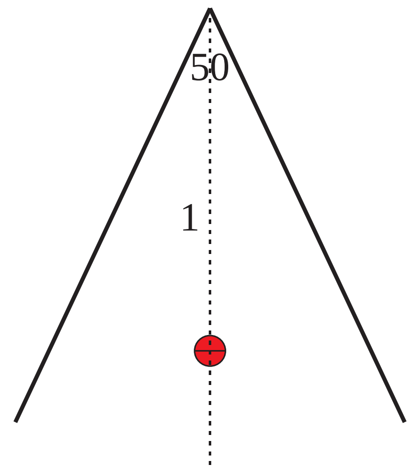
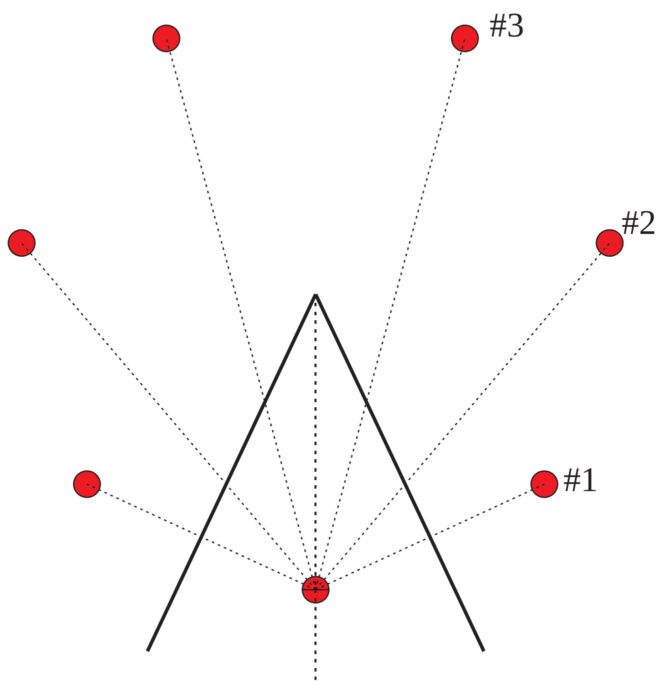
1. Mirrors

You are in the middle of two mirrors set at an angle of 50° and your eyes are 1 meter from the vertex. You can see 6 images of yourself. Here's the problem:



Plot, for each image, the path of the light ray as it travels from you back to your eye and calculate the apparent distance from you to each of these images.

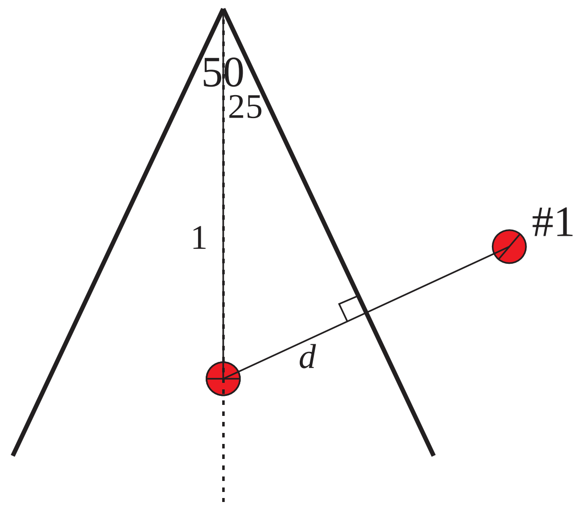


The three images look something like this. The configuration is symmetric so we only need solve the problem for the three reflections in the right-hand mirror.

Skye and Mike working on the Mirror problem. I bought these mirrors in JYSK for $12 each, used duct tape to put them together, and set them up in the classroom. It was worth it—the kids kept going back to look and think. (Photo credit Siobhain)

Image #1

The simplest case is diagramed at the right. If you look orthogonally at the mirror, your image is as far behind the mirror as you are in front of it. Of course the light ray does *not* go through the mirror––it goes from your face to the mirror and then right back to your eye––a distance of $2d$ where from the diagram:



$$\frac{d}{1} = \sin(25)$$

Thus:

$$d = \sin(25)$$

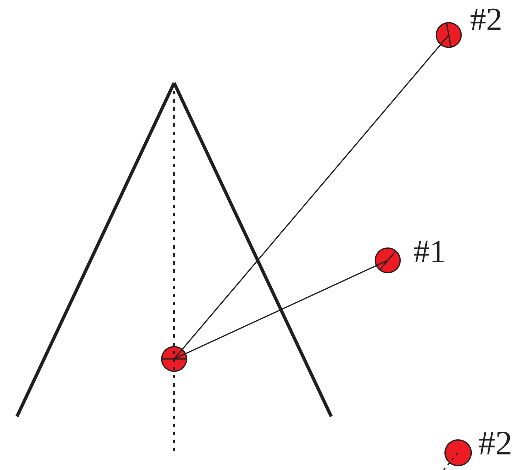
and the light ray travels a total distance

$$D_1 = 2d = 2\sin(25) \approx 0.845.$$

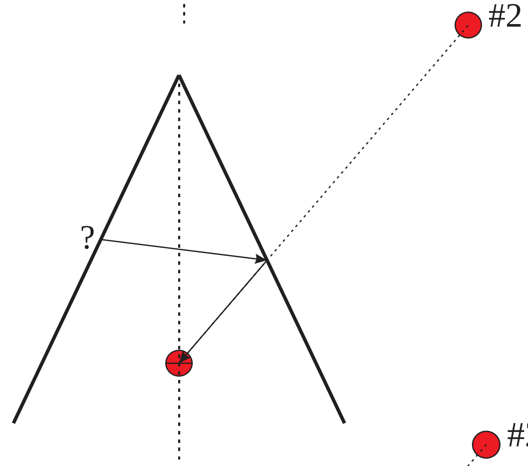
Let’s move on to image #2.

Image #2

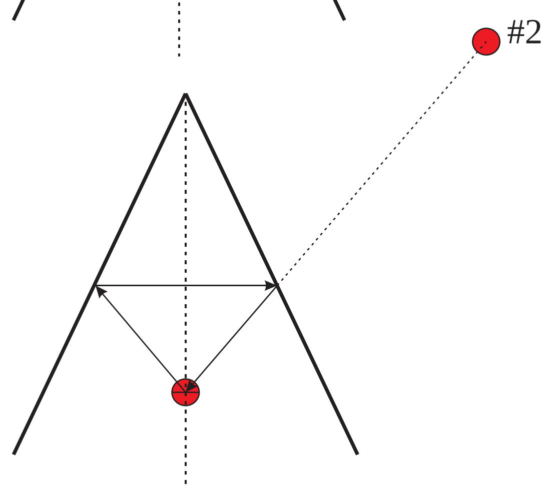
Moving on. Looking a bit to the left of the first image, you see another image that seems a bit farther away. But how does the light ray travel to your eye?



Some students notice that the first two images (the two that are depicted at the right of the mirror triangle) have a different parity. Image #1 is right-left reversed—what you always see when you look in the mirror. But Image #2 is not––it is how others see you. Now the image reverses every time the light ray is reflected, so that means that Image #2 is reflected an even number of times, likely twice. So the light ray has to also hit the other mirror.



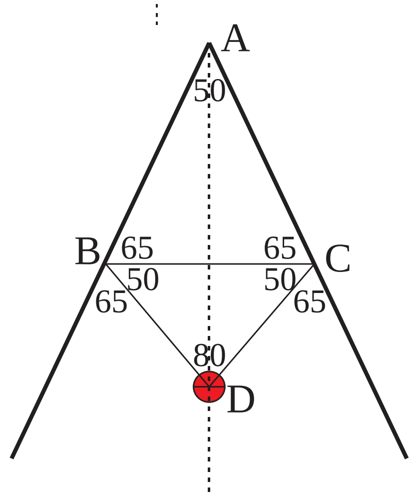
Okay, where has it come from before hitting the mirror on the left? Well you are looking at an image of yourself, so that light-ray has to have come from *you*. If we suppose the light ray reflects exactly twice, it must have come directly from you.



Given this, by symmetry (to maintain the equal-angle principle) the path has to be symmetric about the centre line.

And given that, we should be able to work out the angles and the distance of the image.

Finding the angles for Image #2



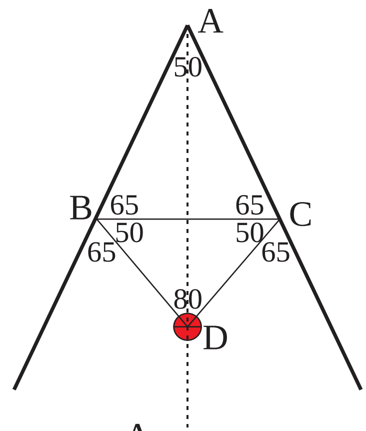
From the fact that the line BC between the two mirrors is perpendicular to the centre line, we can deduce that triangle ABC is isosceles and thus its base angles are equal. Since the vertex angle is 50° the base angles are each 65°. Using the equal angle principle (Snell's Law) we identify two more 65° angles and then the base angles of triangle DBC are both

$$180 - 2(65) = 50°.$$

Finally the angle at D will be

$$180 - 2(50) = 80°.$$

The distance of Image #2.

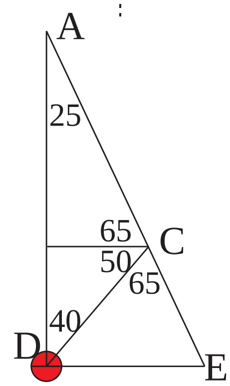


Your distance from Image #2 is the distance the light ray travels in its trip from you back to you. That's the perimeter of triangle BCD.

Now the only distance we are given is your distance from the vertex of the mirror: AD = 1

But that's enough to allow us to calculate all remaining distances. There are however many ways to do this and here I will present three of these. No doubt your students will come up with many variations.

1. Using Pythagoras. Here we work with right-angled triangles but unfortunately AD is not part of any right-angled triangle in the original diagram. The best way to get such a triangle is to draw the line DE perpendicular to AD. Then ADE is a right-angled triangle and hence



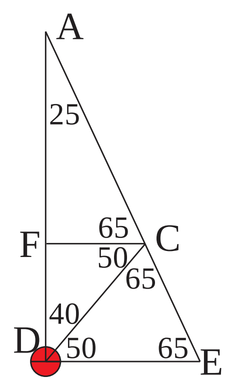
$$\frac{\mathrm{DE}}{\mathrm{AD}} = \tan(25)$$

And since AD = 1,

$$\mathrm{DE} = \tan(25) \approx 0.466$$

Now angle CDE is 50° (since ADE = 90°) so that angle CED will equal 65° and triangle CDE is isosceles with base CE. That implies:

$$\mathrm{DC} = \mathrm{DE} = \tan(25)$$



Now we introduce the point F on the line AD so that DC plus FC is half of the perimeter of triangle BCD. Then we need only find the length of FC. Well DCF is a right-angled triangle so that

$$\frac{\mathrm{FC}}{\mathrm{DC}} = \sin(40)$$

Hence

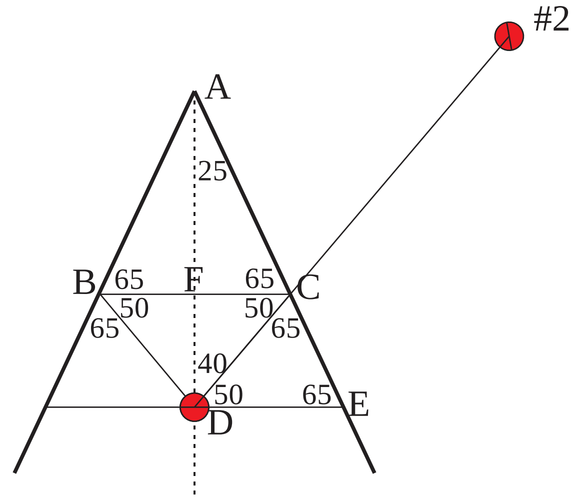
$$\mathrm{FC} = \mathrm{DC}\sin(40)$$

Then half the perimeter of triangle BCD is

$$\mathrm{DC} + \mathrm{FC} = \mathrm{DC} + \mathrm{DC}\sin(40)$$

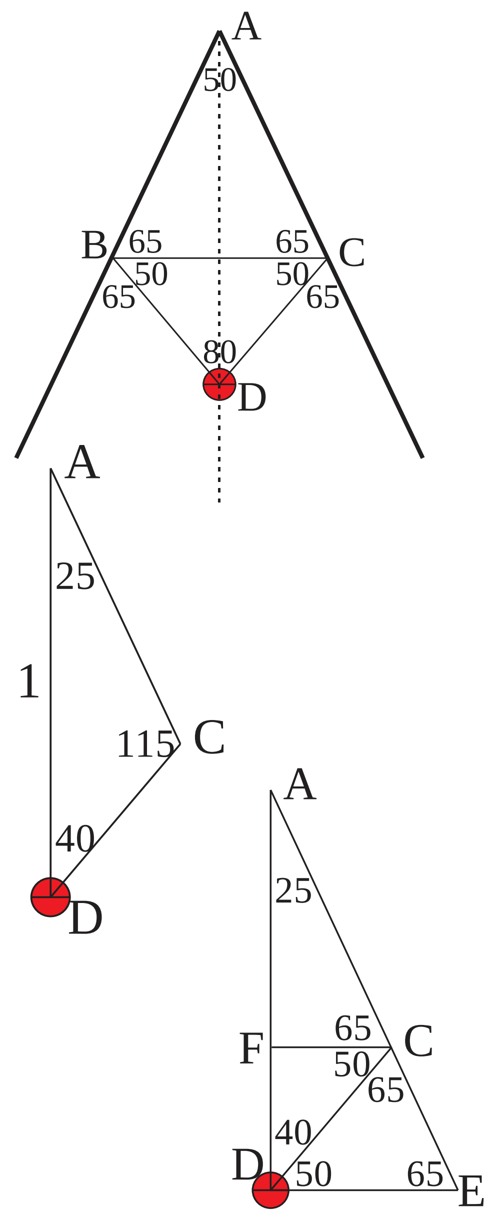
$$= \mathrm{DC}[1 + \sin(40)] = \tan(25)\,[1 + \sin(40)]$$

and the distance of Image #2 is



$$D_2 = 2(\mathrm{DC} + \mathrm{FC})$$

$$= 2\tan(25)\,[1 + \sin(40)] \approx 1.532$$



2. Using the Law of Sines. The students encountered this at the end of Grade 10 but might not have done very much with it, and I have put a primer at the end. The law frees us from having to rely on right-angled triangles.

Recall that we are looking for the distance of Image #2 and that's the perimeter of the triangle BCD. What we know is that AD = 1.

Well given that we can find DC by applying the sine law to the triangle ADC.

$$\frac{\mathrm{DC}}{\sin(25)} = \frac{\mathrm{AD}}{\sin(115)} = \frac{1}{\sin(115)}$$

Thus

$$\mathrm{DC} = \frac{\sin(25)}{\sin(115)} = 0.466$$

From this point we could use the sine law again to get BC from the triangle BCD. But we get a much simpler formula using the previous method to calculate FC. As before, we get:

$$\mathrm{FC} = \mathrm{DC}\sin(40)$$

Then half the perimeter of triangle BCD is

$$\mathrm{DC} + \mathrm{FC} = \mathrm{DC} + \mathrm{DC}\sin(40)$$

$$= (\mathrm{DC})[1 + \sin(40)] = \left(\frac{\sin(25)}{\sin(115)}\right)[1 + \sin(40)]$$

and the distance of Image #2 is

$$D_2 = 2(\mathrm{DC} + \mathrm{FC})$$

$$= 2\left(\frac{\sin(25)}{\sin(115)}\right)[1 + \sin(40)] \approx 1.532$$

Let's compare the two formulae for D_2 that we have so far. The first approach gave us

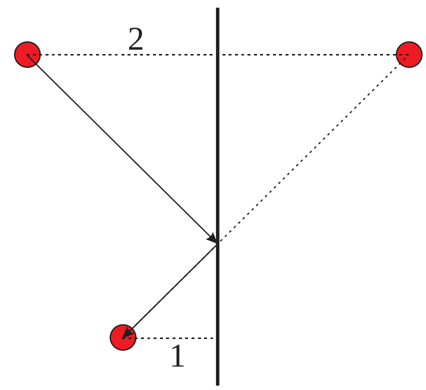
$$D_2 = \tan(25)\,[1 + \sin(40)]$$

So we deduce that

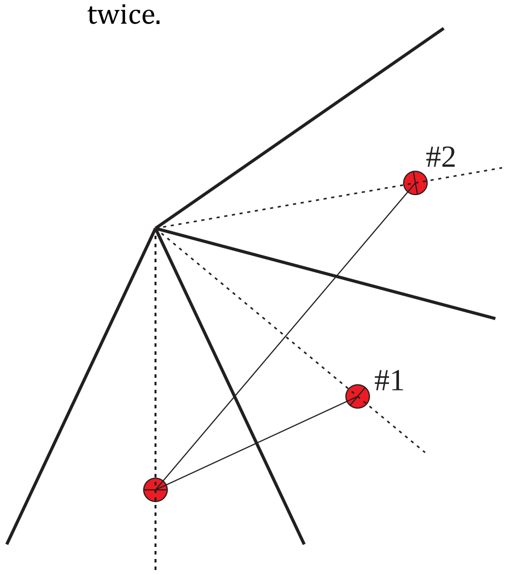
$$\frac{\sin(25)}{\sin(115)} = \tan(25)$$

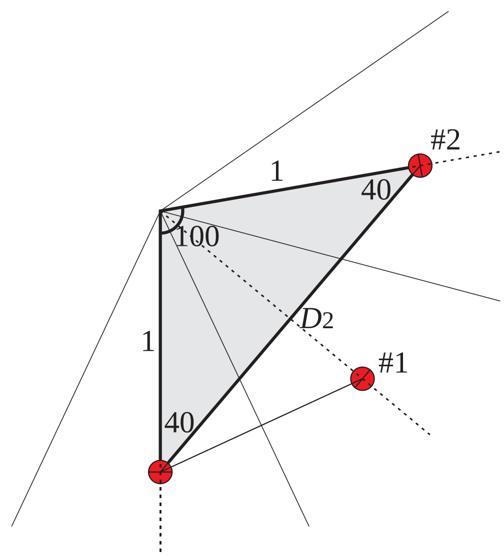
Can you show directly that this holds? [Problem 1]

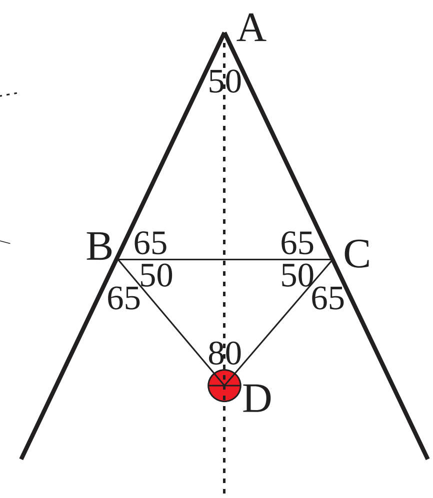
3. Reflecting the entire configuration. Some students will recall this idea from Grade 10 optics. Suppose you are standing 1 m from a wall mirror looking in the mirror at your friend beside you who is 2 m from the mirror. Where does the light-ray from your friend to you strike the mirror? A simple way to get the answer is note that the light-ray will arrive at your eye as if it had come directly from the image of your friend in the mirror. Given that you will see right away that the light ray will hit the mirror at a point that is twice as far from your friend as it is from you.



So let's reflect the mirror configuration in the mirror, and since the light ray from image #2 hits both mirrors, we will need to reflect twice.







The diagram at the left is the twice-reflected configuration and at the right the critical triangle for image #2 is shaded. The angle at the apex is 4×25 = 100° so that the base angles are 40°. The distance D_2 from you to the image is the base of the triangle. The law of sines gives us immediately

$$\frac{D_2}{\sin(100)} = \frac{1}{\sin(40)}$$

Hence

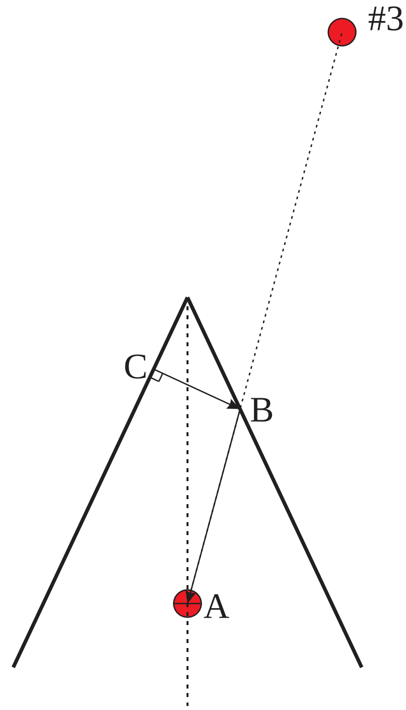
$$D_2 = \frac{\sin(100)}{\sin(40)} \approx 1.532$$

Okay guys, how on earth could that simple expression be the same as those complicated expressions we derived above?

This is actually quite a wonderful diagram and one might well ask: what on earth is the point of doing the previous calculations when there is this elegant argument available?

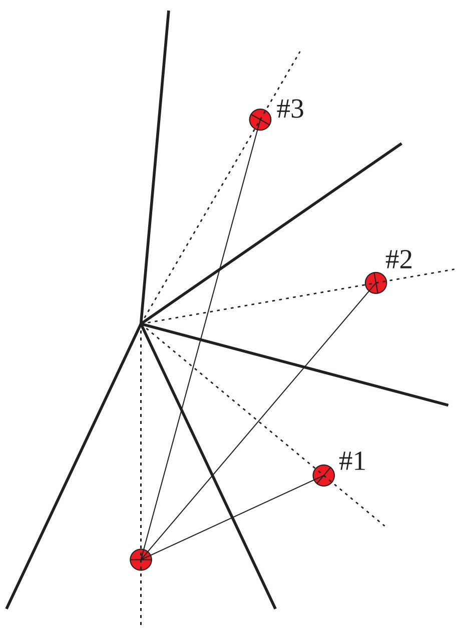
The answer is that there is some good geometric reasoning and "trig practice" in it and there is a certain fascination in seeing a problem done in such different ways particularly when they produce such different expressions for the same answer!

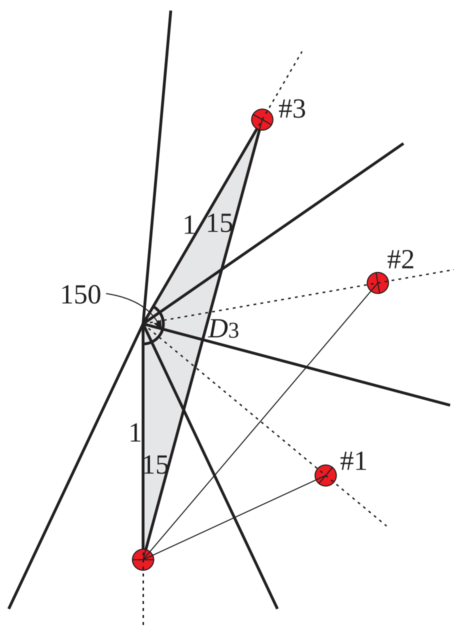
Image #3



We're going to use the reflection principle for this one, but let's just first think about the path. The light ray that enters your eye at A certainly comes directly from the right-hand mirror (because that's the mirror you are looking at). Before that it has to come from the left-hand mirror. But what can be its *previous* path (from A to C)? It didn't come to C directly from A or else it would be Image #2. So it had to come from A via the right-hand mirror. Now the only way *that* could happen (in a total of 3 reflections) is if it followed the same path that brought it from C to A. But that can only be the case if it hits the left-hand mirror at 90°. And that's enough information to work out the angles.

But instead of doing that, we'll hit the reflection diagram.





Take note of the 90° angle at C. You could argue for that directly without the benefit of the above reflection diagram.

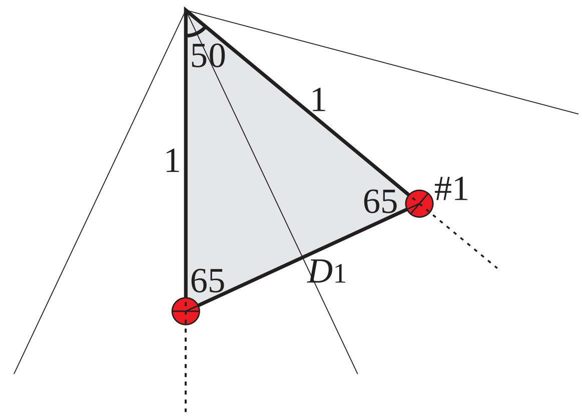
Finally we calculate the distance following the argument for Image #2.

$$\frac{D_3}{\sin(150)} = \frac{1}{\sin(15)}$$

Hence the distance to Image #3 is:

$$D_3 = \frac{\sin(150)}{\sin(15)} \approx 1.932$$

Searching for the pattern



I want to get a better sense of the pattern for the three cases. The simplest versions of the three formulae we have so far are

$$D_1 = 2\sin(25)$$

$$D_2 = \frac{\sin(100)}{\sin(40)}$$

$$D_3 = \frac{\sin(150)}{\sin(15)}$$

There's a clear pattern in the last two. To get the first equation in that pattern I need to use the reflection approach to derive it. The sine law gives me the equation

$$\frac{D_1}{\sin(50)} = \frac{1}{\sin(65)}$$

Hence:

$$D_1 = \frac{\sin(50)}{\sin(65)}$$

And we get the compelling pattern:

$$D_1 = \frac{\sin(50)}{\sin(65)}$$

$$D_2 = \frac{\sin(100)}{\sin(40)}$$

$$D_3 = \frac{\sin(150)}{\sin(15)}$$

Let's dig deeper. The pattern in the numerators is clear enough, but what about the denominators? Well, for example, the above triangle tells us that

$$65 = \frac{180 - 50}{2} = 90 - 25$$

And that works for the others as well.

$$D_1 = \frac{\sin(50)}{\sin(90 - 25)}$$

$$D_2 = \frac{\sin(100)}{\sin(90 - 50)}$$

$$D_3 = \frac{\sin(150)}{\sin(90 - 75)}$$

But we can simplify that as sin(90–x) = cos(x). So:

$$D_1 = \frac{\sin(50)}{\cos(25)}$$

$$D_2 = \frac{\sin(100)}{\cos(50)}$$

$$D_3 = \frac{\sin(150)}{\cos(75)}$$

That's neat. The angle on top is twice the angle on the bottom. Beauty.

The double angle formulae.

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = 2\cos^2(x) - 1$$

But—hey! There's a double angle formula for sin. I know–– it doesn't appear in the Ontario curriculum until grade 12. But surely this is a special enough situation to give students a heads-up!

We get:

$$D_1 = \frac{\sin(50)}{\cos(25)} = \frac{2\sin(25)\cos(25)}{\cos(25)} = 2\sin(25)$$

$$D_2 = \frac{\sin(100)}{\sin(40)} = \frac{2\sin(50)\cos(50)}{\cos(50)} = 2\sin(50)$$

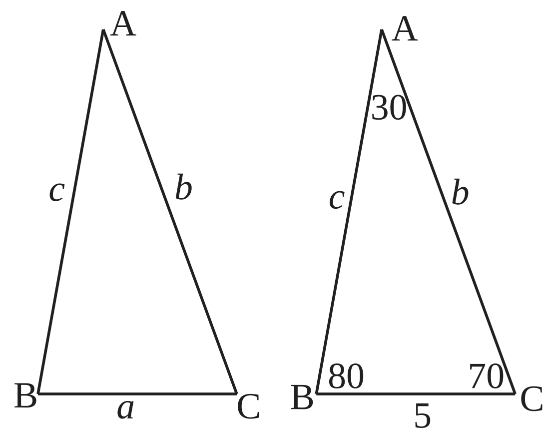
$$D_3 = \frac{\sin(150)}{\sin(15)} = \frac{2\sin(75)\cos(75)}{\cos(75)} = 2\sin(75)$$

Wow. That's bloody amazing. Who would have thought?

The Law of Sines.

The law says that in any triangle the ratio between the sines of the angles and the opposite sides are all the same:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



The triangle in the diagram has angles 30, 80 and 70 as shown. If a = BC = 5, then

$$\frac{a}{\sin A} = \frac{5}{\sin 30} = \frac{5}{1/2} = 10$$

Using this we can find the other two sides:

$$\frac{b}{\sin B} = 10 \qquad \Rightarrow \qquad b = 10\sin 80 \approx 9.85$$

$$\frac{c}{\sin C} = 10 \qquad \Rightarrow \qquad c = 10\sin 70 \approx 9.40$$

Mirrors Problems

1. Show directly that

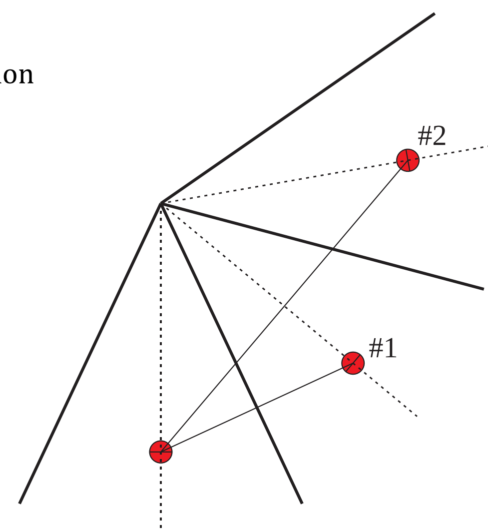
$$\frac{\sin(25)}{\sin(115)} = \tan(25)$$

2. If the mirrors are set at an angle of 40° you actually see 8 images, 4 on each side.

(a) Find formulae for the distance from the observer to each of the 4 images in the right-hand mirror. Use the reflection diagram to help in your explanations. Does the simple mathematical pattern that we found at the end of the main example still work?

(b) Use the reflection diagram to help you draw the path of the light-ray for the 4th image, the one that is farthest from the observer. Is this image left-right reversed? How close does the light-ray get to the vertex? Calculate this minimum distance.

3. Use the Python plotting program to draw the reflection graph for Image #2 (using the 50° angle).



Parabola