Weak Monotone Comparative Statics

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Abstract

We develop a theory of monotone comparative statics based on weak set order, or in short *weak monotone comparative statics*, and identify the enabling conditions in the context of individual choices, Pareto optimal choices for a coalition of agents, and Nash equilibria of games. Compared with the existing theory based on strong set order, the conditions for weak monotone comparative statics are weaker, sometimes considerably, in terms of the structure of the choice environment and underlying preferences of agents. We apply the theory to establish existence and monotone comparative statics of Nash equilibria in games with strategic complementarities and of stable many-to-one matchings in two-sided matching problems, allowing for general preferences that accommodate indifferences and incomplete preferences.

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1 Introduction

Comparative statics in economics concerns how predictions of behavior—be it individual choices, collective or social choices, or equilibria of games—change as economic conditions indexed by some parameters change. In many economic problems, predictions are non-unique, so they are represented by a set $S(t) \subset X$ indexed by a parameter $t \in T$, for some set X of possible predictions. The key question is then: *what would it take for set $S(t)$ to "increase" as $t \in T$ increases.* Although there are typically well-defined orders on X

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and on T, there may be no clear sense of how one set S' "dominates" another S, given the primitive order $\geqslant$ defined on X.[1]

The theory of monotone comparative statics pioneered by Topkis (1979, 1998) and Milgrom and Shannon (1994) focuses on the so-called "strong set order," denoted $\geqslant_{ss}$. Namely, $S' \geqslant_{ss} S$ if, *for any* $x \in S$ *and* $x' \in S'$, $x \vee x' \in S'$ *and* $x \wedge x' \in S$, *where* $x \vee x' := \inf\{x'' \in X : x'' \geqslant x, x'' \geqslant x'\}$ *and* $x \wedge x' := \sup\{x'' \in X : x'' \leqslant x, x'' \leqslant x'\}$, and $\geqslant$ is a partial order on X. This notion of induced set order implies an intuitive property, captured by a weaker notion called "weak set order" and denoted by $\geqslant_{ws}$. Namely, $S' \geqslant_{ws} S$ if, *for each* $x \in S$*, one can find* $x' \in S'$ *such that* $x' \geqslant x$*, and likewise, for each* $x' \in S'$*, one can find* $x \in S$ *such that* $x \leqslant x'$*.* Strong set order is stronger than weak set order, although the economic meaning of the difference may not be easy to interpret or motivate. For ease of our discussion, we refer to *monotone comparative statics in strong set order* as **strong monotone comparative statics** (or **sMCS** in short), whereas the one in weak set order—the focus of this paper— as **weak monotone comparative statics** (or **wMCS** in short).

As shown by Topkis (1979, 1998) and Milgrom and Shannon (1994), in the context of individual choices, the strong set order proves to be an appropriate notion. Intuitive conditions capturing complementarities across alternative choice dimensions and complementarities between them and parameters are known to be sufficient for sMCS of their maximizers and they are also necessary if one insists upon the same properties to hold for every subdomain.[2]

Beyond individual choices, however, strong set order proves less useful. Take Nash equilibria of a game. Topkis (1979, 1998), Vives (1990), Milgrom and Roberts (1990), and Milgrom and Shannon (1994) show that complementarities between one's strategies and those of her opponents as well as a parameter, say t, ensure each player's best-response correspondence to vary monotonically in strong set order with those variables. Yet, this does not lead to the same sort of monotonic shift for Nash equilibria. More specifically, appealing to Tarski (1955)'s fixed-point theorem, one could under suitable conditions guarantee that Nash equilibria contain the largest and smallest elements, each of which varies monotonically with t. This result *does* imply monotone comparative statics in weak set

[1]While monotone selection—i.e., S' declared to dominate S if $x' \geqslant x$ for every $x \in S, x' \in S'$—would be most natural and easy to interpret, monotone selection is rather difficult to achieve for individual choices and virtually impossible beyond individual choices such as for equilibria of games.

[2]See Milgrom and Shannon (1994) for the detail and our discussion in Section 3. See Quah and Strulovici (2009) for a characterization with a weaker condition known as interval dominance for the case in which X is a *chain*, i.e., a totally ordered set.

order but not in strong set order (see Figure 1).[3]

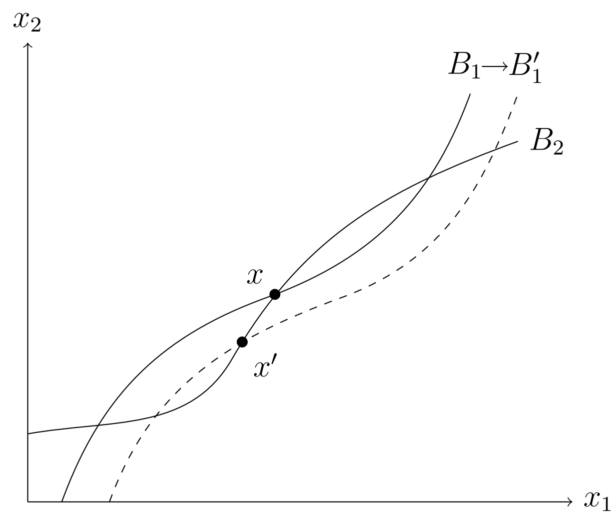

Figure 1: Failure of sMCS.

Note: This figure depicts how equilibria of a game with strategic complementarities may change as player 1's best response shifts out from B_1 to B_1' with a parameter change. The points x and x' are equilibria before and after the change, respectively. However, $x \vee x' = x$ is not an equilibrium after the change.

Consider next a social choice problem. One may be interested in the monotone comparative statics of Pareto optimal choices by a collection of agents, although, to the best of our knowledge and to our surprise, this interesting question has never been asked let alone investigated. Monotone comparative statics is unlikely to hold in strong set order here as well. Under suitable conditions, Pareto optimal choices consists of a union of alternatives that maximize social welfare—a weighted sum of individual utilities—, where the union is taken over all possible welfare weights (and partitions of agents, as will become clear). Neither strong set order nor a lattice property is preserved by the set union operator. Hence, even when the welfare-maximizing alternatives can be made to vary monotonically in strong set order for fixed welfare weights, their union (over all welfare weights) fails to vary monotonically in the same sense. Here again, a more suitable notion of monotonicity is the weak set order.

Finally, consider two-sided matching problems where agents on two sides—e.g., men and women, students and schools, and workers and firms—seek to match across the sides

[3]Short of assuming "uniqueness," no obvious way of strengthening the notion of complementarities either across players' strategies or between their strategies and parameters restores monotone comparative statics of equilibria in strong set order.

stably, i.e. in ways avoiding coalitional deviations, or "blocks." When the participants have *substitutable* preferences, the set of stable matchings can be characterized as a set of fixed points of a certain monotonic operator—which corresponds to Gale and Shapley's deferred acceptance algorithm in a simple setup—, and the stable matchings exhibit monotone comparative statics properties when the market conditions change in terms of agents' preferences and/or their entry or exit. Here again, monotone comparative statics holds in weak set order but not in strong set order.[4]

These observations suggest that, for many problems of interest, monotone comparative statics is feasible only in weak set order. Given this, the current paper asks: *What would it take to guarantee wMCS? Namely, what are the minimal structure of the problem and properties one needs, if the goal is just to establish* $S(t') \geqslant_{ws} S(t)$ *whenever* $t' \geqslant t$ *and nothing more.* We show that the conditions required for monotone comparative statics can be weakened compared to existing conditions, sometimes considerably. Naturally, the notion of complementarities is weakened. More surprisingly, the lattice structure of domain and the images of relevant operators, taken virtually as given by the existing literature, proves not to be essential, and thus can be dispensed with, for results such as existence of equilibria and their monotone comparative statics.

The current paper proceeds as follows. In Section 3, we consider individual choice problems in which an action is chosen to maximize an objective function defined over a lattice, and provide sufficient conditions for their wMCS. In particular, we identify notions of one objective function "dominating" another in some weaker senses—called *weak domination* and *weak interval domination*—than are required for sMCS (see for example Milgrom and Shannon (1994) and Quah and Strulovici (2009)) such that the maximizers of the former dominates the maximizers of the latter in weak set order, and show them to be also necessary if one insists upon the wMCS relation to hold for all subdomains of certain richness.

In Section 4, we consider Pareto optimal choices for a set of agents. Pareto optimal choices are interesting in and of itself, but they can also be a model of behavior by an individual whose preference is not complete (Eliaz and Ok (2006) for instance); such an individual may be seen as balancing multiple, possibly conflicting, complete preferences, each represented by a well-defined utility function. We study conditions on the change of these latter "component" utility functions that give rise to wMCS of the associated Pareto optimal choices. When X is a general lattice, the desired result requires fairly strong

[4] Given the fixed-point characterization of stable matchings, one may expect the set of stable matchings to behave monotonically in weak set order but not in strong set order for a similar reason that the set of Nash equilibria does so.

conditions both on the curvature of individual utility functions and their complementarity properties. When X is totally ordered, however, wMCS is simply ensured by the standard single-crossing property.

Next, Section 5 studies existence of fixed points and their comparative statics. A fixed-point theorem—an essential element of equilibrium analysis—concerns existence of an element $x \in X$ with the property $x \in F(x)$ for some correspondence $F : X \rightrightarrows X$. The fixed-point theorem originally developed by Tarski (1955) and extended by Zhou (1994) to correspondences is particularly useful for economic analysis. For their existence and lattice properties, the theorem requires (see Zhou (1994)) that (i) X be a complete lattice; (ii) $F(x)$ be a complete sublattice of X for each $x \in X$, and (iii) $F(\cdot)$ be nondecreasing in strong set order. For monotone comparative statics—i.e., to establish that the fixed points of correspondence G dominates those of another correspondence F in *weak set order*—the theorem requires (i)-(iii) for both F and G and that (iv) $G(x)$ dominate $F(x)$ in *strong set order* for each x.[5]

Despite its usefulness, the theorem's applicability is limited by conditions (i)-(iv), some or all of which may not hold in many environments of interest. In particular, the lattice structure assumed for X and for the image $F(x)$ of each x is quite restrictive. These conditions can be weakened considerably, if one cares only about *existence* and *wMCS* of fixed points. Theorems 5 (due in large part to Li (2014)) and 6 establish these two results under a new set of conditions: For the existence of a fixed point, our requirements are (i') X is only partially ordered and compact (under a suitable topology); (ii') $F(x)$ is closed for each x; and (iii') $F(\cdot)$ is nondecreasing in weak set order, plus a mild condition to be introduced later. For wMCS of fixed points, our requirement is that (iv') $G(x)$ dominates $F(x)$ in weak set order for each x. The fixed points need not form a complete lattice, but their minimal and maximal points exist, and they in turn exhibit the wMCS property. Further, the computability of a fixed point via an iterative algorithm carries over to our environment albeit with some wrinkles (Theorem 7). Naturally, these results apply to a broader class of games with strategic complementarities than have been identified before (see Vives (1990), Milgrom and Roberts (1990), and Milgrom and Shannon (1994)). An advantage of the present approach is that our class of games exhibits virtually the same set of useful properties as theirs without making strong assumptions on lattices or strong complementarities. For instance, the ability to handle non-lattice environments both in (i') and (ii') means that the same powerful results extend to games (with strategic complementarities) played by agents with incomplete preferences or games played by coalitions

[5]While the comparative statics result is not explicitly stated by Zhou (1994), it is a straightforward implication of his result.

of agents choosing Pareto optimal responses to their opponents which, as argued above, generally do not form lattices.

In Section 6, we study wMCS of stable many-to-one matchings when agents have general preferences. Tarski's fixed-point theorem has been used to prove existence of stable matchings under substitutable preferences (see Adachi (2000), Fleiner (2003), and Hatfield and Milgrom (2005)). That the fixed-point theorem of Section 5 applies to correspondences with non-lattice images means that agents' preferences accommodate very general forms of substitutability, as well as indifferences or even incomplete preferences. Indifferences are natural when agents' preferences arise from coarse priorities; a case in point is public schools that often place many students in the same priority class. Incomplete preferences may arise naturally in a multidivisional firm in which multiple divisions may compete for common resources for hiring workers, or in a medical matching with regional caps, where hospitals in the same region may compete for quotas subject to a cap. We prove existence of a stable matching and its wMCS properties allowing for such general preferences. A key step toward this end is the characterization of a stable matching via a fixed point of a tâtonnement-like operator, and this requires a version of revealed preferences condition. The standard version, known as Weak Axiom of Revealed Preference (WARP), however, may not hold for incomplete preferences. Our characterization thus weakens the notion of revealed preference condition that is compatible with incomplete preferences. This characterization, together with the associated wMCS properties, advances the frontier of matching theory.

2 Preliminaries

Throughout, our domain of choices X is assumed to be *partially ordered set* with regard to some *primitive partial order* $\geqslant$, namely a binary relation that is reflesive, transitive and anti-symmetric on X. This primitive order induces two set orders, *strong set order* $\geqslant_{ss}$ and *weak set order* $\geqslant_{ws}$. We shall use the following terminologies related to these set orders. We say $X'' \subset X$ *strong set dominates* $X' \subset X$ if $X'' \geqslant_{ss} X'$. Similarly, X'' *upper weak set dominates* X', and write $X'' \geqslant_{uws} X'$, if, for each $x' \in X'$, there exists $x'' \in X''$ such that $x'' \geqslant x'$; and X'' *lower weak set dominates* X', and write $X'' \geqslant_{lws} X'$, if for each $x'' \in X''$, there exists $x' \in X'$ such that $x' \leqslant x''$. And, X'' *weak set dominates* X', and write $X'' \geqslant_{ws} X'$, if $X'' \geqslant_{uws} X'$ and $X'' \geqslant_{lws} X'$, as we have already defined in the introduction.

Some, but not all, results invoke additional order properties. We say X is a *lattice* if for any $x, x' \in X$, $x \vee x' \in X$ and $x \wedge x' \in X$, or equivalently if $X \geqslant_{ss} X$. X is a *complete* *lattice* if, for any $X' \subset X$, $\sup_X X' \in X$ and $\inf_X X' \in X$, where $\sup_X X' := \inf\{z \in X : z \geqslant x, \forall x \in X'\}$ and $\inf_X X' := \sup\{z \in X : z \leqslant x, \forall x \in X'\}$. A subset $X' \subset X$ is a *sublattice* of X, if, for any $x, x' \in X'$, $x \wedge_X x' \in X'$ and $x \vee_X x' \in X'$, where $x \wedge_X x' := \sup\{x'' \in X : x'' \leqslant x \text{ and } x'' \leqslant x'\}$ and $x \vee_X x' := \inf\{x'' \in X : x'' \geqslant x \text{ and } x'' \geqslant x'\}$. A subset $X' \subset X$ is a *complete sublattice* of X if $\sup_X Y \in X'$ and $\inf_X Y \in X'$ for all $Y \subseteq X'$.[6] (We will henceforth use $\wedge$ and $\vee$ instead of $\wedge_X$ and $\vee_X$, unless the sup or the inf is being taken over a set other than X.) Finally, a subset X' is a *subinterval* of X if there exist $a \leqslant b, a, b \in X$, such that $X' = \{x \in X : a \leqslant x \leqslant b\}$, denoted equvalently by $[a, b]$.

Finally, some of our results pertaining to existence of maximizers or fixed points invoke topological properties such as compactness of X and upper semicontinuity of an objective function defined on X. Whenever such properties are invoked, we invoke a metrizable *natural topology* under which upper contour sets $U_y := \{x \in X : x \geqslant y\}$, $\forall y \in X$, and lower contour sets $L_y := \{x \in X : x \leqslant y\}$, $\forall y \in X$, are closed, where $\geqslant, \leqslant$ are our primitive partial order.

3 Individual Choices

In this section, we study wMCS of individual choices. Consider an individual who chooses an action x from some set $X' \subset X$ by maximizing an objective function $f : X \to \mathbb{R}$. We are concerned with how her choices

$$M_{X'}(f) := \arg\max_{x \in X'} f(x)$$

change when her objective function f shifts from one function u to another v. In particular, we explore sufficient conditions for the shift to produce wMCS of her choices—or more precisely, $M_{X'}(v) \geqslant_{ws} M_{X'}(u)$—for every subdomain X' within a class $\mathcal{X} \subset 2^X$.

The sufficient conditions we look for should ideally be "tight" or "necessary" in some sense, and this desideratum is provided by the requirement that the conditions be necessary for wMCS for *every* subdomain $X' \subset X$ *within a class* $\mathcal{X} \subset 2^X$. How rich we require that class $\mathcal{X}$ to be involves a tradeoff. If $\mathcal{X}$ is very coarse, then the sufficient conditions become weak, but they could become too dependent on the "details" of the specific subdomain to be of practical value. If $\mathcal{X}$ is very rich, the conditions become detail-free and robust but at the expense of being strong. In this regard, we follow two prominent works by Milgrom

[6]Some other terminologies are used for the same notion: Topkis (1998) uses subcomplete sublattice and Zhou (1994) uses closed sublattice. In particular, the "closedness" of Zhou (1994) should not be confused with the topological "closedness" used in this paper.

and Shannon (1994) and Quah and Strulovici (2009).

Milgrom and Shannon (1994) find conditions that guarantee sMCS on the class $\mathcal{X}_{sublat}$ of all sublattices of X, whereas Quah and Strulovici (2009) find conditions that guarantee sMCS on the class $\mathcal{X}_{subint}$ of all subintervals of X, where X is totally ordered.[7] Obviously, the class of sublattices of X is richer than the class of subintervals of X (note a subinterval is a sublattice).[8] So, the condition for monotone comparative statics with respect to the former class will be more robust, albeit stronger, than that with respect to the latter class.

3.1 Characterization with Respect to Sublattices of X.

Milgrom and Shannon (1994) provide canonical conditions that guarantee sMCS of individual choice on the class $\mathcal{X}_{sublat}$. Specifically, their Theorem 4 proves that the maximizers of v strong set dominate those of u for every sublattice of X if v **MS dominates** u, or $v \geq_{MS} u$:[9] (i) v *single-crossing dominates* u, i.e., for any $x'' > x'$, $u(x'') - u(x') \geqslant (>)0 \Rightarrow v(x'') - v(x') \geqslant (>)0$; and (ii) $f = u, v$ is *quasi-supermodular*: for any $x', x'' \in X$, $f(x'') - f(x' \wedge x'') \geqslant (>)0 \Rightarrow f(x' \vee x'') - f(x') \geqslant (>)0$. Intuitively, (i) means that it pays a decision maker to raise her action under utility function v whenever it does so under utility function u, and (ii) means that raising one component of action by a decision maker increases her incentive to raise another component of her action (in the ordinal sense).

These two conditions combined together imply that: for any $x', x'' \in X$, $x'' \nleqslant x'$,

$$u(x'') \geqslant (>)u(x' \wedge x'') \Rightarrow v(x' \vee x'') \geqslant (>)v(x'). \quad (1)$$

It is immediate that sMCS follow from (1): if $x'' \in M_{X'}(u)$ and $x' \in M_{X'}(v)$ for any sublattice X', then $x' \vee x'' \in M_{X'}(v)$ and $x' \wedge x'' \in M_{X'}(u)$.

We weaken (1) in the following way. We say v **weakly dominates** u, and write $v \succeq_w u$, if, for any $x', x'' \in X$, $x'' \nleqslant x'$,

$$u(x'') \geqslant (>) \max\{u(x' \wedge x''), u(x')\} \Rightarrow \max\{v(x''), v(x' \vee x'')\} \geqslant (>)v(x'). \quad (2)$$

This condition is weaker than MS dominance since the hypothesis of (2) is stronger, and

[7] There is a subtle difference between the two studies: Quah and Strulovici (2009) obtain their characterization by fixing the constraint set X' in two maximization problems under comparison, while Milgrom and Shannon (1994) do so by varying X' (in the strong set order sense) together with the objective function. Our study takes the former approach.

[8] If $x, x' \in [a, b]$, then $x \wedge x', x \vee x' \in [a, b]$.

[9] As mentioned earlier in footnote 7, MS dominance does not quite characterize the sMCS for all sublattices. To be precise, Theorem 4 of Milgrom and Shannon (1994) shows that MS dominance is also necessary (in addition to being sufficient) for $M_{X''}(v) \geqslant_{ss} M_{X'}(u)$ in which X'' strong set dominates X'.

its conclusion is weaker, than that of (1). Therefore, (1), and hence $v \succeq_{MS} u$, implies that $v \succeq_w u$. Note also that weak dominance need not yield sMCS of individual choices. Suppose $x'' \in M_{X'}(u)$ and $x' \in M_{X'}(v)$ for a sublattice X', so the hypothesis of (2) holds; yet we are not assured that $x' \vee x'' \in M_{X'}(v)$.

For wMCS of individual choice on sublattices, however, weak dominance turns out to be just the right condition:

Theorem 1. *Suppose that X is a lattice. Function v weakly dominates u if and only if, for every $X' \in \mathcal{X}_{sublat}$,*

$$M_{X'}(u) \leqslant_{ws} M_{X'}(v) \tag{3}$$

whenever both sets are nonempty.

Proof. **The "only if" direction.** Fix any sublattice $X' \subset X$ and let $z'' \in M_{X'}(u)$ and $z' \in M_{X'}(v)$. Clearly, $u(z'') \geqslant \max\{u(z' \wedge z''), u(z')\}$. Since $v \succeq_w u$, we then have $\max\{v(z''), v(z' \vee z'')\} \geqslant v(z')$. Hence, $M_{X'}(v)$ upper weak set dominates $M_{X'}(u)$. For the lower weak set monotonicity, we invoke the contrapostive involving strict inequalities. Since $v(z') \geqslant \max\{v(z'') \vee v(z' \vee z'')\}$, we must have $\max\{u(z' \wedge z''), u(z')\} \geqslant u(z'')$, proving that $M_{X'}(v)$ lower weak set dominates $M_{X'}(u)$.

The "if" direction. Consider $X' = \{x', x'', x' \wedge x'', x' \vee x''\}$, where $x'' \nleqslant x'$. Suppose first $u(x'') \geqslant \max\{u(x' \wedge x''), u(x')\}$. Then, $\{x'', x' \vee x''\} \cap M_{X'}(u) \neq \varnothing$. We must then have $\max\{v(x''), v(x' \vee x'')\} \geqslant v(x')$, or else $M_{X'}(v)$ does not upper weak set dominate $M_{X'}(u)$. To prove the strict inequality part of the condition, we consider its contrapositive. To this end, suppose $\max\{v(x''), v(x' \vee x'')\} \leqslant v(x')$. Then, $\{x', x' \wedge x''\} \cap M_{X'}(v) \neq \varnothing$. We must then have $\max\{u(x' \wedge x''), u(x')\} \geqslant u(x'')$, or else $M_{X'}(v)$ does not lower weak set dominate $M_{X'}(u)$. This implies that $u(x'') > \max\{u(x' \wedge x''), u(x')\} \Rightarrow \max\{v(x''), v(x' \vee x'')\} > v(x')$. □

3.2 Characterization with Respect to Subintervals of X.

The domain of subintervals is coarser than that of sublattices. Hence, the condition characterizing wMCS in the former domain must be weaker than weak dominance. To describe that condtion, for any $x', x'' \in X$, we let

$$J(x', x'') := \{x \in X : x' \wedge x'' \leqslant x \leqslant x' \vee x''\}$$

denote the smallest subinterval of X containing them. Further, we assume that $M_{X'}(f)$ is well defined for every subinterval X' of X, for $f = u, v$.[10]

We say v **weakly interval dominates** u, or $v \succeq_{wI} u$, if, for any $x', x'' \in X$ such that $x'' \nleqslant x'$, $u(x'') \geqslant u(x)$, and $v(x') \geqslant v(x), \forall x \in J(x', x'')$,

$$u(x'') \geqslant (>) \max_{x \in J(x' \wedge x'', x')} u(x) \Rightarrow \max_{x \in J(x'', x' \vee x'')} v(x) \geqslant (>)\, v(x'). \tag{4}$$

Note that the weak interval dominance is implied by weak dominance: the hypothesis of (4) is stronger, and its conclusion is weaker, than that of (2). The following result shows that weak interval dominance characterizes wMCS of individual choices on every subinterval.

Theorem 2. *Suppose that X is a lattice. Function v weakly interval dominates u if and only if, for every $X' \in \mathcal{X}_{subint}$,*

$$M_{X'}(u) \leqslant_{ws} M_{X'}(v). \tag{5}$$

Proof. **The "only if" direction.** Choose any $z'' \in M_{X'}(u)$ and $z' \in M_{X'}(v)$, and suppose that $z'' \nleqslant z'$. Then, since $v \succeq_{wI} u$ and $u(z'') \geqslant \max_{x \in J(z' \wedge z'', z')} u(x)$, there exists $z''' \in J(z'', z' \vee z'')$ such that $v(z''') \geqslant v(z')$. That X' is an interval and $z', z'' \in X'$ implies $J(z', z'') \subset X'$, which in turn implies $z''' \in J(z'', z' \vee z'') \subset J(z', z'') \subset X'$. We must thus have $z''' \in M_{X'}(v)$, since $v(z''') \geqslant v(z')$ and $z' \in M_{X'}(u)$. Hence, $M_{X'}(v)$ upper weak set dominates $M_{X'}(u)$.

For the lower weak set domination, we consider the contrapositive relation involving strict inequalities. Specifically, choose any $z'' \in M_{X'}(u)$ and $z' \in M_{X'}(v)$, and suppose that $z'' \nleqslant z'$. Then, since $v \succeq_{wI} u$ and $v(z') \geqslant \max_{x \in J(z'', z' \vee z'')} v(x)$, there exists $z''' \in J(z' \wedge z'', z')$ such that $u(z''') \geqslant u(z'')$. For the same reason as above, we have $z''' \in J(z' \wedge z'', z') \subset J(z', z'') \subset X'$. We must then have $z''' \in M_{X'}(u)$, since $u(z''') \geqslant u(z'')$ and $z'' \in M_{X'}(u)$, proving that $M_{X'}(v)$ lower weak set dominates $M_{X'}(u)$.

The "if" direction. Fix any x'', x' with $x'' \nleqslant x'$ such that $u(x'') \geqslant u(x)$ and $v(x') \geqslant v(x), \forall x \in J(x', x'')$. Obviously, $u(x'') \geqslant \max_{x \in J(x' \wedge x'', x')} u(x)$. Suppose to the contrary that $v(x''') < v(x')$, $\forall x''' \in J(x', x' \vee x'')$. Then, $M_{J(x', x'')}(v)$ fails to upper weak set dominate $M_{J(x', x'')}(u)$, a contradiction. Next we prove the strict inequality part of the condition, by considering its contrapositive. Note that $v(x') \geqslant \max_{x \in J(x' \vee x'', x'')} v(x)$. Suppose to the contrary that $u(x''') < u(x'')$, $\forall x''' \in J(x' \wedge x'', x')$. Then, $M_{J(x', x'')}(v)$ fails to lower weak set dominate $M_{J(x', x'')}(u)$, a contradiction. □

Theorem 2 parallels the characterization result in Quah and Strulovici (2009) for a

[10] This is guaranteed if X is compact, X' is closed, and $f = u, v$ is upper semicontinuous, for instance.

totally ordered X. For such X, their *interval dominance order* characterizes sMCS for all subintervals of X.[11] In fact, one can extend their interval dominance order to a general lattice X. For such X, we say v **interval dominates** u, or $v \succeq_I u$, if, for any $x', x'' \in X$, $x'' \nleqslant x'$, such that $u(x'') \geqslant u(x)$ and $v(x') \geqslant v(x), \forall x \in J(x', x'')$,

$$u(x'') \geqslant (>)\, u(x' \wedge x'') \;\Rightarrow\; v(x' \vee x'') \geqslant (>)\, v(x'). \tag{6}$$

This condition reduces to Quah and Strulovici's interval dominance order when X is totally ordered. For the general lattice X, Theorem 13 in Online Appendix D proves that (6) characterizes sMCS for every subinterval of X in strong set order.[12] This condition implies weak interval dominance, and hence yields wMCS.

Corollary 1. *If $v \succeq_I u$, then $v \succeq_{wI} u$.*

Proof. The statement follows from Theorems 2 and 13 (in Online Appendix D). □

3.3 Properties of Individual Choices.

One by-product of the conditions guaranteeing sMCS is that $M_{X'}(f)$ forms a sublattice of X. Specifically, if f is quasi-supermodular and X' is a sublattice, then $M_{X'}(f)$ is a sublattice. The same is not implied by our wMCS conditions, however. In fact, $M_{X'}(f)$ need not (even) be a lattice. The following example illustrates this point.

Example 1. Let $X = [0,1]^2$, $u(x_1, x_2) = -(x_1 + x_2 - t)^2$, and $v(x_1, x_2) = -(x_1 + x_2 - t')^2$, for $0 < t < t' < 1$. Note that v weakly interval dominates u. Indeed, for any subinterval X', $M_{X'}(u)$ is the projection of the hyperplane $\{x \in [0,1]^2 : x_1 + x_2 = t\}$ on X', and likewise $M_{X'}(v)$ is the projection of the hyperplane $\{x \in [0,1]^2 : x_1 + x_2 = t'\}$ on X'. See the blue and red lines in Figure 2. One can easily see that $M_{X'}(v)$ dominates $M_{X'}(u)$ in weak set order but not in strong set order. Note also that neither set forms a lattice, let alone a sublattice of X. Finally, observe that v does not weakly dominate u. Consider a sublattice $Z = \{x', x'', x' \wedge x'', x' \vee x''\}$ consisting of the four dots in Figure 2. Clearly, $M_Z(u) = x''$ and $M_Z(v) = x'$, and they are not weak set ordered.

The following proposition establishes some useful properties of individual choices, which will be referred to in our analysis of games.

[11] Their online appendix considers a general lattice X and provides a set of conditions that are *sufficient* (but not necessary) for sMCS for all subintervals of X,

[12] This characterization means that the current (generalized) interval dominance order is weaker than the sufficient condition provided in Theorem 1 of Quah and Strulovici (2007): their total-order version of interval dominance and their *I-quasisupermodularity*. See Online Appendix D.

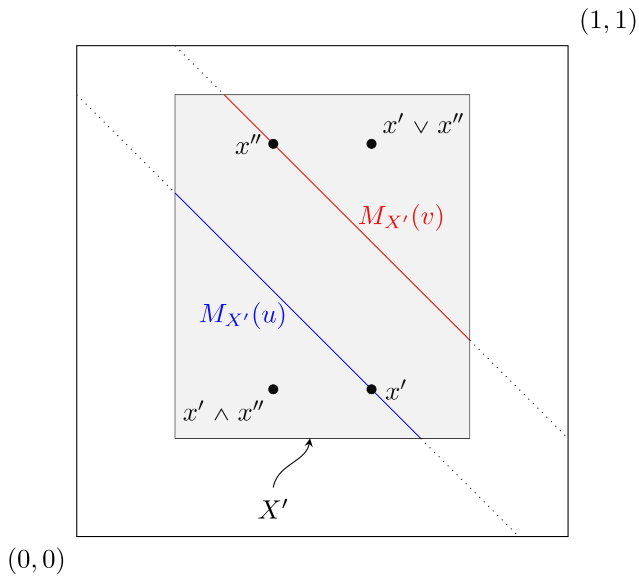

Figure 2: Weakly interval dominating shift that is not weakly dominating

Proposition 1. *Assume X is a partially ordered metric space, and f is upper semicontinuous. Then, for any compact subset X' of X, $M_{X'}(f)$ is nonempty and compact, and admits maximal and minimal points.*[13]

Proof. First of all, the nonemptiness of $M_{X'}(f)$ follows from Weierstrass' extreme value theorem. Let us prove that $M_{X'}(f)$ is closed, and thus compact. Consider any sequence (x_m) with $x_m \in M_{X'}(f), \forall m$, and any limit point x^* of the sequence. We must have $x^* \in X'$ since X' is compact. Also, the upper semicontinuity of f implies that $f(x^*) \geqslant \limsup_{m \to \infty} f(x_m)$, which in turn implies $x^* \in M_{X'}(f)$, as desired. By Theorem 2.3 of Li (2014), the compactness of $M_{X'}(f)$ implies that $M_{X'}(f)$ is *chain complete*: namely, every chain in X has a supremum and an infimum in X. By Zorn's lemma, it then follows that there are maximal and minimal points in $M_{X'}(f)$.[14] $\square$

[13]Minimal points of X' are a set $\{x \in X' : x' \nleqslant x, \forall x' \in X'\}$ and maximal points of X' are a set $\{x \in X' : x' \ngeqslant x, \forall x' \in X'\}$.

[14]Zorn's lemma states that a partially ordered set X' has a maximal element if it satisfies the following property: every chain in X' has an upper bound in X'. The latter property is satisfied if X' is chain-complete. Note that the existence of minimal point obtains easily from reversing a given order.

4 Pareto-Optimal Choices

Consider a set of alternatives X and a finite set I of individuals with utility functions $\mathbf{u} = (u_i)_{i \in I}$, where $u_i : X \to \mathbb{R}$ is a utility function for individual i. We say $y \in X$ *Pareto dominates* $x \in X$ given $\mathbf{u}$ if $u_i(y) \geqslant u_i(x)$ for all $i \in I$ and $u_j(y) > u_j(x)$ for at least one $j \in I$. The set of *Pareto optimal choices* (or POC in short) given $\mathbf{u}$ is the set $P(\mathbf{u}) := \{x \in X : \text{ no } y \in X \text{ Pareto dominates } x \text{ given } \mathbf{u}\}$. We wish to study conditions enabling wMCS of sets $P(\mathbf{u})$ with respect to a change in utility functions $\mathbf{u}$.

Remark 1. Our main interpretation of the set I is a collective of individuals. As mentioned in the introduction, however, one could also interpret I as a single decision maker with incomplete preferences. According to that interpretation, the decision maker I is associated with functions $\mathbf{u} = (u_i)_{i \in I}$, and the decision maker's choice behavior is described by the Pareto optimal choices $P(\mathbf{u})$.[15]

The existence of a Pareto optimal choice follows from standard assumptions.

Proposition 2. *Assume X is compact and u_i is upper semicontinuous for every $i \in I$. Then, the set $P(\mathbf{u})$ is nonempty.*[16]

Proof. See Appendix A. □

4.1 Monotone Comparative Statics of Pareto Optimal Choices

In order to analyze wMCS of Pareto optimal choices, we first present a novel characterization of POC, which we will use later for our wMCS result.

Lemma 1. *Suppose X is compact and convex, and u_i is upper semicontinuous and concave for each $i \in I$. Then, $x \in P(\mathbf{u})$ if and only if there exist strictly positive weights $\lambda := (\lambda_1, ..., \lambda_{|I|}) \in \mathbb{R}^{|I|}_{++}$ and a partition $\mathcal{I} = \{I_1, ..., I_n\}$ of the set of agents I such that*

$$x \in X_n, \text{ where } X_0 := X \text{ and } X_m := \arg\max_{x' \in X_{m-1}} \sum_{i \in I_m} \lambda_i u_i(x') \text{ for all } m = 1, ..., n.$$

Proof. See Appendix A. □

[15] In a context of choice problems under certainty, Ok (2002) provides sufficient conditions for incomplete preferences to be represented as Pareto optimal choices. Characterization results are given by Dubra, Maccheroni and Ok (2004) and Ok, Ortoleva and Riella (2012) for problems with lotteries and uncertainty. See also Eliaz and Ok (2006) who characterize a class of incomplete preferences.

[16] The set of Pareto optimal choices may be empty if X is not compact. For example, let $X = [0, 1)$, $I = \{1\}$, and $u_1(x) = x$. Then there exists no Pareto optimal choice because for any $x \in X$, there exists $x' \in X$ with $x' > x$ and hence $u_1(x') > u_1(x)$.

This characterization is related to the fact that, given the conditions above, every Pareto optimal choice maximizes a weighted sum of utilities (see Mas-Colell et al. (1995)), possibly with zero weights on some individuals. This fact does not provide a characterization, however, since the converse does not hold if there are ties.[17] Instead, our characterization rationalizes POC as serially maximizing welfare. More specifically, an alternative is Pareto optimal if and only if it is a solution to a series of maximization problems, where at each step m, the solutions maximize the weighted total welfare of a subset of players I_m (with all weights strictly positive) among the solutions from the earlier steps.

Our serial welfare maximization is reminiscent of the serial-dictatorship characterization of Pareto efficiency for indivisible object assignment (Abdulkadiroğlu and Sönmez, 1998). Note that indifferences are quite natural in this environment, since an agent is indifferent to alternatives that differ only in others' assignments. In that setting, serial dictatorships correspond to a subclass of the above serial welfare maximization such that the partition $\mathcal{I}$ is made up of all singleton sets so that exactly one individual's utility is maximized at each step.

Building on the characterization in Lemma 1, we now establish a wMCS result for POC. To this end, we indtroduce several conditions. We say that utility functions $\mathbf{v}$ *single-crossing dominates* utility functions $\mathbf{u}$ if v_i single-crossing dominates u_i for each $i \in I$, and $\mathbf{v}$ *increasing-differences dominates* $\mathbf{u}$ if, for each $i \in I$ and $x' > x$, $v_i(x') - v_i(x) \geqslant u_i(x') - u_i(x)$. Obviously, the latter condition implies the former. We say $\mathbf{u}$ is *supermodular* if u_i is supermodular for each $i \in I$: for each $x, x' \in X$, $u_i(x \vee x') - u_i(x) \geqslant u_i(x') - u_i(x \wedge x')$.[18] Just like single-crossing dominance and quasi-supermodularity, increasing-difference dominance and supermodularity guarantee that individual choices exhibit sMCS (Topkis, 1979). We use them to establish wMCS of POC.

Theorem 3. *Suppose X is a compact, convex and complete lattice, and $\mathbf{u}$ and $\mathbf{v}$ are both upper semicontinuous, concave and supermodular. If $\mathbf{v}$ increasing-differences dominates $\mathbf{u}$, then $P(\mathbf{v}) \geqslant_{ws} P(\mathbf{u})$.*

Proof. Suppose $\hat{x} \in P(\mathbf{u})$. Then, by Lemma 1, there exist $\lambda := (\lambda_1, ..., \lambda_{|I|}) \in \mathbb{R}^{|I|}_{++}$ and a

[17] Specifically, merely maximizing weighted welfare is not sufficient for Pareto optimality when welfare weights for some individuals are zero. This is because welfare maximization may not break the ties for agents with non-zero weight in a way that accounts for the utilities of agents with zero welfare weights. On the other hand, it is straightforward to see that maximizing weighted welfare for strictly positive weights is not necessary for Pareto optimality (though it is sufficient).

[18] It is straightforward to see that supermodularity implies quasi-supermodularity.

partition $\mathcal{I} = \{I_1, ..., I_n\}$ of agents I such that

$$\hat{x} \in X_m := \arg\max_{x' \in X_{m-1}} \sum_{i \in I_m} \lambda_i u_i(x'), \forall m = 1, ..., n,$$

where $X_0 := X$.

For the same λ and $\mathcal{I}$, define

$$Y_m := \arg\max_{x'' \in Y_{m-1}} \sum_{i \in I_m} \lambda_i v_i(x''), \forall m = 1, ..., n,$$

where $Y_0 = X$. We prove inductively that $Y_m \geqslant_{ss} X_m$ for all $m = 1, ..., n$. Suppose indeed $Y_{m-1} \geqslant_{ss} X_{m-1}$.[19] Since $\mathbf{u}$ and $\mathbf{v}$ are supermodular and $\mathbf{v}$ increasing-differences dominates $\mathbf{u}$, $\sum_{i \in I_m} \lambda_i u_i(x'')$ and $\sum_{i \in I_m} \lambda_i v_m(x'')$ are supermodular, and the latter increasing-differences dominates the former. Then, by Theorem 4 of Milgrom and Shannon (1994) we have $Y_m \geqslant_{ss} X_m$, so $Y_m \geqslant_{ws} X_m$, for $m = 1, ...n$. Hence, there exists $\hat{y} \geqslant \hat{x}$ such that $\hat{y} \in Y_n$. Applying Lemma 1 again, we conclude that $\hat{y} \in P(\mathbf{v})$. This proves that $P(\mathbf{v})$ upper weak set dominates $P(\mathbf{u})$. A symmetric argument in the reverse order proves the lower weak set domination of $P(\mathbf{u})$ by $P(\mathbf{v})$. □

In the proof of this theorem, the characterization in Lemma 1 plays an important role. Under convexity of X and concavity of each u_i, by this Lemma, $\hat{x}$ is Pareto optimal only if it solves a series of maximization problems for some partition of agents and some profile of weights. Then, supermodularity of $\mathbf{u}$ and $\mathbf{v}$, together with increasing-differences domination of $\mathbf{u}$ by $\mathbf{v}$, guarantees that the weighted sums of u_i's and v_i's inherit the same properties. This implies that, for the same partition and weights, the series of maximization problems associated with $\mathbf{v}$ has a solution $\hat{y} \geqslant \hat{x}$. By Lemma 1 again, $\hat{y}$ is Pareto optimal under $\mathbf{v}$, as desired.

Given Theorem 3, it is natural to ask whether the conditions in its hypothesis can be relaxed. We establish that compactness of X cannot be dispensed with; see the example in Online Appendix E.3. Whether other properties, namely the convexity of X or the concavity or supermodularity or the increasing-differences dominance of the utility functions, can be weakened proved more difficult to resolve. On the one hand, our proof utilizes these conditions in an essential manner. As illustrated earlier, we heavily rely on the characterization of Pareto optimality by a series of maximization problems of a weighted sum of individuals' utilities (Lemma 1), and this characterization does not hold unless X is convex

[19] This holds for $m = 1$—namely, $Y_0 = X \geqslant_{ss} X = X_0$—, since X is a lattice. For $m > 1$, it is an induction hypothesis.

and the utility functions are concave. Moreover, the proof of the theorem builds upon applying a (strong) MCS result of Milgrom and Shannon (1994) to weighted welfares associated with two profiles of utilities $\mathbf{u}$ and $\mathbf{v}$; and supermodularity and increasing-differences dominance are used in a crucial manner for guaranteeing that the weighted welfares satisfy conditions for MCS.[20] On the other hand, we have not found any counterexample when those conditions are dropped. Whether those conditions are tight or not is an interesting but challenging question, and we submit it as an open question.

Fortunately, further progress can be made for a certain natural subdomain. Specifically, we next analyze a case where X is totally ordered as in the case of, for instance, one-dimensional Euclidean space. In that case, we offer a much weaker sufficient condition for wMCS.

To proceed, we first present the following lemma which holds generally for compact X, and not just for totally ordered X.

Lemma 2. *Suppose X is compact. If $y \notin P(\mathbf{u})$, then y is Pareto dominated by some $x \in P(\mathbf{u})$.*

Proof. See Appendix A. □

With this lemma at hand, we are ready to establish wMCS for totally ordered X under the weaker sufficient condition.

Theorem 4. *Suppose that X is compact and totally ordered. If $\mathbf{v}$ single-crossing dominates $\mathbf{u}$, then $P(\mathbf{v}) \geqslant_{ws} P(\mathbf{u})$.*

Proof. By Proposition 2, both $P(\mathbf{u})$ and $P(\mathbf{v})$ are well defined and admit infima and suprema. Any $x < \inf P(\mathbf{u})$ is Pareto dominated, so it must be Pareto dominated by some $x' \in P(\mathbf{u})$ by Lemma 2: $u_i(x') - u_i(x) \geqslant 0$ for all $i \in I$ and $u_j(x') - u_j(x) > 0$ for some $j \in I$. Since $x' > x$ and $\mathbf{v}$ single-crossing dominates $\mathbf{u}$, we must have $v_i(x') - v_i(x) \geqslant 0$ for every $i \in I$ and $v_j(x') - v_j(x) > 0$. Hence, x is also Pareto dominated under $\mathbf{v}$. This proves that $\inf P(\mathbf{u}) \leqslant \inf P(\mathbf{v})$.

The same argument, applied to the contrapositive of the strict inequality part of the single-crossing domination, implies that if $x'' > \sup P(\mathbf{v})$, then $x'' > \sup P(\mathbf{u})$, implying that $\sup P(\mathbf{u}) \leqslant \sup P(\mathbf{v})$.

[20]In particular, one might wonder why Theorem 3 assumes cardinal properties, namely increasing-differences dominance and supermodularity, rather than weaker, ordinal ones, namely single-crossing dominance and quasisupermodularity. The reason is that, as already mentioned, our proof builds upon weighted welfare maximizations for two profiles of utilities $\mathbf{u}$ and $\mathbf{v}$, and the cardinal properties are closed under linear combinations while the ordinal ones are not.

It remains to rule out the possibility that $\inf P(\mathbf{u}) = \inf P(\mathbf{v})$, $\inf P(\mathbf{u}) \notin P(\mathbf{u})$, and $\inf P(\mathbf{v}) \in P(\mathbf{v})$. To this end, suppose $\inf P(\mathbf{u}) \notin P(\mathbf{u})$. Then $\inf P(\mathbf{u})$ is Pareto dominated by some $x' \in P(\mathbf{u})$, and by the same single-crossing argument as above, it follows that $\inf P(\mathbf{u})$ is Pareto dominated by x' under $\mathbf{v}$, so $\inf P(\mathbf{v}) \notin P(\mathbf{v})$. Finally, a similar argument rules out a symmetric possibility with regard to the suprema. Combining the results, we conclude that $P(\mathbf{v})$ weak set dominates $P(\mathbf{u})$. $\square$

4.2 Properties of Pareto Optimal Choices.

Given Theorems 3 and 4, a natural question is whether the same assumptions guarantee sMCS, not just wMCS, of POC. The following example shows that the answer is negative.[21]

Example 2 (Failure of sMCS)**.** Let $X = [0, 6]^2$ endowed with the standard product order, that is, $(x, y) \geqslant (x', y')$ if and only if $x \geqslant x'$ and $y \geqslant y'$.[22] Suppose that $I = \{1, 2\}$ and

$$\begin{cases} u_1(x,y) = -(x-1)^2 - (y-1)^2 \\ u_2(x,y) = -(x-4)^2 - (y-1)^2 \end{cases} \text{and} \begin{cases} v_1(x,y) = -(x-1)^2 - (y-4)^2 \\ v_2(x,y) = -(x-4)^2 - (y-2)^2. \end{cases}$$

Then $\mathbf{u}$ and $\mathbf{v}$ satisfy all the conditions in Theorem 3. Meanwhile, $P(\mathbf{u})$ is a (closed) line segment between $(1, 1)$ and $(4, 1)$ while $P(\mathbf{v})$ is a line segment between $(1, 4)$ and $(4, 2)$, i.e., $P(\mathbf{u}) = \{\lambda(1, 1) + (1-\lambda)(4, 1) : \lambda \in [0, 1]\}$ and $P(\mathbf{v}) = \{\lambda(1, 4) + (1-\lambda)(4, 2) : \lambda \in [0, 1]\}$. The set $P(\mathbf{v})$ dominates $P(\mathbf{u})$ in weak set order but not in strong set order; for instance, $(4, 1) \in P(\mathbf{u})$ and $(1, 4) \in P(\mathbf{v})$, but $(4, 1) \vee (1, 4) = (4, 4) \notin P(\mathbf{v})$. See Figure 3.

As illustrated in Introduction and the proof of Theorem 3, a change in utility functions shifts the set of (serial) maximizers of weighted welfare monotonically in the sense of strong set order given any fixed partition and weights. POC is a *union* of such maximizers over different partitions and weights, however, and this fact causes the failure of strong set monotonicity. In the present example, $(4, 1) \in P(\mathbf{u})$ is the solution to a series of maximization problems with respect to partition $\mathcal{I} = \{\{2\}, \{1\}\}$ (together with arbitrary weights), and the solution given the same weight and partition under $\mathbf{v}$ is $(4, 2)$, which does strong set dominates $(4, 1)$ as singleton-sets. However, $P(\mathbf{v})$ contains $(1, 4)$, a solution with respect to a different partition, $\{\{1\}, \{2\}\}$, and indeed the join $(4, 1) \vee (1, 4) = (4, 4)$ fails to be in $P(\mathbf{v})$. $\square$

[21] While Example 2 is set in a multidimensional setting Theorem 3 assumes, the example in Online Appendix E.3 shows the same conclusion for Theorem 4.

[22] Throughout, when X is a subset of (multi-dimensional) Euclidean space, we endow X with the standard (product) order unless noted otherwise.

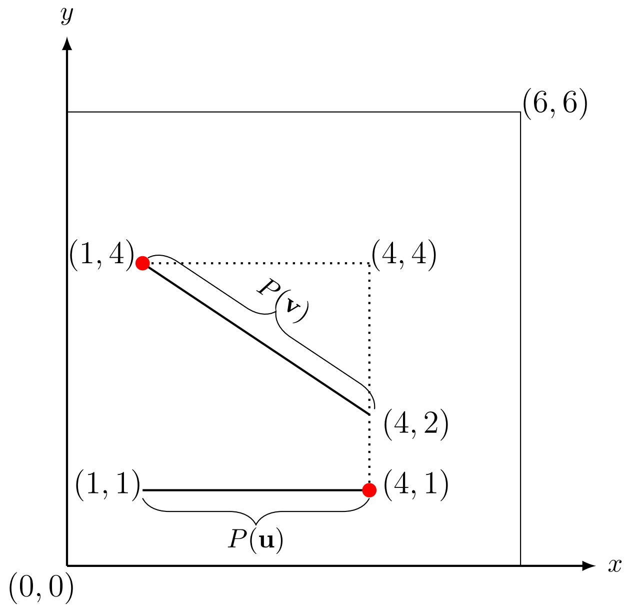

Figure 3: Failure of strong set monotonicity

The above example also demonstrates that the set of Pareto optimal choices does not necessarily form a lattice. In particular, $P(\mathbf{v})$ in that example fails to be a lattice (indeed, it does not even have a smallest or largest point).

Another question of interest is whether in general $P(\mathbf{u})$ is compact (or equivalently closed, given compactness of X). Compactness of POC plays an important role in Section 5.3. In that section, we consider a game where a player is a representative of a collective or an individual with incomplete preferences whose best response is composed of POC. For establishing an existence of a (pure strategy) Nash equilibrium and comparative statics in such a game, we need the best response correspondences to be compact-valued.

The following example shows that $P(\mathbf{u})$ is not necessarily compact in our environment. In fact, $P(\mathbf{u})$ may even fail to have a minimal (or maximal) element (note that any compact set has both minimal and maximal points).[23]

Example 3 (Non-existence of a minimal POC)**.** Let $X = [0, 1]$, $I = \{1, 2\}$, and

$$u_1(x) = \begin{cases} x & \text{if } x \leqslant 1/2 \\ 1 - x & \text{if } x > 1/2, \end{cases} \quad \text{and } u_2(x) = \begin{cases} 2 - x & \text{if } x < 1 \\ 3 & \text{if } x = 1. \end{cases}$$

[23]To see the connection between compactness and existence of maximal/minimal points, refer to the proof of Proposition 1.

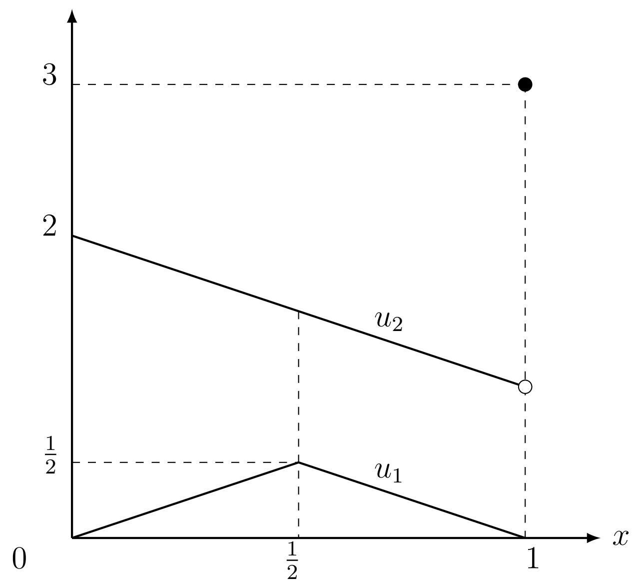

Figure 4: Example with no minimal Pareto optimal choice and failure of compactness.

This environment satisfies all the assumptions made for Proposition 2; In particular, X is compact and both utility functions are upper semicontinuous. See Figure 4. However, $P(\mathbf{u}) = (0, 1/2] \cup \{1\}$, and this set is non-compact and does not have a minimal element.[24] □

This example shows that conditions assumed in Proposition 2 need not guarantee compactness of $P(\mathbf{u})$. Fortunately, some additional regularity conditions lead to compactness (and hence the existence of maximal and minimal points). To state them, for each $i \in I$ and $x \in X$, let $U_{-i}(x) := \{y \in X : u_j(y) \geqslant u_j(x), \forall j \in I \backslash \{i\}\}$ denote the set of alternatives that every agent other than i weakly prefers to x.

Proposition 3. *Suppose that X is compact and convex and that, for each $i \in I$, $u_i(\cdot)$ is continuous and the correspondence $U_{-i}(\cdot)$ is lower hemicontinuous.*[25] *Then, $P(\mathbf{u})$ is compact. In particular, $P(\mathbf{u})$ has minimal and maximal points.*

Proof. See Appendix A. □

[24] An example with no maximal element of $P(\mathbf{u})$ can be obtained from this example by endowing X with the opposite order to the standard one.

[25] Given the compactness of X (which is assumed throughout), $U_{-i}(x)$ is lower hemicontinuous if, for each sequence $(x_n)_n$ with $x_n \in X$ for each $n \in \mathbb{N}$ and converging to x, and for any $z \in U_{-i}(x)$, there exists $(z_n)_n$ with $z_n \in U_{-i}(x_n)$ for each $n \in \mathbb{N}$ that converges to z. Proposition 6 in Online Appendix E.2 provides sufficient conditions for $U_{-i}(x)$ to be lower hemicontinuous.

5 Fixed Point Theorem and Games with Weak Strategic Complementarities

In this section, we present a fixed-point theorem that plays a central role in the remainder of this paper. In addition to establishing the existence of a fixed point, we also offer a new comparative statics theorem for the fixed points and an algorithm to find fixed points. We then apply these results to analyze a new class of games called games with weak strategic complementarities.

Consider a nonempty set X endowed with a partial order $\geqslant$ and a metric which induces a natural topology. Throughout, assume that X is compact with respect to this topology. Let $F : X \rightrightarrows X$ be a self-correspondence, i.e., a correspondence from X to itself. We say that F is *strong set monotonic* if $F(x')$ strong set dominates $F(x)$ for $x' \geqslant x$, *upper weak set monotonic* if $F(x')$ upper weak set dominates $F(x)$ for $x' \geqslant x$, and *lower weak set monotonic* if $F(x')$ lower weak set dominates $F(x)$ for $x' \geqslant x$. Finally, we say that F is *compact-valued* if $F(x)$ is compact for all $x \in X$.

5.1 Fixed Point Theorem.

Here, we provide our fixed-point theorem which will play a central role in the subsequent analysis. In addition, we present a novel comparative statics theorem for the fixed points. We first define $X_+ := \{x \in X : \exists y \geqslant x \text{ s.t. } y \in F(x)\}$ to be the set of points whose image includes a weakly higher point than that point, and similarly define $X_- := \{x \in X : \exists y \leqslant x \text{ s.t. } y \in F(x)\}$. Now we are ready to present a formal statement of our fixed-point theorem.

Theorem 5 (Fixed-Point Theorem)**.** *Suppose X is compact, a self-correspondence $F : X \rightrightarrows X$ is nonempty-valued, upper (resp., lower) weak set monotonic and compact-valued, and X_+ (resp., X_-) is nonempty. Then, F has a fixed point. Moreover, there exists a maximal (resp., minimal) fixed point, that is, a fixed point x for which there is no other fixed point y with $y > x$ (resp., $y < x$).*

Proof. See Appendix B. □

Before proceeding, it is instructive to compare this theorem with Zhou (1994)'s fixed-point theorem, which extends Tarski (1955)'s fixed-point theorem to accommodate correspondences. First, we require that X be partially ordered. This condition is considerably weaker than the complete lattice condition required by Tarski (1955) or Zhou (1994). Second, we do not require $F(x)$ to be a complete sublattice of X, as is assumed by Zhou

(1994). Third, we require F to be weak set monotonic instead of strong set monotonic as in Zhou (1994). Finally, the nonemptiness of X_+ (or X_-) is trivially satisfied both in Tarski (1955) and Zhou (1994) because they restrict their attentions to the case where X is a complete lattice and hence where there exist smallest and largest points. Meanwhile, our theorem requires two topological conditions—compactness of X and compact-valuedness of F—absent in Tarski (1955) and Zhou (1994).

Compared with the fixed-point theorem of Tarski (1955) or Zhou (1994), Theorem 5 dispenses with some restrictive order-theoretic assumptions but adds the aforementioned topological assumptions. Since these latter conditions are satisfied in many economic applications, the current theorem will be useful in many settings in which Tarski (1955) or Zhou (1994) cannot be applied. In fact, Theorem 14 in Online Appendix F shows that, in many problems of interest, the conditions in Theorem 5 are weaker than those of Zhou (1994)'s theorem. Furthermore, the following examples show that our conditions cannot be dispensed with:

- **Compactness of X**: Let $X = [0, 1)$. The correspondence $F : X \rightrightarrows X$ with $F(x) = \{\frac{1}{2} + \frac{1}{2}x\}, \forall x \in [0, 1)$ satisfies all conditions except for compactness of X and admits no fixed point.

- **Compact-valuedness of F**: Let $X = [0, 1]$. The correspondence $F : X \rightrightarrows X$ with $F(x) = (0, 1)\backslash\{x\}, \forall x \in [0, 1]$ satisfies all conditions except for compact-valuedness of F and admits no fixed point.

- **Nonemptiness of X_+**: Let $X = \{(0, 1), (1, 0)\}$. The correspondence $F : X \rightrightarrows X$ with $F((0, 1)) = \{(1, 0)\}$ and $F((1, 0)) = \{(0, 1)\}$ satisfies all conditions except for nonemptiness of X_+ and admits no fixed point.

- **Upper weak set monotonicity**: Let $X = \{(0, 0), (0, 1), (1, 0)\}$. The correspondence $F : X \rightrightarrows X$ with $F((0, 0)) = \{(0, 1), (1, 0)\}$, $F((0, 1)) = \{(1, 0)\}$, and $F((1, 0)) = \{(0, 1)\}$ satisfies all conditions except for upper weak set monotonicity (although it satisfies lower weak set monotonicity) and admits no fixed point.

- **Lower weak set monotonicity and minimal fixed point**: Let $X = [0, 1]$. The correspondence $F : X \rightrightarrows X$ with $F(x) = [x, 1]$ for $x \in (0, 1]$ and $F(0) = [1/2, 1]$ is upper weak set monotonic but not lower weak set monotonic, while satisfying all other conditions for Theorem 5. The set of fixed points is $(0, 1]$ and contains a maximal element but not a minimal one.

While the conditions required in Theorem 5 are typically weaker than those in extant results, the conclusions obtained are also weaker. Unlike Tarski's fixed-point theorem and Zhou (1994)'s extension, fixed points need not form a complete lattice in the current case, and the set of fixed points may not even have the largest or the smallest element. Still, it is worth pointing out that the theorem shows that the set has a maximal or minimal point.[26]

Remark 2. After proving Theorem 5, we became aware of an earlier contribution by Li (2014), who established the existence of a fixed point under the same set of assumption as ours. We fully acknowledge his prior contribution here. Meanwhile, a few remarks are in order. First, our proof is different from, and arguably simpler than, his; see Appendix B. Second, we establish the existence of maximal and minimal fixed points, a property that Li (2014) did not show. Finally, we also establish a comparative statics result on the fixed points, to be presented below as Theorem 6, which is novel to our knowledge.

An important benefit of the fixed-point theorem is the ease with which it can be adapted for monotone comparative statistics. For each self-correspondence F, let $\mathcal{E}(F)$ denote the set of fixed points of F.

Theorem 6 (Comparative Statics)**.** *Suppose X is compact, both self-correspondences F and G are upper (resp., lower) weak set monotonic and compact-valued, and X_+ (resp., X_-) is nonempty for both F and G. If $G(x) \geqslant_{uws} F(x)$ (resp., $G(x) \geqslant_{lws} F(x)$) for all $x \in X$, then $\mathcal{E}(G) \geqslant_{uws} \mathcal{E}(F)$ (resp., $\mathcal{E}(G) \geqslant_{lws} \mathcal{E}(F)$).*

Proof. Consider any $x^* \in \mathcal{E}(F)$. Define correspondence G^* by $G^*(x) := G(x)_{\geqslant x^*}$ for $x \in X_{\geqslant x^*}$, where for any $X' \subseteq X$ and $x \in X$, $X'_{\geqslant x} := \{x' \in X' : x' \geqslant x\}$. Clearly, G^* is compact-valued. That $x^* \in F(x^*)$ and that $G(x) \geqslant_{uws} F(x)$ for each $x \in X$ imply that for any $x \geqslant x^*$, there is some $x' \in G^*(x)$, that is, G^* is a nonempty-valued self-correspondence defined on $X_{\geqslant x^*}$. Moreover, G^* is upper weak set monotonic since, for any $x, x' \in X_{\geqslant x^*}$ with $x' \geqslant x$ and any $y \in G^*(x) \subset G(x)$, there exists some $y' \in G(x')$ such that $y' \geqslant y (\geqslant x^*)$ so $y' \in G(x')_{\geqslant x^*} = G^*(x')$. Since G^* satisfies all the conditions for Theorem 5, there must exist a fixed point $\tilde{x} \in G^*(\tilde{x})$, which means that $\tilde{x} \in G(\tilde{x})$ and $\tilde{x} \geqslant x^*$. This completes the proof for the "upper" version of the statement. The proof of the "lower" version is symmetric. □

[26] The set of maximal or minimal fixed points is not necessarily compact. See Example 8 in Online Appendix F.2.

5.2 Iterative Algorithm for Finding a Fixed Point.

An important benefit of a monotonic operator is that it gives rise to a constructive method for finding a fixed point. A classic result in the standard environment of Tarski and Zhou is: given some additional continuity property of F which holds trivially if X is finite (a practical situation), the highest fixed point can be obtained by iteratively applying the highest selection from the correspondence starting from the highest point $\overline{x} := \sup X$ (and a symmetric result holds for the lowest fixed point). This property, known as Kleene's fixed-point theorem (see Baranga (1991) for instance) and also established for supermodular games by Milgrom and Roberts (1990) and Milgrom and Shannon (1994), is very convenient in practice.

We show that a similar property holds if X satisfies the hypotheses of Theorem 5, albeit with some qualifications. We say F is **upper hemi-order-continuous** if, for any sequence $((x_n, y_n))_{n\in\mathbb{N}}$ converging to (x, y), where $(x_n)_n$ is either monotone nondecreasing or monotone nonincreasing, and $y_n \in F(x_n)$ for each $n \in \mathbb{N}$, we have $y \in F(x)$.[27]

Theorem 7. *Suppose X is compact, a self-correspondence F is upper weak set monotonic, compact-valued, and upper hemi-order continuous, and X_+ is nonempty.*

(i). For every $x \in X_+$ there exists a weakly increasing sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_1 = x$ and and $x_{n+1} \in \{y \in X : y \in F(x_n), y \geqslant x_n\}$, for each $n \in \mathbb{N}$; and its limit $x_* = \lim_{n\to\infty} x_n$ is well defined and is a fixed point of F.

(ii). Suppose $G : X \rightrightarrows X$ satisfies the properties of Theorem 5, is upper weak set monotonic and upper hemi-order continuous, and $G(x)$ upper weak set dominates $F(x)$ for each x. Then, for each fixed point x_F of F, there is a fixed point x_G of G with $x_G \geqslant x_F$ that can be found by an upward iterative procedure starting with $x_1 = x_F$ for G.[28]

A symmetric conclusion holds if F is lower weak set monotonic and lower hemi-order-continuous, and X_- is nonempty.

Proof. Given the symmetry, we only prove (i) and (ii). First, since $x_1 \in X_+$, there exists $x_2 \in \{y \in F(x_1) : y \geqslant x_1\}$. By upper weak set monotonicity of F, if $x_{n+1} \in F(x_n)$ and if $x_{n+1} \geqslant x_n$, for any $n \in \mathbb{N}$, then there must exist $x_{n+2} \in \{y \in F(x_{n+1}) : y \geqslant x_{n+1}\}$. We

[27] Note that the condition is weaker than upper hemi-continuity since the condition is required only for $(x_n)_n$ that is monotone. The condition can be seen also as a generalization of the order continuity defined for a function to a correspondence. See Milgrom and Roberts (1990) for an order-continuous function.

[28] More specifically, there exists a weakly increasing sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_1 = x_F$ and $x_{n+1} \in \{y \in X : y \in G(x_n), y \geqslant x_n\}$, for each $n \in \mathbb{N}$; and its limit $x_G = \lim_{n\to\infty} x_n$ is well defined and is a fixed point of G.

thus obtain a nondecreasing sequence of $\{x_n\}_{n\in\mathbb{N}}$. Since X is a compact metric space, the increasing sequence has a limit $x_* = \lim_{n\to\infty} x_n$. By the upper hemi-order-continuity of F, $x_* \in F(x_*)$, proving (i). The proof of (ii) follows the same argument, once we redefine the starting point $x_1 = x_F$ of the iterative procedure for operator G. $\square$

Recall that upper hemi-order-continuity is trivially satisfied if X is finite. Hence, Theorem 7 suggests a convenient and fast algorithm to identify a fixed point for finite X, even without the standard set of assumptions required by the traditional Tarski approach.

One may recall that in the setting of Tarski and Zhou, a monotonic algorithm starting from the largest and smallest elements finds the largest and smallest fixed points, respectively, and may wonder if maximal and minimal points can be found in this way in our context. The following example provides a negative answer to that question.[29]

Example 4. Suppose $X = \{1,2,3\} \times \{1,2\}$ and $F : X \rightrightarrows X$ is defined by: $F((1,1)) = \{(1,2),(2,1)\}$, $F((2,1)) = \{(1,2),(3,2)\}$, $F((1,2)) = \{(2,1),(3,2)\}$, $F((2,2)) = \{(2,2),(3,2)\}$, $F((3,1)) = \{(3,2)\}$, and $F((3,2)) = \{(3,2)\}$. Note that F is both upper and lower weak set monotonic. There are two fixed points $\{(2,2),(3,2)\}$. Suppose that one iterates F as suggested in Theorem 7, starting with the lowest point $x_1 = (1,1)$. Then, no matter which point one chooses along the iteration, the only fixed point one can reach is $(3,2)$. But this is not a minimal fixed point; $(2,2)$ is the unique minimal point and smaller than $(3,2)$. The minimal fixed point $(2,2)$ cannot be reached from any iterative application of F starting from $(1,1)$. See Figure 5.

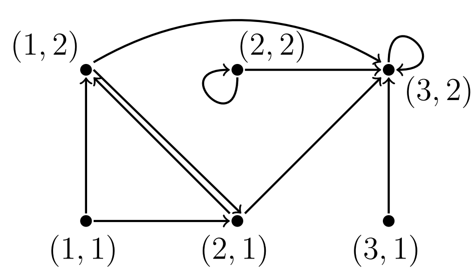

Figure 5: Every iteration fails to reach a minimal fixed point.

5.3 Games with Weak Strategic Complementarities

In this part, we apply our monotone comparative statics results and the fixed-point theorem to strategic environments to establish the existence and comparative statics of Nash

[29]Example 9 in Online Appendix F.3 illustrates additional difficulty with iterative procedures.

equilibrium. Our equilibrium theory is reminiscent of that of Milgrom and Shannon (1994) who use their sMCS result to establish analogous results to ours—i.e., wMCS—in games with strategic complementarities. Our approach applies to a broader class of games, called *games with weak strategic complementarities*, in which the best response correspondences are required to be only weak set monotonic. Given our results in the previous sections, the existence and comparative statics of Nash equilibrium can be established even in games where only weak set monotonicity (and not strong set monotonicity) holds. As mentioned earlier, games with weak strategic complementarities even allow for players who have incomplete preferences or players who are representatives of coalitions each of whom follows the Pareto criterion to make a choice for a coalition he or she belongs to.

Consider a normal-form game $\Gamma = (I, X, (B_i)_{i\in I})$, where I is a finite set of players, $X := \times_{i\in I} S_i$ is a cartesian product of strategy sets S_i, and $B_i : S_{-i} \rightrightarrows S_i$ is a correspondence interpreted as the best response correspondence for player i. We assume that S_i is partially ordered for each i and any cartesian product, e.g., X or S_{-i}, is partially ordered by the product order based on the relevant partial orders. We further assume that each S_i is a compact metric space inducing a natural topology and let X be endowed with the product topology. Finally, we assume that each B_i is a nonempty-valued and compact-valued correspondence. We call these **basic properties**.

Remark 3. Importantly, we do not necessarily require that the best response correspondence B_i be based on maximization of a utility function $u_i : X \to \mathbb{R}$,

$$B_i(s_{-i}) := \arg\max_{s_i\in S_i} u_i(s_i, s_{-i}). \tag{7}$$

Indeed, our analysis is applicable when, for instance, the best response correspondence is defined as the set of Pareto optimal choices by a group of agents or by a set of "multi-selves" in the case of an agent with incomplete preferences (see Proposition 4 below). At the same time, we also note that the best response correspondence based on utility maximization (7) satisfies our requirements, namely nonemptiness and compactness, as long as u_i is upper semicontinuous in s_i.[30] For the case of Pareto optimal choices, recall that Propositions 2 and 3 establish nonemptiness and compactness.

A strategy profile $s = (s_i)_{i\in I}$ is a Nash equilibrium if $s_i \in B_i(s_{-i})$ for every $i \in I$.

We call Γ **a game with weak strategic complementarities** if it satisfies the basic properties and the following conditions:

[30] The compactness can be seen as follows. Let $v_i^* := \max_{s_i\in S_i} u_i(s_i, s_{-i})$, and let $\{s_i^n\}_{n\in\mathbb{N}}$ be a sequence with $s_i^n \in B_i(s_{-i})$ for each $n = 1, \ldots,$ converging to some $s_i^* \in S_i$. Then, by the upper semicontinuity, $u_i(s_i^*, s_{-i}) \geqslant \limsup_{n\to\infty} u_i(s_i^n, s_{-i}) = v_i^*$.

(1) for each $i \in I$, B_i is upper weak set monotonic;

(2) there exists $\underline{s} = (\underline{s}_i)_{i \in I} \in X$ such that for each i, $s'_i \in B_i(\underline{s}_{-i})$ for some $s'_i \geqslant \underline{s}_i$.

Conditions (1) and (2) correspond to those required by the Fixed Point Theorem (Theorem 5) for general correspondences. Condition (1) is satisfied if players are economic agents who possess the preferences we imposed for the comparative statics results in the previous section, as will be discussed later. Condition (2) is vacuously satisfied if there exists a smallest element in each player's strategy space, e.g., if the strategy space is a lattice.

The definition of games with weak strategic complementarities is general and rather abstract. The following proposition offers two sufficient conditions for a game to exhibit weak strategic complementarities.

Proposition 4. *A game $\Gamma = (I, X, (B_i)_{i \in I})$ is a game with weak strategic complementarities if it satisfies the basic properties and, for each player $i \in I$,*

(i). $B_i(s_{-i})$ is given as the solution to utility maximization (7), S_i is a lattice, and $u_i(\cdot, s'_{-i})$ weakly interval dominates $u_i(\cdot, s_{-i})$ for any $s'_{-i} \geqslant s_{-i}$, or

(ii). $B_i(s_{-i})$ is given as the set of Pareto optimal choices for a collection of payoff functions $\mathbf{u}_i(s_{-i})$,[31] S_i is a lattice, and $\mathbf{u}_i(s'_{-i})$ dominates $\mathbf{u}_i(s_{-i})$ whenever $s'_{-i} \geqslant s_{-i}$, in the sense of Theorem 3 or Theorem 4.

With these preliminary concepts and results at hand, we now provide general existence and comparative statics results:

Theorem 8. *(i). A game $\Gamma = (I, X, (B_i)_{i \in I})$ with weak strategic complementarities has a nonempty set of Nash equilibria.*

(ii). Suppose that $\Gamma = (I, X, (B_i)_{i \in I})$ and $\Gamma' = (I, X, (B'_i)_{i \in I})$ are both games with weak strategic complementarities, and $B_i(s_{-i}) \geqslant_{uws} B'_i(s_{-i})$ for every $i \in I$ and $s_{-i} \in S_{-i}$. Then, the set of Nash equilibria in Γ upper weak set dominates the set of Nash equilibria in Γ'. (A symmetric result based on the lower weak set comparison also holds.)

Proof. Note first that $B_i(s_{-i})$ is nonempty and compact. Due to this property and properties (1) and (2) of games with weak strategic complementarities, the mapping $F : X \rightrightarrows X$ defined by $F(s) := (B_i(s_{-i}))_{i \in I}$ satisfies the requirement of Theorem 5. We thus conclude

[31] That is, for each $i \in I$ we have $\mathbf{u}_i(s_i, s_{-i}) := (u_{ij}(s_i, s_{-i}))_{j \in J_i}$ with a certain index set J_i, and $B_i(s_{-i})$ is the set of Pareto optimal choices for agents in J_i with utility functions $(u_{ij}(s_i, s_{-i}))_{j \in J_i}$.

that there exists a fixed point $s^* \in F(s^*)$, which means that the set of Nash equilibria is nonempty. Moreover, we note that $F(s)$ upper weak set dominates $F'(s) := (B'_i(s_{-i}))_{i \in I}$ for each $s \in X$. Thus, all the conditions for Theorem 6 are satisfied, which implies the comparative statics conclusion of item 2 of the theorem. □

6 Application to Matching Theory

In this section, we apply our theory to matching problems. As we will demonstrate below, the techniques we developed in the previous sections prove useful for analyzing stable matching under weaker assumptions than have been employed by the existing research. We first establish the existence of a stable matching building on our fixed-point theorem (Theorem 5). We then obtain comparative statics of stable matchings based on our general wMCS result for fixed points (Theorem 6).

The main departure from the existing literature is the generality of agents' choice correspondences we allow for. Specifically, we relax the two main assumptions in the literature; WARP and substitutability. These relaxed assumptions allow for indifferences or even incompleteness of preferences. This generality plays an important role in our main applications.

6.1 A Motivating Example

We begin with a simple example that illustrates why a choice correspondence that does not satisfy WARP may arise in matching settings.

Example 5. Consider a firm f with two divisions, δ and δ'. The firm is subject to a budget constraint that compels it to hire at most one worker across the divisions, but the firm does not have strict preferences over which division hires a worker when both divisions have applicants. Each division has its own preferences over the workers. There are 3 workers, w, w' and w'', who are all acceptable to both divisions, and division δ' prefers w'' to w'. Then, if workers w and w' apply to divisions δ and δ', respectively, then the choice of the firm from this set of applications $\{(w, \delta), (w', \delta')\}$ would be either (w, δ) or (w', δ'), where (w, δ), for instance, denotes a contract specifying a matching between w and δ. If w'' applies to δ' in addition, then the firm faces a set of applications $\{(w, \delta), (w', \delta'), (w'', \delta')\}$ and chooses either (w, δ) or (w'', δ').

A few points are worth noting. First, a multidivisional organization facing such a constraint as in this example may exhibit a choice behavior described by a (multi-valued) correspondence rather than a function. This is the case if the organization does not have strict preferences over different ways to resolve preference conflicts among its divisions. In the above example, either (w, δ) or (w', δ') can be chosen from the set of applications $\{(w, \delta), (w', \delta')\}$.

Second, a multidivisional organization's choice behavior may violate WARP and thus not admit any complete (possibly weak) preference relation that rationalizes it. To see this, note that if the choice of firm f were rationalizable by a complete (possibly weak) preference relation, then the firm must be indifferent between (w, δ) and (w', δ') because both of them are chosen from $\{(w, \delta), (w', \delta')\}$. However, (w', δ') is never chosen from $\{(w, \delta), (w', \delta'), (w'', \delta')\}$ even though (w, δ) is. As we argue below, the associated choice correspondence fails WARP. Intuitively, incompleteness of the underlying preferences arises naturally in a multidivisional firm because the firm simply lacks a criterion to compare placement in different divisions (e.g., between (w, δ) and (w'', δ')).

To our knowledge, few papers in matching research allow for choice correspondences,[32] and none that we know of accommodates incomplete preferences. However, such preferences are quite natural, as suggested by the above example of a firm with multiple divisions. Similar issues may arise in matching problems with constraints such as Japanese medical match (Kamada and Kojima, 2015). In that problem, the government imposes a maximum number of doctors to be placed in each region of the country. If the government does not specify how many positions each hospital in a given region must give up to satisfy the joint constraint, the choice behavior of the set of hospitals in the region cannot be described by a single-valued function, and the resulting choice correspondence (for the region) cannot necessarily be rationalized by a complete preference relation, just as in the case of a multidivisional organization in Example 5. As we show below, the theory of matching, both in terms of existence and comparative statics, generalizes to such settings.

6.2 Model and Results

We now present our model. There are a finite set F of firms and a finite set W of workers, as well as a finite set X of contracts. Each contract $x \in X$ is associated with one firm $x_F \in F$ and one worker $x_W \in W$. We will often write x to denote a singleton set $X' = \{x\}$. Given a set $X' \subset X$ of contracts, let $X'_f = \{x \in X' : x_F = f\}$ and $X'_w = \{x \in X' : x_W = w\}$ denote the sets of contracts firm f and worker w are involved with within X', respectively.

[32] There are, of course, some exceptions. A number of papers—for instance, Erdil and Ergin (2008) and Abdulkadiroğlu, Pathak and Roth (2009)—consider matching under responsive preferences with ties on the side of schools. In many cases, however, tie-breaking allows the problem to be reduced to the case with strict priorities.

A set of contracts $X' \subset X$ will be called an **allocation** if it contains at most one contract for each worker.

Each agent $a \in F \cup W$ is endowed with a **choice correspondence**: $C_a : 2^X \rightrightarrows 2^X$ where, for each $X' \subseteq X$, $C_a(X')$ is a nonempty family of subsets of X'_a. Any element of $C_a(X')$ represents a set of contracts agent a chooses from X'. The choice correspondence C_a induces the **rejection correspondence** $R_a : 2^X \rightrightarrows 2^X$, defined by $R_a(X') = \{Z : Z = X'_a \backslash Y \text{ for some } Y \in C_a(X')\}$.

For any pair of allocations X' and X'', we say that agent a **weakly prefers** X'' to X' if $X''_a \in C_a(X'_a \cup X''_a)$, and write $X'' \succeq_a X'$.[33] We say that a **strictly prefers** X'' to X' if $X'' \succeq_a X'$ but not $X' \succeq_a X''$, and write $X'' \succ_a X'$.

We focus on the many-to-one matching setup by assuming that the choice correspondence of each worker w satisfies the following properties: for any $X' \subseteq X$, (i) $X'' \in C_w(X')$ implies $X'' \subseteq X'_w$ and $|X''| \leqslant 1$; and (ii) $X'' \in C_w(X')$ if $\varnothing \not\succ_w X''$, and $\{x'\} \not\succ_w X''$ for any $x' \in X'_w$. In words, each worker chooses at most one contract (possibly a null set) and any such that is not dominated by any other contract (including remaining unemployed).

An economy is summarized as a tuple $\Gamma = (F, W, X, (C_a)_{a \in F \cup W})$. An allocation Z is **stable** if

(i). (Individual Rationality) $Z_a \in C_a(Z)$ for every $a \in F \cup W$, and

(ii). (No Blocking Coalition) $Z_f \in C_f(Z \cup U(Z))$ for every $f \in F$, where $U(Z) := \{x \in X : x \succ_{x_W} x', \forall x' \in Z_{x_W}\}$.[34]

The key method for analyzing stable allocations is to associate them with fixed points of suitably-defined correspondence (see Adachi (2000), Fleiner (2003), Echenique and Oviedo (2004, 2006), and Hatfield and Milgrom (2005), for example). WARP has been crucial for this purpose.[35] Formally, a preference relation for agent $a \in F \cup W$ satisfies WARP if and only if the associated choice correspondence C_a satisfies the following two conditions (see Kreps (1988), for instance):

(i). **Sen's** α: $Y \in C_a(X'')$ and $Y \subset X' \subset X'' \Longrightarrow Y \in C_a(X')$, and

(ii). **Sen's** β: $Y, Y' \in C_a(X')$ and $Y \in C_a(X'')$ for $X' \subset X'' \Longrightarrow Y' \in C_a(X'')$.

[33] This is the so-called "Blair order" introduced by Blair (1988).

[34] In Online Appendix G.1, we consider an alternative notion of stability and its relation with the present stability notion under Sen's α or WARP.

[35] See Hatfield and Milgrom (2005), Che, Kim and Kojima (2019), and Aygün and Sönmez (2013), among others. We note that authors have invoked WARP under different names; the first two sets of authors call it Revealed Preference, while the last set of authors, who highlight the importance of the condition, call it Irrelevance of Rejected Contracts.

In words, Sen's α states that an optimal choice from a "bigger" set must be an optimal choice from a "smaller" set that contains it. Sen's β attributes non-uniqueness of choice to indifferences: if multiple alternatives are optimal from a smaller set and one of them is still optimal from a bigger set, the other(s) must also be optimal from the bigger set. While the former remains compelling, the latter can easily fail in the context of multidivisional organizations or of incomplete preferences. For instance, in our motivating example, the firm chooses either (w, δ) or (w', δ') from $\{(w, \delta), (w', \delta')\}$, and it chooses (w, δ) from $\{(w, \delta), (w', \delta'), (w'', \delta')\}$, but it never chooses the contract (w', δ') from the latter set, violating Sen's β. Therefore, we shall relax WARP by dispensing with Sen's β, insisting only on Sen's α. Indeed, Sen's α is compatible with a wide variety of preferences with indifferences or even incompleteness.[36]

We now proceed with a fixed-point characterization of stable allocations. Let $C_F(X') := \{\cup_{f \in F} Y_f : Y_f \in C_f(X'), \forall f \in F\}$ and $R_F(X') := \{\cup_{f \in F} Y_f : Y_f \in R_f(X'), \forall f \in F\}$. Define C_W and R_W analogously. Then, a fixed-point mapping (or correspondence) $T : 2^X \times 2^X \rightrightarrows 2^X \times 2^X$ is defined as follows: For each $(X', X'') \in 2^X \times 2^X$, $T(X', X'') = (T_1(X''), T_2(X'))$, where

$$T_1(X'') = \{\tilde{X} \in 2^X : \tilde{X} = X \backslash \tilde{Y} \text{ for some } \tilde{Y} \in R_W(X'')\},$$
$$T_2(X') = \{\tilde{X} \in 2^X : \tilde{X} = X \backslash \tilde{Y} \text{ for some } \tilde{Y} \in R_F(X')\}.$$

Intuitively, we can think of T as iterating on sets X' and X'' of contracts available respectively to firms and workers. For each pair (X', X''), T_1 returns sets of contracts that are available to the firms after removing contracts workers reject out of X'', while T_2 returns sets of contracts that are available to the workers after removing contracts rejected by firms out of X'. Mapping T is is similar to fixed-point mappings used in the existing literature such as Hatfield and Milgrom (2005), except that it is generalized to handle choice correspondences rather than choice functions.

Theorem 9. *Suppose that C_a satisfies Sen's α for each $a \in F \cup W$. Then, there exists a stable allocation Z if and only if (X', X'') is a fixed point of T, where $Z \in C_F(X') \cap C_W(X'')$.*

Proof. See Appendix C. □

As will become clear, our fixed-point characterization is crucial for both existence and comparative statics of stable allocations. We first use it and apply Theorem 5 to establish

[36] Eliaz and Ok (2006) introduce an axiom called weak axiom of revealed non-inferiority (WARNI) that are consistent with incomplete preferences. In Online Appendix G.2, we show that Sen's α is implied by WARNI.

existence of stable allocations. To this end, we consider a partially ordered set $(2^X, \geqslant)$, where the order $\geqslant$ is given by "set inclusion" operator; i.e., $X'' \geqslant X'$ if $X'' \supset X'$. The associated upper and lower weak set orders over families of sets of contracts are defined based on this primitive (set inclusion) order. The monotonicity of correspondence $f : 2^X \rightrightarrows 2^X$ is defined accordingly: that is, f is upper weak set monotonic if for $X' \subset X'' \subset X$, $Y' \in f(X')$ implies there exists $Y'' \supset Y'$ such that $Y'' \in f(X'')$; and similarly for lower weak set monotonicity. For the product set $2^X \times 2^X$, we endow the following order: $(X'', Y'') \geqslant (X', Y')$ if $X'' \supset X'$ and $Y'' \subset Y'$. The monotonicity of correspondence $f : 2^X \times 2^X \rightrightarrows 2^X \times 2^X$ is then defined according to this order.

The next step is to invoke an appropriate assumption on agents' choice correspondences to ensure that $T = (T_1, T_2)$ is weak set monotonic. Specifically, we assume that, for each $a \in F \cup W$, the choice correspondence $C_a(\cdot)$ is **weakly substitutable**, i.e., R_a *is weak set monotonic.* A standard notion of substitutability considers a choice function—rather than a choice correspondence—and requires the associated rejection function to be monotonic (e.g., Hatfield and Milgrom (2005)). One way to generalize this notion to choice correspondences would be to require that the rejection correspondences to be complete sublattice valued and monotonic in the strong set order—the condition Che, Kim and Kojima (2019) labels **substitutability**. However, this condition proves too restrictive to accommodate even the most common form of indifferences:

Example 6. A firm f has one position and is willing to fill it via any one of three contracts, x, y, and z. The resulting rejection correspondence is not sublattice-valued: $R_f(\{x, y\}) = \{\{x\}, \{y\}\}$ and $\{x\} \vee \{y\} = \{x, y\} \notin R_f(\{x, y\})$. It is not strong set monotonic, either: $\{y, z\} \in R_f(\{x, y, z\})$, $\{x\} \in R_f(\{x, y\})$, and $\{y, z\} \vee \{x\} = \{x, y, z\} \notin R_f(\{x, y, z\})$. We thus conclude that C_f is not substitutable. Nevertheless, R_f is weak set monotonic, as can be checked easily, so C_f is weakly substitutable.

By contrast, weak substitutability is quite weak. Nevertheless, it is sufficient for existence, as we show now.

Theorem 10. *Suppose that C_a satisfies Sen's α and is weakly substitutable for each $a \in F \cup W$. Then, a stable allocation exists.*

Proof. See Appendix C. □

To the best of our knowledge, the current existence result is the most general of its kind, requiring very weak preferences conditions that allow for both indifferences and incompleteness.[37] At the same time, an astute reader may notice that no claim is made in the above theorem about the existence of worker- and firm-optimal stable allocations, which are often shown to exist under substitutible preferences. Indeed, such "side-optimal" allocations are not guaranteed to exist in the presence of indifferences.[38] Formally, this can be attributed to the fact that our fixed-point theorem (Theorem 5) does not guarantee the lattice structure for the fixed-point set.[39]

We now turn to our marquee result: monotone comparative statics of stable allocations. To this end, we say that choice correspondence C_a is **weakly more permissive** than C'_a if, for each set of contracts X', $R_a(X') \leqslant_{ws} R'_a(X')$. In words, an agent with C_a rejects fewer contracts than an agent with C'_a.[40] We let $\succeq_a$ and $\succeq'_a$ denote the (possibly incomplete) preferences associated with C_a and C'_a, respectively; and similarly for T and T'.[41]

Theorem 11. *Consider two economies* $\Gamma = (F, W, X, (C_a)_{a \in F \cup W})$ *and* $\Gamma' = (F, W, X, (C'_a)_{a \in F \cup W})$ *such that* C_w *is weakly more permissive than* C'_w *for each* $w \in W$ *while* C'_f *is weakly more permissive than* C_f *for each* $f \in F$*. Then,*

(i). for each stable allocation Z *in* Γ*, there exists a stable allocation* Z' *in* Γ' *such that* $Z_f \succeq_f Z'_f$ *for each* $f \in F$ *and* $Z'_w \succeq'_w Z_w$ *for each* $w \in W$*, and*

(ii). for each stable allocation Z' *in* Γ'*, there exists a stable allocation* Z *in* Γ *such that* $Z_f \succeq_f Z'_f$ *for each* $f \in F$ *and* $Z'_w \succeq'_w Z_w$ *for each* $w \in W$*.*

Proof. See Appendix C. □

[37]One may wonder if our general approach based on choice correspondenses is important for existence of stable allocation. It may be tempting instead to work with choice functions obtained after breaking ties in some manner. Indeed, such an approach works if, for instance in school choice, schools have responsive preferences with ties (i.e., coarse priorities); one can then arbitrarily break ties in schools' preferences and apply the existence result with strict preferences. This "trick" does not work, however, if school preferences are non-responsive (but are weakly substitutable), since breaking ties in a arbitrary manner may not preserve the substitutability for the resulting choice function. For example, consider a set of contracts $X = \{x, y\}$ and a firm f which has the following choice correspondence: $C_f(\{x, y\}) = \{\{x, y\}, \{x\}, \{y\}\}$; $C_f(\{z\}) = \{\{z\}, \varnothing\}$ for $z = x, y$. Suppose tie-breaking selects the choice function $\tilde{C}_f$: $\tilde{C}_f(\{x, y\}) = \{\{x, y\}\}$; $\tilde{C}_f(\{z\}) = \{\varnothing\}$ for $z = x, y$. This function violates substitutability, even though the original choice correspondence is weakly substitutable. Tie-breaking is even less useful for comparative statics, since no simple tie-breaking method may "discover" the entire set of stable allocations, which is required for monotone comparative statics.

[38]Recall Example 6. Suppose every worker prefers to work for f instead of being unemployed. Then, there are three stable allocations; f hiring any one of three workers. None of them is worker optimal.

[39]On the other hand, substitutable preferences guarantee existence of side optimal stable matching even in the presence of indifferences. See Che, Kim and Kojima (2019).

[40]For any worker w, a change in choice correspondences has a particularly simple form. Specifically, suppose that C_w is weakly more permissive than C'_w. Then, for any $X' \subseteq X$, either $C_w(X') = C'_w(X')$ or $C'_w(X') = \{\varnothing\}$. See Online Appendix G.3 for detail.

[41]Recall that $\succeq_a$ and $\succeq'_a$ are the preferences defined by Blair (partial) order.

The basic idea of the proof is to utilize the fixed-point characterization of stable allocations by the mapping T. We first establish that the fixed-point mapping "shifts up" in the weak set order sense with the change of choice correspondences. By Theorem 6, this implies that the set of fixed points "increases" in the weak set order. This gives rise to the desired comparative statics properties of stable allocations.

Theorem 11 generalizes various comparative statics results in the existing literature from the cases of choice functions to choice correspondences.[42] As such, it implies a number of standard results. For instance, a stable allocation becomes more favorable to one side when it becomes more "scarce" or when there is more competition from the other side:

Corollary 2. *Suppose that a worker exits the market or a new firm enters the market. Then, for each stable allocation in the original market, there exists a stable allocation in the new market in which all the remaining workers are weakly better off and all the existing firms are weakly worse off. A symmetric result, though in the opposite direction, holds if a worker enters the market or a firm exits a market.*

Proof. See Appendix C. □

The entry/exit of agents in this Corollary corresponds to their choice correspondences becoming more/less permissive. For instance, an agent exiting a market corresponds to that agent having a less permissive correspondence than before (in fact, she rejects every contract). Therefore, all remaining agents from the same side become weakly better off and those from the opposite side become weakly worse off in some new stable allocation by Theorem 11.

Aside from these standard comparative statics, the generality of Theorem 11 enables us to obtain new kinds of comparative statics results. For instance, if the internal constraint of a multidivisional firm is relaxed (e.g., a hiring budget increases), then all the workers are made weakly better off while all the other firms are made worse off in at least one new stable matching. A similar monotone comparative statics holds in matching with constraints. Suppose, for example, in the Japanese medical matching, the maximum number of doctors that can be hired by hospitals in a region increases. Then, the choice correspondence representing that region becomes more permissive, so the doctors are weakly better off in at least one (weakly) stable matching. These new comparative statics results are formalized and proven in Online Appendices G.4 and G.5.

[42] There are many comparative statics results for choice functions in various formulations and generality. See Gale and Sotomayor (1985*a*,*b*), Crawford (1991), Konishi and Ünver (2006), Echenique and Yenmez (2015), and Chambers and Yenmez (2017), for instance.

6.3 Applications

The present framework subsumes environments beyond those analyzed in existing research. Let us describe two applications of our approach in informal manners here. The formal treatments are relegated to Online Appendices G.4 and G.5.

(i). **Multidivional Organization:** Consider an organization, such as large firms, that has multiple divisions.[43] Such an organization may face a total hiring budget and may decide to allot positions across divisions within that budget. Given the allotted positions, each division chooses the best applicants according to its own linear preference order. The firm with multiple divisions described in Example 5 is a concrete example.

In Online Appendix G.4, we construct a choice correspondence that captures these features. The organization's choice is not necessarily described as a function, but as a correspondence—the organization may find indifferent or incomparable two allotments of positions across different divisions as long as both of them satisfy the organization's internal constraint. This feature leads to the failure of conditions assumed in existing studies, but we show that the organization's choice correspondence still satisfies both Sen's α and weak substitutability. Hence, Theorems 10 and 11 allow us to establish the existence of a stable matching as well as wMCS property.

(ii). **Matching with Constraints:** Consider a problem of matching with constraints, such as medical match faced with a government-imposed cap on the number of doctors in each region or in each medical specialty. Kamada and Kojima (2017) present a model of matching with constraints, introduce a concept called weak stability, and establish the existence of a weakly stable matching.[44]

We prove the existence of a weakly stable matching as a corollary of Theorem 10. The basic idea of the proof is to associate the model of matching with constraints with an auxiliary model of matching with contracts between doctors and the "hospital side," a consortium that jointly chooses among applicants to different hospitals.[45] Intuitively,

[43]This class of choice correspondences considered here is similar in spirit to a multidivisional choice function with flexible allotments analyzed by Hatfield, Kominers and Westkamp (2017), but neither is more general than the other.

[44]Alternative concepts of stability, including weak stability, are defined by Kamada and Kojima (2015, 2017, 2018). Weak stability has advantages over others such as existence under mild conditions and an axiomatic characterization (Kamada and Kojima, 2017).

[45]To our knowledge, Kamada and Kojima (2015) is the first to associate matching with constraints to matching with contracts, and this technique has been used in subsequent studies such as Kamada and Kojima (2018, 2019), Goto et al. (2014*a*), Goto et al. (2014*b*), and Kojima, Tamura and Yokoo (2018).

we exploit the fact that the hospital side's choice behavior under constraints works in a manner that is analogous to that of a multidivisional organization. Choice behavior of the hospital side is not necessarily a function but a correspondence because there is some degree of freedom as to how many positions are allotted to different hospitals given the joint constraint. These features can be readily incorporated into our model. More formally, we verify that the hospital side's choice correspondence satisfies both Sen's α and weak substitutability. Moreover, we establish that a matching is weakly stable in the given model of matching with constraints if and only if a corresponding allocation in the auxiliary model of matching with constraints is a stable allocation. These results imply that a weakly stable matching exists.

While the existence of a weakly stable matching has been established before, our technique allows us to obtain a novel comparative statics result with respect to changes in constraints. While such results were hitherto unvailable, they are a natural consequence of our approach and can be obtained as a corollary of Theorem 11.

However, our approach is different from theirs in at least two respects. First, all the other works focus on choice functions rather than choice correspondences, making it impossible to connect their approach to weak stability. Second, the class of constraints we consider are more general than those studied in any of the above papers. Both of these differences are crucial for our analysis, and our analysis capitalizes heavily on the generality of the present model which allows for choice correspondences under Sen's α and weak substitutability.

A Proofs for Section 4

Proof of Proposition 2. To prepare for the proof of the proposition, we begin with the following lemma.

Lemma 3. *$x \in P(\mathbf{u})$ if and only if $x \in \Phi(x)$, where $\Phi(x) := \bigcap_{i \in I} \Phi_i(x)$ and $\Phi_i(x) :=$ $\arg\max_{y \in U_{-i}(x)} u_i(y)$.*[46]

That is, $x \in P(\mathbf{u})$ if and only if x is a fixed point of correspondence Φ. The proof of this lemma is straightforward; $x \in \bigcap_{i \in I} \Phi_i(x)$ if and only if there exist no i and y such that $u_i(y) > u_i(x)$ and $u_j(y) \geqslant u_j(x)$ for all $j \neq i$.

We now prove the proposition: nonemptiness of $P(\mathbf{u})$. To this end, fix any $x_0 \in X$. Let $I = \{1, \ldots, n\}$ and define x_i and X_i recursively as follows: $X_i = \cap_{j \in I} \{\tilde{x} \in X \mid u_j(\tilde{x}) \geqslant u_j(x_{i-1})\}$ and $x_i \in \arg\max_{\tilde{x} \in X_i} u_i(\tilde{x})$. Note that the existence of the maximizer x_i is guaranteed by the assumption that u_1 is USC and the fact that u_j being USC for all j implies X_i is closed and thus compact since X is compact. We shall show that x_n is a fixed point of Φ, which by Lemma 3 implies x_n is Pareto optimal. To do so, observe that for all $i \in I$,

$$u_i(x_n) = \cdots = u_i(x_i) \geqslant u_i(x_{i-1}) \geqslant \cdots \geqslant u_i(x_1) \geqslant u_i(x_0). \tag{8}$$

We thus have $x_n \in U_{-i}(x_n) \subset X_i, \forall i \in I$, which implies $x_n \in \Phi_i(x_n), \forall i \in I$, so $x_n \in \Phi(x_n)$, as desired. □

Proof of Lemma 1. To begin, we note that u_i is continuous for each $i \in I$ since u_i is concave and USC.

To proceed with the proof, we work with the utility space. Define the set

$$\mathcal{U} := \{(u_1, ..., u_{|I|}) \in \mathbb{R}^{|I|} : \exists x \in X \text{ such that } u_i \in [\min_{y \in X} u_i(y), u_i(x)], \forall i \in I\}$$

of utility vectors (note that $\min_{y \in X} u_i(y)$ exists because u_i is continuous and X is compact). Since u_i is concave for each i, $\mathcal{U}$ is convex. Further, since each u_i is continuous and X is compact, $\mathcal{U}$ is compact, so it is closed and bounded. Likewise, let

$$\mathcal{U}(P(\mathbf{u})) := \{u \in \mathcal{U} : \text{ there exists } x \in P(\mathbf{u}) \text{ such that } u_i(x) = u_i \text{ for every } i \in I\}$$

be the utility vectors associated with all the Pareto optimal choices. It sufficies to prove that $u \in \mathcal{U}(P(\mathbf{u}))$ if and only if there exists $\lambda := (\lambda_1, ..., \lambda_{|I|}) \in \mathbb{R}^{|I|}_{++}$ and a partition

[46] Recall that $U_{-i}(x)$ is defined prior to Proposition 3.

$\mathcal{I} = \{I_1, ..., I_n\}$ of I such that

$$u \in \mathcal{U}_m := \underset{u \in \mathcal{U}_{m-1}}{\arg\max} \sum_{i \in I_m} \lambda_i u_i, \tag{9}$$

where $\mathcal{U}_0 := \mathcal{U}$ (the reason for this is that each $u \in \mathcal{U}_m$ is attained by some $x \in X$).

The "if" direction is obvious (see Proposition 16.E.2 of Mas-Colell et al. (1995)). Thus, we shall show the "only if" direction in the remainder. To do so, first notice that, by convexity of $\mathcal{U}$ and Proposition 16.E.2 of Mas-Colell et al. (1995), there exists $\lambda^1 \in \mathbb{R}_+^{|I|}$ with $\lambda^1 \neq 0$ such that $u \in \arg\max_{u' \in \mathcal{U}} \lambda^1 \cdot u' := \hat{\mathcal{U}}_1$. Let $I_1 := \{i \in I : \lambda_i^1 > 0\}$. Note that I_1 is nonempty because $\lambda^1 \neq 0$.

Let $\tilde{\mathcal{U}}_1$ be the projection of $\hat{\mathcal{U}}_1$ to $I \backslash I_1$, that is, $\tilde{\mathcal{U}}_1 := \{u \in \mathbb{R}^{|I \backslash I_1|} : \exists v \in \hat{\mathcal{U}}_1, \forall i \in I \backslash I_1, u_i = v_i\}$. $\hat{\mathcal{U}}_1$ is convex because $\mathcal{U}$ is convex and $\lambda^1 \cdot u$ is a linear function of u, so $\tilde{\mathcal{U}}_1$ is convex as well. Now, denoting the projection of u to $I \backslash I_1$ as $\tilde{u}^1$ and applying Proposition 16.E.2 of Mas-Colell et al. (1995) to $\tilde{\mathcal{U}}_1$, we conclude that there exists $\lambda^2 \in \mathbb{R}_+^{|I \backslash I_1|}$ with $\lambda^2 \neq 0$ such that $\tilde{u}^1 \in \arg\max_{u' \in \tilde{\mathcal{U}}_1} \lambda^2 \cdot u' =: \hat{\mathcal{U}}_2$. Let $I_2 := \{i \in I \backslash I_1 : \lambda_i^2 > 0\}$. Note that I_2 is nonempty because $\lambda^2 \neq 0$.

Proceeding inductively, we obtain a partition of I, $I_1, I_2, \ldots, I_n$. Set $\mathcal{I} = \{I_1, \ldots, I_n\}$. Also, define $\lambda \in \mathbb{R}_{++}^{|I|}$ by setting $\lambda_i = \lambda_i^m$ for each $i \in I_m$. Then, by construction of $\mathcal{I}$ and λ, it follows that $u \in \mathcal{U}_m$ for each m (where $\mathcal{U}_m$ is defined by equation (9) given $\mathcal{I}$ and λ). □

Proof of Lemma 2. Consider any x_0 that is not Pareto optimal. One can find x_n as in the proof of Proposition 2. Then, x_n is Pareto optimal while it must Pareto dominate x_0 by (8). □

Proof of Proposition 3. Given the continuity of $u_i, \forall i \in I$, it is routine to see that $U_{-i}(\cdot)$ is upper hemicontinuous for each $i \in I$. Since $U_{-i}(\cdot)$ is also lower hemicontinuous, it is continuous. Since u_i is continuous, by Berge's theorem of maximum, $\Phi_i(\cdot)$ is upper hemicontinuous.

To prove the compactness of $P(\mathbf{u})$, it suffices to show that $P(\mathbf{u}) = \bigcap_{i \in I} \Phi_i(x)$ is closed since $X \supset P(\mathbf{u})$ is compact. To this end, consider any sequence $(x_m)_{m \in \mathbb{N}}$ with $x_m \in P(\mathbf{u})$ for every $m \in \mathbb{N}$ that converges to x. Since $x_m \in P(\mathbf{u})$, by the characterization in Lemma 3, $x_m \in \Phi_i(x_m)$ for all $i \in I$. Since $\Phi_i(\cdot)$ is upper hemicontinuous and $x_m \to x$ as $m \to \infty$, we must have $x \in \Phi_i(x)$ for all $i \in I$. We thus have $x \in P(\mathbf{u})$, proving that $P(\mathbf{u})$ is closed. □

B Proof of Theorem 5

The existence of a fixed point follows from Corollary 3.7 of Li (2014). Here we provide a simpler independent proof. Our proof builds on Theorem 1.1 of Smithson (1971), which introduces the following condition:

> **Condition III.** Let $F : X \rightrightarrows X$ and let C be a chain in X. Suppose that there is a nondecreasing function $g : C \to X$ such that $g(x) \in F(x)$ for all $x \in C$. If $x_0 = \sup_X C$, then there exists $y_0 \in F(x_0)$ such that $g(x) \leqslant y_0$ for all $x \in C$.

Theorem 1.1 of Smithson (1971) is reproduced as follows (with the terminologies comparable to those of the present paper):

Theorem 12 (Smithson (1971))**.** *Let X be a (nonempty) partially ordered set in which each nonempty chain has a least upper bound. Suppose a self-correspondence $F : X \rightrightarrows X$ is upper weak set monotonic and X_+ is nonempty. Further, F satisfies Condition III. Then, F has a fixed point.*

Note first that since X is a compact metric space, it is chain complete by Theorem 2.3 of Li (2014), which implies that each nonempty chain has a least upper bound. The crucial part of proof is that the compactness of X, together with closed-valuedness of F, implies condition III.

Lemma 4. *Given the conditions of Theorem 5, F satisfies Condition III.*

Proof. Let X, F, C, g, and $x_0 = \sup_X C$ as stated in the hypothesis of Condition III. Define correspondence $H : X \rightrightarrows X$ as follows: for each $x \in C$,

$$H(x) := \{y \in F(x_0) : y \geqslant g(x)\}.$$

We observe that $H(x)$ is a closed set for each x. This is because $H(x) = F(x) \cap G(x)$ where $G(x) := \{y \in X : y \geqslant g(x)\}$, $F(x)$ is a closed set by assumption, $G(x)$ is a closed set by the assumptio of natural topology, and an intersection of two closed sets is closed.

Since X is compact by assumption, and a closed subset of a compact space is compact, $H(x)$ is compact. Now,

Claim 1. *For any finite subset C' of C, $\cap_{x \in C'} H(x) \neq \varnothing$.*

Proof. Let $C' = \{x_1, x_2, \ldots, x_n\}$ where $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$. Then, by upper weak set monotonicity of F, for each $y_n \in F(x_n)$, there exists $y_0 \in F(x_0)$ with $y_n \leqslant y_0$. In particular, take $y_n = g(x_n)$, and we obtain $y_0 \geqslant g(x_n)$ for some $y_0 \in F(x_0)$. Because g is nondecreasing, this implies $y_0 \geqslant g(x)$ for each $x \in C'$. Therefore $y_0 \in \cap_{x\in C'} H(x)$. □

Thus, by finite intersection property, we conclude that $\cap_{x\in C} H(x)$ is nonempty. This concludes the proof. □

Lemma 4 and Theorem 12 imply that F has a fixed point. We next prove the existence of a maximal fixed point.

Lemma 5. *A maximal fixed point exists.*

Proof. Let $\mathcal{E}(F)$ denote the set of all fixed points for F. Observe first that X_f is nonempty due to Theorem 5. Consider any chain $X_c \subseteq X_f$. We show below that X_c has an upper bound in X_f, which will imply by Zorn's lemma that X_f has a maximal point.

To begin, let $X'_{\geqslant x} := X' \cap \{x' \in X : x' \geqslant x\}$ for any $X' \subseteq X$ and $x \in X$. Note that for any closed set X', $X'_{\geqslant x}$ is closed as it is an intersection of two closed sets. Note also that since X is chain complete, there is a supremum of X_c, denoted y, in X. Then, for each $x \in X_c$, $F(y)_{\geqslant x}$ is closed and nonempty due to the fact that $x \in F(x)$, $y \geqslant x$, and F is upper weak set monotonic. Consider now a collection of sets $(F(y)_{\geqslant x})_{x\in X_c}$ and observe that it satisfies the finite intersection property (that is, any finite subcollection has non-empty intersection). The compactness of X then implies that $\cap_{x\in X_c} F(y)_{\geqslant x}$ is nonempty, which in turn implies that $F(y)_{\geqslant y}$ is also nonempty since $F(y)_{\geqslant y} = \cap_{x\in X_c} F(y)_{\geqslant x}$. Let us define a correspondence $G(x) := F(x)_{\geqslant y}$. By the fact that $F(y)_{\geqslant y}$ is nonempty and F is upper weak set monotonic, G is a closed-valued, nonempty self-map on subspace $X_{\geqslant y}$, so it admits a fixed point in $X_{\geqslant y}$ by Theorem 5. Clearly, this point is also a fixed point of F and thus an upper bound of X_c, as desired. □

The proof for the existence of a fixed point and a minimal fixed point under the alternative assumptions is symmetric and thus omitted.

C Proofs for Section 6

Proof of Theorem 9. For any set $X' \subseteq X$, the (strict) upper contour set of X' for workers is denoted as $U(X') := \{x \in X : x \succ_{x_W} x', \forall x' \in X'_{x_W}\}$.

The "only if" direction. Consider any stable allocation Z, and let $X' = Z \cup U(Z)$ and $X'' = X \backslash U(Z)$. We prove that (X', X'') is a fixed point of T.

By stability of Z, we have $Z_f \in C_f(X'), \forall f \in \tilde{F}$ and thus $Z \in C_F(X')$, which means $U(Z) \in R_F(X')$ or $X'' = X\backslash U(Z) \in T_2(X')$.

Observe next that for each $w \in W$, $X'' = X \backslash U(Z)$ implies there is no $x \in X''_w$ such that $x \succ_w Z_w$. Thus, we have $Z_w \in C_w(X'')$ for each $w \in W$ or $Z \in C_W(X'')$. Letting $\tilde{Y} = X'' \backslash Z$, we have $\tilde{Y} \in R_W(X'')$. Note also that $\tilde{Y} = X'' \backslash Z = X'' \cap Z^c = X \cap U(Z)^c \cap Z^c = X \cap (Z \cup U(Z))^c = X \backslash X'$. Thus, $X' = X'' \backslash \tilde{Y}$, which means $X' \in T_1(X'')$, as desired.

The "if" direction. Consider any (X', X'') such that $(X', X'') \in T(X', X'')$, that is, $X' \in T_1(X'')$ and $X'' \in T_2(X')$. Then, $X \backslash X' \in R_W(X'')$ and $X \backslash X'' \in R_F(X')$. Letting $\tilde{Y} = X \backslash X'$ and $Z = X'' \backslash \tilde{Y}$, we have $Z \in C_W(X'')$. Let us show that Z is a stable allocation.

Note first that $Z = X'' \backslash \tilde{Y} = X'' \cap \tilde{Y}^c = X'' \cap X' = X' \cap (X \backslash X'')^c = X' \backslash (X \backslash X'')$, which means that $Z \in C_F(X')$ since $X \backslash X'' \in R_F(X')$.

It is clear that Z is an allocation, since $Z \in C_W(X'')$ implies that Z contains at most one contract for each worker $w \in W$. Also, given $Z_w \in C_w(X'')$ and $Z \subset X''$, Sen's α implies that $Z_w \in C_w(Z)$, i.e., Z is individually rational for w. The individual rationality for firms is implied by the absence of blocking coalitions, which we will show below.

To show that Z admits no blocking coalition, suppose for contradiction that there exists $f \in F$ such that $Z_f \notin C_f(Z \cup U(Z))$. Note that $U(Z) \subseteq X \backslash X''$ since, given $Z_w \in C_w(X'')$, any $x \succ_w Z_w$ for each $w \in W$ cannot belong to X''. Then, $Z \cup U(Z) \subseteq X'$ since $Z \subseteq X'$ and $U(Z) \subseteq X \backslash X'' \subseteq X'$. Given this and the assumption that $Z_f \notin C_f(Z \cup U(Z))$, Sen's α implies $Z_f \notin C_f(X')$, a contradiction. Now that $Z_f \in C_f(Z \cup U(Z))$, Sen's α implies $Z_f \in C_f(Z)$, i.e., the individual rationality for firms □

Proof of Theorem 10. We first prove the following claim:

Claim 2. *Suppose C_a is weakly substitutable for each $a \in F \cup W$. Then, T is both upper and lower weak set monotonic.*

Proof. To prove the upper weak set monotonicity of T, consider any $(X', X'') \leqslant (Y', Y'')$, and any $(\tilde{X}', \tilde{X}'')$ such that $\tilde{X}' \in T_1(X', X'')$ and $\tilde{X}'' \in T_2(X', X'')$. Then, there are some $\hat{Y}' \in R_W(X'')$ and $\hat{Y}'' \in R_F(X')$ such that $\tilde{X}' = X \backslash \hat{Y}'$ and $\tilde{X}'' = X \backslash \hat{Y}''$. Since $X'' \supset Y''$ and R_W is lower weak set monotonic, we can find $\hat{Z}' \subset \hat{Y}'$ such that $\hat{Z}' \in R_W(Y'')$. Also, since $X' \subset Y'$ and R_F is upper weak set monotonic, we can find $\hat{Z}'' \supset \hat{Y}''$ such that $\hat{Z}'' \in R_F(Y')$. Letting $\tilde{Y}' = X \backslash \hat{Z}'$ and $\tilde{Y}'' = X \backslash \hat{Z}''$, we have $\tilde{Y}' \in T_1(Y', Y'')$ and $\tilde{Y}'' \in T_2(Y', Y'')$. Also, $\tilde{Y}' \supset \tilde{X}'$ and $\tilde{Y}'' \subset \tilde{X}''$ or $(\tilde{Y}', \tilde{Y}'') \geqslant (\tilde{X}', \tilde{X}'')$, proving the upper weak monotonicity of T.

The proof for the lower weak monotonicity is analogous and hence omitted. □

To complete the proof of the theorem, we endow the family of subsets of contracts with the discrete topology. Then, it is straightforward to see that this set is nonempty, partially ordered and compact. Moreover, the self-correspondence T is upper weak set monotonic by Claim 2, and it is clearly closed-valued. Furthermore, set $X' = X'' = \varnothing$. Then, there exist $\tilde{X} \in T_1(X'')$ and $\tilde{Y} \in T_2(X')$ such that $\tilde{X}$ and $\tilde{Y}$ are weakly larger than X' and X'', respectively. Therefore, by Theorem 5, there exists a fixed point (X', X'') of T. Finally by Lemma 9, we conclude that there exists a stable allocation. □

Proof of Theorem 11. We first establish the following result:

Lemma 6. *$T(X', X'') \geqslant_{ws} T'(X', X''), \forall (X', X'')$.*

Proof. To prove that T lower weak set dominates T', consider any (X', X'') and $(\tilde{X}', \tilde{X}'')$ such that $(\tilde{X}', \tilde{X}'') \in T(X', X'')$, which means that there are some $Y' \in R_W(X'')$ and $Y'' \in R_F(X')$ such that $\tilde{X}' = X \backslash Y'$ and $\tilde{X}'' = X \backslash Y''$.

Since C_w being weakly more permissive than C'_w for each $w \in W$ implies $R'_W(X'')$ upper weak set dominates $R_W(X'')$, there is some $\tilde{Y}' \in R'_W(X'')$ such that $Y' \subset \tilde{Y}'$. Also, since C'_f being weakly more permissive than C_f for each $f \in F$ implies $R_F(X')$ lower weak set dominates $R'_F(X')$, there is some $\tilde{Y}'' \in R'_F(X')$ such that $\tilde{Y}'' \subset Y''$. Letting $\hat{X}' = X \backslash \tilde{Y}'$ and $\hat{X}'' = X \backslash \tilde{Y}''$, we have found $\hat{X}' \in T'_1(X', X'')$ and $\hat{X}'' \in T'_2(X', X'')$ such that $\hat{X}' \subset \tilde{X}'$ and $\hat{X}'' \supset \tilde{X}''$, as desired.

Proving that T upper weak set dominates T' is analogous and hence omitted. □

We only provide the proof for (i) while the proof for (ii) is omitted since it is analogous. Let Z be a stable allocation in economy Γ. By the "only if" part of Theorem 9, there exists a fixed point (X', X'') of T such that $Z \in C_F(X') \cap C_W(X'')$. Because T (upper) weak set dominates T' by Lemma 6, Theorem 6 implies that there exists a fixed point $(\tilde{X}', \tilde{X}'')$ of T' such that $(X', X'') \geqslant (\tilde{X}', \tilde{X}'')$ or $X' \supset \tilde{X}'$ and $X'' \subset \tilde{X}''$. By the "if" part of Theorem 9, there exists a stable allocation Z' in economy Γ' such that $Z' \in C'_F(\tilde{X}') \cap C'_W(\tilde{X}'')$. Therefore, for each $f \in F$, $Z'_f \subseteq \tilde{X}'_f \subseteq X'_f$ and thus $Z_f \cup Z'_f \subseteq X'_f$. Given this and $Z_f \in C_f(X')$, Sen's α implies $Z_f \in C_f(Z_f \cup Z'_f)$, meaning $Z_f \succeq_f Z'_f$. Also, for each $w \in W$, $Z_w \subseteq X''_w \subseteq \tilde{X}''_w$ and thus $Z_w \cup Z'_w \subseteq \tilde{X}''_w$. Given this and $Z'_w \in C'_w(\tilde{X}'')$, Sen's α implies $Z'_w \in C'_w(Z_w \cup Z'_w)$, meaning $Z'_w \succeq'_w Z_w$. □

Proof of Corollary 2. Letting $\tilde{\Gamma}$ denote the original economy, suppose that a worker $\tilde{w}$ exists or a firm $\tilde{f}$ enters the market, which results in a new economy $\tilde{\Gamma}'$. Let W and F denote the set of all workers and all firms including $\tilde{w}$ and $\tilde{f}$, respectively. Let $\tilde{C}_a$ denote the choice correspondence of each agent $a \in W \cup F$. Now, in order to apply Theorem 11, let us define the two economies Γ and Γ' as follows: in Γ, $C_{\tilde{f}}(X') = \{\varnothing\}, \forall X' \subset X$ while $C_a = \tilde{C}_a$ for all $a \neq \tilde{f}$; in Γ', $C'_{\tilde{w}}(X') = \{\varnothing\}, \forall X' \subset X$ while $C_a = \tilde{C}_a$ for all $a \neq \tilde{w}$.

First, the sets of stable allocation in $\tilde{\Gamma}$ and $\tilde{\Gamma}'$ coincide with those in Γ and Γ', respectively. Second, C_w is weakly more permissive than C'_w for each $w \in W$ while C'_f is weakly more permissive than C_f for each $f \in F$. Thus, the desired result follows from applying Theorem 11. □

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D Supplemental Results for Section 3

As in Section 3, we assume that $M_{X'}(f)$ to be well defined for every subinterval X' of X, for u, v. Recall that v *interval dominates* u, or $v \succ_I u$, if, for any $x', x'' \in X$, $x'' \not\leqslant x'$ such that $u(x'') \geqslant u(x)$ and $v(x') \geqslant v(x), \forall x \in J(x', x'')$,

$$u(x'') \geqslant (>) u(x' \wedge x'') \;\Rightarrow\; v(x' \vee x'') \geqslant (>) v(x').$$

We first note that this notion reduces to Quah and Strulovici (2009)'s interval dominance order when X is totally-ordered (the case they focused on). To avoid confusion, we say v *QS interval dominates* u if, for any $x', x'' \in X$, $x' < x''$ such that $u(x'') \geqslant u(x), \forall x \in [x', x'']$,

$$u(x'') \geqslant (>) u(x') \Rightarrow v(x'') \geqslant (>) v(x').$$

Lemma 7. *Assume that X is totally ordered. Then, the interval dominance and QS interval dominance are equivalent.*

Proof. Clearly, the QS interval dominance implies the interval dominance. To show the converse, consider any x', x'', $x'' \not\geqslant x'$ such that $u(x'') \geqslant u(x), \forall x \in [x', x'']$. We must have $x'' > x'$ since X is totally ordered. The result would be immediate if $x' \in \arg\max_{x \in [x', x'']} v(x)$. Let us thus assume $x' \notin \arg\max_{x \in [x', x'']} v(x)$. Since $M_{X'}(f)$ is well defined for every subinterval X', there exists some $\hat{x} \in [x', x'']$ such that $\hat{x} \in \arg\max_{x \in [x', x'']} v(x)$, which means $v(\hat{x}) \geqslant v(x), \forall x \in [\hat{x}, x'']$. The interval dominance then implies $v(x'') = v(\hat{x} \vee x'') \geqslant v(\hat{x}) > v(x')$, as desired. □

Now consider any complete lattice X (that is not necessarily totally ordered). The following characterization holds.

Theorem 13. *Assume X is a complete lattice. Function v interval dominates u if and only if, for every subinterval X' of X,*

$$M_{X'}(u) \leqslant_{ss} M_{X'}(v). \tag{10}$$

Proof. **The "only if" direction.** Suppose to the contrary that $z'' \in M_{X'}(u)$ and $z' \in M_{X'}(v)$ for some subinterval X', but either $z'' \vee z' \notin M_{X'}(v)$ or $z'' \wedge z' \notin M_{X'}(u)$. Clearly, $u(z'') \geqslant u(x)$ and $v(z') \geqslant v(x), \forall x \in J(z', z'')$. Since $v \succeq_I u$, $u(z'') \geqslant u(z' \wedge z'') \Rightarrow v(z' \vee z'') \geqslant v(z')$, so $z'' \vee z' \in M_{X'}(v)$. Hence, it must be $z'' \wedge z' \notin M_{X'}(u)$, or $u(z'') > u(z' \wedge z'')$. Again by interval dominance, this means $v(z' \vee z'') > v(z')$, which yields a contradiction.

The "if" direction. Consider any x'', x', $x'' \nleqslant x'$, and $u(x'') \geqslant u(x)$ and $v(x') \geqslant v(x)$ for all $x \in J(x', x'')$. Since $x'' \in M_{J(x',x'')}(u)$ and $x' \in M_{J(x',x'')}(v)$, (10) implies that $x' \wedge x'' \in M_{J(x',x'')}(u)$ and $x' \vee x'' \in M_{J(x',x'')}(v)$, which means $u(x' \wedge x'') \geqslant u(x'')$ and $v(x' \vee x'') \geqslant v(x')$. Then, (6) follows. □

In the multidimensional setup, Quah and Strulovici (2007) consider an additional condition, *I-quasisupermodularity*, to obtain sMCS result: u is *I-quasisupermodular* if, for any $x', x'' \in X$ such that $u(x') \geqslant u(x), \forall x \in [x' \wedge x'', x']$, $u(x') \geqslant (>) u(x' \wedge x'')$ implies $u(x' \vee x'') \geqslant (>) u(x'')$. They then establish the following result:

Proposition 5. *Assume that X is a lattice. If $v : X \to \mathbb{R}$ QS interval dominates $u : X \to \mathbb{R}$ and if either u or v is I-quasisupermodular, then* (10) *holds.*

We show that the interval dominance condition is weaker than QS interval dominance plus I-quasisupermodularity.

Lemma 8. *Assume that X is a lattice. If $v : X \to \mathbb{R}$ QS interval dominates $f : X \to \mathbb{R}$ and either u or v is I-quasisupermodular, then v interval dominates u.*

Proof. Consider any $x', x'' \in X$ with $x'' \nleqslant x'$ such that $u(x'') \geqslant u(x)$ and $v(x') \geqslant v(x)$ for all $x \in J(x', x'')$. Suppose that $u(x'') \geqslant u(x' \wedge x'')$, so $v(x'') \geqslant v(x' \wedge x'')$ due to the fact that v QS interval dominates u. We aim to show that $v(x' \vee x'') \geqslant v(x')$ while $u(x'') = u(x' \wedge x'')$ (that is, $u(x'') \ngtr u(x' \wedge x'')$).

Assume first that v is I-quasisupermodular. Since $u(x'') \geqslant u(x), \forall x \in [x' \wedge x'', x'']$, QS interval dominance of v over u implies $v(x'') \geqslant v(x), \forall x \in [x' \wedge x'', x'']$. Given this, v being I-quasisupermodular implies $v(x' \vee x'') \geqslant v(x')$. If $u(x'') > u(x' \wedge x'')$, then $g(x'') > v(x' \wedge x'')$ and thus $v(x' \vee x'') > v(x')$, which is a contradiction to the hypothesis that $v(x') \geqslant v(x), \forall x \in J(x', x'')$.

Assume next that u is I-quasisupermodular. Consider any $x \in [x' \wedge x'', x'']$. Since $u(x'') \geqslant u(y), \forall y \in [x' \wedge x, x'']$ and since u is I-quasisupermodular, we have $u(x'') = u(x'' \vee x) \geqslant u(x)$. Since this inequality holds for any $x \in [x' \wedge x'', x'']$ and since $u(x'') \geqslant u(x' \wedge x'')$, applying I-quasisupermodularity of u once again implies $u(x' \vee x'') \geqslant u(x')$. Given this, QS interval dominance of g over f implies $v(x' \vee x'') \geqslant v(x')$. Again, a contradiction arises if $u(x'') > u(x' \wedge x'')$, since then $u(x' \vee x'') > u(x')$ so $v(x' \vee x'') > v(x')$. □

E Supplemental Results for Section 4

E.1 Role of Compactness for Lemma 2

To highlight the role of compactness for Lemma 2, we present the following example.

Suppose $X = (0,1)$ with the Euclidean topology, so X is not compact. Suppose

$$u_1(x) = \begin{cases} 2 - x & \text{if } x < 1/2 \\ 3 - x & \text{if } x \geqslant 1/2. \end{cases} \quad \text{and } u_2(x) = 1 - x \text{ for all } x.$$

See Figure 6. Note that $P(\mathbf{u}) = \{\frac{1}{2}\}$. Any $x \in (0, \frac{1}{2})$ is Pareto dominated by $x' \in (0, x)$, but is not Pareto dominated by an alternative in $P(\mathbf{u})$, contrary to Lemma 2.

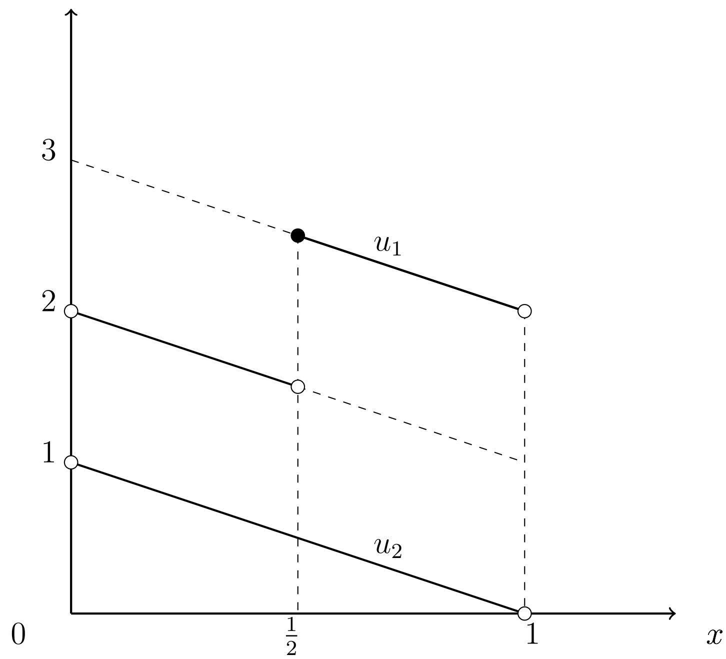

Figure 6: Failure of Lemma 2 for non-compact X.

E.2 Sufficient Conditions for Lower Hemi-Continuity of a Correspondence

Lower hemicontinuity of $U_{-i}(\cdot)$ can be further clarified and motivated by the following sufficient conditions:

Proposition 6. *Utility functions* $\mathbf{u}$ *defined on a convex set* X *are regular if* u_i *is upper hemicontinuous for each* $i \in I$ *and either (i) for each* $i \in I$, u_i *is strictly quasi-concave,*[47] *or (ii) for each* $i \in I$, *the correspondence* $U_{-i}(\cdot)$ *is continuous in the Hausdorff topology.*[48]

Proof. To prove (i), for any $i \in I$, consider a sequence $\{x_n\}$ converging to x and suppose $y \in U_{-i}(x)$. We will show that there exists a sequence $\{y_n\}$ that converges to y and $y_n \in U_{-i}(x_n)$ for each n. To begin, if $y = x$, then the conclusion is obvious by setting $y_n = x_n$ for each n. So let us assume $y \neq x$.

Now, consider $z_m := \lambda_m x + (1 - \lambda_m)y$, where $\lambda_m \in (0, 1)$ converges monotonically to 0 as $m \to \infty$. Since each utility function u_j, $j \neq k$, is strictly quasi-concave, $y \neq x$, and $u_j(y) \geqslant u_j(x)$ because $y \in U_{-i}(x)$, we have that $u_j(z_m) > \min\{u_j(y), u_j(x)\} = u_j(x)$ for each $j \neq k$. This property, the upper hemicontinuity of the utility functions, and the assumption that $x_n \to x$ imply that, for each $m \in \mathbb{N}$, there exists $N(m) \in \mathbb{N}$ such that $z_m \in U_{-i}(x_n)$ for all $n > N(m)$. Without loss of generality, take $N(m)$ to be strictly increasing in m so that $N(m) \to \infty$ as $m \to \infty$. Let $m(n) := \sup\{m \in \mathbb{N} : n > N(m)\}$ whenever the set is nonempty, and let n_0 be the smallest integer such that the set $\{m \in \mathbb{N} : n_0 > N(m)\}$ is nonempty (note that n_0 exists because for any $n > N(1)$, the set includes 1 by definition). Note that $m(n)$ is a finite integer because $N(m)$ is strictly increasing and hence the set $\{m \in \mathbb{N} : n > N(m)\}$ is a finite set and that $m(n) \to \infty$ as $n \to \infty$. Now, define $y_n := x_n$ for $n < n_0$ and $y_n = z_{m(n)}$ for all $n \geqslant n_0$. Then, $y_n \in U_{-i}(x_n)$ for each n, and $y_n \to y$. We have thus proven that $U_{-i}(\cdot)$ is LHC.

For (ii), consider again, for any $i \in I$, a sequence $\{x_n\}$ converging to x, and suppose $y \in U_{-i}(x)$. By the convergence of $U_{-i}(x_n)$ in Hausdorff topology, $\mathrm{d}_H(U_{-i}(x_n), U_{-i}(x)) \to 0$ as $n \to \infty$. Since $y \in U_{-i}(x)$, this implies that for any $\epsilon > 0$, $\inf_{z \in U_{-i}(x_n)} d(z, y) < \epsilon/2$ for any sufficiently large n, so there exists $y_n \in U_{-i}(x_n)$ with the property that $d(y_n, y) < \epsilon$ for any sufficiently large n. This proves that $U_{-i}(\cdot)$ is LHC. □

E.3 Role of Compactness for wMCS of POC

Let $X = (0, 1)$ with the Euclidean topology as well as the standard order and

$$u_1(x) = \begin{cases} 2 - x & \text{if } x < 1/2 \\ 3 - x & \text{if } x \geqslant 1/2, \end{cases} \quad \text{and } u_2(x) = 1 - x \text{ for all } x,$$

[47] That is, for any $x, x' \in X$ with $x \neq x'$ and $\lambda \in (0, 1)$, $u_i(\lambda x + (1 - \lambda)x') > \min\{u_i(x), u_i(x')\}$.

[48] More precisely, the continuity in Hausdorff topology means $\mathrm{d}_H(U_{-i}(x), U_{-i}(x')) \to 0$ as $d(x, x') \to 0$, where $d(\cdot, \cdot)$ is the metric defined on X and $\mathrm{d}_H(\cdot, \cdot)$ is the Hausdorff metric: for $Y, Z \subset X$, $\mathrm{d}_H(Y, Z) := \max\{\sup_{y \in Y} \inf_{z \in Z} d(y, z), \sup_{z \in Z} \inf_{y \in Y} d(y, z)\}$.

$$v_1(x) = \begin{cases} x & \text{if } x < 1/4 \\ \frac{1}{2} - x & \text{if } x \in [1/4, 1/2) \\ \frac{1}{2} + x & \text{if } x \geqslant 1/2 \end{cases} \quad \text{and } v_2(x) = \begin{cases} x & \text{if } x < 1/4 \\ \frac{1}{2} - x & \text{if } x \in [1/4, 1/2) \\ \frac{1}{4}(x - \frac{1}{2}) & \text{if } x \geqslant 1/2 \end{cases}.$$

Observe that $\mathbf{v}$ single-crossing dominates $\mathbf{u}$ but X is not compact. Observe also that $P(\mathbf{u}) = \{\frac{1}{2}\}$ while $P(\mathbf{v}) = \{\frac{1}{4}\}$, so $P(\mathbf{v})$ fails to weak set dominate $P(\mathbf{u})$. See Figure 7.

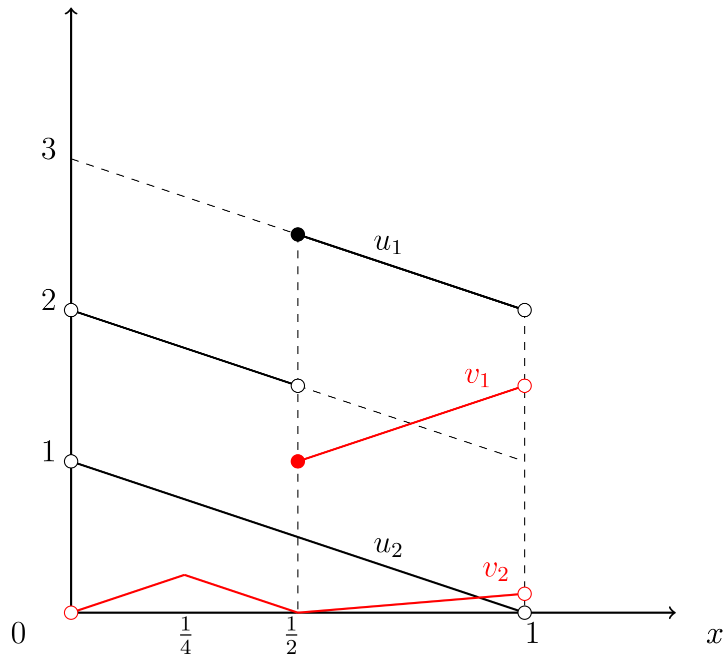

Figure 7: wMCS of POC in Theorem 4 fails due to non-compactness of X.

This example can be slightly modified to show that the conclusion of Theorem 4 cannot be strengthened to give the sMCS of POC. To do so, let $X = [0, 1]$ (so that X is now compact) and observe that $P(\mathbf{u}) = \{0, \frac{1}{2}\}$, $P(\mathbf{v}) = \{\frac{1}{4}, 1\}$. The domination relation between $P(\mathbf{u})$ and $P(\mathbf{v})$ holds in the weak set order, but not in the strong set order; $\frac{1}{2} \wedge \frac{1}{4} \notin P(\mathbf{u})$ and $\frac{1}{2} \vee \frac{1}{4} \notin P(\mathbf{v})$.

F Supplemental Results for Section 5

F.1 Comparison of Conditions between Theorem 5 and Zhou (1994)'s theorem

Among the advantages of our fixed-point theorem compared to previous results of Tarski and Zhou (1994) is the fact that we impose only weak assumptions regarding order structures. At the same time, our theorem requires certain topological conditions which the existing results do not impose. A natural question is how restrictive those additional topological conditions are. They turn out to be mild in many, if not all, environments of interest, as formally stated in the following theorem.

Theorem 14. *Suppose X is (i) a subset of $\mathbb{R}^n$ (endowed with Euclidean topology); or (ii) a set of bounded nonnegative measures defined on a finite set, endowed with weak convergence topology; or (iii) a subset of a family of equicontinuous and pointwise bounded functions $\mathcal{F} \subset C[\Theta]$ defined on compact metric space Θ endowed with topology induced by uniform norm. Then, the following results hold.*

- *If X is a complete lattice, then X is compact.*
- *If Y is a complete sublattice of X, then Y is closed.*

Proof. (i) and (ii) follow from Frink (1942), who proves that a complete lattice is compact in the interval topology, since the Euclidean topology and weak convergence topology on measures defined on finite sample space reduce to the interval topology.[49]

For (iii), the space X is a subset of $\mathcal{F} \subset C(\Theta)$, which is a complete lattice. By the Arzela-Ascoli's theorem, $\mathcal{F}$ is relatively compact under the uniform convergence topology. Hence, for both results, it suffices to show that X is closed. Consider any sequence $\{x_n\}$, $x_n \in X$, that converges to x. We show that $x \in X$. To this end, let $z_n := \sup\{x_k | k = n, n+1, ...\}$. Now consider $x'; = \inf\{z_n | n = 1, ...\}$. Since X is a complete lattice, x' is well defined and contained in X. Further, z_n is nonincreasing, so z_n converges to x', i.e., $x' = \lim_{n\to\infty} z_n$. It suffices to show therefore that $x' = x$, or z_n converges to x. To this end, note first that since $x_k \to x$ in uniform norm, for any $\epsilon > 0$, there exists N large enough such that, for any $k \geqslant N$, we have $||x_k - x|| < \epsilon$. It thus follows that

$$\begin{aligned}
||z_n - x|| &= \sup_{\theta\in\Theta} |z_n(\theta) - x(\theta)| \\
&= \sup_{\theta\in\Theta} |\sup_{k\geqslant n} x_k(\theta) - x(\theta)| \\
&\leqslant \sup_{\theta\in\Theta} \sup_{k\geqslant n} |x_k(\theta) - x(\theta)| \\
&= \sup_{k\geqslant n} \sup_{\theta\in\Theta} |x_k(\theta) - x(\theta)| \\
&= \sup_{k\geqslant n} ||x_k - x|| < \epsilon.
\end{aligned}$$

[49] Theorem 2.3.1 of Topkis (1998) shows the result and for the Euclidean space. For the converse, Birkhoff (1967) (and Theorem 2.3.1 of Topkis (1998) for the Euclidean space) shows that a lattice that is compact in its interval topology is complete.

□

Theorem 14 demonstrates that the conditions required by Theorem 5 are typically weaker than those required by Zhou's theorem. In fact, the proof of Theorem 5 for cases (i) and (ii) makes it clear that, due to Frink (1942), the desired conclusions hold generally under interval topology, even beyond cases (i) and (ii). However, the same conclusions do not hold for every space X, as illustrated by the following example:

Example 7. Let $\mathcal{P}$ be the set of all nonnegative measures defined on the Borel sets in $[0,1]$ such that $P([0,1]) \in [0,1]$ for all $P \in \mathcal{P}$. Endow $\mathcal{P}$ with the weak convergence topology and the partial order $\sqsupset$ such that $P' \sqsupset P$ if $P'(E) \geqslant P(E)$ for each Borel set $E \subset [0,1]$. In this space, there is no relationship between compactness and complete lattice.

A complete lattice need not be either closed or compact: Consider the following subset $\mathcal{P}'$ of $\mathcal{P}$ defined by

$$\mathcal{P}' = \{\overline{P}\} \cup \{\underline{P}\} \cup \left(\cup_{k=1}^{\infty} P^k\right),$$

where $\overline{P}(E) = \lambda(E)$, the Lebesque measure of E, $\underline{P}(E) = 0$, for all Borel sets $E \subset [0,1]$, and for each $k = 1, ...,$

$$P^k(E) := \begin{cases} \lambda(E) & \text{if } E \subset (\frac{i-1}{2^k}, \frac{i}{2^k}), i \text{ odd } \leqslant k; \\ 0 & \text{if } E \subset (\frac{i-1}{2^k}, \frac{i}{2^k}), i \text{ even } \leqslant k; \end{cases}$$

One can see that $\mathcal{P}'$ is a complete lattice: no two elements in $\cup_{k=1}^{\infty} P^k$ are ordered, so any subset of that set has $\overline{P}$ as the least upper bound and $\underline{P}$ as the greatest lower bound. At the same time, we can see that P^k converges to P^* in weak topology, where $P^*(E) = \frac{1}{2}\lambda(E)$. This can be seen by the fact that the cumulative distribution functions associated with P^k converges to P^* pointwise, which is sufficient for weak convergence. Since $P^* \notin \mathcal{P}'$, the set $\mathcal{P}'$ is not closed. Since $\mathcal{P}$ (endowed with weak convergence topology) is Hausdorff, closedness is necessary for compactness. Hence, $\mathcal{P}'$ is not compact.

A compact subset of $\mathcal{P}$ need not be a lattice: Consider

$$\mathcal{P}'' = \{P^*\} \cup \left(\cup_{k=1}^{\infty} P^k\right).$$

Since $\mathcal{P}$ is compact by Alaoglu's theorem and since $\mathcal{P}''$ is closed as seen above, $\mathcal{P}''$ is compact. Yet, the set is not even a lattice.

F.2 Existence of Minimal/Maximal Fixed Points

Example 8 (Non-compactness of the (maximal or minimal) fixed-point set)**.** Consider a domain $X = [0,1]^2$ and let

$$A = \{(x,y) \in X \,|\, x + y = 1\},$$
$$B = \{(x,y) \,|\, x + y = 3/4,\ x \in [1/4, 1/2]\} \cup \{(x,0) \,|\, x \in [1/2,1]\} \cup \{(0,y) \,|\, y \in [1/2,1]\}.$$

Define

$$F(x,y) = \begin{cases} A & \text{if } x + y \geqslant 1 \text{ and } (x,y) \neq (1/2,1/2) \\ B & \text{otherwise.} \end{cases}$$

This correspondence satisfies all the conditions for Theorem 5, being both upper and lower weak set monotonic (in the usual vector-space order). The set of maximal fixed point is $\{(x,y) \,|\, x+y = 1 \text{ and } (x,y) \neq (1/2,1/2)\}$, which is not closed (and thus not compact). The set of minimal fixed point is $\{(x,y) \,|\, x + y = 3/4,\ x \in (1/4,1/2)\} \cup \{(1/2,0)\} \cup \{(0,1/2)\}$, which is not closed either.

F.3 Example of difficulty for iterative algorithms

Even when an iterative procedure can find an extremal fixed point, it may not be easily computable. More specifically, a minimal fixed point may not be reached for some selection from the correspondence, even if the selection is restricted to be among the minimal points of the correspondence (which is sufficient for reaching the smallest—and hence minimal—fixed point in the settings of Tarski and Zhou).

Example 9. Suppose $X = \{1,2\}^2$. Suppose $F : X \rightrightarrows X$ is defined by: $F((1,1)) = \{(1,2),(2,1)\}, F((2,1)) = \{(2,1),(2,2)\},\ F((1,2)) = \{(2,2)\}, F((2,2)) = \{(2,2)\}$. Note that F is both upper and lower weak set monotonic. There are two fixed points $\{(2,1),(2,2)\}$. If one iterates F on an arbitrary selection of a minimal point of F, then one could proceed as follows: starting at $x_1 = (1,1)$, then proceeding to $x_2 = (1,2) \in F((1,1))$, and finally terminating at a fixed point $x_3 = (2,2) \in F((1,2))$, which is clearly not a minimal fixed point. See Figure 8.

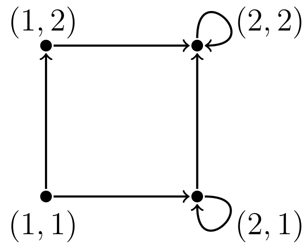

Figure 8: Fixed points reached are sensitive to selection.

G Supplemental Results for Section 6

G.1 An alternative stability notion

We consider an alternative definition of a stable allocation. More specifically, consider the following condition of no blocking coalition:

(ii') (No Blocking Coalition) There exist no $f \in F$ and allocation $Y \subseteq X$ such that $Y_f \succ_f Z_f$ and $x \succ_{x_W} Z_{x_W}$ for each $x \in Y_f \backslash Z_f$.[50]

The condition (ii') is based on the pairwise comparison between the two alternatives available to the firm, that is, Z is considered to admit no blocking coalition if it is not dominated by any other allocation Y available to the firm. Though this condition looks similar to the one often adopted in the existing literature, our view is that there can be multiple ways to extend the stability notion when one tries to accommodate general indifferent/incomplete preference. In fact, our condition (ii) implies condition (ii') if Sen's α holds, so our stability notion is stronger than the one based on (ii'). To see this, suppose that (ii) holds and consider any $Y \subseteq X$ such that $x \succ_{x_W} Z_{x_W}$ for each $x \in Y_f \backslash Z_f$. Then, $Z_f \cup Y_f \subseteq Z \cup U(Z)$. Since $Z_f \in C_f(Z \cup U(Z))$ by condition (ii), this and Sen's α imply $Z_f \in C_f(Z_f \cup Y_f)$. Therefore, we cannot have $Y_f \succ_f Z_f$, so condition (ii') holds, as desired. The following example shows that the converse need not hold, however:

Example 10. Suppose that there are one firm, f, three contracts x, y, and z associated with f, and three workers x_W, y_W, and z_W associated with x, y, and z, respectively. The firm's choice correspondence is given as follows: $C_f(\{x, y, z\}) = \{\{x\}\}$; $C_f(\{x, y\}) = \{\{x\}, \{y\}\}$; $C_f(\{x, z\}) = \{\{x\}\}$; $C_f(\{y, z\}) = \{\{y\}\}$; $C_f(\{\tilde{x}\}) = \{\{\tilde{x}\}\}$ for $\tilde{x} = x, y, z$. Each worker strictly prefers working for f to being unemployed. One stable allocation, based on

[50]Note that the relationship $\succ$ here is the Blair order.

(ii'), is $Z := \{y\}$, since there exists no set of contracts Y such that $Y \succ_f Z$. However, Z is not stable according to our notion based on (ii) since $U(Z) = \{x, z\}$ and thus $C_f(Z \cup U(Z)) = \{\{x\}\}$.

In the above example, C_f violates Sen's β, as can be checked easily. If each firm's choice correspondence satisfies both Sen's α and β (or equivalently WARP), then the two stability notions are equivalent:

Lemma 9. *If each C_f satisfies WARP, then the stability notion based on condition (ii) is equivalent to the one based on (ii').*

Proof. It suffices to prove that (ii') implies (ii) under WARP. Consider any f and Z satisfying (ii'). Consider $Z' \in C_f(Z \cup U(Z))$. Then, by Sen's α, $Z' \in C_f(Z \cup Z')$, which implies $Z_f \in C_f(Z \cup Z')$ since otherwise we would have $Z' \succ_f Z_f$ and $x \succ_{x_W} Z_{x_W}$ for each $x \in Z'_f \backslash Z_f$, violating (ii'). This implies $Z_f \in C_f(Z \cup U(Z))$ by Sen's β. □

G.2 Sen's α and WARNI

Consider the following condition due to Eliaz and Ok (2006) adapted to our matching environment. We say that choice correspondence C_f satisfies **WARNI (weak axiom of revealed non-inferiority)** if, for any $X' \subseteq X$ and $Z \subseteq X'$, if, for every $Y \in C_f(X')$, there exists $X'' \subseteq X$ with $Z \in C_f(X'')$ and $Y \subseteq X''$, then $Z \in C_f(X')$. Eliaz and Ok (2006) show that WARNI implies that the choice correspondence can be rationalized by an acyclic, if possibly incomplete, binary relation. The following result establishes that WARNI implies Sen's α.

Proposition 7. *If C_a satisfies WARNI, then it satisfies Sen's α.*

Proof. Consider any $X' \subset X''$ and $Z \in C_a(X'')$ with $Z \subseteq X'$. Note that for any $Y \in C_a(X')$, we have $Y \subset X' \subset X''$, which means the hypothesis of WARNI is satisfied. Thus, $Z \in C_a(X')$, as required by Sen's α. □

The following example demonstrates that the converse of Proposition 7 does not hold:

Example 11. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, and the choice correspondence C_f of firm f is defined as follows:

(i). $\{x_1, x_2\} \in C_f(X')$ if and only if $\{x_1, x_2\} \subseteq X'$ and $\{x_3, x_4\} \nsubseteq X'$,

(ii). $\{x_3, x_4\} \in C_f(X')$ if and only if $\{x_3, x_4\} \subseteq X'$ and $\{x_5, x_6\} \nsubseteq X'$, (iii). $\{x_5, x_6\} \in C_f(X')$ if and only if $\{x_5, x_6\} \subseteq X'$ and $\{x_1, x_2\} \nsubseteq X'$,

(iv). $\{x\} \in C_f(X')$ for every $x \in X'$, and

(v). no other set is in $C_f(X')$.

By inspection, one can verify that C_f satisfies Sen's α. Meanwhile, the choice correspondence C_f violates WARNI. To see this point, consider $X' = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $Z = \{x_1, x_2\}$. Any $Y \in C_f(X')$ is a singleton set, i.e., a set of the form $\{x\}$. Thus, the hypothesis part of WARNI, $Z = \{x_1, x_2\} \in C_f(\{x\} \cup \{x_1, x_2\})$, is satisfied for X' and Z. However, $\{x_1, x_2\} \notin C_f(X')$ by definition.

Note that the choice correspondence C_f in this example features a cyclic binary relation $\{x_1, x_2\} \succ_f \{x_5, x_6\} \succ_f \{x_3, x_4\} \succ_f \{x_1, x_2\}$.[51] This example suggests that our theory based on Sen's α might prove useful even in applications in which WARNI fails and, related, the choice behavior may not even be rationalizable by acyclic preference relations.

G.3 Implication of a change in workers' choice correspondence

Lemma 10. *If C'_w is weakly more permissive than C''_w, then, for any $X' \subseteq X$, either $C'_w(X') = C''_w(X')$ or $C''_w(X') = \{\varnothing\}$.*

Proof. We only need to show that if $C''_w(X') \neq \{\varnothing\}$ (that is, some contract in X' is acceptable to w according to $\succeq'_w$), then $C'_w(X') = C''_w(X')$. Consider any $x \in C'_w(X')$ and note that $X' \backslash \{x\} \in R'_w(X')$. Given that $C''_w(X') \neq \{\varnothing\}$ implies $|\tilde{X}| = |X'| - 1$ for all $\tilde{X} \in R''_w(X')$, we must have $x \in C''_w(X')$ since otherwise there would be no $\tilde{X} \in R''_w(X')$ such that $X' \backslash \{x\} \subseteq \tilde{X}$. Conversely, consider any $x \in C''_w(X')$. Since $X' \backslash \{x\} \in R''_w(X')$ and since there has to be some $\tilde{X} \in R'_w(X')$ such that $\tilde{X} \subseteq X' \backslash \{x\}$ with $|\tilde{X}| \geqslant |X'| - 1$, we must have $X' \backslash \{x\} \in R'_w(X')$ or $x \in C'_w(X')$, as desired. □

G.4 Multidivisional organizations (internal constraints)

Consider an organization that has multiple divisions. The organization does not have a strict preference relation over outcomes, and its choice behavior is not described by a single-valued choice function. Rather, the organization has a choice correspondence. We continue to refer to the organization as a hospital for consistency, but such a multidivisional structure

[51] We note that Eliaz and Ok (2006) show that WARNI implies that a choice correspondence can be rationalized by a transitive (and hence acyclic) binary relation.

is prevalent in many organizations, ranging from for-profit firms to non-profit organizations and government.

Formally, we assume that the hospital h is associated with a finite set of **divisions** Δ_h and an **internal constraint** $f_h : \mathbb{Z}_+^{|\Delta_h|} \to \{0,1\}$ such that $f_h(w) \geqslant f_h(w')$ whenever $w \leqslant w'$ and $f_h(0) = 1$, where the argument 0 of f_h is the zero vector and $\mathbb{Z}_+$ is the set of nonnegative integers. The interpretation is that each coordinate in w corresponds to a division of the firm, and that the number in that coordinate represents the number of doctors matched to that division. We say that w is **feasible** if $f_h(w) = 1$ and w is infeasible if $f_h(w) = 0$. The monotonicity property of f_h means that if w' is feasible then any w with a weakly fewer doctors in each division must be feasible for the hospital as well. Let $\Delta := \bigcup_{h \in H} \Delta_h$.

Internal constraints in organizations may represent budget constraints and availability of office space and other resources. The hospital may be able to use some resources in a flexible manner across divisions, but the profile of the numbers of the hire in different divisions needs to satisfy the overall constraints represented by the internal constraint f_h.

For each hospital h and its internal constraint f_h, we define a correspondence, called quasi-choice correspondence, $\tilde{\mathrm{C}}_h : \mathbb{Z}_+^{|\Delta_h|} \rightrightarrows \mathbb{Z}_+^{|\Delta_h|}$ by $\tilde{\mathrm{C}}_h(w) = \{w' : w' \leqslant w, f_h(w') = 1, \text{ and } (\nexists w'' \leqslant w, w' < w'', f_h(w'') = 1)\}$, that is, the set of all vectors that are weakly smaller than w, feasible, and maximal among all vectors that are weakly smaller than w and feasible.

We assume that each hospital h has a choice correspondence $\mathrm{C}_h(\cdot)$ over all subsets of $D \times \Delta_h$. Each division $\delta \in \Delta_h$ of the hospital has a preference relation $\succ_\delta$ over the set of doctors and the outside option, $D \cup \{\varnothing\}$. For any $X' \subset D \times \Delta_h$, let $w(X') := (w_\delta(X'))_{\delta \in \Delta_h}$ be the vector such that $w_\delta(X') = |\{(d,\delta) \in X' : d \succ_\delta \varnothing\}|$. For each X', the choice correspondence $\mathrm{C}_h(X')$ is defined by

$$\mathrm{C}_h(X') = \left\{ X'' : \exists w \in \tilde{\mathrm{C}}_h(w(X')), X'' = \bigcup_{\delta \in \Delta_h} \{(d,\delta) \in X' : |\{d' \in D : (d',\delta) \in X', d' \succeq_\delta d\}| \leqslant w_\delta\} \right\}. \tag{11}$$

That is, in any of the chosen subsets of contracts, there exists a vector $w \in \tilde{\mathrm{C}}_h(w(X'))$ such that each division $\delta \in \Delta_h$ chooses its w_δ most preferred contracts from acceptable contracts in X'.

A matching problem with multidivisional hospitals is defined by a tuple $\Gamma = (D, H, (\Delta_h)_{h \in H}, (\succ_a)_{a \in D \cup \Delta}, (f_h)_{h \in H})$.

Claim 3. *Choice correspondence* $\mathrm{C}_h(\cdot)$ *defined by relation (11) satisfies Sen's* α.[52]

Proof. Consider any $Y \subset X' \subset X''$ such that $Y \in C_h(X'')$. Then clearly Y is individually rational for divisions. Also, by construction, the set Y has the property that $w(Y)$ is a maximal vector among those that are weakly smaller than $w(X'')$, and for each division δ, δ is matched under Y to its $w_\delta(Y)$ most preferred contracts among those in X'' by construction. Given that $w(X') \leqslant w(X'')$ and $Y \subset X'$, Y satisfies the same property with respect to X'. Thus, $Y \in C_h(X')$, as desired. □

Claim 4. *Choice correspondence* $\mathrm{C}_h(\cdot)$ *defined above satisfies the weak substitutes condition.*

Proof. Let us first show that the rejection correspondence $R_h(\cdot)$ associated with $\mathrm{C}_h(\cdot)$ satisfies upper weak set monotonicity. Let X' and X'' be two sets of contracts, with $X' \subseteq X''$, and $Y' \in \mathrm{C}_h(X')$. Then there exists $w' \in \tilde{\mathrm{C}}_h(w(X'))$ such that, for each $\delta \in \Delta_h$,

$$Y'_\delta = \{(d,\delta) \in X' : |\{d' \in D | (d',\delta) \in X', d' \geq_\delta d\}| \leqslant w'_\delta\}.$$

By the definition of $w(\cdot)$ and the assumption that $X' \subseteq X''$ it follows that $w(X') \leqslant w(X'')$, so there exists $w'' \in \tilde{\mathrm{C}}_h(w(X''))$ such that $w'' \geqslant w'$. Let $Y'' \in \mathrm{C}_h(X'')$ be the chosen set of contracts associated with w'', so for each $\delta \in \Delta_h$,

$$Y''_\delta = \{(d,\delta) \in X'' : |\{d' \in D | (d',\delta) \in X'', d' \geq_\delta d\}| \leqslant w''_\delta\}.$$

Consider two cases.

(i). Suppose $w''_\delta > w'_\delta$. Then $w'_\delta = w_\delta(X')$ because otherwise $w'_\delta < w_\delta(X')$ and $f_h(w'_\delta + 1, w_{-\delta}) \geqslant f_h(w'') = 1$, contradicting the maximality of w'. Therefore, every contract in X' of the form (d,δ) such that $d >_\delta \varnothing$ is in Y'.

(ii). Suppose $w''_\delta = w'_\delta$. Then, by the definition of $\mathrm{C}_h(\cdot)$, any contract $(d,\delta) \in X' \backslash Y'$ is also in $X'' \backslash Y''$—recall that the division δ chooses its $w''_\delta = w'_\delta$ most preferred contracts from X'_δ and X''_δ at Y' and Y'', respectively, and $X'_\delta \subseteq X''_\delta$.

Therefore $(X' \backslash Y') \subseteq (X'' \backslash Y'')$ as desired. The proof for lower weak set monotonicity is analogous and hence omitted. □

It follows from those claims that a stable allocation exists.

[52]We note that the choice correspondence considered here does not necessarily satisfy WARP. See Example 5 for a choice correspondences within the class considered here that violates WARP.

Corollary 3. *A stable allocation exists in any matching problem with multidivisional hospitals.*

We say that an internal constraint f'_h is **weakly more permissive** than constraint f_h if $f'_h(w) \geqslant f_h(w)$ for every w. With this notion at hand, we are now ready to present a comparative statics result with respect to constraints.

Corollary 4. *Consider two matching problems with multidivisional hospitals* $\Gamma = (D, H, (\Delta_h)_{h\in H}, (\succ_a)_{a\in D\cup\Delta}, (f_h)_{h\in H})$ *and* $\Gamma' = (D, H, (\Delta_h)_{h\in H}, (\succ_a)_{a\in D\cup\Delta}, (f'_h)_{h\in H})$ *such that* f'_h *is weakly more permissive than* f_h *for each* $h \in H$*. Then,*

(i). for each stable matching μ *in* Γ*, there exists a stable matching* μ' *in* Γ' *such that* $\mu'_d \succeq_d \mu_d$ *for each* $d \in D$*, and*

(ii). for each stable matching μ' *in* Γ'*, there exists a stable matching* μ *in* Γ *such that* $\mu'_d \succeq_d \mu_d$ *for each* $d \in D$*.*

Proof. For each h, let C_h and C'_h be given by relation (11) with the corresponding quasi-choice rules $\tilde{\mathrm{C}}_h$ and $\tilde{\mathrm{C}}'_h$ defined by $\tilde{\mathrm{C}}_h(w) = \{w' : w' \leqslant w, f_h(w') = 1, \text{ and } (\nexists w'' \leqslant w, w' < w'', f_h(w'') = 1)\}$ and $\tilde{\mathrm{C}}'_h(w) = \{w' : w' \leqslant w, f'_h(w') = 1, \text{ and } (\nexists w'' \leqslant w, w' < w'', f'_h(w'') = 1)\}$. By inspection, it follows that C'_h is weakly more permissive than C_h for each h. This fact and Theorem 11 imply the desired conclusion. □

G.5 Matching with constraints

In this section, we consider a model of matching with constraints (Kamada and Kojima, 2015, 2017, 2018). Based on our fixed-point characterization and comparative statics results, we reproduce an existing result and obtain a new result.

Let there be a finite set of doctors D and a finite set of hospitals H. Each doctor d has a strict preference relation $\succ_d$ over the set of hospitals and the option of being unmatched (being unmatched is denoted by $\emptyset$). For any $h, h' \in H \cup \{\emptyset\}$, we write $h \succeq_d h'$ if and only if $h \succ_d h'$ or $h = h'$. Each hospital h has a strict preference relation $\succ_h$ over the set of subsets of doctors. For any $D', D'' \subseteq D$, we write $D' \succeq_h D''$ if and only if $D' \succ_h D''$ or $D' = D''$. We denote by $\succ = (\succ_i)_{i\in D\cup H}$ the preference profile of all doctors and hospitals.

Doctor d is said to be **acceptable** to h if $d \succ_h \emptyset$. Similarly, h is acceptable to d if $h \succ_d \emptyset$.

Each hospital $h \in H$ is endowed with a **capacity** q_h, which is a nonnegative integer. We say that preference relation $\succ_h$ is **responsive with capacity** q_h (Roth, 1985) if (i). For any $D' \subseteq D$ with $|D'| \leqslant q_h$, $d \in D \backslash D'$ and $d' \in D'$, $(D' \cup d) \backslash d' \succeq_h D'$ if and only if $d \succeq_h d'$,

(ii). For any $D' \subseteq D$ with $|D'| \leqslant q_h$ and $d' \in D'$, $D' \succeq_h D' \backslash d'$ if and only if $d' \succeq_h \varnothing$, and

(iii). $\varnothing \succ_h D'$ for any $D' \subseteq D$ with $|D'| > q_h$.

In words, preference relation $\succ_h$ is responsive with a capacity if the ranking of a doctor (or the option of keeping a position vacant) is independent of her colleagues, and any set of doctors exceeding its capacity is unacceptable. We assume that preferences of each hospital h are responsive with some capacity q_h.

A **matching** μ is a mapping that satisfies (i) $\mu_d \in H \cup \{\varnothing\}$ for all $d \in D$, (ii) $\mu_h \subseteq D$ for all $h \in H$, and (iii) for any $d \in D$ and $h \in H$, $\mu_d = h$ if and only if $d \in \mu_h$. That is, a matching simply specifies which doctor is assigned to which hospital (if any).

A **feasibility constraint** is a map $f : \mathbb{Z}_+^{|H|} \to \{0, 1\}$ such that $f(w) \geqslant f(w')$ whenever $w \leqslant w'$ and $f(0) = 1$, where the argument 0 of f is the zero vector and $\mathbb{Z}_+$ is the set of nonnegative integers. The interpretation is that each coordinate in w corresponds to a hospital, and the number in that coordinate represents the number of doctors matched to that hospital. $f(w) = 1$ means that w is **feasible** and $f(w) = 0$ means it is not. If w' is feasible then any w with a weakly fewer doctors in each hospital must be feasible, too. In this model, we say that matching μ is **feasible** if and only if $f(w(\mu)) = 1$, where $w(\mu) := (|\mu_h|)_{h \in H}$ is a vector of nonnegative integers indexed by hospitals whose coordinate corresponding to h is $|\mu_h|$. The feasibility constraint distinguishes the current environment from the standard model. We allow for (though do not require) $f((|q_h|)_{h \in H}) = 0$, that is, it may be infeasible for all the hospitals to fill their capacities. In order to guarantee that all feasible matchings respect capacities of the hospitals, we assume that $f(w) = 1$ implies $w \leqslant (|q_h|)_{h \in H}$. A matching problem with constraints is summarized by $\Gamma = (D, H, (\succ_a)_{a \in D \cup H}, (q_h)_{h \in H}, f)$.

To accommodate the feasibility constraint, we introduce a new stability concept that generalizes the standard notion. For that purpose, we first define two basic concepts. A matching μ is **individually rational** if (i) for each $d \in D$, $\mu_d \succeq_d \emptyset$, and (ii) for each $h \in H$, $d \succeq_h \emptyset$ for all $d \in \mu_h$, and $|\mu_h| \leqslant q_h$. That is, no agent is matched with an unacceptable partner and each hospital's capacity is respected.

Given matching μ, a pair (d, h) of a doctor and a hospital is called a **blocking pair** if $h \succ_d \mu_d$ and either (i) $|\mu_h| < q_h$ and $d \succ_h \emptyset$, or (ii) $d \succ_h d'$ for some $d' \in \mu_h$. In words, a blocking pair is a pair of a doctor and a hospital who want to be matched with each other (possibly rejecting their partners in the prescribed matching) rather than following the proposed matching.

Definition 1. Fix a feasibility constraint f. A matching μ is **weakly stable** if it is feasible, individually rational, and if (d, h) is a blocking pair then (i) $f(w(\mu) + e_h) = 0$ and (ii) $d' \succ_h d$ for all doctors $d' \in \mu_h$.

The notion of weak stability relaxes the standard definition of stability by tolerating certain blocking pairs, but impose restrictions on what kind of blocking pairs can remain. Kamada and Kojima (2017) provide a detailed discussion and axiomatic characterization of weak stability, so we refer interested readers to that paper.

Theorem 15. *A weakly stable matching exists.*

Proof. We relate our model to the matching model with contracts in the previous subsection. Let there be two types of agents, doctors in D and the "hospital side". Note that we regard the entire hospital side, instead of each hospital, as an agent in this model (thus there are $|D| + 1$ agents in total). There is a set of contracts $X = D \times H$.

For each doctor d, her preferences $\succ_d$ over $(\{d\} \times H) \cup \{\emptyset\}$ are given as follows.[53] We assume $(d, h) \succ_d (d, h')$ in this model if and only if $h \succ_d h'$ in the original model, and $(d, h) \succ_d \emptyset$ in this model if and only if $h \succ_d \emptyset$ in the original model.

We define a correspondence, called quasi-choice correspondence, $\tilde{\mathrm{C}}_H : \mathbb{Z}_+^{|H|} \rightrightarrows \mathbb{Z}_+^{|H|}$ by $\tilde{\mathrm{C}}_H(w) = \{w' : w' \leqslant w, f(w') = 1, \text{ and } (\nexists w'' \leqslant w, w' < w'', f(w'') = 1)\}$, that is, the set of all vectors that are weakly smaller than w, feasible, and maximal among all vectors that are weakly smaller than w and feasible.

For the hospital side, we assume that it has preferences and its associated choice correspondence $\mathrm{C}_H(\cdot)$ over all subsets of $D \times H$. For any $X' \subset D \times H$, let $w(X') := (w_h(X'))_{h \in H}$ be the vector such that $w_h(X') = |\{(d, h) \in X' | d \succ_h \varnothing\}|$. For each X', the choice correspondence $\mathrm{C}_H(X')$ is defined by

$$\mathrm{C}_H(X') = \left\{ X'' : \exists w \in \tilde{\mathrm{C}}_H(w(X')), X'' = \bigcup_{h \in H} \{(d, h) \in X' : |\{d' \in D | (d', h) \in X', d' \succeq_h d\}| \leqslant w_h\} \right\}. \tag{12}$$

That is, in any of the chosen subsets of contracts, there exists a vector $w \in \tilde{\mathrm{C}}_H(w(X'))$ such that each hospital $h \in H$ chooses its w_h most preferred contracts from acceptable contracts in X'.

[53] We abuse notation and use the same notation $\succ_d$ for preferences of doctor d both in the original model with constraints and in the associated model with contracts.

Claim 5. *Choice correspondence* $\mathrm{C}_H(\cdot)$ *defined above satisfies Sen's* α *and weak substitutability.*

Proof. We note that the choice correspondence given by relation (12) is within the class of choice correspondences given by relation (11), with the "hospital side" H in (12) taking the role of the multidivisional hospital in (11) and each hospital in H in (12) taking the role of divisions of the hospital in (11). Thus, Sen's α and weak substitutability follow from Claims 3 and 4, respectively. □

Given any individually rational set of contracts X', define a **corresponding matching** $\mu(X')$ in the original model by setting $\mu_d(X') = h$ if and only if $(d, h) \in X'$ and $\mu_d(X') = \emptyset$ if and only if no contract associated with d is in X'. For any individually rational X', $\mu(X')$ is well-defined because each doctor receives at most one contract at such X'.

Claim 6. X' *is a stable allocation in the associated model with contracts if and only if the corresponding matching* $\mu(X')$ *is a weakly stable matching in the original model.*

Proof. **The "only if" direction.** Suppose that X' is a stable allocation in the associated model with contracts and denote $\mu := \mu(X')$. Individual rationality of μ is obvious from the construction of μ. Suppose that (d'', h'') is a blocking pair of μ. This implies that $(d'', h'') \in U(X')$. Then, because X' is a stable allocation, it must then follow that (a) $f(w(X') + e_{h''}) = 0$ and (b) $|\{d' \in D : (d', h'') \in X', d' \succeq_{h''} d''\}| > w_{h''}(X')$. To show this, note first that the individual rationality of X' implies the existence of $w \in \tilde{\mathrm{C}}_H(w(X'))$ such that for each $h \in H$,

$$X'_h = \Big\{(d, h) \in X' : |\{d' \in D \,|\, (d', h) \in X', d' \succeq_h d\}| \leqslant w_h\Big\},$$

which then implies that for each $h \in H$, $w_h = w_h(X')$ (since $w_h \leqslant w_h(X')$ and the cardinality of the set in the RHS of the above equality cannot exceed w_h). Thus, we must have $\tilde{\mathrm{C}}_H(w(X')) = \{w(X')\}$. Now let $X'' = X' \cup \{(d'', h'')\}$. Suppose for contradiction that (a) does not hold, which implies that $f(w(X'')) = 1$ so $\tilde{\mathrm{C}}_H(w(X'')) = \{w(X'')\}$. Then $w(X')$ is not maximal given $X' \cup U(X')$, a contradiction to stability of X'. Suppose for another contradiction that (a) does hold but (b) does not. Since $\tilde{\mathrm{C}}_H(w(X')) = \{w(X')\}$, this implies $\tilde{\mathrm{C}}_H(w(X'')) = \{w(X')\}$. Given this and the fact that $|\{d' \in D : (d', h'') \in X', d' \succeq_{h''} d''\}| \leqslant w_{h''}(X')$, for any $Y'' \in \mathrm{C}_H(X'')$, we must have $(d'', h'') \in Y''$. This implies $X' \notin \mathrm{C}_H(X' \cup U(X'))$, a contradiction.

The "if" direction. Suppose that X' is not a stable allocation in the associated model with contracts and denote $\mu := \mu(X')$. If X' is not individually rational, then clearly

μ is not individually rational in the original problem with constraints. Thus, suppose that X' is individually rational and that $X' \notin \mathrm{C}_H(X' \cup U(X'))$. First, note that for any $(d, h) \in U(X')\backslash X'$, $(d, h) \succ_d X'_d$, so $h \succ_d \mu_d$ in the matching problem with constraints. If there exists any d such that $(d, h) \in U(X')\backslash X'$ and $d \succ_h d'$ for for some $d' \in \mu_h$, then clearly (d, h) is the kind of block for μ in the original matching model with constraints which makes μ fail weak stability. So, for all d with $(d, h) \in U(X')\backslash X'$, suppose that $d' \succ_h d$ for all $d' \in \mu_d$. Then the only way that $X' \notin \mathrm{C}_H(X' \cup U(X'))$ is that $w(X')$ is not maximal, so there exists $(d, h) \in U(X')\backslash X'$ such that $w(X' \cup \{(d, h)\}) = w(X') + e_h = w(\mu) + e_h$ is feasible, that is, $f(w(\mu) + e_h) = 1$. This and the fact that $d \succ_h \varnothing$ imply that μ is not weakly stable, as desired. $\square$

Theorem 10 and Claims 5 and 6 complete the proof. $\square$

We say that constraint f' is **weakly more permissive** than constraint f if $f'(w) \geqslant f(w)$ for every w. With this notion at hand, we are now ready to present a comparative statics result with respect to constraints.

Theorem 16. *Consider two matching problems with constraints* $\Gamma = (D, H, (\succ_a)_{a \in D \cup H}, (q_h)_{h \in H}, f)$ *and* $\Gamma' = (D, H, (\succ_a)_{a \in D \cup H}, (q_h)_{h \in H}, f')$ *such that* f' *is weakly more permissive than* f*. Then,*

(i). for each weakly stable matching μ *in* Γ*, there exists a weakly stable matching* μ' *in* Γ' *such that* $\mu'_d \succeq_d \mu_d$ *for each* $d \in D$*, and*

(ii). for each weakly stable matching μ' *in* Γ'*, there exists a weakly stable matching* μ *in* Γ *such that* $\mu'_d \succeq_d \mu_d$ *for each* $d \in D$*.*

Proof. By Claim 6, the sets of weakly stable matchings in Γ and Γ' correspond to stable matchings in the associated matching problems with contracts with the hospital sides' choice correspondences C_H and C'_H, respectively, where C_H and C'_H are given by relation (12) with the corresponding quasi-choice rules $\tilde{\mathrm{C}}_H$ and $\tilde{\mathrm{C}}'_H$ defined by $\tilde{\mathrm{C}}_H(w) = \{w' : w' \leqslant w, f(w') = 1, \text{ and } (\nexists w'' \leqslant w, w' < w'', f(w'') = 1)\}$ and $\tilde{\mathrm{C}}'_H(w) = \{w' : w' \leqslant w, f'(w') = 1, \text{ and } (\nexists w'' \leqslant w, w' < w'', f'(w'') = 1)\}$. By inspection, it follows that C'_H is weakly more permissive than C_H. This fact, Claim 6, and Theorem 11 imply the desired conclusion. $\square$