F Derivation of Real Rate Functional

This section derives the real interest rate functional given in Equation (25). We start
from the characterization of optimal consumption dynamics contained in the Online
Appendix C.1. Namely, we use (C.9) and (C.13) to integrate across all households $j$:

$$
\frac{d}{dt} \int_j c_t(a_j, z_j) \, dj = \int_{\tilde{N}_j} \left( \partial_a c_t(a_j, z_j) s_t(a_j, z_j) + \partial_t c_t(a_j, z_j) + \sum_{z' \neq z_j} \lambda_{z_jz'} [c_t(a_j, z') - c_t(a_j, z_j)] \right) \, dj
+ \int_{j : z' \neq z_j} \lambda_{z_jz'} [c_t(0, z') - c_t(0, z_j)]
$$

(F.1)

where the $d\tilde{N}_j$ terms vanish by the exact law of large numbers (Duffie and Sun, 2007,
2012). The first integral on the right-hand side is over unconstrained households
($j : u$), while the second integral is over constrained households ($j : c$). Note that the
above equation must be equal to zero, since $\int_j c_t(a_j, z_j) \, dj = 1$, by market clearing.

Dividing by $u''(c_t(a_j, z_j))$ in (C.7) and using CRRA preferences, we obtain:

$$
\frac{1}{\gamma} (\rho - r) c_t(a_j, z_j) = \sum_{z' \neq z_j} \frac{1}{u''(c_t(a_j, z_j))} \left[ u'(c_t(a_j, z')) - u'(c_t(a_j, z_j)) \right]
\left[ u'(c_t(a_j, z')) - u'(c_t(a_j, z_j)) \right]
+ \partial_t c_t(a_j, z_j) + s_t(a_j, z_j) \partial_a c_t(a_j, z_j)
$$

(F.2)

Integrating over all unconstrained agents and using (F.1) to substitute for $\partial_t c_t(a_j, z_j) + s_t(a_j, z_j) \partial_a c_t(a_j, z_j)$ yields

$$
\frac{1}{\gamma} (\rho - r) \int_{j : u} c_t(a_j, z_j) \, dj = \int_{j : u} \left( \sum_{z' \neq z_j} \frac{1}{u''(c_t(a_j, z_j))} \left[ u'(c_t(a_j, z')) - u'(c_t(a_j, z_j)) \right] \right) \, dj
- \int_j \sum_{z' \neq z_j} \lambda_{z_jz'} (c_t(a_j, z') - c_t(a_j, z_j)) \, dj
$$

(F.3)
Moreover, constrained agents consume their current income $z_j$. Hence, adding and subtracting $-\frac{1}{\gamma}(\rho - r)\int_{j \in c} z_j dj$ to the equation above and rearranging yields an expression for the interest rate:

$$r = \rho + \gamma \frac{\int_{j \in u} \sum_{z' \neq z_j} \lambda_{z_j z'} \frac{[u'(c_t(a_j, z')) - u'(c_t(a_j, z_j))]}{u'(c_t(a_j, z_j))} \lambda_{z_j z'} (c_t(a_j, z') - c_t(a_j, z_j)) dj - \int_{j \in c} \sum_{z' \neq z_j} \lambda_{z_j z'} (c_t(a_j, z') - c_t(a_j, z_j)) dj}{1 - \int_{j \in c} z_j dj}$$

(F.4)

Using CRRA utility, and the fact that $\lambda_{z_j z_j} = \sum_{z' \neq z_j} \lambda_{z_j z'}$, we may write the above expression as

$$r = \rho - \frac{\int_{j \in u} c(a_j, z_j) \left[ \sum_{z'} \lambda_{z_j z'} \left( \frac{c_t(a_j, z')}{c_t(a_j, z_j)} \right)^{-\gamma} \right] dj + \gamma \int_{j \in c} c(a_j, z_j) \left[ \sum_{z'} \lambda_{z_j z'} \frac{c_t(a_j, z')}{c_t(a_j, z_j)} \right] dj}{1 - \int_{j \in c} z_j dj}$$

(F.5)

Relative to the representative agent economy, the sum differs by two terms: the (i) marginal utility variation due to income risk for unconstrained agents, and (ii) consumption variation due to income risk for both constrained and unconstrained agents, multiplied by the coefficient of relative risk aversion. All of these terms are scaled by one minus the total income holdings of constrained agents (which is trivially less one since aggregate consumption is equal to one). The interest rate can be written as a functional in terms of aggregate states by replacing $c_t(a_{jt}, z_{jt})$ with $c(\omega_{jt}, z_{jt}, \Omega_t)$. Equation (25) then follows directly.

G Household Problem with Diffusion Process

This section sets up an economy in which income follows a diffusion process. We derive as an auxiliary result that $r_t < \rho$ for all $t \geq 0$ in this economy.

Concretely, we assume that household income follows a diffusion process given by

$$dz_{jt} = \mu_z(z_{jt}) dt + \sigma_z(z_{jt}) dB_{jt}$$

(G.6)

where $B_{jt}$ is adapted Brownian motion, independent across $j$, and $\mu_z(\cdot) : \mathbb{R} \to \mathbb{R}$ and $\sigma_z(\cdot) : \mathbb{R} \to \mathbb{R}^+$ are twice-differentiable functions. We further assume that (G.6) admits a stationary distribution. The household problem now satisfies the following
HJB equation:

\[
\rho V_t(a, z) - \partial_t V_t(a, z) = \max_c \left( \frac{c^{1-\gamma}}{1-\gamma} + \partial_a V_t(a, z) \left[ r_t a + z - \tau_t(z) - c \right] + \mu_z \partial_z V_t(a, z) + \frac{1}{2} \sigma_z^2 \partial_{zz} V_t(a, z), \right)
\]

(G.7)

together with the boundary condition \( \partial_a V_t(0, z) \geq (z - \tau_t(z))^{-\gamma} \). A solution to the HJB equation alongside (12) solves the household problem. The associated KFE equation is:

\[
\partial_t g_t(a, z) = -\partial_a [g_t(a, z) c_t(a, z)] - \partial_z [\mu_z(z) g_t(a, z)] + \frac{1}{2} \sigma_z^2 (g_t(a, z)) \]

(G.8)

**Expected Consumption Dynamics.** We now derive the expected consumption dynamics for unconstrained households. Following exactly the same steps outlined in Online Appendix C.1 for the case in which income follows a Poisson process, we can derive an Euler equation for unconstrained households:

\[
(\rho - r_t) u'(c_t(a, z)) = \mu_z(z) u''(c_t(a, z)) \partial_z c_t(a, z) + \frac{1}{2} \sigma_z^2(z) (u''(c_t(a, z)) \partial_{zz} c_t(a, z) + u'''(c_t(a, z)) (\partial_z c_t(a, z))^2) + u''(c_t(a, z)) [\partial_t c_t(a, z) + \varsigma_t(a, z) \partial_a c_t(a, z)]
\]

(G.9)

We can also use Ito’s lemma on \( c_t(a_{jt}, z_{jt}) \) to obtain

\[
dc_t(a_{jt}, z_{jt}) = [\partial_t c_t(a_{jt}, z_{jt}) + \varsigma_t(a_{jt}, z_{jt}) \partial_a c_t(a_{jt}, z_{jt})] dt + [\mu_z(z_{jt}) \partial_z c_t(a_{jt}, z_{jt}) + \frac{1}{2} \sigma_z^2(z_{jt}) \partial_{zz} c_t(a_{jt}, z_{jt})] dt + \sigma_z(z_{jt}) \partial_a c_t(a_{jt}, z_{jt}) dB_{jt}
\]

(G.10)

Taking expectations of the above equation, combining it with (G.9), and imposing that \( u \) is isoleastic with curvature parameter \( \gamma \) yields the expected consumption dynamics for unconstrained households:

\[
\frac{E_t[dc_{jt}]}{c_{jt} dt} = \frac{1}{\gamma} (r_t - \rho) + \gamma \frac{1}{2} \sigma_z^2(z_{jt}) \left( \frac{\partial_z c_t(a_{jt}, z_{jt})}{c_t(a_{jt}, z_{jt})} \right)^2
\]

(G.11)
Constrained households simply consume their income. Hence, their consumption dynamics are
\[ dc_{jt} = [\mu_z(z_{jt})]dt + \sigma_z(z_{jt})dB_{jt} \] (G.12)

The expected consumption dynamics of constrained households are therefore given by
\[ \mathbb{E}_t[dc_{jt}] = \mu_z(z_{jt}) \] (G.13)

**Derivation of Interest Rate Functional.** Integrating over the consumption dynamics of unconstrained households and making use of the fact that
\[ \int_j \frac{dc_{jt}}{dt} dj = 0 \]

yields
\[ 0 = \int_j \frac{1}{\gamma}(r_t - \rho)c_{jt} dj + \int_j \frac{\gamma + 1}{2} c_t(a_{jt}, z_{jt}) \left( \frac{\sigma_z(z_{jt})\partial_z c_t(a_{jt}, z_{jt})}{c_t(a_{jt}, z_{jt})} \right)^2 dj 
+ \int_{j:c} \mu_z(z_{jt})c_t(a_{jt}, z_{jt}) \] (G.14)

where we have used (G.11) and (G.13). Finally, imposing market clearing \( \int_j c_{jt} dt = 1 \) yields
\[ r_t = \rho - \frac{\gamma(\gamma + 1)}{2} \int_j \frac{c_t(a_{jt}, z_{jt})\partial_z c_t(a_{jt}, z_{jt})}{c_t(a_{jt}, z_{jt})} \] (G.15)

Note that this implies that \( r_t < \rho \) for all \( t \geq 0 \) (not just in steady-state) if no households are constrained, or if \( \int_{j:c} \mu_z(z_{jt}) dj > 0 \), so that constrained households expect their income to increase, on average. We may also write the formula analogously as the one in the main text for the Poisson income process (25):
\[ 0 = \mathcal{C}_t^n(r_t - \rho) + \mathcal{C}_t^n\mathbb{E}_t^n \left[ \frac{\gamma + 1}{2} \sigma_z^2(z) \left( \frac{\partial_z c_t(a, z)}{c_t(a, z)} \right)^2 - \mu_z(z) \right] + \bar{E}_t[\mu_z(z)] \] (G.16)
H Additional Details on Long-Run Anchoring

In this section, we demonstrate how the monetary authority can eliminate all dynamic equilibria that converge to the high inflation steady-state, leaving only a unique equilibrium that leads to the saddle-path stable, low-inflation steady-state. Concretely, suppose the monetary authority has the power to coordinate private sector beliefs about long-run inflation. Under such a setting we envisage two pillars of central bank policy: (i) a path or rule for short-term nominal interest rates $i_t$, and (ii) a long-run inflation target $\pi^*$. Whereas the interest rate is a policy tool that the central bank directly implements by intervening in appropriate markets or paying interest on reserves, the long-run inflation target is no more than an attempt to coordinate beliefs.

If:

(i) the long-run inflation target and the long-run nominal interest rate $(\pi^*, i^*)$ are set to be consistent with the equilibrium real rate at the saddle-path steady state, $i^* - \pi^* - g = r^*_H$;

(ii) fiscal policy follows a constant deficit policy or a passive interest payment reaction rule with $\phi_s < 1$, so that the high real rate, low inflation steady-state is saddle-path stable;

(iii) private sector beliefs about long-run inflation are consistent with the central bank’s target,

then there is a unique real equilibrium and the price-level and inflation are pinned down for all $t$. The third of these conditions is a big “if”, and there is no fundamental reason to expect it to hold. However the key point is that managing long-run inflation expectations is sufficient to pin down the price level and inflation in the short-run. If the central bank is successful at convincing the private sector to coordinate on a long-run inflation target, then this is sufficient to eliminate any indeterminacy about inflation at all points in time. Note that anchoring long-run inflation expectations at $\pi^*$ does not assume away the issue of price-level determination in the short-run. Both the initial price level and subsequent inflation remain endogenous and depend on monetary policy, fiscal policy and private sector behavior.

Even with long-run inflation anchoring, fiscal policy remains an essential component of price-level determination. Coordinating long-run expectations only uniquely determines the price-level in the short-run if fiscal policy acts in a way that ensures
the saddle-path stability of the low-inflation steady state. Such fiscal policy settings are the same as those required for uniqueness in the case with persistent surpluses.

I Proof for the Model With Long-Term Debt

Proposition 5. The household budget constraint follows (6) and the real government budget constraint follows (18) for \( t > 0 \). Moreover, the price of long-term debt satisfies the following differential equation for \( t > 0 \):

\[
\frac{\dot{q}_t}{q_t} + \frac{\chi - \delta q_t}{q_t} = i_t \tag{I.17}
\]

Proof. We define the auxiliary variable \( u = A_{jt}' \). Note that this implies \( dA_{jt}' = u \). Hence, the households HJB equation is given by:

\[
\tilde{\rho}V_t(A^l, A^s, z) - \partial_t V_t(A^l, A^s, z) =
\]

\[
\max_{c,u} \frac{e^{1-\gamma}}{1-\gamma} + \tilde{s}_t \partial_{A^l} V_t(A^l, A^s, z) + \partial_{A^l} V_t(A^l, A^s, z) u + \sum_{z' \neq z} \lambda_{zz'}[V_t(A^l, A^s, z') - V_t(A^l, A^s, z)]
\]

where

\[
\tilde{s}_t := i_t A^s + (\chi - \delta q_t) A^l + (z - \tau(z)) P_t y_t - P_t \tilde{c}_t - q_t u
\]

The first-order condition with respect to \( u \) is given by:

\[
q_t \partial_{A^s} V_t(A^l, A^s, z) = \partial_{A^l} V_t(A^l, A^s, z) \tag{I.18}
\]

We may differentiate with respect to time to obtain:

\[
q_t \partial_{A^l,t}^2 V_t(A^l, A^s, z) + \partial_t q_t \partial_{A^s} V_t(A^l, A^s, z) = \partial_{A^l,t} V_t(A^l, A^s, z) \tag{I.19}
\]

The envelope condition for the HJB with respect to \( A^l \) is:

\[
\tilde{\rho} \partial_{A^l} V_t - \partial_{t,A^l}^2 V_t = \tilde{s}_t \partial_{A^l}^2 V_t + (\chi - \delta q_t) \partial_{A^l} V_t + u \partial_{A^l}^2 V_t + \sum_{z' \neq z} \lambda_{zz'}[\partial_{A^l} V_t - \partial_{A^l} V_t] \tag{I.20}
\]
Similarly, the envelope condition for the HJB with respect to $A^s$ is:

$$\tilde{\rho}\partial_{A^s}V_t - \partial_{t,A^s}^2 V_t = \tilde{s}_t\partial_{A^s}^2 V_t + i_t\partial_{A^s}V_t + u\partial_{A^s,A^s}^2 V_t + \sum_{z' \neq z} \lambda_{z'} [\partial_{A^s}V_t - \partial_{A^s}V_t]$$  \hspace{1cm} (I.21)

Multiplying (I.21) by $q_t$, subtracting Equation (I.20) from (I.21) and using (I.18) and (I.19) yields:

$$(q_i t - (\chi - \delta q_t) - \partial_t q_t)\partial_{A^s} V_t = 0$$  \hspace{1cm} (I.22)

By market clearing, we must have $\partial_{A^s} V_t > 0$ (otherwise no long-term debt would be purchased in equilibrium). Hence, we have the arbitrage relationship:

$$\frac{\dot{q}_t}{q_t} + \frac{\chi - \delta q_t}{q_t} = i_t$$  \hspace{1cm} (I.23)

Differentiating $B_t = q_t B_t^l + i_t B_t^s$ and using the (E.57) yields (15), which can be written in real terms. This completes the proof.

J Supplement on Wealth Distribution and MPCs

This section provides some additional detail on the MPCs in the calibrated steady-state. Figure 14a shows the dependence of marginal propensities to consume on real assets, disaggregated by the highest and lowest income draws. The plotted MPCs are

![Figure 14: MPCs in the calibrated steady-state](image)

(a) MPCs by income \hspace{2cm} (b) Distribution of MPCs

Figure 14: MPCs in the calibrated steady-state
Figure 15

Note: Impulse responses to a temporary increase in the wealth tax, with the proceedings distributed lump-sum, for various values of the wealth tax. In all experiments, the wealth tax is levied on the top 10% of the wealth distribution, the proceeds of which are redistributed lump-sum to the bottom 60%.

the quarterly marginal propensities to consume from an unanticipated $500 income gain.

MPCs are not monotonically decreasing in real assets because there is a borrowing wedge. Households with zero assets therefore have a high marginal propensity to consume because of the discontinuous cost of borrowing (Kaplan and Violante, 2014). Note that the MPCs of high income households lie uniformly below the MPCs of low income households.

Figure 14b plots the distribution of MPCs in the calibrated steady-state. A large number of households have an MPC of around 0.15 and hold zero assets. The average MPC in the economy is 0.14, which is in line with commonly estimated values for marginal propensities to consume (Jappelli and Pistaferri, 2010).
K  Inflationary Effects of Pure Redistribution

A comparison of the heterogeneous agent and representative agent economies in the preceding experiments suggests that redistribution itself has effects on the price level and inflation that are independent of the overall level of surpluses and nominal government debt. To emphasize the inflationary effects of redistribution, Figure 15 shows simulations from purely redistributive shocks. We consider one-time wealth taxes levied on the top 10% of the wealth distribution, the proceeds of which are redistributed lump-sum to the bottom 60%. Although these shocks do not entail any new issuance of government debt or any change in primary deficits, they do cause a period of inflation. The redistribution causes upward pressure on consumption because low-wealth households have higher average MPCs than high wealth households. Equilibrium is achieved through a period of higher real interest rates. The corresponding lower government revenues require a downward revaluation in real debt through a jump in the price level.

Inflationary Effects of Proportional Wealth Taxes. We contrast this experiment with another version of wealth taxation. Consider an economy where the government levies a proportional wealth tax at a rate of $\tau_b$ so that total primary surpluses are $s^* + \tau_b b_t$ (where $s^*$ are surpluses net of revenue from the wealth tax). The real government budget constraint becomes:

$$db_t = [(r_t - \tau_b)b_t - s^*]dt. \quad (K.24)$$

The wealth tax appears in the household budget constraint in a similar fashion, as it increases the after-tax real rate paid to the government, $r_t - \tau_b$. Changes in $\tau_b$ therefore only affect the inflation rate through the Fisher equation, but otherwise leave the real economy and the initial price level unchanged.

L  Endogenous Output

In this subsection, we outline an economy in which labor is a variable input in production. Next, we discuss how endogenous output affects price level and inflation dynamics in response to unanticipated shocks.
L.1 Set-Up

**Households.** The set-up of the household problem closely follows that of the main text. However, we assume that households choose real consumption flows $\tilde{c}_{jt}$ and hours worked $\ell_{jt}$ to maximize

$$\mathbb{E}_0 \int e^{-\rho t} \left[ \frac{\tilde{c}_{jt}^{1-\gamma}}{1-\gamma} - \phi_t^{1-\gamma} \frac{\ell_{jt}^{1+\psi}}{1+\psi} \right] dt$$

where the expectation is taken with respect to households’ efficiency units of labour $z_{jt}$. The exponent $\psi > 0$ is the inverse of the Frisch elasticity of labor supply. The term $\phi_t$ is a time-varying constant that augments the labor disutility in order to allow the economy to be consistent with balanced growth when $\gamma \neq 1$. Concretely, we assume that

$$\phi_t = \tilde{\phi} e^{gt}$$

where $\tilde{\phi} > 0$ and $g > 0$ is the growth rate of the economy. This formulation implies that a stationary equilibrium exists. Moreover, the distribution of hours across households is constant in the stationary equilibrium.\(^{42}\) The households nominal budget constraint therefore satisfies

$$dA_{jt} = [i_t A_{jt} + (1 - \tau_{1t}) z_{jt} P_t w_t \ell_{jt} - P_t \tilde{c}_{jt} + P_t \tau_{0t}] dt$$

where $w_t$ is the real wage rate for effective labor services at time $t$, $\tau_{0t}$ is a lump-sum payment and $\tau_{1t}$ is a constant proportional tax rate. We assume that $\tau_{0t}$ grows at a rate $g > 0$ in order to ensure that a stationary equilibrium exists:

$$\tau_{0t} = \tilde{\tau}_0 e^{gt}$$

Finally, the stochastic process for $z_{jt}$ and the definition of de-trended real variables for the evolution of real debt are identical to those of the main text.

\(^{42}\)We intentionally assume separability between hours and consumption in the instantaneous utility function so as to maximize comparability between the economy with endogenous output presented in this subsection and the endowment economy presented in the main text. In particular, the endowment economy can be closely approximated for large $\psi$ and a given calibrated $\tilde{\phi}$. We note, however, that preferences by King et al. (1988) leave the key mechanisms unaffected.
Firms. We assume that perfectly competitive firms hire labor to produce output $y_t$ with the constant returns to scale (CRS) production function

$$y_t = \Theta_t L_t$$  \hspace{1cm} (L.29)

where $\Theta_t$ is aggregate total factor productivity that grows at a rate $g > 0$ and $L_t$ are total effective hours:

$$L_t := \int_j z_{jt} \ell_{jt} dj$$  \hspace{1cm} (L.30)

CRS implies that the real wage rate $w_t$ is equal to $\Theta_t$ for all $t \geq 0$.

Government. The dynamics for government debt are given by

$$dB_t = [i_t B_t - s_t P_t y_t] dt$$  \hspace{1cm} (L.31)

where $s_t$ is the ratio of primary surpluses to output and is determined by the $\tau_{0t}$ and $\tau_{1t}$ as

$$s_t = \frac{\tau_{0t}}{y_t} + \int_{j \in [0,1]} \tau_{1t} w_t z_{jt} \ell_{jt} dj$$  \hspace{1cm} (L.32)

De-trended real government debt then follows

$$db_t = [r_t b_t - s_t] dt$$  \hspace{1cm} (L.33)

We do not consider unanticipated changes in the nominal rate in this section. Consequently, we assume an interest rate peg $i_t = i^*$ without loss of generality in analyzing real dynamics.

Calibration. Our calibration sets $\psi = 2$, so that the intensive-margin Frisch elasticity of labor supply is equal to one-half, in line with the recommendation of Chetty et al. (2011). Moreover, we calibrate $\phi$ so as to set total hours worked equal to unity. Allowing labor to adjust on the intensive margin provides additional insurance to households. As such, the discount rate increases to 6.1% p.a. (relative to 2.8% p.a. from the calibration in the main text) in order to match a debt-to-annual GDP ratio of 1.10. The values for the remaining parameters remain unchanged from Table 1.
**L.2 Quantitative Exercise**

We consider the economy’s response to an increase in deficits. First, we consider the economy’s response to a permanent change in $\tau_{0t}$ from 0.333 to 0.340, keeping $\tau_{1t}$ fixed. Second, we consider a permanent change in $\tau_{1t}$ from 0.300 to 0.307, keeping $\tau_{0t}$ fixed. These changes amount to a change in deficits from 3.3% to 4% of GDP, if output was unchanged (in line with the analysis of Section 5.5).

An increase in deficits due to a tax cut results in a smaller jump in the initial price level, relative to the transfer expansion case. The main reason is that lower taxation increases the labor supply (whereas a transfer expansion lowers it). The corresponding rise in output raises tax revenues and attenuates the long-run increase.
In both economies, however, real output eventually declines relative to the representative agent benchmark. In order to understand this result, consider the tax cut experiment. There are two forces that contribute to an increase in labor supply. First, the tax cut directly raises the return to working, as explained above. Second, households in the new steady-state hold lower amounts of wealth, on average. This gives rise to positive wealth effects that also expands total hours worked. However, the new steady-state features a lower long-run real rate – a force only present in the heterogeneous agent economy. The reduction in the real rate increases consumption state-by-state due to the intertemporal savings motive, thereby reducing total hours worked. This last force is sufficiently strong that it counteracts the positive effect on output due to the lower tax rate and the change in the wealth distribution. Consequently, in the long-run output falls and deficits rise relative to the representative agent economy.

\[\text{43} \] The tax cut also increases precautionary motives by amplifying the volatility of post-tax earnings, in line with the reasoning of Section 5.5. The real interest rate therefore decreases relatively less. Since the government now finances its debt at a higher cost, this a force that contributes to a larger initial jump in the price level. However, this mechanism is dominated by labor-supply channel.
References


