Further Validation of Equations for Motorcycle Lean on a Curve

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Abstract

Previous studies have reported and validated equations for calculating the lean angle required for a motorcycle and rider to traverse a curved path at a particular speed. In 2015, Carter, Rose, and Pentecost reported physical testing with motorcycles traversing curved paths on an oval track on a pre-marked range in a relatively level parking lot. Several trends emerged in this study. First, while theoretical lean angle equations prescribe a single lean angle for a given lateral acceleration, there was considerable scatter in the real-world lean angles employed by motorcyclists for any lateral acceleration level. Second, the actual lean angle was nearly always greater than the theoretical lean angle.

This prior study was limited in that it only examined the motorcycle lean angle at the apex of the curves. The research reported here extends the previous study by examining the accuracy of the lean angle formulas throughout the curves. The degree to which these equations can be used to model the development of lean as the rider enters a curve is evaluated. The prior study was also limited in that it only examined maneuvers on an oval track in a flat parking lot. The current study examines the accuracy of the theoretical lean angle formulas on a mountainous highway with curves of varying radius and changing banking and slope. The real-world data presented in this study is also utilized in conjunction with the lean angle formula to examine the interplay between the geometry of a curve, the motorcycle speed, and the rider’s skill level.

Introduction

Three basic factors limit the speed at which a motorcyclist can traverse a curve. The first of these is the limit of the available friction between the motorcycle tires and the roadway. The second is a geometric limit that is defined by the lean angle at which components of the motorcycle (a foot peg, for instance) come into contact with the roadway or at which the geometry of the tire prevents additional leaning. The third is the limit imposed by the rider’s psychological limits - their willingness to approach either the geometric or friction limits of their motorcycle [Hugemann, 2013]. Previous studies by Rose [2014] and Carter [2015] have described methods for analyzing each of these limits.

Of relevance to the present study is the fact that many riders will reach a psychological limit on their willingness to increase the lean angle of their motorcycle before they reach either the friction limit of their tires or the geometric limit of their motorcycle [Bartlett, 2011; Hugemann, 2013]. Watanabe and Yoshida found that the maximum lean angles utilized by novice riders were typically in the range of 15 to 25 degrees and those used by experienced riders were in the range of 34 to 40 degrees [Watanabe and Yoshida, 1973]. These results imply that the experienced riders used maximum lean angles that would approach the lean angle limits of many motorcycles, whereas novice riders stopped well short of the motorcycle limits. The middle values of these lean angle ranges imply that on a flat curve with a 250-foot radius, an experienced rider would be willing to lean far enough to traverse the curve at a speed of 53 mph whereas a novice rider would only be willing to lean far enough to traverse the curve at a speed of 37 mph. This further implies that the speed at which a motorcyclist can successfully follow a particular curved path depends on their own skill level and their willingness limits.

Motorcycle Lean on a Curve

The lean angle required for a motorcyclist to traverse a particular curved path will be the angle that brings the overturning moment generated by the tire frictional forces into balance with the opposing moment generated by the tire forces perpendicular to the road surface. The required lean angle increases with increasing speed and decreasing path radius. Fricke [2010] and Cossalter [2006] report that the lean angle of a motorcycle for a particular path and speed can be calculated with the following equation:

$$\theta = \tan^{-1} \left( \frac{v^2}{\mu g r} \right)$$  \hspace{1cm} (1)
In this equation, $\theta$ is the lean angle of the motorcycle, $v_{mc}$ is the motorcycle’s velocity, $g$ is the gravitational acceleration, and $r$ is the path radius. Equation (1) yields the lean angle relative to gravity or relative to the vertical. For a flat roadway, this will also be the lean angle relative to the roadway. However, if the motorcycle is traversing a curve with superelevation, the lean angle relative to the roadway will be different than what Equation (1) yields. When the curve is banked, Equation (2) can be used to obtain the lean angle relative to the roadway. 

$$\theta_{road} = \tan^{-1}\left(\frac{v_{mc}^2}{gr}\right) - \phi$$

Equations (1) and (2) assume the motorcycle is traveling a constant speed over the distance the radius is measured. Second, they assume that the motorcycle and its rider have the same lean angle. This will often be an accurate assumption, but sometimes a rider leans more or less than they lean the motorcycle. Finally, Equations (1) and (2) assume that the motorcycle tires have no width, such that the portion of the tires contacting the roadway does not change as the motorcycle and rider lean. In reality, as the motorcycle leans, the portion of the tire contacting the road changes and the contact patch moves in the direction of the lean. This results in the actual lean angle required for a particular curve being higher than that predicted by Equations (1) or (2).

Cossalter showed that the additional lean angle required due to the tire width could be calculated using Equations (3) and (4).

$$\theta = \theta_{Equation(1)} + \Delta \theta_{tire width}$$

$$\Delta \theta_{tire width} = \sin^{-1}\left(\frac{\frac{t}{2} \times \sin(\theta_{Equation(1)})}{h - \frac{t}{2}}\right)$$

In these equations, $t$ is the tire width and $h$ is the combined motorcycle and rider center of gravity height. For purposes of this study, the average of the front and rear tire widths was used. The center of gravity height of the motorcycle was estimated using Equation (5), which is from Cossalter [2002]. In this equation, $WB$ is the wheelbase of the motorcycle. The rider’s seated center of gravity height was estimated to be at his navel. The combined center of gravity height for the motorcycle and rider was calculated using Equation (6).

$$CG Height_{MC} = 0.3705 \times WB$$

$$CG Height_{Combined} = \frac{CG Height_{MC} \times W_{MC} + CG Height_{R} \times W_{R}}{W_{Total}}$$

The results presented later in this paper depend, to some degree on this estimate of the center of gravity height. Thus, additional discussion may be warranted. Cossalter’s equation [Equation (5)] was based on tests conducted with two supersport motorcycles. Foale presented the center of gravity heights for 39 motorcycles [2006]. For the sport motorcycles in Foale’s dataset, the center of gravity height was on average 38.9 percent of the wheelbase, generally consistent with, but slightly higher than, Cossalter’s data. The standard deviation on this was approximately 4.6 percent of the wheelbase. DiTallo and his colleagues presented the center of gravity heights for 25 additional motorcycles [2017]. The center of gravity height for the sport motorcycles in this dataset was on average 36.9 percent of the wheelbase. The standard deviation on this was approximately 5.2 percent of the wheelbase. Thus, Foale [2006] and DiTallo [2017] provide additional validation that the center of gravity height estimate used here is reasonable.

### Physical Testing

On August 9, 2017, the authors conducted physical testing using a 2007 Suzuki GSX-R750 motorcycle (Figure 1). This testing utilized a single experienced rider who was a motorcycle safety instructor through the Motorcycle Safety Foundation (MSF). The rider traversed the route with the goal of maintaining safety, varying his speed in accordance with the characteristics of the roadway. No special instructions were given to the rider in terms of how he should lean his body relative to the motorcycle.

This testing involved the rider driving the motorcycle westbound along County Road 95 between Frazier Park, California and State Highway 33. Seven curves were identified for analysis on this section of roadway. The rider also drove southbound along Highway 33 and three additional curves were identified for analysis on this section. Three of the ten curves were digitally mapped with a Faro laser scanner. One of these three was also mapped with a total station. The remaining seven were mapped with aerial imagery. The images below depict the geometry of the curve that was mapped with both a total station and a Faro scanner. Figure 2 is an aerial photograph showing this curve.

This photograph is oriented such that north is up on the page. Riders traveling southbound through this curve would first traverse from the top left to the top right of this photograph and then exit the curve traveling from right to left across the bottom of image. Figure 3 is a photograph showing the geometry of the curve from a ground level perspective. Riders traveling southbound through this curve would traveling towards the viewer of this image. Figures 4 shows the 45,784,170 scan data points that were captured in the area of this curve.
Throughout the entire route, the path, speed, and lean angle of the motorcycle were continuously recorded using a Racelogic VBOX that measured speed, position, and roll angle at 20 Hz. The VBOX system was a VB20SL3 model that utilized two GPS antenna. A metal crossbar was strapped to the rear of the motorcycle and the GPS sensors were magnetically attached to this crossbar, near its outer extents (Figures 1 and 5). The motorcycle was scanned with a Faro laser scanner both before and after the ride, so that any displacement of this bar that might occur over the course of the ride could be quantified. The VBOX data logger was carried in the rider’s backpack. This testing was also captured with a video camera and a GoPro camera attached to a chase vehicle and with two GoPro cameras attached to the rider (helmet and chest). Figure 6 contains images from the chase camera with the speed and lean angle of the motorcycle overlaid onto the images. All of these cameras were recording at a rate of 30 frames per second. At the time of the testing, the road surface was dry.

### Analysis

The VBOX data from each of the 10 curves was analyzed to determine the motorcycle’s actual path radius, speed, and lean angle relative to gravity (rather than relative to the road surface). For evaluation of Equations (1) and (3), these values were tabulated for the motorcycle’s entire traversal of each curve. The path radius was calculated using positional data from the VBOX. The equations involved in this calculation began with the following equation:

\[
r^2 = (x - h)^2 + (y - k)^2
\]

In this equation, \( h \) and \( k \) are the coordinates for the center of a circle with a radius, \( r \). The coordinates \( x \) and \( y \) are on the circle. The VBOX data included coordinates for the position of the antenna at each sample. For a motorcycle traversing a curved path, three points were used to determine the radius of the path between the three points - the downstream point \( (x_{i-1}, y_{i-1}) \), the middle point \( (x_i, y_i) \), and the upstream point...
$\left( x_{i-1}, y_{i-1} \right)$. The calculated radius is assigned to the middle of the three points. This process involved the following steps: (a) The downstream and middle points for a segment of the VBOX data was entered into the following equation:

$$
\left( x_i - h \right)^2 + \left( y_i - k \right)^2 = \left( x_{i-1} - h \right)^2 + \left( y_{i-1} - k \right)^2
$$

(8)

(b) The middle and upstream points were entered into the following equation:

$$
\left( x_i - h \right)^2 + \left( y_i - k \right)^2 = \left( x_{i+1} - h \right)^2 + \left( y_{i+1} - k \right)^2
$$

(9)

(c) Equations (8) and (9) represent a system of linear equations with two equations and two unknowns, $h$ and $k$. Expanding the terms in these equations and rearranging to create linear equations with unknown variables $h$ and $k$ yields Equations (10) and (11).

$$
A - B \cdot h - C \cdot k = 0
$$

(10)

$$
D - E \cdot h - F \cdot k = 0
$$

(11)

In Equation (10), the constants $A$, $B$, and $C$ are:

$$
A = \left[ x_{i-1}^2 - x_i^2 + y_{i-1}^2 - y_i^2 \right]
$$

$$
B = \left[ 2x_{i-1} - 2x_i \right]
$$

$$
C = \left[ 2y_{i-1} - 2y_i \right]
$$

In Equation (11), the constants $D$, $E$, and $F$ are:

$$
D = \left[ x_i^2 - x_{i+1}^2 + y_i^2 - y_{i+1}^2 \right]
$$

$$
E = \left[ 2x_i - 2x_{i+1} \right]
$$

$$
F = \left[ 2y_i - 2y_{i+1} \right]
$$

Solving equation (10) and (11) for the unknown variables, $h$ and $k$, yields:

$$
k = \frac{A - B \cdot h}{C}
$$

(12)

$$
h = \frac{C \cdot D - F \cdot A}{C \cdot E - F \cdot B}
$$

(13)

(d) The radius of the path at the middle point was then calculated with Equation (7).

When the authors instrumented the motorcycle, they attempted to position the antenna bracket perpendicular to the vertical axis of the motorcycle, such that the lean angle recorded by the VBOX system accurately represented the lean angles of the motorcycle. The authors then scanned the motorcycle in its instrumented state. The scan data was examined, and resulted in a lean angle offset of 2.2 degrees. This offset was subtracted from the VBOX lean angle measurements.

Adjustments were also made to compensate for the primary antenna placement relative to the center of gravity of the motorcycle. When viewed from behind, the antenna on the left side of the motorcycle was used to measure the position and speed, while the right antenna was used only to measure lean angle. To adjust the radius as measured by the primary VBOX antenna to the center of gravity of the motorcycle / rider combination, an adjustment was calculated using Equation 14.

$$
\Deltar = \sqrt{d_i^2 + d_i^2 \sin \left( \theta_{\text{VBOX}} + \tan^{-1} \left( \frac{d_f}{d_r} \right) \right)}
$$

(14)

In this equation, $d_i$ is the lateral distance from the motorcycle vertical axis to the antenna, and was equal to 19.685 inches (0.5 meters), while $d_f$ is the vertical distance from the motorcycle / rider center of gravity to the primary antenna, which was estimated to be 13.5 inches (0.34 meters). For left hand turns, this adjustment was positive and resulted in a path radius for the center of the motorcycle greater than the path radius of the primary antenna. For right hand turns, this adjustment was negative and resulted in a smaller path radius for the center of the motorcycle compared to the primary antenna.

**Results**

The positional data from the VBOX was analyzed using Equations (7) through (12) to determine the instantaneous radius of the path of the motorcycle at each point along each curve. To eliminate excessive noise due to the sensitivity of the radius calculation to small changes in positional data, the upstream and downstream points were selected to be ½ second before and after the time of interest (10 samples before and 10 samples after the point of interest).

Figure 8 depicts a comparison of the VBOX measured lean angle (dashed black) to the lean angle calculated with Equation (1) (green) and with Equation (3) (red) for Turn 4. In this figure and subsequent figures, positive values for lean angle indicate a leftward lean, while negative values indicate a rightward lean. Examination of this graph reveals that Equation (1) tends to underestimate the fully developed lean for each curve. Equation (3), on the other hand, closely predicts the lean angle throughout the course of this curve.

Figures 9 through 18 depict the VBOX measured lean angle in dashed black and lean angle calculated with Equation (3) in...
FIGURE 8  Comparison of VBOX Lean Data to Lean Angles Calculated with and without Consideration of Tire Width (Turn 4)

FIGURE 9  Comparison of VBOX Lean Data to Calculated Lean Angle (Turn 1)
FIGURE 10  Comparison of VBOX Lean Data to Calculated Lean Angle (Turn 2)

FIGURE 11  Comparison of VBOX Lean Data to Calculated Lean Angle (Turn 3)
FIGURE 12  Comparison of VBOX Lean Data to Calculated Lean Angle (Turn 4)

FIGURE 13  Comparison of VBOX Lean Data to Calculated Lean Angle (Turn 5)
FIGURE 14  Comparison of VBOX Lean Data to Calculated Lean Angle (Turn 6)

FIGURE 15  Comparison of VBOX Lean Data to Calculated Lean Angle (Turn 7)
FIGURE 16  Comparison of VBOX Lean Data to Calculated Lean Angle (Turn 8)

FIGURE 17  Comparison of VBOX Lean Data to Calculated Lean Angle (Turn 9)
**FIGURE 18** Comparison of VBOX Lean Data to Calculated Lean Angle (Turn 10)

**FIGURE 19** VBOX Lean Data to Calculated Lean Angle with Reduced Satellite Timeframes (Turn 2)
red, along with the difference between the calculated and the measured values in blue. These graphs show that the calculated lean angle is typically within 3 degrees of the measured lean angle, and is seldom greater than 5 degrees different than the measured lean angle. Overall, the calculated lean angles closely follow the shape of the actual lean angle curves.

Table 1 lists the average differences between the measured lean angle and the lean angle calculated with Equation (3). The difference was calculated in two ways - first, simply as the average of the point-to-point differences, with consideration for the positive and negative signs associated with left and right turns, and second, as the average of the absolute value of the difference. Examination of the data during Turns 2 and 8 showed two short sections of variance significantly greater than the averages in Table 1. Analysis of these sections of data revealed positional data error due to a reduction in the number of GPS satellites in view during those specific sections of the turn. The GPS location data is used to calculate the instantaneous radius and lean angle during the turn.

During the testing, the VBOX unit was typically acquiring data from six or seven satellites. During Turns 2, 6 and 8, the number of satellites in view briefly dropped to four. This was not unexpected because the testing area was a mountainous roadway with areas where satellite signals could be blocked by terrain.

<table>
<thead>
<tr>
<th>Turn Number</th>
<th>Average Difference (deg)</th>
<th>Average of Absolute Value of Difference (deg)</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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</tr>
<tr>
<td>2</td>
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</tr>
<tr>
<td>3</td>
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</tr>
<tr>
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</tr>
<tr>
<td>5</td>
<td>0.10</td>
<td>0.89</td>
</tr>
<tr>
<td>6</td>
<td>0.10</td>
<td>0.89</td>
</tr>
<tr>
<td>7</td>
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<td>0.82</td>
</tr>
<tr>
<td>8</td>
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<td>1.18</td>
</tr>
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</tr>
<tr>
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<td>1.14</td>
</tr>
<tr>
<td>All Data</td>
<td>-0.10</td>
<td>0.94</td>
</tr>
</tbody>
</table>

The accuracy of a GPS system generally increases when more satellites are in view [Garmin, 2017]. According to the VBOX VB20SL3 User Guide, quality signal reception is dependent on the VBOX receiving signals from at least five satellites [Racelogic, 2012]. Figures 19 and 20 contain plots for Turns 2 and 8 with green vertical lines added to represent the
timeframe during each turn when data was collected with only four satellites in view. The timeframe during Turn 6 when only four satellites were in view was very short and did not have a significant effect on the accuracy of the data.

**Conclusions**

1. Using Equation (3) to account for the tire width of the motorcycle resulted in a more accurate prediction of the lean angle than using Equation (1), which does not account for the tire width.
2. Equation (3) typically predicts the actual lean angle within 3 degrees (sometimes underestimating and sometimes overestimating). The difference between calculated and actual is seldom greater than 5 degrees.
3. Equation (3) reasonably models the buildup of lean angle through the progression of a curve (in the time domain). Thus, Equation (3) could be used to compare the rate at which a motorcyclist would need to lean when traversing a curve at various speeds.

**References**


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