instance, since every element of \( \mathcal{R} \) assigns a lower precise probability to \( X \) than to \( Z \), Alice thinks that \( X \) is less likely than \( Z \). However, since some credence function assigns \( .3 \) to both \( X \) and \( Y \), Alice has no precise opinion as to whether \( X \) or \( Y \) is strictly more likely.

### 3.2. Interpretation

There are two importantly distinct ways to interpret how \( \mathcal{R} \) represents Alice’s doxastic state:

**Determinate.** Alice determinately identifies with the set of probability functions \( \mathcal{R} \).

**Indeterminate.** It’s determinate that Alice’s credence function is a member of \( \mathcal{R} \), but it’s indeterminate which member of \( \mathcal{R} \) it is.

On the first option, there’s no indeterminacy at all. Alice’s doxastic state simply is the set \( \mathcal{R} \). On the second, there’s no fact of the matter what, exactly, her doxastic state is.

Imprecise credences fit naturally with the *Indeterminate* interpretation. After all, there is a clear analogy to supervaluationist semantics. On the supervaluationist approach, vague terms have *admissible* and *inadmissible* precisifications—roughly, reasonable and unreasonable ways of completely disambiguating the term. If, on every admissible precisification, a proposition comes out true, then it’s determinately true. If, on every admissible precisification, it comes out false, then it’s determinately false. If it comes out true on some but not other precisifications, then it is indeterminate whether it is true or false.

As Rinard [2015: 2; my minor changes] points out, imprecise credences behave much like vague concepts under supervaluationism:

Functions excluded from your [representor] are inadmissible precisifications. Whatever is true according to all functions in your representor is determinately true; if something is true on some, but not all functions in your representor, then it’s indeterminate whether it’s true. … If different functions in your set assign different values to some proposition \( X \), then for each such value, it’s indeterminate whether that value is your credence in \( X \).

The orthodox interpretation is structurally supervaluationist, as it treats each function in Alice’s representor in the same way as supervaluationism treats precisifications in vagueness. So, the *Indeterminate* reading is at least a natural one and worth exploring further in the context of accuracy-first epistemology.

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7 There is more structure to imprecise credal states than simply the intervals assigned to propositions. Suppose that Alice and Bob both have credences between \( .2 \) and \( .3 \) in \( X \) and between \( .3 \) and \( .4 \) in \( Y \). However, for every \( b \) in Alice’s representor, \( b(Y) = b(X) + .1 \), whereas, for every \( b \) in Bob’s representor, \( b(X) + b(Y) = .6 \). Alice and Bob have different opinions about \( X \) and \( Y \) even though they assign them the same interval-valued credence. Thanks to a referee for this example.

8 Aside from Rinard [2015], not much has been written on whether to endorse *Determinate* or *Indeterminate*. Levi [1985] distinguishes between what he calls *imprecise* and *indeterminate* credences. However, his terminology doesn’t align with ours. For him, an agent with imprecise credences is someone whose actual (point-valued) credence exists but cannot be measured with full precision, while indeterminate credences are those that, even when precisely measured, should be represented as a set of probability measures. Levi doesn’t, as far as I know, ever endorse either thesis directly.
3.3. The Incompatibility Arguments Revisited

Let’s return to the incompatibility arguments. They show that if we assign a score of $m$ to $[0.05, 0.95]$ and $s$ to $0.5$ then we have no good options.

This is a big problem on the determinate view. If Alice’s doxastic state in mystery coin is a set, then we need some way to score that set as a whole. That is, we need some way of saying when an accuracy-seeking agent should prefer being in the set-valued doxastic state to being in a precise credal state.

However, on the indeterminate view, Alice’s attitude toward Heads in mystery coin isn’t really the interval $[0.05, 0.95]$. More precisely, her doxastic state isn’t some set of functions that assign credence between $0.05$ and $0.95$ to Heads. Instead, it’s indeterminate what her credal state is. So, assigning a score to the whole interval is a category mistake. There’s no fact of the matter as to how inaccurate Alice is, since it’s indeterminate which credence function is hers.

Indeterminate thus denies the presupposition of the incompatibility argument that there’s some way or other to score a representor as a whole. However, by leaving indeterminate an agent’s inaccuracy score, it doesn’t tell us why imprecise credences might be at all desirable to an accuracy-firster, or why an imprecise agent must obey the Principal Principle.

The answer that I’ll develop below is based on the claim that Alice should be permitted to have indeterminate alethic values. Although all that she should care about, epistemically speaking, is accuracy, there isn’t any single way that she cares about accuracy, or any precise notion of accuracy that she places above all others. If her values are indeterminate, she can end up with indeterminate credences as a result.

4. Imprecise Epistemic Values

At first blush, it seems that AFE has settled the question of epistemic value. All that an agent should care about is having her doxastic state come close to matching actual truth-values. Epistemology thereby becomes a matter of determining what sorts of epistemic actions and policies are most truth-conducive. However, as William James famously argued, much remains undetermined [1896: sec. VII]:

Believe truth! Shun error!— these, we see, are two materially different laws; and by choosing between them we may end by coloring differently our whole intellectual life. We may regard the chase for truth as paramount, and the avoidance of error as secondary; or we may … treat the avoidance of error as more imperative, and let truth take its chance.

Notice that, in the context of full belief, each commandment is individually easy to satisfy. An agent can believe all truths by believing every proposition, yet she is thereby sure to violate the commandment to shun error. Likewise, an agent can suspend belief about each proposition and avoid error, yet she gives up the chance to believe truths.

A similar lesson applies to precise credences. An agent with credences all close to 0 or 1 has the chance to be extremely accurate, but she risks great inaccuracy. In turn, an agent with credences closer to the middle of the spectrum protects herself from alethic disasters, but she has no chance for very low inaccuracy. Deciding on a credence requires striking a balance between these two commandments.

We’ll apply a broadly Jamesian theme to develop an accuracy-first approach to imprecise credences. We’ll first see two ways in which alethic values can rationally
differ. Then we’ll see that an agent’s credences can be affected either by her utility function or by the method that she uses to choose a credence from her options. If it’s rationally permissible not to have determinate epistemic values, then it’s rationally permissible to have imprecise credences. Thus, indeterminate values generate indeterminate credences.

4.1. Scoring Rules and Epistemic Value

As mentioned in section 2.3.2, accuracy-firsters use proper scoring rules to quantify epistemic disutility [Joyce 2009; Pettigrew 2016a; Konek and Levinstein forthcoming]. Scoring rules are purely alethic, in the sense that they are truth-directed and measure divergence between credence and truth-value. Nonetheless, they encapsulate different values. First, they disagree about the rank-order of epistemic options. That is, they disagree about which credence functions an agent should prefer at which worlds. Second, they disagree about the cardinal level of epistemic risk involved in epistemic decisions. However, no single scoring rule in particular seems uniquely best. The upshot is that if no one scoring rule is privileged then the notion of accuracy itself is indeterminate.

4.2. Examples

Although infinitely many different proper scoring rules satisfy the constraints of AFE, we’ll focus on three popular measures that will do the job.\(^9\) That is, these measures will, when combined with the right decision-theoretic principles, underwrite arguments for probabilism, conditionalisation, the Principal Principle, etc.

- **Brier score**: \(BS(x, i) = (i - x)^2\)
- **Log score**: \(\log(x, i) = -\ln(|(1 - i) - x|)\)
- **Spherical score**: \(\text{Sph}(x, i) = 1 - (1 - i - x)/(x^2 + (1 - x)^2)^{1/2}\)

Let’s now see how these scoring rules encode different alethic values.

4.3. Ordinal Differences

We begin with ordinal differences. Consider this situation:

- **Lottery**: An urn contains four balls: A, B, C, and D. As a matter of fact, A is chosen.

Alice, Bob, and Carol have the credences shown in Table 1 over which ball was selected. Who among the three is more accurate than whom? Or (to put the question differently) who is—from a purely alethic perspective—best off, epistemically speaking?

It’s hard to say. Alice has the lowest credence in the true proposition A, so in that respect she’s doing the worst. However, Bob has a high credence in the false proposition D, whereas Alice’s highest credence in a false proposition is .49. So, it’s

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\(^9\) For more, see Joyce [2009].
unclear whether Alice is more inaccurate than Bob. What about Alice versus Carol? Again, it’s unclear. Carol is more accurate than Alice on A, C, and D, but is much less accurate on B. No ordering leaps out as the one upon which all rational agents must agree.

As you may suspect, our scoring rules also disagree about the ordering. On the BRIER SCORE, Carol is the least inaccurate, followed by Alice and then Bob. On the LOG SCORE, Carol is the least inaccurate, followed by Bob and then Alice. And, on the SPHERICAL SCORE, Alice is the least inaccurate, Carol is second, and Bob is last. Philosophical reflection is unlikely to make a definitive case for which way to order our participants.

So, agents with different truth-directed proper scoring rules will disagree about which credence function is the most preferable. More abstractly, agents who differ in their alethic values can disagree about their epistemic preferences even when the state of the world is known.

If different orderings are reasonable, there’s a further question of whether rational agents are nonetheless obligated to make up their minds as to who is more accurate than whom. More generally, we have this:

TOTALITY. For any world w and any two credence functions b₁ and b₂, a rational agent would either prefer to have b₁ to b₂ as her credence function, prefer b₂ to b₁, or be indifferent between b₁ and b₂.

According to this principle, agents must have a single epistemic disutility function that completely ranks each credence function at each world.

However, TOTALITY is not a compelling rational requirement. Fundamentally, seeking accuracy means that you prefer higher credences in truths and lower credences in falsehoods. That may well be rationally required. But that preference leaves much to be determined, and there isn’t any compelling reason to force agents into a total preference ranking. Without specific reasons to the contrary, we can assume that some rational agents may not have definitive views about whether Alice, or Bob, or Carol is best-off in this situation.

### 4.4. Cardinal Comparisons

Scoring rules also disagree on the amount of epistemic risk involved in epistemic decisions. Let’s see how the Brier and Log Scores the evaluate a credence in a single proposition H. Suppose that Alice has credence .01 and Bob has credence .001.

<table>
<thead>
<tr>
<th></th>
<th>BS(x, 1)</th>
<th>BS(x, 0)</th>
<th>Log(x, 1)</th>
<th>Log(x, 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>.01</td>
<td>.98</td>
<td>10⁻³</td>
<td>4.6</td>
</tr>
<tr>
<td>Bob</td>
<td>.001</td>
<td>.998</td>
<td>10⁻⁶</td>
<td>6.9</td>
</tr>
</tbody>
</table>
Consider the ratio of how much Alice stands to gain or lose in inaccuracy if she were to switch to Bob’s credence. On the Brier Score, if she adopted Bob’s credence and $H$ turned out false, her score would decrease by $10^{-3} - 10^{-6}$. If $H$ turned out true, she would increase her score by .998 – .98. The ratio of possible inaccuracy gain to loss is approximately 18:1. That’s risky, but not nearly as risky as the same change in credence is on the Log Score—namely, around 256:1. So, the Log Score is much more sensitive to small changes in credence around 0.

Either of these risk profiles can seem reasonable. Both Alice and Bob are very confident that $\neg H$ in absolute terms. Since $.01 - .001 \approx 0$, this way of looking at their credences pushes us to assign relatively similar scores. On the other hand, Alice is ten times more confident in $H$ than Bob is. Looked at in this way, it seems that Alice should count as much more accurate than Bob when $H$ is true.

This point about the difference in risk-profile generalises: If an agent moves her credence in proposition $X$ from $x$ to $x\pm \varepsilon$, how much she stands to gain or lose if $X$ is true will depend both on the value of $x$ and the scoring rule.\footnote{Each scoring rule is generated by an underlying measure on the unit interval that represents the importance of having a credence on the correct side of a given point. Some rules care more about the middle region (Spherical), some care more about the ends (Log), and some don’t privilege any region (Brier). See Levinstein [2017].} One can get some sense of the different risk profiles by examining Figure 1.

So, seeking truth and avoiding error can be balanced in different ways. Rational agents need not make up their minds about the exact details. However, we still have work to do before such imprecise values can lead to imprecise credences.

4.5. An Oddity for Precise Bayesianism

If different alethic values are permissible, it seems that such values would naturally influence epistemic behaviour. For instance, it seems that agents who use the Log Score would be more skittish about lowering a credence from $.01$ to $.001$ than would agents who use the Brier Score.

However, the story is more complicated than that. The problem is that propriety severely limits the role that scoring rules can play in precise Bayesian epistemology.\footnote{See Horowitz [2018] for another discussion of Jamesian goals in the context of AFE.} Suppose that Bob has credence function $b$ and at first uses the Brier Score.

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**Figure 1.** The Local Brier, Log, and Spherical Scores. The ascending curves represent $J(x, 0)$, and the descending curves represent $J(x, 1)$ for the respective scoring rules. Note that the Brier and Spherical Scores are bounded, but the Log Score is unbounded.
Bob then has a change of heart, and decides that the Spherical Score captures his values better. How does his epistemic behaviour change? Answer: not at all! For $b$ is still the function that he expects to minimise inaccuracy. So, despite the change in value, Bob sticks with $b$.

This apparent epiphenomenalism of value is more surprising when we consider the effects of such values on learning. Suppose that Carol begins life as an epistemic risk-taker. Her credence function ‘learns’ quickly: without much evidence, she tends to arrive at credences close to 1 or 0. So, although she can quickly become accurate, she also risks massive inaccuracy. Carol uses the Brier Score, reflecting her risk-tolerance.

Over time, Carol grows more conservative. She just doesn’t have the same tolerance for error that she previously had. Carol switches to the risk-averse logarithmic score. What happens? Again, nothing. By Carol’s lights, planning to update her current credence function by conditionalization minimizes expected inaccuracy. So, the change in value doesn’t manifest in any change in behaviour.

Since Bayesianism at its core asks one to be probabilistically coherent and to (plan to) update by conditionalization, how could your alethic values influence your epistemic behaviour? The answer, I think, is that they can affect how you choose a credence to have in the first place.\footnote{More carefully, values affect which credences it’s rational to start with in a situation. I make no commitment to doxastic voluntarism.} When looking at your evidence, you don’t always have any credence—precise or imprecise—at all. You need to examine the available options and to pick one that’s attractive. How you select your doxastic state can depend on your values. We now discuss how this might work.

### 4.6. Two-Stage Conception of Evidence

To see how agents may end up with a particular doxastic state, it’s natural to adopt a common two-stage conception of evidence. To spell this out, let’s return to our mystery coin example. Let $E$ be the proposition that the chance of Heads is between .05 and .95.

Suppose that Alice is rational but has not yet formed a credence (precise or imprecise) toward Heads. Epistemologists disagree about what Alice should think when she learns $E$ and wants to form a credence. Perhaps she’s obligated to have credence of exactly .5. Perhaps she can rationally adopt any credence between .05 and .95. Or perhaps she should end up in some imprecise credal state.

The important point now is that, regardless of what she should do, we can reconstruct Alice’s selection of a doxastic state as follows. First, $E$ eliminates from rational contention all probability functions that assign credence less than .05 or greater than .95 to $H$. Second, Alice selects her credal state (precise or imprecise) from the remaining functions.

More generally, the first stage rules out at least a subset of credences incompatible with the evidence.\footnote{The first stage imposes no constraints if the agent has no evidence relevant to the proposition in question. She then selects her credence from the entire interval.} For our purposes, we’ll say that stage 1 rules out any
credences outside the interval in which the chance is known to fall. At the second stage, the rational agent picks a credal state from whatever is left, using some rational method or other (to be discussed below).

It may seem that this thesis requires us to deny unique objective Bayesianism. That is, it seems to fit naturally with this:

**Non-uniqueness.** For some bodies of evidence $E$, there is sometimes no precise credence function that uniquely responds in the objectively most epistemically rational way to $E$.

However, even austere objective Bayesians have adopted a two-stage conception of evidence [Jaynes 1973; Williamson 2010]. On their view, the first stage may leave multiple options on the table, but the second stage always narrows the set of rational choices to exactly one because only one method of selection is permitted. So, although the two-stage conception might not be compatible with every view of how evidence works, it’s a big tent.

### 4.7. Selection Rules

On our example, after the first stage, Alice has to select from the probability functions that assign anything between .05 and .95 to Heads. We now examine her selection process. Let’s begin with two temporary simplifying assumptions. We’ll assume, first, that Alice ends up with a precise credence, and, second, that she uses the Brier Score.

It may seem that Alice’s only option is credence $.5$ since her evidence doesn’t favour Heads over $\neg$Heads. But that’s too quick. If she selects $.5$, she’s sure to have a Brier-inaccuracy of $.25$ regardless of whether Heads.

However, if she adopts $.4$ and $\neg$Heads, she obtains a score of $.4^2 = .16$. Of course, she risks a worse score of $(1-.4)^2 = .36$. So, while $.5$ minimises the loss in the worst-case scenario, it also maximises the loss in the best-case scenario.

Her ultimate credence therefore depends on her appetite for epistemic risk. This appetite is partly reflected in the *selection rule* that she uses—that is, her policy for choosing a credence from those allowed by the evidence.

Although there are infinitely many potential selection rules, we review three. Let $O$ be the set of available credences after the first stage of the evidential process. Alice can pick a credence via these:

**MINIMAX:** Select the credence that has the best worst-case outcome. That is,

$$\arg\min_{x \in O} \max_{i = 0, 1} BS(x, i)$$

**MINIMIN:** Select a credence that has the best best-case outcome: That is,

$$\arg\min_{x \in O} \min_{i = 0, 1} BS(x, i)$$

**HURWICZ**: Select the credence that has the best weighted average of the best- and worst-case outcomes, with weights given by $\lambda \in [0, 1]$. That is,

$$\arg\min_{x \in O} (\lambda \max_{i = 0, 1} BS(x, i) + (1 - \lambda) \min_{i = 0, 1} BS(x, i))$$

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14 One may worry that this first stage isn’t based on accuracy considerations. However, following one’s evidence is generally a good means toward accuracy, and evidence will eliminate certain credences as good strategies for obtaining accuracy. In this case, any credence outside the [.05, .95] interval is objectively epistemically dominated [Pettigrew 2013], and is therefore an objectively irrational strategy for pursuing truth. Note that Alice can be aware of such dominance even before she has a credence in Heads.

15 This same two-tier conception can be applied to selection of entire credence functions instead of a single credence.

16 For simplicity, we state the rules as pertaining to individual credences, not entire credence functions.
Each rule encodes a different risk-management policy. **MINIMAX** heeds the commandment to *Shun error!* It chooses a credence based only on the maximum possible inaccuracy. In **MYSTERY COIN**, it recommends credence of .5 under the Brier Score. **MINIMIN** zealously seeks to *Believe truth!* Here, it recommends either a credence of .05 or .95. **HURWICZ** seeks a balance between the two commandments.\(^\text{17}\) Depending on \(\lambda\), it could recommend any credence in \([.05, .95]\).\(^\text{18}\)

Although **MINIMIN** is too extreme to be a rational policy, various versions of **HURWICZ** are arguably rational. The risk-aversion of **MINIMAX** leads to potentially undue scepticism, since it requires the most agnostic credence that remains among any set of options. Given our goal of finding a place for imprecise credences in accuracy-first epistemology, I’ll assume that no single rule is mandatory.

However, even if a single selection rule were privileged, different scoring rules disagree about which credence function is more accurate than which others at various worlds. Therefore, selection rules that appeal to best- and worse-case outcomes will yield different results, depending on the utility function.\(^\text{19}\)

### 4.8. Epistemic Value and Imprecise Credence

It should now be clear how indeterminate alethic values can lead to indeterminate credences. Suppose that an agent cares only about accuracy, but that at least one of the following claims is true:

- i. Her values don’t single out a unique selection rule.
- ii. Her values don’t single out a unique scoring rule.

If either holds, there won’t always be a unique credence function that best reflects her epistemic values. All credence functions compatible with her epistemic values have equal claim. She therefore will not have reason to adopt any one candidate over another.

Note that our supervaluationist interpretation of imprecise credences is key. Because the agent’s values are not fully determinate, various scoring rules paired with various selection rules can be equally reasonable precisifications of those values. In turn, if she doesn’t eliminate all but one compatible credence function from

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\(^{17}\) See Pettigrew [2016b] and Konek [forthcoming].

\(^{18}\) An agent can use a selection rule to choose her ur-prior (i.e. the credence function that she uses at the start of her epistemic life). She can also use it to choose a credence in a proposition toward which she doesn’t have any attitude, even if she already has credences in other propositions. The latter case adds a complication. It’s unsettled how adding new propositions into your credence function’s domain affects global accuracy. Different answers arise, e.g., depending on whether we use the version of the Brier Score (above) that averages local accuracy, or an alternative version, that sums up total accuracy (corresponding roughly to average and total utilitarianism). I won’t provide an answer, but instead note that this provides a third way in which epistemic values could affect epistemic behaviour. See Carr [2015], Perez Carballo [2018], and Pettigrew [2018].

\(^{19}\) This can arise in two ways. If the agent uses, say, both the Brier Score and an asymmetric rule (such that \(J(5.1) \neq J(5.0)\)), then the Hurwicz criterion can lead to different selections under the same \(\lambda\). (To see why, notice that the Hurwicz criterion is essentially an expectation in this case.) The second way occurs when selecting multiple credences at once. Suppose that either Red, Green, or Blue will be drawn from an urn, and that the evidence requires credence between \([.2, .3]\) in Red, \([.5, .7]\) in Green, and \([.0, .3]\) in Blue. Hurwicz\(_2\) will select credence .7 in Green under the Brier Score, but .668 under the Log Score.
her doxastic state, distinct credence functions will be equally good precisifications of her views.

4.8.1. Scoring Imprecise Credences

We began with a puzzle that suggested that, from an accuracy-first perspective, imprecise credences were at best unmotivated and disconnected from the goal of pursuing truth. We’ve already seen the quietistic response available on our indeterministic picture (section 3.3). In MYSTERY COIN, Alice doesn’t think that $[.05, .95]$ is less inaccurate than $.5$. Instead, her views about the inaccuracy of any credence in the interval $[.05, .95]$ and $.5$ are indeterminate. On some precisifications of her credal state, she expects $.23$ to be more accurate than $.5$. On others, she expects it to be less inaccurate. Each $x$ in $[.05, .95]$ is, on some precisification, one that minimises expected inaccuracy by her lights.

We can put that point slightly differently: because it’s indeterminate whether $.23$ or $.5$ is the credence that she thinks does best at optimising her epistemic values, it’s indeterminate whether $.23$ or $.5$ really is her credence. Now, one might object that if it’s indeterminate which $x$ in $[.05, .95]$ is really Alice’s credence, it’s also indeterminate whether she prefers $.05 − \epsilon$ to $.5$. After all, according to the precisification that assigns $.05$ to $H$, $.0499$ is expected to be less inaccurate than $.5$. That’s true, but it misses the point. On no precisification of her epistemic values is $.0499$ a maximally good credence to have.

We must also explain why imprecise credences don’t lead to permitted violations of the Principal Principle. If Alice learns that the chance of Heads is $x$, it’s determinate that obeying the Principal Principle maximises objective expected epistemic utility according to every strictly proper scoring rule [Pettigrew 2013]. If Alice’s values are determinately rational, then, according to every precisification of her values, she should obey the Principal Principle. Every reasonable way of pursuing truth and avoiding error leads to deferring to chance. Even though precisifications of her rational epistemic values sometimes disagree, they all agree when the chances are known.

One might raise a worry along the following lines. Suppose that Alice and Bob both know that the chance of Heads is $.5$. Alice adopts credence $.5$, while Bob ends up in an imprecise state of $[.05, .95]$. Carol learns the chance of Heads as well, and she is deciding whether to end up like Alice or Bob. Even though Bob’s credal state is indeterminate, Carol can—let’s suppose—take a brain scan of Bob and Alice and decide whether to switch her brain state to match either of theirs. Since the brain is a physical object, the brain state is determinate. If there’s no fact of the matter as to whether Alice or Bob is more accurate, why should Carol prefer to be like Alice instead of like Bob?

The answer is that it’s not agents or brains that are evaluated for accuracy, strictly speaking. Instead, accuracy is what gives value to doxastic states. From an epistemic point of view, Carol shouldn’t judge directly whether she would prefer to be Alice or Bob. Instead, she should judge which credal states agree with her epistemic values. If her values are determinately rational, she’ll baulk at any states that don’t assign credence $.5$ to Heads. Any epistemic values and risk-profiles that lead to other credences are determinately irrational. All three agents have decisive
reason to eliminate any credence functions from their doxastic state that cannot be generated through rational values and attitudes toward risk.20

5. Comparison to Alternatives

We’ve seen how we can generate imprecise credences from imprecise values, and how imprecise credences are compatible with accuracy-first epistemology. Before concluding, I briefly compare the view developed with some alternatives.

5.1. Pointwise Conditionalisation

On the orthodox imprecise view, an agent should always plan to update her doxastic state by pointwise conditionalization. That is, if her representor is $\mathcal{R}$, she should plan to change her doxastic state, upon learning $E$, to $\mathcal{R}_E = \{c(\cdot | E) : c \in \mathcal{R}\}$.

What’s the justification for this updating rule? In the case of precise Bayesianism, conditionalization can be justified through appeal to expected utility maximization. Roughly, on any strictly proper scoring rule, updating by conditionalization minimizes expected inaccuracy.

The same cannot be said for pointwise conditionalization in the imprecise case. On the indeterminate view, Alice is not, strictly speaking, in credal state $\mathcal{R}$ or $\mathcal{R}_E$ at any point. So, there’s no way that $\mathcal{R}_E$ itself could be a state that minimizes expected inaccuracy.

Instead, we have to be more subtle. Suppose that, should Alice learn $E$, she plans to be in a doxastic state representable by $\mathcal{R}'$. What this means is that Alice’s conditional plan—her plan about what to do, should she learn $E$—rules out all credence functions except those found in $\mathcal{R}'$. If $\mathcal{R}' \neq \mathcal{R}_E$, is there something rationally wrong with her plans?

Not necessarily. Suppose that $c \in \mathcal{R}$, but that $c(\cdot | E) \notin \mathcal{R}'$. Bad move, according to $c$. However, if $c' \in \mathcal{R}$ and $c'(\cdot | E) \in \mathcal{R}'$, it’s not determinately true that Alice’s plan fails to minimize expected inaccuracy. Therefore, it’s not determinately true that she did anything irrational.

One might object that Alice should determinately be an expected inaccuracy minimizer. After all, minimization of expected disutility is the determinately rational thing to do. However, if Alice’s plans rule out all and only credence functions not in $\mathcal{R}_E$, it’s still indeterminate whether she really minimized expected inaccuracy. According to any $c \in \mathcal{R}$, minimizing expected inaccuracy amounts to excluding all states except $c(\cdot | E)$.

This position raises a further subtlety about the rationality of imprecise credences. Recall what we said above about the Principal Principle. We argued that in FAIR COIN, in which Alice knows the chance of Heads is .5, she should determinately have credence .5 in Heads and not credence [.05,.95]. However, in that case it appeared at first that having credence [.05,.95] is not determinately irrational, because it’s indeterminate whether Alice’s credence is really .5. Why can we say that Alice shouldn’t have credence [.05,.95] in FAIR COIN but that she is permitted not to update by pointwise conditionalization?

20 See section 5.1 for related discussion.
In *FAIR COIN*, all rationally permissible epistemic values and epistemic risk profiles rule out every credence other than .5. Alice therefore has *decisive epistemic reason* to have credence .5 and no other credence. In so far as she cares about pursuing truth and avoiding error, .5 alone is her best option.

In the case of conditionalization, Alice does *not* have decisive epistemic reason *not* to exclude $c(\cdot | E)$ from her doxastic state. Assuming that all credence functions in her representor were rational to begin with, only some of them will say that she shouldn’t exclude $c(\cdot | E)$. Therefore, by her lights, it’s indeterminate whether her plan should rule out $c(\cdot | E)$.\(^{21}\)

Note that this view also *precludes* Alice from planning to allow new credence functions into her representor. Suppose that there’s no $c \in \mathcal{R}$ such that $c(\cdot | E) = b(\cdot | E)$. Then, by her lights, it’s determinately irrational not to exclude $b(\cdot | E)$ should she learn $E$, since every credence function in $\mathcal{R}$ agrees on that fact. Thus, her conditional plan for what to do upon learning $E$ should be a subset of $\mathcal{R}_E$.\(^{22}\)

### 5.2. Imprecise Credences and Permissive Bayesianism

One alternative to imprecise Bayesianism is *permissivism*. Permissivists think that sometimes evidence doesn’t single out a unique precise credence function as the maximally rational option. That is, Permissivists and Imprecise-Credencers agree with NON-UNIQUENESS above.

Permissivists and Imprecise-Credencers disagree, however, about what’s rational to do in those situations. Unlike Imprecise-Credencers, Permissivists think that an agent must pick a single precise credence function from the set of rational options.

The main reason to favour imprecise credences over permissivism is that, on the latter view, agents must be either more or less confident than the evidence requires. If .1 to .8 is permitted by the evidence, then there’s no evidential reason to be so bold as to have a credence of .7, or to be so timid as to have a credence of .5. Permissivists nonetheless require agents to adopt a single credence function even though the evidence doesn’t privilege that credence function above the others.

To make matters worse, once agents adopt a single credence, they are committed to a variety of additional opinions. Suppose that Alice and Bob share evidence that makes any credence in $[\frac{2}{3}, \frac{3}{2}]$ toward Heads maximally rational. Alice ends up with credence .24, and Bob ends up with credence .29. Both Alice and Bob recognise that they chose their credences from a set of options that were rationally on a par.

However, Alice thinks that .24 maximises expected epistemic utility on *every* strictly proper rule. That is, she expects that, on any reasonable measure, .24 is less inaccurate than .29. Still, she recognises that the evidence itself provides no reason to form this opinion over Bob’s view that his credence is more accurate. So, (i) she realises that, had she had a different epistemic utility function or selection rule before she chose .24, she would have ended up with .29, (ii) she currently thinks that, even on that alternative utility function, .24 is better than .29, and (iii) the

\(^{21}\) Thanks to an anonymous referee for pressing this point.

\(^{22}\) For a similar view with different motivation, see Weatherson [2007].
evidence doesn’t support the claim that .24 is more accurate than .29 over the claim that .29 is more accurate than .24.

That may be all right if Alice were merely permitted to adopt opinions that the evidence doesn’t uniquely support. The problem is that permissivism mandates that she form such unsupported opinions. So long as she ends up in some precise doxastic state or other, she expects that her own credence function is the least inaccurate on every strictly proper measure.

Imprecise credences allow for more modesty. If Alice has .24 and .29 in her representor, then there’s no fact of the matter as to which one she expects to do better. Even though Alice determinately expects to be least inaccurate—since each function in her representor does—she doesn’t determinately think that any particular credence in [.2,.3] is more accurate than any other. Since the evidence itself doesn’t objectively support any credence in the interval over any other, this seems like a superior response. She isn’t required to form opinions that exceed the evidence in this way.

6. Conclusion

Imprecise credences look attractive from an evidential perspective, but they also appear incompatible with accuracy-first epistemology. Appearances are deceiving. Different ways of valuing the truth—for instance, different scoring and selection rules—lead to different credences when evidence is unspecific. If agents have indeterminate values, they’ll in turn have indeterminate credences. Imprecise credences thus do not conflict with accuracy-first epistemology but naturally emerge from it.23

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