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CHAPTER 1

Quantities, Numbers, & Counting

It all starts with quantities . . .

A child starts his trip into mathematics by learning to identify quantities of objects. Children are essentially born with the ability to distinguish small quantities of one, two, or three objects directly by sight.

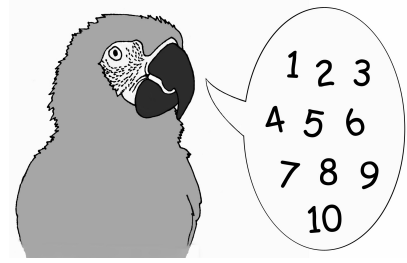
There is an important difference between working with quantities of objects and working with the associated numbers. It is one thing for a child to see that when two apples are put with three apples that there is a collection of five apples, and quite another to understand that when the number two is added to three the answer is five.

If you ask a child entering kindergarten to add 5 + 2, it is not unusual for a child of that age to do this by thinking of five things, perhaps fingers, and then putting two more of the same thing with them. That is, the child understands how to do the adding with quantities, and has not yet learned to work directly with numbers. This is perfectly fine, and is just part of the process all children go through in learning mathematics.

Basic number understanding, identification, and counting are where everything starts for you as a teacher. Naming the quantity of things and counting things out in front of your child can start at any age. This practice helps lay the groundwork for number familiarity and acceptance of numbers as part of the normal day to day world.

Apply counting out loud to the simple things you see in the world, and have your child apply it doing everyday things. You can count shoes going on feet, vegetables being cut in the kitchen, food being placed on a plate, steps being climbed to get upstairs, nails being hammered into a board, red cars being passed on the road, and so on.

Avoid having your child simply memorize the sequence of numbers as they are counted out. It may be impressive to friends and relatives that your child can count to 10 at a very early age, but it serves no good if your child is only parroting a sequence of meaningless words.



While learning the numbers, it helps to compare the relative sizes of numbers. It is very easy to ask additional questions, such as who has more of something, or who is winning or losing, or how much one person or team is ahead or behind in some game.

When doing comparisons, compare collections of objects where the objects are essentially identical. Little children often get confused between the concepts of quantity (how many) and size (how much length, area or volume), so you need to be a bit careful.

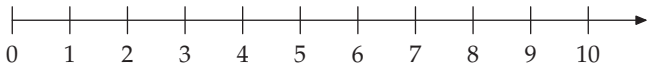
For example, if one person has a single new long pencil and the

other person has two well-used short pencils, little children are very tempted (quite reasonably) to say that the single pencil represents more pencils.

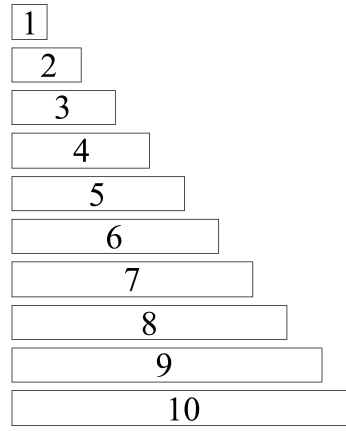
Two invaluable visual aids at this stage of a child's developing quantity and number understanding are a number chart (usually from 1 to 100) and a number line.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

A number line is particularly useful in discussions comparing sizes of numbers. Your child will learn to associate being larger with being farther to the right on the number line, and being smaller with being farther to the left.



Number strips are another extremely helpful manipulative for learning the relationship between sizes and numbers. They are also helpful for learning addition and subtraction, and for developing a feeling for the numbers on a number line. There is a description for making these strips in Section 14.2: *Number strips* in the *Manipulatives* Chapter.



As your child is able to count up to 6, 10, or 20, you can start playing some traditional games together. Games such as Parcheesi, Sorry, Dominoes, and Chutes and Ladders can be a lot of fun, and of course they provide great practice. For games that involve two dice, such as Parcheesi, if your child does not feel comfortable with all of the adding involved, then just count all of the dots on the two dice.

Some of these games can be played with almost no counting whatsoever. For example, if the game involves rolling one die, then your child can move his game piece one place forward for each dot showing on the die. One game of this type is Mancala. This game involves moving small quantities of beads, but no counting is needed. It is described further in the section *Mancala* in Chapter 15: *Learning Games and Activities*

1.1 A lot to learn

LESSON

Your child has more to learn than you may realize.

Developmental steps There are many developmental steps that your child must go through before being completely ready to count and understand what it means.

1 to 1 principle When counting objects, a single number is associated with each single object in the group.

Ordered numbers principle The numbers used for counting must be used in a standard, fixed order. That is, counting is done 1, 2, 3, 4, and not 1, 5, 2, 4.

Last number principle When counting a group of objects, the number associated with the last item counted is the number of objects in the group. It is one thing to associate one number with each item counted, it is another to realize that the last number used is the total number of objects.

Universal principle Counting can be applied to any group of objects, independent of the type of objects they are. Counting can be used with fingers, apples, people, cars, or any other collection of things.

Unordered objects principle The objects in a group may be counted in any order, and the count will always be the same. If you have a red, a green, and a blue ball, you will count three balls whether the first ball counted is the red one, the green one, or the blue one.

Most children will work through these developmental steps somewhere between the ages of 2 and 3.

These are not things you can teach your child by explaining them. What you can do is count in front of your child, and with your child, whenever the possibility presents itself. Give your child lots of opportunities to develop these principles internally.

Ultimately, you must wait for your child to be ready.

So much to learn Once your child is able to count, it is important that you understand just how much there is for your child to learn. Your child will be learning to say, hear, write, and read the names and symbols for the different numbers, as well as learning to recognize and associate the quantities linked with those numbers.

To give you a taste of how much effort is involved for a child learning new symbols and how to use them, educator Rick Garlikov proposes that we invent ten new names for the digits, say "*a*" through "*j*" in place of 0 through 9.

Now try to count by *h*'s to the number *ha* (count by 7's to 70).

$$h, be, cb, ci, df, ec, ej, fg, gd, ha$$

$$7, 14, 21, 28, 35, 42, 49, 56, 63, 70$$

As another example, figure out that $db + ij = bca$. Although it is the same thing, notice how much harder this is for us than the usual calculation $31 + 89 = 120$.

It may be automatic for us to see $1+9$ and think 10, but look how much harder it becomes when you see the same thing written as $b + j = ba$. There is a lot that is easy for us that your child has to work for.

CHAPTER 2

Finger Adding and Subtracting

Children start learning the basic adding and subtracting facts by first building up a solid feeling for how the numbers are connected in the counting sequence. This is accomplished by doing lots of counting, as in the last chapter, and by doing lots of the initial adding with any manipulatives that are available. Of course, the manipulative that is most easily available is your child's fingers.

It is essential that your child knows and feels what the numbers mean in terms of quantities, rather than having them be abstract symbols that are manipulated and memorized. For these early adding and subtracting problems, do not rush to get your child to simply memorize the answers. Spend lots of time doing examples with groups of objects so that your child understands these operations.

The initial adding and subtracting steps discussed in this chapter are aimed at providing your child with understandable and dependable methods for doing adding calculations on their fingers. The steps in the subsequent chapter provide structural techniques that your child can use to replace the finger methods and gain greater speed.

Subtracting can be thought of as "take away" or as a "difference." Both viewpoints are important, and should be used and practiced by your child. For example, to subtract 6 from 9 you can write 9 – 6, and talk about starting with 9 things and *taking away* 6 of them to see what you have left. You can also discuss the *difference* between 9 and 6 by how much distance, space, or room there is between 6 and 9.

While introducing adding and subtracting, use words that are more natural. Use **and** and not the more mysterious words **adding**, **plus**, and **sum**. Similarly, use **take away** and **difference** instead of **subtracting** and **minus**. Gradually introduce the more technical words as your child becomes more comfortable with the operations.

As you will see in the next chapter, some subtraction facts are more easily learned thinking of them as taking away, and others as differences. For now, just make sure your child is comfortable with both ways of thinking of subtraction.

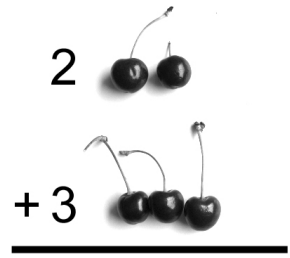
GAME

Hidden Objects: Start with any number of objects; for example, 6 coins. Take turns with your child removing some number of the 6 coins without letting the other person see. The other person is challenged to figure out how many coins were removed by looking at how many are remaining. A variation of this game is to let your child see how many coins are taken away and have your child figure out how many are remaining.

For some children, subtracting, especially the concept of "taking

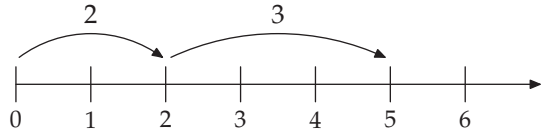
away," may seem more natural or easier to catch on to than adding. If this is the case for your child, feel free to start with subtraction.

If you do any written work with your child at this level, consider accompanying the work with pictures of the corresponding number of objects. For example, I have added cherries in the following addition problem to give something concrete to count:



Alternatively, instead of pictures added to the written problem, provide manipulatives for your child to set up the problem and then count.

Another method for illustrating addition is to use a ruler or a number line on a piece of paper. Using this, addition is seen as moving the corresponding number of steps to the right along the number line.



At the back of this book, Chapter 16: *Alternative Methods To Fingers*, is a chapter on alternatives to working with fingers for addition and subtraction. You may want to explore some of the techniques mentioned there to see if you would like to use any of them instead of, or in addition to, the techniques in this chapter.

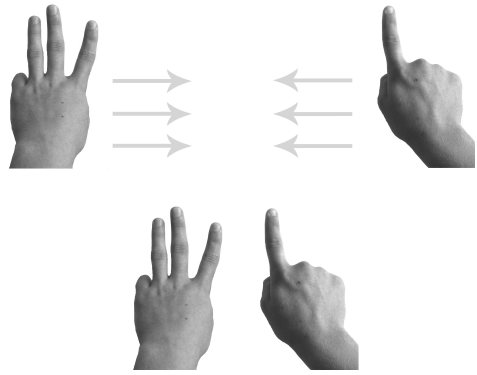
2.1 Adding – totals up to 5

LESSON

Finger addition with totals up to 5.

Practice

Fingers Put up the number of fingers on each hand for the numbers being added. First show them apart, and then bring them together and have your child count the total.



Piles of things You can also do this counting/adding with everyday things such as pennies on a table, a small set of beads, or whatever is handy. Make a point of doing addition whenever the opportunity should arise – "Look, I have 3 flowers and you have 1 flower, so how many do we have together?"

It is easy Approach this as a simple thing. Your child won't think it is any harder than counting if no suggestion of difficulty is made.

Patience Do this until your child is very familiar with the problems, and has little need to see fingers or other manipulatives.

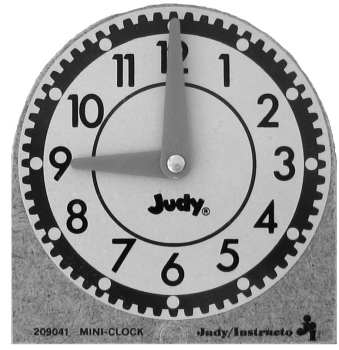
11.2 Clock face reading

LESSON Reading the time from a clock face.

Practice **Areas of confusion** There are a couple of typical areas of confusion. The first is remembering which hand is the minute hand and which is the hour hand. Another is figuring out which hour the hour hand is pointing to. A third problem is learning that the clock face numbers are not only used to indicate the hour, but also, somewhat cryptically, the minutes.

Establish early and often that the hour hand is the shorter of the two hands, and the minute hand is the longer of the two hands. There generally is little additional confusion if the clock happens to have a second hand, because the second hand moves so quickly compared to the others.

Teaching clocks Most educational supply stores sell clock faces that have hands that are easy to move around to whatever time you want to talk about. You can, of course, just use any wall clock or similar clock, but the hands are usually much harder to move around, and they are also easier to break.



You can also simply draw pictures of clock faces with various times on them, or get math work books that have pages of questions about telling time.

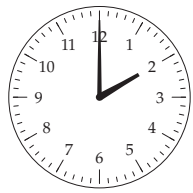
Four steps Reading the time from a clock face occurs in four steps.

1. Read the time exactly on the hour.
2. Read quarter hours and half hours
3. Read with minutes that are a multiple of five
4. Read all times

During these steps be sure to teach each skill in two ways – by showing a clock face and asking what time it shows, and by giving a time and asking your child to set the clock hands to show that time.

Step 1 Read the time exactly on the hour.

Start by telling time for exact hours where the minute hand is pointing at the 12. Do lots of examples, especially some with 12 o'clock, where the hour hand is also pointing at the 12.



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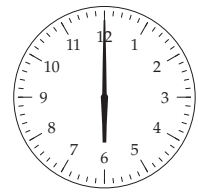

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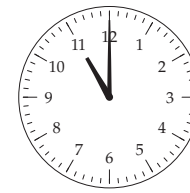


11:00

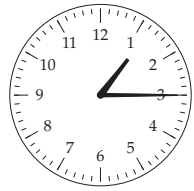
Step 2 Read quarter hours and half hours.

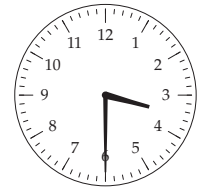
Usually children quickly learn that the minute hand pointing at the 3 means quarter past, pointing at the 6 means half past, and pointing at the 9 means quarter till.

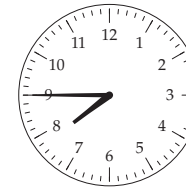
What is often difficult for children is deciding which hour it is when the hour hand is not pointing straight at one number.

Approach this by first teaching your child the direction, "clockwise," of hand movement around a clock. The hands move from a number to the next bigger number (except when moving from 12 to 1).

After your child is confident with clockwise movement, then, when the hour hand is not pointing directly at a number, you can ask which number the hour hand is leaving, or which number was the last one it pointed at. This will tell your child the hour.


1:15


3:30


7:45

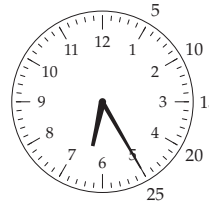
As your child gets comfortable with this step, start referring to these times sometimes in terms of 15, 30, or 45 minutes. For example, mention that a quarter past three can also be described as 15 minutes past three.

Step 3 Read the minutes that are a multiple of five.

To do this step your child needs to either be able to do multiplication by 5 (Section 4.7), or skip count by 5 (Section 4.5).

If your child is using skip counting, then there are two strategies that can be used. The more straight-forward approach for skip counting is to always skip count by 5's around the clock until reaching where the minute hand is pointing.

The second approach is to use knowledge of the values for 15, 30, and 45 minutes from the quarter and half hours. For example, if the minute hand is pointing at the 8, your child can start at the 6 and say "30, 35, 40" and get to 40 much more quickly.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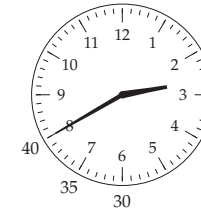

2:40

The practice in this step is great for working on skip counting and multiplying by 5. Similarly, if you take a time such as 3:40 and ask your child to position the hands of the clock, this is excellent practice for Section 5.5: *Dividing by 10 and 5* in the *Dividing* Chapter.

Step 4 Read all times.

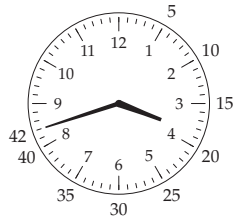
For this step you will need a clock that has tick marks for all of the minutes.

The method for figuring out how many minutes it is past the hour is to know how many minutes it is to the last multiple of five, and then add on the additional minutes. For example, to read 3:42 on the clock face, your child would start at the 8 on the clock face and say "40," and then add the additional two minutes to get 42.

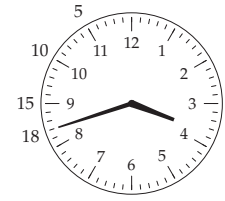
As before, there may be occasional confusion about which hour it is. Follow the directions given in step 2 for understanding which hour the hour hand most recently pointed to.

After getting comfortable and practiced with telling how many minutes it is after the hour, have your child practice describing some of the times as a certain number of minutes before the next hour. For example, 3:42 can also be described as "18 minutes before 4."

Giving the number of minutes before the hour is not usually done by subtracting from 60, though it can be done that way. Usually, the child will count by 5's backward from the 12 on the clock, and then add the few extra minutes as needed. For example, for 3:42 the child would count back "5, 10, 15" and then add 3 more to get to the 42 minute mark.



42 after 3



18 before 4