

A Fiscal Theory of Monetary Policy with Partially-Repaid Long-Term Debt

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Abstract

I construct a simple model with sticky prices and interest rate targets, closed by fiscal theory of the price level with long-term debt and fiscal and monetary policy rules. Fiscal surpluses rise following periods of deficit, to repay accumulated debt, but surpluses do not respond to arbitrary unexpected inflation and deflation, so fiscal policy remains active. This specification avoids many puzzles and counterfactual predictions of standard active-fiscal specifications. The model produces reasonable responses to fiscal and monetary policy shocks, including smooth and protracted disinflation following monetary or fiscal tightening.

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1 Introduction

This paper advances the fiscal theory of monetary policy, combining four ingredients that bring us towards an empirically realistic model useful for policy analysis.

A “fiscal theory of monetary policy” uses the new-Keynesian/DSGE ingredients, including intertemporal optimization, potentially rational expectations, market clearing, price stickiness, and a central bank that follows an interest rate target, but a model that substitutes active fiscal for active monetary policy to select equilibria, following the fiscal theory of the price level. I develop the model and I analyze of the effects of fiscal and monetary policy shocks.

The model in this paper has four central ingredients. The first ingredient, and the central innovation of this paper, is a fiscal policy process in which the government repays deficits with subsequent surpluses, partly or in full. Yet fiscal policy remains “active,” in the sense of Leeper (1991): The government does not respond to changes in the value of debt that result from arbitrary unexpected inflation or deflation. Therefore, the government debt valuation equation determines unexpected inflation, and provides the extra forward-looking root needed to uniquely determine equilibria.

One way to view the fiscal policy specification is that the surplus responds to a latent state variable, which is what the value of debt would be for one value of unexpected inflation. This state variable accumulates deficits, and responds to variation in the real rate of interest, but this state variable and therefore surpluses ignore changes in the value of nominal debt that come from arbitrary unexpected inflation or deflation, so fiscal policy remains active.

In equations, and in the simplest case, I write

$$s_{t+1} = \alpha v_{t+1}^* + u_{s,t+1} \quad (1)$$

$$\eta v_{t+1}^* = v_t^* - s_{t+1} \quad (2)$$

$$\rho v_{t+1} = v_t - (\pi_{t+1} - E_t \pi_{t+1}) - s_{t+1} \quad (3)$$

$$u_{s,t+1} = \rho_s u_{s,t} + \varepsilon_{s,t+1}. \quad (4)$$

The variable s_t is the real primary surplus, $u_{s,t}$ is a persistent fiscal disturbance, v_t^* is the new state variable, v_t is the value of debt, and $\rho = e^{-r}$ is a constant of linear approximation, with units of the steady-state real discount rate. In (3), the value of debt increases with time, via ρ , is paid down by surpluses, and is decreased by unexpected inflation. In (2), the state variable v_t^* evolves similarly to the value of debt v , but it does not respond to unexpected inflation. The modified discount rate $\eta < \rho$ allows a specification in which the government partially repays debts. I

derive these equations below, discuss them more fully, and generalize them substantially.

A second and equivalent way to view this fiscal policy specification is that fiscal policy disturbances $u_{s,t} + \alpha v_t^*$ have an s-shaped moving average representation, so that today's deficits lead to expectations of future surpluses. Here we think of surpluses responding to their own past and the path of real interest rates, rather than responding to debt or a state variable. The latent state variable v_t^* simply lets us represent an s-shaped moving average in a convenient VAR form.

We are familiar with specifications of this sort in which surpluses may respond to the value of debt itself, i.e.

$$s_{t+1} = \gamma v_{t+1} + u_{s,t+1} \quad (5)$$

together with the debt accumulation equation (3). If surpluses respond to the value of debt, $\gamma > 0$, then today's deficits are repaid by future surpluses, and the real value of nominal government debt equals the present value of future surpluses for any initial price level or unexpected inflation. Fiscal policy is passive. Unexpected inflation must be pinned down elsewhere in the model, e.g. by active monetary policy.

This familiar framework seems to leave us a conundrum. If we make fiscal policy active with $\gamma = 0$, and maintaining the usual assumption of positively autocorrelated disturbances $u_{s,t}$, a host of starkly counterfactual predictions emerge, most damningly perhaps that higher surpluses raise rather than pay down the value of debt. I list the others below. If we make $\gamma > 0$, we fix these predictions but fiscal policy is passive.

But this familiar analysis makes an implicit assumption, that surpluses respond equally to debts accumulated from previous deficits as they do to changes in the real value of debt induced by unplanned, unexpected, undesired, or multiple-equilibrium inflation and deflation.

One way to view the specification (1)-(4) is that it breaks this latter assumption. Active fiscal policy requires only that governments refuse to respond to arbitrary, off-equilibrium, values of unexpected inflation and deflation. An active-fiscal government may commit to repay its debts, accumulated from past deficits. It may commit to repay changes in the value of debt that come from changing real returns. It may commit to repay changes in the value of debt coming from specified and state-contingent paths of unexpected inflation.

Another way to view (1)-(4) is that it modifies the second assumption in the above framework, that the surplus disturbances are positively autocorrelated. Directly specifying a disturbance with an s-shaped moving average preserves active fiscal policy, and also allows us to model a government that borrows and then repays debts, in full or in part.

Once one considers its possibility, this separation between surplus responses to past deficits, and surplus responses to changes in the value of debt due to unexpected inflation and

deflation, is not unreasonable or artificial. Yes, governments often raise surpluses after a time of deficits. Governments often raise revenue from debt sales, and the value of debt increases after such sales, which essentially proves that investors believe surpluses will rise to pay off new debts. Raising surpluses after such debts have been incurred makes good on the explicit or implicit promise made when borrowing, and sustains the reputation needed for future borrowing. We see many institutions in place to try to guarantee or pre-commit to repayment, rather than default or inflation, which allow the government to borrow in the first place.

But the same government may well refuse to accommodate arbitrary unexpected inflation and deflation. Pre-committing *not* to raise taxes or cut spending to pay an unexpected deflation-induced windfall to nominal bondholders, or pre-committing *not* to cash in on unexpected inflation-induced debt devaluation, allows the government to produce a stable price level.

And we can see institutions and reputations at work to make these commitments as well. A gold standard is a commitment to raise taxes to buy gold in the event of inflation, rather than enjoy the bounty of an inflation-induced debt reduction. An inflation target signals the government's commitment to raise taxes to pay off nominal debts at the inflation target, neither more nor less. Many economists call now for governments to commit to helicopter-drop un-backed fiscal stimulus in event of deflation. This policy represents a refusal to passively adapt surpluses to deflation, but to repay debts in normal circumstances. A fiscal rule that raises taxes in response to inflation, and runs such stimulus in the event of deflation, but still pays off debts incurred from past deficits should the price level come out on target, is exactly the sort of policy I describe. Benhabib, Schmitt-Grohé, and Uribe (2002), for example, advocate that governments undertake unbacked fiscal expansion to escape a deflationary liquidity trap, and commit to such expansion ahead of time to avoid such traps.

Even without formal commitments, it is sensible to expect governments to refuse to react to changes in the value of debt induced by arbitrary unexpected inflation and deflation. Should a 50% cumulative deflation break out, likely in a severe recession, does anyone expect the US government to sharply raise taxes or drastically cut spending, to pay an unexpected, and, it will surely be argued, undeserved, windfall to nominal bondholders?

Cochrane (2017a) and Cochrane (2018) argue that this expectation is why the standard new-Keynesian prediction of a large deflationary shock at the zero bound, and the old-Keynesian predictions of a “deflation spiral,” did not happen in 2008-2009. People simply did not believe the “passive” fiscal requirements of those models, that the government would respond to deflation with extreme austerity. Conversely, “fiscal austerity” is a common response to inflation.

Jacobson, Leeper, and Preston (2019) clearly describe the Roosevelt administration's actions in 1933 as such a refusal to accommodate deflation, and to convince people that the government would undertake an unbacked fiscal expansion. Eggertsson (2008) gives a similar account of the history. His theoretical account is different, focusing on expectations of monetary policy after the end of the zero bound, but even that account must have the same fiscal underpinnings, "passively" achieved: The government would not pay off the deflation-induced windfall to bondholders. The United Kingdom's fiscal policy in centuries leading up to the famous restoration of gold parity in the 1920s is a good example in the opposite direction. The government suspended convertibility to gold during wars, and there was some inflation. Rather than pay off its nominal debt at the new lower value, the government refused to respond to that inflation, and repaid debt as if the inflation had not happened. By doing so, it established a reputation that enhanced its ability to borrow vast amounts in the first place, and held down inflation during the period of suspended convertibility. Though the UK also suffered the economic consequences of high marginal tax rates during restoration, viewed in intertemporal terms, it was not as senseless a policy as Keynes famously argued. Whether wise or not, it was a policy followed for centuries and not some-other-planet artificial policy for us to assume holds today, at least in part, spirit or expectation.

If we think of the specification as simply modifying standard fiscal theory to include a surplus process with an s-shaped moving average representation, perhaps it is even easier to accept. Many alleged pathologies of fiscal theory models turn out to stem from the positive autocorrelation of the surplus process, an auxiliary assumption, not the necessary active-fiscal theory assumption that surpluses do not respond to arbitrary unexpected inflation. Anyone borrowing money promises subsequent repayment. An s-shaped surplus process rather than an AR(1), is, if you stop think about it, the most natural assumption to start with.

As mentioned briefly above, a positively autocorrelated surplus process leads to a wide number of pathological predictions. These have been mis-interpreted as fiscal theory criticisms, or evidence that we must specify $\gamma > 0$ and hence passive fiscal policy. By allowing a fiscal theory with s-shaped surplus process, we can address those criticisms, and produce a fiscal theory with greater chance of broad empirical success. Each criticism also now becomes a piece of evidence that a s-shaped surplus process is an important assumption to include, not that the fiscal theory is wrong.

First, and most obviously, regressions of surpluses on debt of the form (5) produce a positive regression regression coefficient γ . (Bohn (1998), and below.) A model with positively autocorrelated surplus does not produce this coefficient. But surplus process with an s-shaped

moving average does produce this coefficient, even in an active-fiscal model. The government borrows, the value of debt rises, then the government repays. We measure a $\gamma > 0$, though the true response is $\alpha > 0$, to v^* not $\gamma > 0$, to v . Equivalently, a VAR of surpluses and debt produces an s-shaped response function, in which a near-term decline in surplus is followed by a long-term rise. This is direct evidence of a s-shaped moving average.

Second, models with positively autocorrelated surpluses predict that inflation occurs with deficits, and deflation occurs with surpluses. In such a model, a negative shock to surpluses is a negative shock to expected future surpluses, and thus to the present value of surpluses, which causes inflation. This is precisely the opposite pattern we see in data across the business cycle, where deficits, recession, and less inflation go hand in hand. An s-shaped moving average breaks the tie between current and present value of surpluses and this prediction.

Third, and damningly, models with positively autocorrelated surpluses predict that a higher surplus *raises* the value of the debt. A higher surplus forecasts higher future surpluses, and the value of the debt is the present value of subsequent surpluses. Yet higher surpluses in the data unequivocally pay down the value of the debt (Canzoneri, Cumby, and Diba (2001)). A surplus process with an s-shaped moving average reproduces that fact and solves their puzzle, in an active-fiscal model: A higher surplus corresponds to a decrease in present value of subsequent surpluses, and hence a decline in the value of debt.

Fourth, with positively autocorrelated surpluses, all deficits are paid for by devaluing outstanding debt via inflation (or default), and none are paid for by selling new debt. Running a deficit involves selling *less* real debt. With an s-shaped moving average, deficits are financed by borrowing. Bond buyers will only hand over resources to finance today's deficits if they are convinced that the bonds will be paid off by future surpluses.

Fifth, models with positively autocorrelated surpluses predict counterfactually high risk and risk premiums on government bonds. A positively autocorrelated, volatile, procyclical surplus implies that government debt, the present value of surpluses, is as risky as stocks or more so, with returns that are volatile, procyclical, and hence should have a large mean. Actual government bond returns are quiet (low volatility), countercyclical (they do well in recessions) and carry a very low mean. (Jiang et al. (2019).) An s-shaped surplus process resolves the puzzle. When the “dividend,” surplus, falls, the “price,” present value of subsequent surpluses, rises. The overall return need not move at all, nor offer a positive compensation for risk.

At bottom, difficulty in accepting an s-shaped surplus process and its ability to remedy these puzzles may reflect a fundamental misconception. Yes, the value of debt obeys an equation, value of debt equals present value of primary surpluses, that looks much like stock market

valuation equation, price equals present value of dividends. But the government debt equation describes the *total value* of debt, not the value per share. Surpluses and deficits, and consequent increases and decreases in value, can and largely do consist of greater and lesser total shares, not greater and lesser payments per share.

Once you see that surpluses and deficits are issuances and retirement of *debt*, it is perfectly natural that one writes down and allows a process in which today's borrowing is at least potentially matched by tomorrow's repayment, an s-shaped moving average, and a positively autocorrelated process such as an AR(1) seems like a strong and unnatural restriction. By contrast it is also natural to model corporate dividends per share as a persistent process that does not have an s-shaped moving average, generating volatile, procyclical and high-mean returns. A similar valuation

Long-term debt with a geometric maturity structure is the second key ingredient in this paper. Long-term debt helps the model to produce a negative response of inflation to higher interest rates, without specifying a negative fiscal shock contemporaneous to the monetary policy shock. With long-term debt, higher nominal interest rates lower the nominal value of debt. If there is no change to surpluses or real interest rates, the real value of debt is unchanged. At the original price level, people want to buy more debt, and to do so they demand fewer goods and services. The price level declines so debt regains its real value. Real interest rate and endogenous surplus variation alter this underlying mechanism in interesting ways, but the basic mechanism remains.

The geometric maturity specification bridges the gap between Sims (2011) and Cochrane (2017b), who specify perpetuities, and the bulk of the literature discussed below which uses one-period debt. US debt is poorly modeled as a perpetuity, but more poorly modeled as overnight debt. Thus, for a quantitatively reasonable model that can produce a negative response of inflation to interest rates, it is important to model debt that is long term but not too long term.

Long-term debt is especially important to apply the model at relatively high frequencies. Long-term debt allows a fiscal shock to slowly devalue debt via a persistent inflation rather than an unrealistic price-level jump.

Discount rate variation is the third key ingredient in this paper. The fiscal theory relates unexpected inflation to revisions in the present value of future surpluses. But changes in present values can result from discount rate changes as well as changes in expected future surpluses. When the discount rate for government debt declines, the same surpluses become more valuable, a deflationary force. People want to buy government bonds. To do so, they demand fewer goods and services, lowering the price level. Cochrane (2019) finds that most time-series varia-

tion in unexpected US inflation comes from discount rates, not changes in expected surpluses. When inflation falls in a recession, for example, with large deficits, real interest rates decline. This decline raises the value of government debt and accounts for all of the persistent deflation in a recession. Viewed in ex-post terms, lower rates of return rather than larger surpluses or larger growth rates bring debt back down again.

Discount rate variation is also important to the model's predictions of the effects of fiscal and monetary policy under sticky prices. For example, a rise in nominal interest rates with sticky prices raises real interest rates. Higher real interest rates raise the discount rate for government debt, an inflationary force.

Policy rules are a fourth key ingredient. I specify that both the interest rate target and the surplus respond to inflation and output. This ingredient is utterly standard in current new-Keynesian and fiscal theory models, but it plays a novel role here. The monetary policy rule introduces interesting dynamics to the response to monetary shocks, and it smooths out the inflationary consequences of fiscal shocks. The fiscal policy rule adds another mechanism by which higher interest rates can lower inflation. Higher interest rates raise future inflation. Higher future inflation induces higher future surpluses, and hence low current inflation. This is a helpful novel ingredient in the long quest to overturn the Fisherian properties of rational expectations models.

The main results of this paper are responses to persistent fiscal and monetary policy shocks. A negative shock to fiscal surpluses leads to a protracted inflation, and via the Phillips curve it leads to an output expansion. When monetary policy endogenously reacts to inflation, it moderates that inflation, but spreads the inflation further forward. Today's deficits do lead to a long string of future surpluses, and the surplus seems to respond to debt. Both observations could lead one to falsely infer a passive fiscal regime.

A monetary policy shock – a persistent rise in interest rates that does not shock fiscal surpluses – leads to a protracted disinflation, and an output decline. The protracted disinflation, overcoming the “Fisherian” prediction of related models that higher interest rates raise inflation, is important. Endogenous surplus variation amplifies the deflationary impact of monetary policy. Endogenous discount rate variation buffers it. The rise in nominal rates raises real rates which discount surpluses more highly, an inflationary force.

As the final ingredients, I use the most standard IS and Phillips curves, despite their well-known empirical shortcomings. This specification allows me to focus on the effects of the novel fiscal specification, in a familiar playground, but limits its direct empirical applicability. Since the IS and Phillips curves are so stylized, and shrinking from the formidable identification and

estimation difficulties of all such models, I do not proceed to estimation and testing, or of more rigorously comparing impulse responses to even my own atheoretical VAR estimates (Cochrane (2019)). But the fact that a fiscal theory of monetary policy model can be so easily built, can surmount classic criticisms, and produces reasonable responses, avoiding pathological predictions such as price level jumps and surpluses that raise rather than lower the value of debt, is already important news.

In this way, this paper's point is also methodological. You may have recognized problems with standard new-Keynesian active-money model and you may have seen how fiscal theory can repair such model's pathologies in a simple way. (My own list features Cochrane (2011), and Cochrane (2017a), Cochrane (2018) on zero bound issues.) But you may have been daunted by theoretical controversies that pervade the literature, the impression that the model would make immediately silly predictions such as the above list, or by the impression that you would have to *do* something fundamentally hard and different. This paper shows by example that constructing realistic fiscal theory of monetary policy models is a nearly trivial modification of the parallel construction of new-Keynesian DSGE models.

The questions one is led to ask, and the answers, are potentially quite different. For example, one is led to ask for the response of monetary policy shocks holding fiscal policy constant in some sense. This question is likely to have a quite different answer than a question that includes the usual passive-fiscal contraction, and it is likely to require quite different identification assumptions for empirical validation.

Similarly, while solving such models is a trivial adaptation of standard new-Keynesian / DSGE techniques, just which model one should solve – which IS and Phillips curves, and which currently popular ingredients such as financial frictions and heterogeneous constrained agents – remains a large and fertile field of investigation.

1.1 Literature

The fiscal theory of monetary policy, uniting fiscal theory with interest rate targets and sticky prices, is a recent development in a long literature on the fiscal theory of the price level. Sims (2011) and Cochrane (2017b) are immediate antecedents, including sticky prices, long-term debt, and computing the response to monetary and fiscal policy shocks. This paper both builds on and simplifies those models. Both models specify perpetuities and an exogenous AR(1) surplus, and calculate somewhat different responses.

A variety of more complex models combining (sometimes time-varying) fiscal price determination and interest rate targets have been built. While these models have included many

of the features here, none has included them all and especially the surplus process with an s-shaped moving average, or the equivalent distinction between reacting to debt and reacting to unexpected inflation. Thus, they all embody the AR(1) surplus conundrum.

Leeper, Traum, and Walker (2017) is a detailed a sticky-price model allowing fiscal theory solution, aimed at evaluating the output effects of fiscal stimulus. But they specify fiscal policy as an AR(1) (p. 2416) along with one-period debt. Their paper, and others that include a surplus that responds to output, includes the above-mentioned indirect mechanism that buffers the AR(1) surplus conundrum somewhat: A deficit leads to inflation, which raises output, which raises tax revenues, and leads to higher surpluses. But that mechanism is not necessarily large enough, for example, to allow all debts to be repaid.

The comprehensive survey in Leeper and Leith (2016) studies the standard IS-Phillips curve model of this paper, including a monetary policy rule with endogenous responses, but again surpluses can only respond to the full value of debt, leading to passive policy, or respond not at all and then follow an AR(1).

Bianchi and Melosi (2017) specify that taxes follow an AR(1) that responds to output. Their model switches between a passive-fiscal regime that responds to debt and an active-fiscal regime that does not (Equation (6) p. 1041). Government spending also follows an AR(1) that responds to output (p. 1040). They also use one-period debt. Their paper is centrally about the absence of deflation in response to a preference shock, and how expectations of a switch between regimes affects responses to shocks. They don't analyze monetary policy shocks.

Moreover, like many Markov-switching regimes papers, their fiscal regime is not obviously fiscal. Does inflation or the real value of government debt explode in expectation if the economy starts at an off-equilibrium inflation rate? Davig and Leeper (2006) point out that an economy which appears even to have both policies passive can be determinate, if people expect at some future date to switch to one or the other active regime. Similarly an apparently fiscal regime may actually be determinate by an expected switch to active money. Much new-Keynesian literature at the zero bound, where monetary policy must be locally passive, gains determinacy by imagining a switch to active monetary policy at the end of the zero bound.

Relative to this literature, while the present paper extends the list of ingredients in the fiscal policy specification, it pares back ingredients in other dimensions. More ingredients do not necessarily make a better model. Much of this paper's contribution is to show mechanisms at work in more complex, more realistic, but more obscure models, by seeing those mechanisms in a simpler environment. For this reason, this paper builds to its main, already simple, model, with a sequence of even simpler models. Its aim is to provide a counterpart to the standard three-

equation new-Keynesian model, a workhorse for understanding mechanisms, not a counterpart to the medium and large scale models that match data, but whose mechanisms are somewhat of a black box. Before we Markov-mix regimes, and add a zero bound additional frictions or other model complexity, let's figure out how monetary and fiscal policy work in a pure fiscal regime, and let us specify that regime in a sensible way.

Bianchi and Melosi (2019) model the distinction between “regular budget” which is passive-fiscal and an “emergency budget” of unbacked fiscal expansion to fight deflation, mirroring the actions and language of the Roosevelt Administration as described in Jacobson, Leeper, and Preston (2019). This specification also produces a regime that is at least partly active-fiscal, yet with debts repaid.

If one wishes to test active-fiscal vs. active-money regimes, or to measure which regime the economy is in at a given time, one must face up to the theorem that active-fiscal vs. active-money regimes cannot be distinguished based on time-series drawn from equilibrium (Cochrane (1998)). Each model only describes expectations of how fiscal or monetary authorities would react *off* equilibrium.

Identification always requires assumptions, so this theorem mainly puts the spotlight on the assumptions. The usual auxiliary assumption, as in the above papers, is that surpluses follow a policy of the form $s_t = \gamma v_t + u_{s,t}$, with time-series restrictions on $u_{s,t}$, plus responses to other endogenous variables. With that auxiliary assumption, $\gamma = 0$ cannot produce an s-shaped response function, cannot describe a government that borrows and commits to repay debts without inflation, and produces the above puzzles. So naturally one finds a passive-fiscal regime, or much of the Markov state in that regime. But, as the example of this paper shows, there is no reason to specify that an active-fiscal government cannot follow an s-shaped surplus policy. With the possibility of an active-fiscal s-shaped response, that avoids the above counterfactual time-series predictions, we may find quite a bit more frequent active-fiscal regimes.

Bohn (1998), for example, finding a positive coefficient γ writes “The positive response of the primary surplus to changes in debt also shows that U. S. fiscal policy is satisfying an intertemporal budget constraint.” The fact is confirmed, but the implication is false. Bohn makes (implicitly) an additional assumption that the γ coefficient also measures the response of surpluses to changes in the value of debt brought about by arbitrary unexpected and off-equilibrium inflation and deflation.

Canzoneri, Cumby, and Diba (2001) point out that an AR(1) surplus process, discounted at a constant rate, with active fiscal policy, predicts that higher surpluses raise the value of debt, where in the data higher surpluses pay down the debt. They are aware that an s-shaped moving

average surplus process resolves the puzzle, writing

“NR [active-fiscal] regimes offer a rather convoluted explanation that requires the correlation between today’s surplus innovation and future surpluses to eventually turn negative. We will argue that this correlation structure seems rather implausible in the context of an NR regime, where surpluses are governed by an exogenous political process.”

But an s-shaped surplus process, which results from any borrowing with promise of repayment, is natural and sensible, no matter how one specifies the government’s reaction to unexpected inflation and deflation. Even “political processes” must obey the constraints imposed by market prices, one of which is that people will not lend you money if you do not promise to pay it back. Rather, their evidence that surpluses pay down debts, and deficits increase the value of debt, is straightforward evidence that the surplus process does have this s-shaped moving average. The passive-fiscal regime they advocate has such an s-shaped moving average, induced by $\gamma > 0$. And that same equilibrium s-shaped moving average can represent active fiscal policy. Simply write a specification, as here, in which surpluses do not react to off-equilibrium inflation.

Cochrane (2001) advocates a surplus process with an s-shaped moving average representation, in order to fix some of the pathologies of the usual AR(1) specification and the Canzoneri, Cumby, and Diba (2001) puzzles in particular. That paper also shows how one will incorrectly measure an AR(1) surplus by leaving debt out of the VAR.

Jiang et al. (2019) point out that with a positively correlated surplus process, government bonds should have volatile and procyclical returns, and hence a large expected return. They estimate a VAR that excludes the value of debt as a forecasting variable, contra the advice and counterexample in Cochrane (2001). This crucial omission means they cannot measure the s-shaped surplus response. They pronounce a puzzle to be fixed by very large liquidity premiums in government bonds. They do not resolve the predictions of volatile government bond returns, volatile inflation, surpluses that are positively correlated with inflation, and surpluses that raise the value of debt in their model. These predictions remain even with large but steady liquidity premiums in government bonds.

These remarks are not criticisms. I just need to establish that this apparently simple and perhaps seemingly obvious analysis in this paper is actually novel.

I focus much discussion of the effect of monetary policy shocks on how the model can overcome the “Fisherian” prediction that higher interest rates lead to uniformly higher inflation, and instead produce a negative inflation response to higher interest rates. This has been a robust prediction of rational expectations models (Uribe (2018), Cochrane (2018)), and tough to

eradicate. This paper shows how long-term debt and fiscal policy which responds positively to output and inflation *can* overcome the Fisherian prediction. I do not mean to imply thereby that we know the Fisherian prediction is false. Evidence that higher interest rates reduce inflation, especially without contemporaneous fiscal contractions, is weak, and Uribe adds evidence on the other side. Different parameterizations of this model are also consistent with Fisherian responses.

2 Ingredients

The central model of this paper appears in Section 4. I build to that model with a few simpler models. These models explore the key ingredients and important intuition of the final model in simple settings, and they exhibit deficiencies that motivate the more complex ingredients of the final model.

2.1 Linearized debt identities

I use a set of linearized identities to express the fiscal side of the model including long-term debt and time-varying bond returns. These identities are derived in Cochrane (2019) which also describes some of their operation in more detail.

Start with the linearized evolution of the real value of nominal government debt,

$$\rho v_{t+1} = v_t + r_{t+1}^n - \pi_{t+1} - s_{t+1}. \quad (6)$$

Log debt at the end of period $t + 1$, v_{t+1} , is equal to its value at the end of period t , v_t , increased by the log nominal return on the portfolio of government bonds r_{t+1}^n less inflation π_{t+1} , and less the real primary surplus¹ s_{t+1} . The parameter $\rho = e^{-r}$ is a constant describing the linearization point. In the case of one-period debt, the ex-post nominal return equals the interest rate, $r_{t+1}^n = i_t$. Long-term debt shows up in this accounting by the presence of an ex-post nominal return r_{t+1}^n not equal to the interest rate. If long term bond prices fall, r_{t+1}^n is negative and the market value of debt v_{t+1} falls.

The main model below uses (6), as the matrix solution method implicitly solves it forward.

¹Precisely, s_{t+1} is ρ times the real primary surplus divided by steady state debt. For brevity, I refer to s_{t+1} as simply the “surplus.” Cochrane (2019) defines v_t as the log debt to GDP ratio, s_t as the surplus to GDP ratio, scaled by the average debt/GDP ratio, and adds GDP growth g_{t+1} on the right hand side of (6), to accommodate stochastic growth. For simplicity I abstract from stochastic growth here, but it is an important transformation for empirical work. In empirical work, I infer the surplus from (6) so its definition only matters if one wants to construct it from other data sources.

However, it is useful to solve (6) forward explicitly, as this step allows us to solve simpler models analytically and thereby to understand the mechanisms behind computed solutions of the larger model.

Iterating (6) forward, we have a present value identity,

$$v_t = \sum_{j=1}^{\infty} \rho^{j-1} s_{t+j} - \sum_{j=1}^{\infty} \rho^{j-1} (r_{t+j}^n - \pi_{t+j}) + \lim_{T \rightarrow \infty} \rho^T v_{t+T}. \quad (7)$$

This identity holds ex-post, so it also holds in expectation. Equilibrium requires that the expected value of the last term is zero. This is achieved by active or passive fiscal policy assumptions, and in computations by choosing only stable solutions. Then, the log value of government debt is the expected present value of future surpluses, discounted at the expected real return on the portfolio of government bonds.

Taking time $t + 1$ innovations $\Delta E_{t+1} \equiv E_{t+1} - E_t$ and rearranging, we have an innovation identity,

$$\Delta E_{t+1} \pi_{t+1} - \Delta E_{t+1} r_{t+1}^n = - \sum_{j=0}^{\infty} \rho^j \Delta E_{t+1} s_{t+1+j} + \sum_{j=1}^{\infty} \rho^j \Delta E_{t+1} (r_{t+1+j}^n - \pi_{t+1+j}). \quad (8)$$

A decline in the present value of surpluses must correspond to a lower real value of the debt. This reduction can come about by unexpected inflation, or by a decline in nominal long-term bond prices and therefore a low return r_{t+1}^n .

The second term on the right hand side captures discount rate effects. If expected real returns on government bonds rise, the present value of surpluses declines, which requires either inflation or negative bond returns. Cochrane (2019) finds that such discount rate variation is the dominant source of movement in US time series data on inflation.

When does a decline in the present value of surpluses result in inflation vs. a decline in nominal bond prices? To eliminate the bond return here, and to include long-term debt in the model below, we need bond pricing equations. Assuming a geometric maturity structure, in which the face value of maturity j debt declines at the rate ω^j , we have a second linearized identity

$$r_{t+1}^n = \omega q_{t+1} - q_t \quad (9)$$

where q_t is the price at time t of the portfolio of government bonds. We can solve this identity

forward to express the bond price as the inverse of its future returns,

$$q_t = - \sum_{j=0}^{\infty} \omega^j r_{t+1+j}^n. \quad (10)$$

Taking innovations ΔE_{t+1} of this equation we have we have

$$\Delta E_{t+1} r_{t+1}^n = - \sum_{j=1}^{\infty} \omega^j \Delta E_{t+1} r_{t+1+j}^n = - \sum_{j=1}^{\infty} \omega^j \Delta E_{t+1} [(r_{t+1+j}^n - \pi_{t+1+j}) + \pi_{t+1+j}]. \quad (11)$$

An unexpectedly low bond return $\Delta E_{t+1} r_{t+1}^n$ corresponds mechanically to higher expected future nominal returns, as bond prices are the inverse of bond yields. And nominal expected returns equal real returns plus inflation. Cochrane (2019) finds that variation in expected inflation is usually the dominant term in this decomposition.

We can eliminate the bond return from (8)-(11) to yield an identity linking inflation, surpluses, and discount rates:

$$\sum_{j=0}^{\infty} \omega^j \Delta E_{t+1} \pi_{t+1+j} = - \sum_{j=0}^{\infty} \rho^j \Delta E_{t+1} s_{t+1+j} + \sum_{j=1}^{\infty} (\rho^j - \omega^j) \Delta E_{t+1} (r_{t+1+j}^n - \pi_{t+1+j}). \quad (12)$$

This equation emphasizes how, in the presence of long-term debt $\omega > 0$, a fiscal shock can be met by drawn out inflation rather than a price level shock, which devalues long-term bonds. This scenario is more realistic than one-period price-level jumps or i.i.d. inflation. Real expected return variation adds a discount rate effect.

2.2 Simplest FTMP model

Start with flexible prices, a constant real interest rate and expected bond return $E_t r_{t+1}^n = E_t \pi_{t+1}$, one-period debt $\omega = 0$, and exogenous policy processes $\{s_t\}, \{i_t\}$. The model is composed of only the Fisher equation, i.e. the linearized intertemporal first-order condition

$$i_t = E_t \pi_{t+1}, \quad (13)$$

the debt accumulation equation (6) and transversality condition. As above, we can solve the debt accumulation equation forward with the transversality condition, and take innovations to

complete the model with the inflation identity (8) or (12), which here simplify to

$$\Delta E_{t+1}\pi_{t+1} = -\Delta E_{t+1} \sum_{j=0}^{\infty} \rho^j s_{t+1+j}. \quad (14)$$

(With one period debt, $i_t = r_{t+1}^n$ so $\Delta E_{t+1}r_{t+1}^n = 0$. These are deviations from steady state, so a term r on the right hand side of (13) is zero.)

If the central bank sets an interest rate target $\{i_t\}$, the bank determines expected inflation by (13), even if that bank has no power to control fiscal policy $\{s_t\}$. The central bank remains powerful in this model.² Fiscal policy then determines unexpected inflation, the instantaneous response of inflation to a shock $\Delta E_{t+1}\pi_{t+1}$, via (14), and that's all it does. For this latter step to work, we need "active" fiscal policy, that surpluses do not respond to unexpected inflation in such a way that (14) holds for any value of unexpected inflation.

But this frictionless model is Fisherian: A rise in interest rates, with no change in surpluses, produces a rise in expected inflation one period later via $i_t = E_t\pi_{t+1}$, and it produces no change in current inflation $\Delta E_{t+1}\pi_{t+1} = 0$.

The frictionless new-Keynesian counterpart produces a negative inflation response, by implicitly assuming a coincident fiscal contraction, a negative shock in (14), achieved by "passive" fiscal authorities. The fiscal theory of monetary policy can produce the same response, but would call it a coordinated fiscal and monetary shock.

In response to a fiscal shock, this model can only produce one period of inflation, a price-level jump that devalues outstanding short-term (by assumption) bonds. If monetary policy chooses to follow this inflation by raising interest rates, we will see a persistent inflation, but that is an endogenous effect of monetary policy not essential to the fiscal policy shock. If we wish to describe events such as the 1970s as a response to fiscal events, it would be more attractive to have a model in which fiscal policy shocks directly lead to a drawn-out inflation response.

²How does the central bank set an interest rate target? Even in this cashless and frictionless model, the central bank can set interest rates by varying the quantity of debt, without changing surpluses. Briefly, for example, writing the nonlinear valuation identity with a constant interest rate as

$$\frac{B_{t-1}}{P_t} = E_t \sum_{j=0}^{\infty} \beta^j s_{t+j},$$

and hence

$$\frac{B_{t-1}}{P_{t-1}} E_{t-1} \left(\frac{P_{t-1}}{P_t} \right) = E_{t-1} \sum_{j=0}^{\infty} \beta^j s_{t+j},$$

then a change in B_{t-1} at time $t-1$ with no change in surpluses changes expected inflation $E_{t-1}(P_{t-1}/P_t)$ and therefore the nominal interest rate. See Cochrane (2017b) Section 2.4 for an extended discussion. Alternatively, one may appeal to the Woodford (2003) cashless limit.

2.3 Long-term debt

Now, add long-term debt with a geometric maturity structure, keeping for now a constant real interest rate and flexible prices. This model can generate lower inflation when the nominal interest rate rises, with no change in surpluses.

The model consists of the Fisher equation (13)

$$i_t = E_t \pi_{t+1},$$

and (12), which simplifies to

$$\sum_{j=0}^{\infty} \omega^j \Delta E_{t+1} \pi_{t+1+j} = - \sum_{j=0}^{\infty} \Delta E_{t+1} s_{t+1+j} \quad (15)$$

in place of (14). With long-term debt $\omega > 0$, shocks to the present value of surpluses in (15) now change a weighted average of current and expected future inflation.

When surpluses do not move, (15) introduces an important link between changes in expected inflation at different dates. Consider a persistent monetary policy shock – a persistent positive change in $i_t = E_t \pi_{t+1}$, starting at $i_1 = E_1 \pi_2$, with no change in fiscal policy. From (15),

$$\Delta E_1 \pi_1 = - \sum_{j=1}^{\infty} \omega^j \Delta E_1 \pi_{1+j}. \quad (16)$$

If the terms $\Delta E_1 \pi_{1+j}$ for $j \geq 1$ are, on average, positive, then $\Delta E_1 \pi_1 < 0$. In this way, with long-term debt, a positive persistent monetary policy shock that raises long-run inflation induces a negative initial inflation response, which briefly overturns the otherwise Fisherian properties of this frictionless model. (This result is a different s-shaped response function.) This observation boils down a large effort in more complex models (Sims (2011), Cochrane (2017b), Cochrane (2018)). The mechanism does not require sticky prices.

For example, suppose the central bank creates a monetary disturbance that follows an AR(1),

$$i_t = \rho_i i_{t-1} + \varepsilon_{i,t}, \quad (17)$$

with no change in surpluses. After one period, expected inflation responds positively to a unit shock $\varepsilon_{i,1} = 1$:

$$\Delta E_1 \pi_{1+j} = \Delta E_1 i_j = \rho_i^{j-1}; \quad j = 1, 2, 3\dots$$

However, from (16), with long-term debt the impact effect of a higher interest rate is negative:

$$\Delta E_1 \pi_1 = -\frac{\rho\omega}{1-\rho\omega}.$$

With short-term debt $\omega = 0$, a fiscal shock leads immediately to inflation at time 1 by (15). If the central bank chooses to follow that event with higher interest rates, the inflation will persist, but that is an independent choice and has no effect on immediate inflation $\Delta E_1 \pi_1$. With long term debt $\omega > 0$, a fiscal shock in (15) may give rise to a persistent smaller rise in inflation, or even a rise only in future expected inflation with no current inflation at all. Since monetary policy controls expected inflation, monetary policy chooses whether to smooth forward the fiscal shock, or not to do so. Contrary to the impression one gets with short-term debt models, then, fiscal theory can produce a persistent inflation response to fiscal shocks.

As we shorten the time interval, and then move to continuous time without price level jumps, the effective ω rises, and the instantaneous inflation term drops out altogether. All fiscal and monetary shocks now result in changes in expected future inflation alone.

We get useful intuition by bringing bond returns back in to the analysis, expressing this model's fiscal condition as the pair (8)-(11), which specialize to

$$\Delta E_{t+1} \pi_{t+1} - \Delta E_{t+1} r_{t+1}^n = - \sum_{j=0}^{\infty} \Delta E_{t+1} s_{t+1+j} \quad (18)$$

$$\Delta E_{t+1} r_{t+1}^n = - \sum_{j=1}^{\infty} \omega^j \Delta E_{t+1} \pi_{t+1+j}. \quad (19)$$

With short-term debt $\omega^j = 0$ there is no surprise bond return $\Delta E_{t+1} r_{t+1}^n = 0$. A price-level jump is the only way to devalue short-term debt. With long-term debt, a decline in bond prices and negative return r_{t+1}^n can reduce the value of long-term bonds, to bring the value of debt back in line with the present value of surpluses. And, with constant real returns, the decline in bond prices must come from expected future inflation, (19). Thus, a rise in expected future inflation can devalue long-term bonds, both when they are eventually redeemed with nominal dollars, and in their time- $t+1$ market value. Expression (19) essentially marks the future inflation of (15) to market, to produce a present value identity (18) at time $t+1$.

Likewise, when the central bank raises interest rates and thereby expected future inflation, with short-term debt and no change in surpluses, this action has no effect on the value of government debt, $r_{t+1}^n = i_t$, and therefore no effect on current inflation in (18). With long-term debt, when the central bank raises expected future inflation, that lowers long-term bond prices

and the bond return r_{t+1}^n . With no change on the right hand side of (18), immediate inflation must decline.

2.4 Policy rules

Next, I add policy rules whereby monetary and fiscal policy respond to endogenous variables and suffer persistent disturbances. I return to the flexible-price model with one-period debt, and I do not yet introduce the s-shaped fiscal policy process described in the introduction. Policy rules introduce dynamics of their own, best studied first in isolation.

A fiscal policy rule, in which surpluses respond positively to inflation, gives a separate mechanism by which higher interest rates can temporarily lower inflation. If monetary policy produces higher future inflation, as above, that inflation leads to higher surpluses, which lower near-term inflation, also overturning the Fisherian prediction of the simple model.

A monetary policy rule, in which the interest rate responds positively to inflation, naturally spreads forward the inflationary consequences of a fiscal shock.

The model is

$$i_t = E_t \pi_{t+1} \quad (20)$$

$$i_t = \theta_i \pi_t + u_{i,t} \quad (21)$$

$$s_t = \theta_s \pi_t + u_{s,t} \quad (22)$$

$$\Delta E_{t+1} \pi_{t+1} = -\Delta E_{t+1} \sum_{j=0}^{\infty} \rho^j s_{t+1+j}. \quad (23)$$

The presence of rules and disturbances now lets us ask more sophisticated policy questions: What is the effect of a monetary policy disturbance $u_{i,t}$, holding constant fiscal policy disturbances $u_{s,t}$, but allowing the endogenous response of fiscal policy through the effects of inflation on surpluses? What is the effect of a fiscal policy disturbance $u_{s,t}$ that includes the predictable fiscal and monetary policy responses to inflation, here, and output, later?

Substituting the fiscal rule (22) into the inflation identity (23), we obtain

$$\Delta E_{t+1} \pi_{t+1} = -\theta_s \sum_{j=0}^{\infty} \rho^j \Delta E_{t+1} \pi_{t+1+j} - \Delta E_{t+1} \sum_{j=0}^{\infty} \rho^j u_{s,t+1+j}. \quad (24)$$

The second term on the right hand side is what we had before – an increase in future surpluses causes an unexpected deflation today. The first term on the right hand side is the novelty: If expected future inflation rises, that will increase future surpluses, and cause a decline in inflation

today.

To generate a simple example focusing on the dynamics induced by the policy rules, let both disturbances follow $AR(1)$ processes

$$u_{i,t+1} = \rho_i u_{i,t} + \varepsilon_{i,t+1}$$

$$u_{s,t+1} = \rho_s u_{s,t} + \varepsilon_{s,t+1}.$$

After some algebra, relegated to the Appendix, we can find the response of inflation to monetary and fiscal policy shocks. The impact response is

$$\Delta E_{t+1}\pi_{t+1} = \frac{1}{1 - \rho\theta_i + \theta_s} \left[-\frac{\rho\theta_s}{1 - \rho\rho_i} \varepsilon_{i,t+1} - \frac{1 - \rho\theta_i}{1 - \rho\rho_s} \varepsilon_{s,t+1} \right] \quad (25)$$

while the response of future expected inflation for $j \geq 1$ is

$$\begin{aligned} \Delta E_{t+1}\pi_{t+1+j} &= \frac{1}{\theta_i - \rho_i} \left[\frac{(1 - \rho\theta_i)}{(1 - \rho\theta_i + \theta_s)} \frac{(1 - \rho\rho_i + \theta_s)}{(1 - \rho\rho_i)} \theta_i^j - \rho_i^j \right] \varepsilon_{i,t+1} - \\ &\quad - \frac{(1 - \rho\theta_i)}{(1 - \rho\theta_i + \theta_s)(1 - \rho\rho_s)} \theta_i^j \varepsilon_{s,t+1}. \end{aligned} \quad (26)$$

The big news is the first term of (25): the immediate response of inflation to the monetary policy shock can be negative, even with flexible prices and short term debt.

Figure 1 presents these responses to interest rate shocks. Start with the case of no policy responses $\theta_s = 0, \theta_i = 0$, but still persistent shocks, $\rho_i = 0.9$. Focusing on the monetary policy shock, (25) and (26) reduce to

$$\Delta E_{t+1}\pi_{t+1} = 0 \times \varepsilon_{i,t+1}$$

$$\Delta E_{t+1}\pi_{t+1+j} = \rho_i^{j-1} \varepsilon_{i,t+1}.$$

As the figure shows, nothing happens in the first period. Then inflation follows an $AR(1)$ pattern. This is the “Fisherian” pattern we noticed in section 2.2.

The line labeled “ $\theta_s = 0.9, \theta_i = 0$ ” now adds the fiscal policy response to this simple model. Now (25) and (26) reduce to

$$\begin{aligned} \Delta E_{t+1}\pi_{t+1} &= -\frac{1}{1 + \theta_s} \frac{\rho\theta_s}{1 - \rho\rho_i} \varepsilon_{i,t+1} \\ \Delta E_{t+1}\pi_{t+1+j} &= \rho_i^{j-1} \varepsilon_{i,t+1}. \end{aligned}$$

Adding the fiscal response does not change at all the path of expected inflation after period 1,

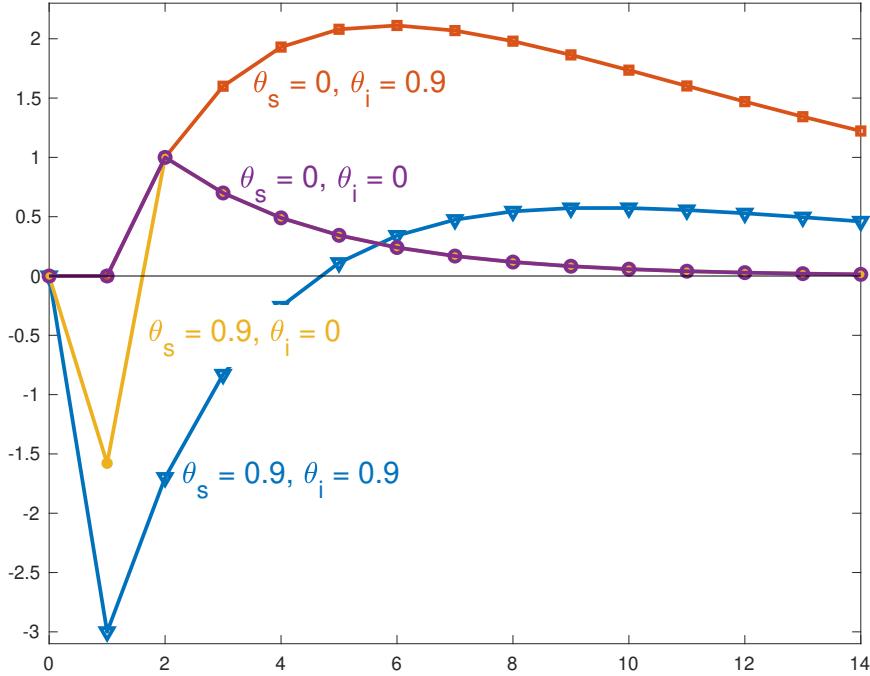


Figure 1: Responses of inflation to a monetary policy shock. Model with short-term debt, flexible prices, and policy responses. Parameters are $\rho_i = \rho_s = 0.7$, $\theta_i = \theta_s = 0.9$ except as indicated.

but it gives a one-period decline $\Delta E_{t+1}\pi_{t+1}$. Thus, the *fiscal* policy response to inflation means that a *monetary* policy shock can lower inflation, even in this model with flexible prices and one-period debt, with much the same pattern that long-term debt offers. Since the interest rate is still expected inflation $i_t = E_t\pi_{t+1}$ here, the period of low inflation coincides with the higher interest rate, so we can say that higher interest rates produce temporarily lower inflation.

The line labeled “ $\theta_s = 0, \theta_i = 0.9$ ” removes this fiscal policy response, but adds the monetary policy response to inflation. Equations (20) and (21) imply the equilibrium condition

$$E_t\pi_{t+1} = \theta_{i,\pi}\pi_t + u_{i,t}$$

so the monetary policy rule adds persistence. In this case (25) and (26) reduce to

$$\begin{aligned}\Delta E_{t+1}\pi_{t+1} &= 0 \times \varepsilon_{i,t+1} \\ \Delta E_{t+1}\pi_{t+1+j} &= \frac{\theta_i^j - \rho_i^j}{\theta_i - \rho_i} \varepsilon_{i,t+1}.\end{aligned}$$

With no fiscal policy response, the impact effect on inflation is still zero, and the subsequent effect is still Fisherian, all positive. The combination of a monetary policy response and a persis-

tent shock give a pretty hump-shaped response function, itself an important ingredient towards eventually matching data.

Adding both policy responses in the line labeled “ $\theta_s = 0.9, \theta_i = 0.9$,” we obtain a response of inflation to a monetary policy shock that combines the negative early response with drawn out dynamics, producing a long period of reduced inflation.

Before cheering too loudly, remember the equilibrium condition $i_t = E_t \pi_{t+1}$ is still part of this model. So, throughout the period of disinflation in this model, unlike the previous case, the actual interest rate moves down, one period ahead of the graphed inflation. The endogenous response of interest rate to inflation is so strong that a positive monetary policy shock lowers the actual interest rate. This is a common feature of new-Keynesian models. That doesn't make it any more pleasant. It means that the model can really not say to have overcome the Fisherian prediction that inflation and interest rates move in the same direction. Still, the progress on the inflation response is welcome, and this is one of the mechanisms that will contribute to the final response functions which break the constant real rate assumption.

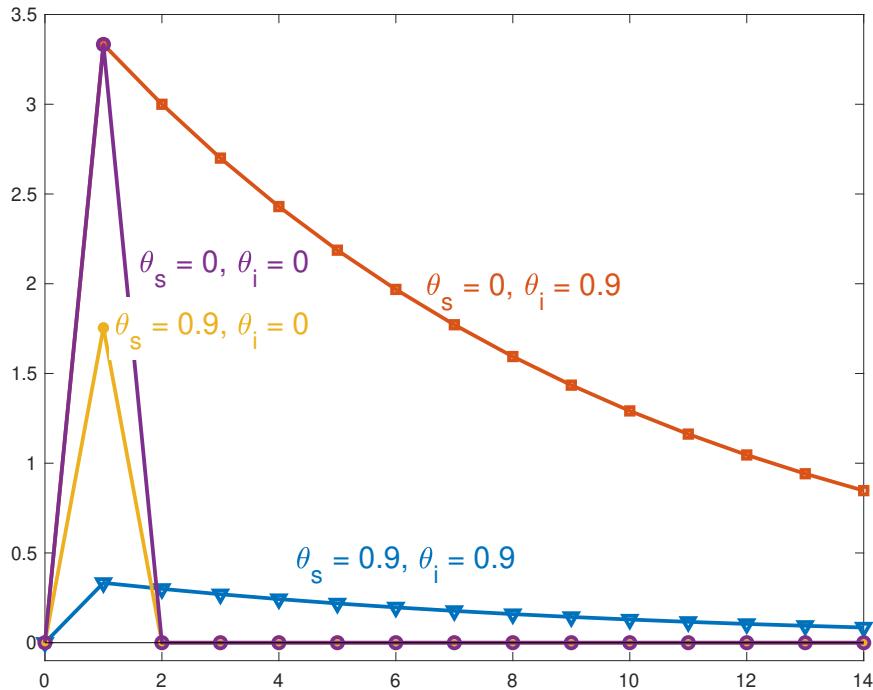


Figure 2: Responses of inflation to a fiscal policy shock. Model with short-term debt, flexible prices, and policy responses. Parameters are $\rho_i = \rho_s = 0.7$, $\theta_i = \theta_s = 0.9$ except as indicated.

Figure 2 presents responses to a fiscal policy shock $\varepsilon_{s,t} = -1$ for the same model. I present a negative, inflationary fiscal shock as it tells an easier story. In the case with no responses,

$\theta_s = \theta_i = 0$, we see one period of unexpected inflation, and that's all.

Adding the fiscal policy response, $\theta_s = 0.9$ with $\theta_i = 0$, cuts this unexpected inflation in half. The first-period inflation raises that period's surplus via the policy rule, mitigating the fiscal shock. This mitigation of underlying shocks by policy rules is a mechanism that occurs in several places.

Adding a monetary policy response $\theta_i = 0.9$ with $\theta_s = 0$ leaves the initial shock the same, but again leads to drawn out dynamics. The first period inflation π_1 leads to higher i_1 , which leads to higher π_2 , and so forth. In this case, the higher subsequent inflation has no effect on the impact inflation π_1 . Long-term debt changes that effect, with the higher future inflation reducing current inflation.

With both fiscal and monetary policy rules, $\theta_s = 0.9$ and $\theta_i = 0.9$, the inflation response is dramatically smaller and also drawn out. Now, the drawn-out inflation generated by $\theta_i = 0.9$ also induces a persistent rise in surpluses. This endogenous rise in surpluses greatly offsets the negative surplus disturbance. A long-term debt effect would lower inflation still. In this way, by drawing out the inflation response to a fiscal shock, monetary policy also reduces the inflationary consequences of that shock.

It is a truism that monetary and fiscal policy must be considered together, but these examples embody that truism in a specific and useful way.

2.5 Fiscal policy

To understand the fiscal policy specification in a simple environment, let fiscal policy follow

$$s_{t+1} = \alpha v_{t+1}^* + u_{s,t+1} \quad (27)$$

$$\eta v_{t+1}^* = v_t^* - s_{t+1} \quad (28)$$

$$\rho v_{t+1} = v_t - (\pi_{t+1} - E_t \pi_{t+1}) - s_{t+1} \quad (29)$$

$$u_{s,t+1} = \rho_s u_{s,t} + \varepsilon_{s,t+1} \quad (30)$$

Equation (29) is the debt evolution equation (6), with one period debt so $i_t = r_{t+1}^n$, the Fisher equation $i_t = E_t \pi_{t+1}$ and a constant interest rate. The v^* term in the surplus policy rule (27) and (28) is a central innovation. This specification gives us a fiscal policy that is active, and rules out multiple equilibria, but nonetheless partially pays off debts accumulated from past deficits.

As one way to see how (28) works, compare it to (29). The surplus responds to a version of debt in which the ex-post real return is replaced by the expected real return, zero here, and the debt growth rate ρ^{-1} is increased to η^{-1} . The first modification means that the surplus does

not respond to unexpected and thus multiple equilibrium inflation. But v^* still accumulates past deficits, so surpluses do rise to pay off debts accumulated from past deficits. The second modification, $\eta > \rho$, allows the government to partially repay debt and partially inflate. With $\eta = \rho$, all debts are paid off, $\Delta E_{t+1} \sum_{j=0}^{\infty} s_{t+1+j} = 0$ in response to any fiscal shock, and there is no unexpected inflation at all. With $\eta > \rho$, part of a surplus shock is repaid by future surpluses, and part by unexpected inflation. Intuitively, with $\eta > \rho$, we can write (28) as

$$[\rho + (\eta - \rho)] v_{t+1}^* = v_t^* - s_{t+1}.$$

The variable v_{t+1}^* is what the value of the debt would be if someone came along with an extra surplus $(\eta - \rho)v_{t+1}^*$. Surpluses react as if that gift had happened, i.e. they react less than fully to the actual deficit-induced rise in debt.

The debt accumulation equation (29) remains a part of the model. Fiscal policy is still active, in that this equation, solved forward, determines unexpected inflation.

To see these points more explicitly, substitute (27) into (28), to write the v^* process as

$$v_{t+1}^* = -\frac{\frac{1}{\eta+\alpha}}{1 - \frac{1}{\eta+\alpha}L} u_{s,t+1}. \quad (31)$$

Substituting (31) back into (27) we obtain the univariate moving average representation of the surplus process

$$\begin{aligned} s_{t+1} &= a(L)u_{s,t+1} = \left(1 - \alpha \frac{\frac{1}{\eta+\alpha}}{1 - \frac{1}{\eta+\alpha}L}\right) u_{s,t+1} \\ &= u_{s,t+1} - \alpha \left(\frac{1}{\eta+\alpha}\right) u_{s,t+1} - \alpha \left(\frac{1}{\eta+\alpha}\right)^2 u_{s,t} - \alpha \left(\frac{1}{\eta+\alpha}\right)^3 u_{s,t-1}.. \end{aligned} \quad (32)$$

A deficit shock, a negative $u_{s,t}$, is followed by a string of small positive surpluses, which pay back some of the debt. We can also write (32) as an ARMA(1,1):

$$s_{t+1} = a(L)u_{s,t+1} = \frac{\eta}{\eta + \alpha} \left(\frac{1 - \frac{1}{\eta}L}{1 - \frac{1}{\eta+\alpha}L} \right) u_{s,t+1}. \quad (33)$$

The AR(1) shock $u_{s,t+1} = (1 - \rho_s L)^{-1} \varepsilon_{s,t+1}$ adds additional dynamics, producing an extended period of deficits followed by surpluses.

In (32) and (33), we see that we can also regard the pair (27) and (28) as a way to write compactly and intuitively a surplus process with an s-shaped moving average representation, in which a government running deficits promises future surpluses in order to borrow and not just

devalue outstanding debts. In this interpretation, the variable v_t^* is just a latent state variable without further economic meaning. Surpluses do not “react to” v_t^* per se, they react to the past deficits that v_t^* summarizes. We could just write the moving average (32) or (33) as the surplus process, with no v^* . Introducing v_t^* writes an otherwise complex and potentially non-invertible MA process in a VAR(1) form, which is convenient for the usual VAR(1)-based matrix model solution technique.

The total response of the surplus $\sum_{j=0}^{\infty} \rho^j \Delta E_1 s_{1+j}$ to a unit transitory shock $u_{s,1} = 1$, $u_{s,2} = u_{s,3} = \dots = 0$ (i.e. with $\rho_s = 0$) is given by $a(\rho) = \sum_{j=0}^{\infty} \rho^j a_j$. (Hansen and Sargent (1980).) With persistent disturbances $\rho_s > 0$ the response is $a(\rho)/(1 - \rho_s \rho)$. From (32) or (33), we have

$$a(\rho) = \frac{\eta - \rho}{\eta - \rho + \alpha}.$$

For $\eta = \rho$, then, we have $a(\rho) = 0$ all debts are repaid, and there are no inflationary surplus shocks, $\Delta E_{t+1} \sum_{j=0}^{\infty} \rho^j s_{t+1+j} = 0$, in response to any (even persistent) shock $u_{s,t+1}$. Raising η , and thereby allowing $a(\rho) > 0$ lets us model a surplus process in which the government promises to pay back only $1 - a(\rho)$ part of the deficit, and let unexpected inflation soak up the rest $a(\rho)$ by devaluing outstanding bonds. As $\eta \rightarrow \infty$, $v_t^* \rightarrow 0$, and $a(\rho) \rightarrow 1$. In this limit, we recover the usual sort of AR(1) surplus process, in which all of a surplus shock is financed by inflating away current debt.

In this schema, $a(\rho)$ between 0 and 1 offers us a model in which government meets a fiscal shock in part by borrowing, promising future real surpluses, and raising the value of debt, and in part by inflating away outstanding debt. One can see many avenues for generalization, in particular a time- or state-dependent split between inflation and future surpluses.

Next, examine the moving averages (31), (32). The condition for the moving average representation of v^* to converge, in (31), is

$$\frac{1}{\alpha + \eta} < 1. \tag{34}$$

Since $\eta > \rho$, the usual condition $\alpha > 1 - \rho$, roughly that the government pays at least the interest on accumulated debt, is sufficient. Typically, the coefficient $1/(\eta + \alpha)$ is a number close to but below one, so the tail of moving average coefficients in (32) is very long.

The numerator coefficient of the surplus moving average polynomial (33) is $1/\eta$. Thus for values of η equal to or just above ρ , the coefficient $1/\eta$ can equal or exceed one. In this case the moving average representations (32) and (33) are not invertible. The moving average representation is always non-invertible when the government pays all its debts, $a(\rho) = 0$. There is nothing

theoretically wrong with a non-invertible moving average forcing process. But a non-invertible representation means that we cannot measure the s_t process from data on the history of s_t .

In sum them, viewed either in the form (32) or (33), this surplus process is of a type common, but all too frequently mistreated, in macroeconomics and finance, especially when we are interested in its long-run properties. In the form (32), we see a big movement in one direction, followed by a long string of offsetting movements in other direction. In (33) we see the classic ARMA(1,1) with nearly-canceling and near-unit coefficients. The subtleties of measuring the surplus response of this model is good news. They mean that we need not be too quickly discouraged by empirical work on surpluses that does not treat these subtleties with care.

3 Puzzles resolved

This section documents the counterfactual predictions of an AR(1) or similar surplus process, as outlined in the introduction, and how the s-shaped surplus process described in the last section resolves them. Together they stress how important it is to include an s-shaped surplus process in a fiscal-monetary model, to avoid all of these counterfactual predictions. They also offer multifaceted evidence, beyond direct estimates, that the surplus process does have an s-shaped moving average. For simplicity, I explore these issues using the case of one-period debt and a constant expected return.

Write the surplus process in moving-average representation

$$s_t = a(L)\varepsilon_t = \sum_{j=0}^{\infty} a_j \varepsilon_{s,t-j}.$$

An AR(1) is the special case

$$s_t = \frac{1}{1 - \rho_s L} \varepsilon_t = \sum_{j=0}^{\infty} \rho_s^j \varepsilon_{s,t-j}.$$

3.1 Estimates

Table 1 presents three vector autoregressions involving surpluses and debt. Here, v_t is the log market value of US federal debt divided by consumption, scaled by the consumption/GDP ratio. I divide by consumption to focus on variation in the debt rather than cyclical variation in GDP. π is the log GDP deflator, g_t is log consumption growth, r_t^n is the nominal return on the government bond portfolio, i_t is the three month treasury bill rate and y_t is the 10 year government bond

yield. I infer the surplus s from the linearized identity, allowing growth,

$$\rho v_{t+1} = v_t + r_{t+1}^n - \pi_{t+1} - g_{t+1} - s_{t+1}.$$

Cochrane (2019) describes the data and VAR in more detail.

	s_t	v_t	π_t	g_t	r_t^n	i_t	y_t	$\sigma(\varepsilon)$	$\sigma(s)$
VAR $s_{t+1} =$	0.35	0.043	-0.25	1.37	-0.32	0.50	-0.04	4.75	6.60
std. err.	(0.09)	(0.022)	(0.31)	(0.45)	(0.16)	(0.46)	(0.58)		
$v_{t+1} =$	-0.24	0.98	-0.29	-2.00	0.28	-0.72	1.60		
std. err.	(0.12)	(0.03)	(0.43)	(0.61)	(0.27)	(0.85)	(1.04)		
Small VAR $s_{t+1} =$	0.55	0.027						5.46	6.60
std. err.	(0.07)	(0.016)							
$v_{t+1} =$	-0.54	0.96							
std. err.	(0.11)	(0.02)							
AR(1) $s_{t+1} =$	0.55							5.55	6.60
std. err.	(0.07)								

Table 1: Surplus forecasting regressions. Variables are s = surplus, v = debt/GDP, π = inflation, g = growth, i = 3 month rate, y = 10 year yield. Sample 1947-2018.

The first group of regressions in Table 1 presents the surplus and value regressions in a larger VAR. The surplus is moderately persistent (0.35). Most importantly, the surplus responds to the value of the debt (0.043). This coefficient is measured with a t statistic of barely 2, with simple OLS standard errors. However, this point estimate confirms estimates such as Bohn (1998). The point here is that the point estimate for the remaining calculations conforms to the facts as viewed by the literature, so we can see explicitly the consequences of that estimate. Debt is very persistent (0.98), and higher surpluses pay down debt (-0.24).

The second group of estimates presents a smaller VAR consisting of only surplus and value. The coefficients are similar to those of surplus and value in the larger VAR, and we will see that this smaller VAR contains most of the message of the larger VAR.

The third estimate is a simple AR(1). Though the small VAR and AR(1) have the same own coefficient 0.55, we will see how they differ crucially on long-run properties.

Figure 3 presents responses of these VARs to a 1% deficit shock at time 0. Here I allow all variables to move contemporaneously to the deficit shock. The central point shows up right away: *The VAR shows an s-shaped surplus moving average.* The initial 1.0% deficit is followed by two more periods of deficit, for a cumulative 1.75% deficit. But then the surplus response turns positive. The many small positive surpluses chip away at the debt, until the sum of surpluses in

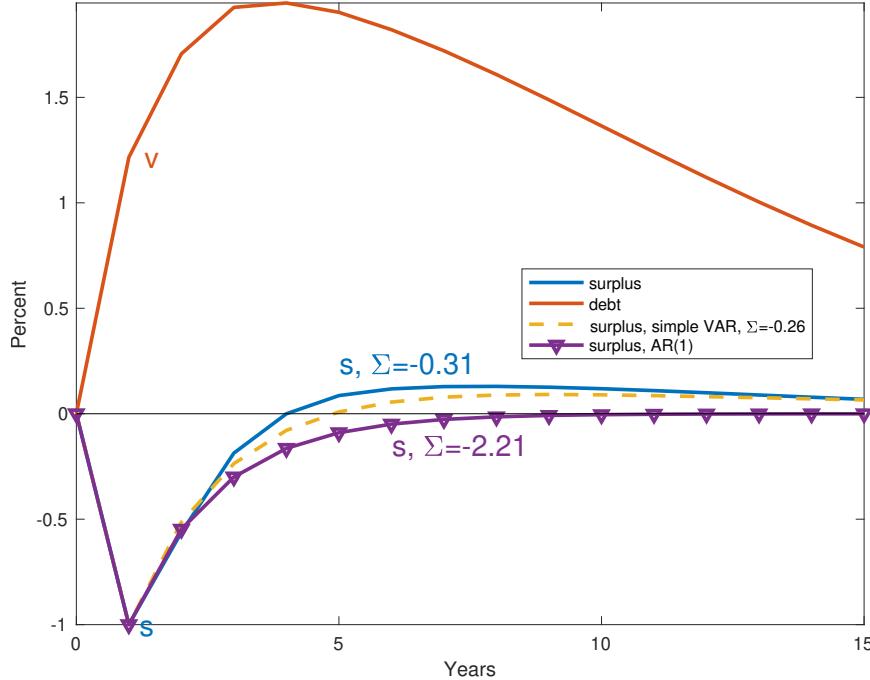


Figure 3: Responses to 1% deficit shocks.

response to the shock is only $-a(1) = \sum_{j=0}^{\infty} s_{1+j} = -0.31\%$. The $a(1)$ is not equal to zero, but it is not one or greater than one either.

Mechanically, the surplus response function comes from the surplus response to the value of debt. The value of debt jumps up when surplus jumps down. Shocks to the surplus and value of debt are strongly negatively correlated, itself below a piece of evidence for an s-shaped response. When the government runs a deficit, the value of debt rises. Surpluses respond to the greater value of debt, and slowly bring down the value of debt. Thus, the s-shaped surplus response estimate may not be statistically strong here, but it is robust and intuitive, as the ingredients come from the negative sign of the regression of surplus on debt, the persistent debt response, and the pattern that higher surpluses bring down the value of debt.

The simple VAR shows almost exactly the same surplus response as the full VAR, emphasizing how the response comes just from the intuitive features of that VAR. The point estimate of the sum of coefficients in the simple VAR is smaller, $-a(1) = -0.26$. The simple VAR surplus response crosses that of the full VAR and continues to be larger past the right end of the graph.

The simple AR(1) response looks almost the same – but it does not rise above zero. It would be very hard to tell univariate and VAR surplus responses apart based on autocorrelations or short-run forecasting ability emphasized in statistical tests. But the long-run implications are

dramatically different. For the AR(1), we have $-a(1) = -2.21$. Where a simpleminded constant discount rate model, fed the VAR process, predicts 0.26%-0.31% inflation in response to a 1% fiscal shock, the AR(1) predicts 2.28% inflation, and the same increase in bond return volatility.

	$\sum_{j=0}^{\infty} s_{1+j}$	$+\sum_{j=0}^{\infty} \omega^j \pi_{1+j}$	$+\sum_{j=0}^{\infty} g_{1+j}$	$-\sum_{j=1}^{\infty} (1 - \omega^j)(r_{1+j}^n - \pi_{1+j}) = 0$
VAR	-0.31	-0.04	-0.25	-(-0.60) = 0
Small VAR	-0.26			
AR(1)	-2.21			

Table 2: Decomposition of debt repayment, in response to a 1% deficit shock.

Table 2 gives a more detailed look. We can read the unexpected inflation identity (12), generalized to include growth, and expressed as

$$\Delta E_1 \sum_{j=0}^{\infty} s_{1+j} + \Delta E_1 \sum_{j=0}^{\infty} \omega^j \pi_{1+j} + \Delta E_1 \sum_{j=0}^{\infty} g_{1+j} - \Delta E_1 \sum_{j=1}^{\infty} (1 - \omega^j)(r_{1+j}^n - \pi_{1+j}) = 0$$

as a decomposition of how the debt/GDP ratio is restored after a deficit shock. Its content is $\Delta E_1 \lim_{T \rightarrow \infty} v_{1+T} = 0$, so it expresses how that happens. It says, when there is a surprise deficit $\Delta E_1 s_1$, debt/GDP eventually returns to where it was 1) by running future surpluses 2) by inflating away the value of outstanding bonds, so debt/GDP doesn't rise in the first place, 3) by high GDP growth and/or 4) by low real returns on government bonds.

Which is it? According to Table 2 the initial 1% deficit, or 1.75% cumulative deficit, is paid by future surpluses to the point that only 0.31% increased debt/GDP remains. Unexpected inflation, devaluation of outstanding government bonds, accounts for essentially nothing. Growth makes matters worse. In this estimate, deficits occur with lower growth, so growth raises the debt/GDP ratio further. Lower bond returns come to the rescue, with 0.60% of the increased debt/GDP ratio resolved by lower real returns on government debt.

These results depend on the shock. Cochrane (2019) studies responses to inflation shocks, for example. Inflation is more important there! The large return effect here is suspiciously fragile. It comes from the implied regression coefficients of real returns on surpluses and value of debt, and most of the negative returns occur at very long horizons as the latter slowly settle down.

For this purpose, however, I stop with the surplus shock. The response function is s-shaped, and $a(1)$ is substantially below one. Responses to alternative shocks give different allocations between inflation, growth, and returns, but do not much change the s-shaped surplus response.

3.2 The correlation of inflation and surpluses

With one-period debt and a constant expected return, the unexpected inflation identities (8) and (12) now give us

$$\Delta E_{t+1} \pi_{t+1} = -\Delta E_{t+1} \sum_{j=0}^{\infty} \rho^j s_{t+1+j} = -\Delta E_{t+1} \sum_{j=0}^{\infty} \rho^j \rho_s^j s_{t+1} = -a(\rho) \varepsilon_{s,t+1} = -\frac{1}{1-\rho\rho_s} \varepsilon_{s,t+1}. \quad (35)$$

With $a(\rho) > 0$, shocks to inflation are negatively correlated with shocks to surpluses.

This point may be even clearer in the nonlinearized version of the government debt valuation equation. With

$$\frac{B_t}{P_{t+1}} = E_{t+1} \sum_{j=0}^{\infty} \beta^j s_{t+1+j} \quad (36)$$

we have

$$\frac{B_t}{P_t} \Delta E_{t+1} \left[\frac{P_t}{P_{t+1}} \right] = \Delta E_{t+1} \sum_{j=0}^{\infty} \beta^j s_{t+1+j} = a(\beta) \varepsilon_{s,t+1} = \frac{1}{1-\beta\rho_s} \varepsilon_{s,t+1}.$$

This prediction is exactly the opposite of the sign we typically see in the data – inflation often comes in booms with surpluses, and disinflation in recessions with deficits. More generally, there is little correlation between inflation and deficits, across time or countries.

By contrast, consider the important case $a(\rho) = 0$, in which case debts are fully repaid. Now a string of positive short-run a_j , representing the fact that surpluses and deficits are persistent, is met by a long string of negative a_j in the long run as illustrated in Figure 3. These coefficients need not be individually large. Such an s-shaped moving average representation removes the prediction that inflation must come with deficits. Today's deficit need have nothing to do with the discounted sum of future deficits. Adding other shocks, an $a(\rho) > 0$ but small as in Figure 3 can still remove the prediction of a strong correlation between deficits and inflation, where $a(1) > 1$ for the AR(1) dominates other sources of variation. (One could go the opposite direction with $a(\rho) < 0$ to generate the opposite prediction, but in fact discount rates account for the negative correlation of surpluses with inflation.)

A correlation of deficits with inflation is possible. The surplus process does not have to be s-shaped. Extreme inflations correlate with deficits, and some cross-country experience lines up inflation and devaluation with deficits. For example, in Jiang (2019b), Jiang (2019a), an AR(1) surplus assumption seems to work. But a correlation of deficits with inflation is not a necessary prediction of the theory.

3.3 Surpluses and the change in the value of the debt.

How do surpluses affect the value of debt? With an AR(1) surplus and constant expected return $i_t = E_t \pi_{t+1}$, the value of debt is

$$v_t = E_t \sum_{j=1}^{\infty} \rho^{j-1} E_t s_{t+j} = \frac{\rho_s}{1 - \rho \rho_s} s_t.$$

A higher surplus $s_{t+1} > s_t$ raises the value of debt $v_{t+1} > v_t$. That's natural if you think of surpluses as dividends. A higher dividend typically raises the subsequent stock market value, since it forecasts higher subsequent dividends. But it is a remarkable prediction for government debt, where we usually presume that deficits are financed by borrowing, which raises the value of debt, and surpluses pay down that debt. And indeed the data scream the latter pattern, as Canzoneri, Cumby, and Diba (2001) document. But if today's deficit predicts higher subsequent surpluses - if the moving average is s-shaped – then the prediction is reversed.

We can state and analyze the point more simply and generally in innovation form. Take innovations of the flow identity (6) to write

$$\rho \Delta E_{t+1} v_{t+1} = \Delta E_{t+1} v_t - \Delta E_{t+1} \pi_{t+1} - \Delta E_{t+1} s_{t+1} \quad (37)$$

$$= 0 + [a(\rho) - 1] \varepsilon_{s,t+1} \quad (38)$$

If $a(\rho) > 1$, as is the case with an AR(1), then a surprise surplus at time $t + 1$ implies higher future surpluses, and raises the value of debt.

Does that fact mean that we cannot think of debt as the present value of surpluses, or that fiscal policy must be passive? No. It means that the surplus process has an s-shaped moving average. If $a(\rho) = 0$, then a surprise surplus at time $t + 1$ implies an exactly offsetting decline in future surpluses, so a rise in surplus lowers the value of the debt one for one. This is what happens to you when you take out a mortgage, or to a government that repays its debts. The s-shaped surplus moving average solves the value of debt puzzle.

For a deeper insight, review the debt accumulation equation itself, (6),

$$\rho v_{t+1} = v_t + i_t - \pi_{t+1} - s_{t+1}.$$

This equation seems to state already that a higher s_{t+1} lowers the value v_{t+1} . How does the AR(1) example reverse that prediction? Because with $a(\rho) > 1$, it states that inflation π_{t+1} moves at the same time, in the opposite direction (more surplus, less inflation) and by a greater quantity. But

in the case $a(\rho) = 0$, inflation is completely unaffected by the surplus shock and the conventional reading of the equation applies.

This point gives us insight into the interesting intermediate case, which is captured by the surplus model with $\eta > \rho$. If $a(\rho)$ is a small but positive number, still less than one, then a deficit shock corresponds to some inflation, and some increase in the value of debt, prompted by partial repayment.

3.4 Financing deficits - revenue or inflation

When the government runs a deficit, it has to get the resources from somewhere. Usually, we think that the government borrows to finance a deficit. Such borrowing results in a larger value of debt. And to borrow, the government must promise to repay, to run an s-shaped surplus. Equation (38) captures this intuition with $a(\rho) = 0$.

That story can't work for an AR(1), with $a(\rho) = 1/(1 - \rho\rho_s) > 1$, or other surplus process with positive moving average coefficients. So how does the government finance a deficit in this case? By inflation (or more generally, default).

Suppose the government decides at $t + 1$ to run an unexpected deficit. At the beginning of period $t + 1$, inflation π_{t+1} devalues the outstanding real debt that must be rolled over. In real terms this is equivalent to a partial default. For $a(\rho) > 1$, the government sells *less* debt v_t than previously planned. The inflation-induced devaluation is even larger than the deficit. For $a(\rho) = 1$, the inflation-induced devaluation is equal to the deficit, so the deficit is exactly financed by devaluation. And if $0 < a(\rho) < 1$, then the deficit is partially financed by inflation, and partially financed by borrowing.

Most deficits in US data are clearly financed by borrowing more debt, another piece of evidence that $a(\rho)$ if nonzero is a small number.

3.5 The risk and premium of government debt

The ex-post real return on government debt in this simple example (constant real rate, one-period debt) is

$$r_{t+1} = i_t - \pi_{t+1} = -\Delta E_{t+1}\pi_{t+1} = -a(\rho)\varepsilon_{s,t+1}.$$

An AR(1), with $a(\rho) = 1/(1 - \rho\rho_s)$, predicts that the standard deviation of inflation, and therefore of ex-post bond returns, is larger than that of surpluses. Surpluses are already volatile. From Table 1, the surplus has a 6.6% annual standard deviation, and the AR(1) leaves a 5.5% residual standard deviation. The Table 1 estimate implies a $5.5/(1-0.55) = 12.2\%$ standard de-

vation of unexpected inflation, and of real one-year treasury bill returns. In fact, unexpected inflation and real treasury bill returns are vastly less volatile. In the same data, unexpected inflation has a 1.1% standard deviation.

A smaller $a(\rho)$ solves that puzzle. With $a(\rho) = 0$, unexpected inflation in this simple model is always zero, and government bonds are risk free in real terms, for any volatility of surpluses themselves.

Surpluses are procyclical, falling in recessions at the same time as consumption falls and the stock market falls. A volatile, procyclical, positively autocorrelated surplus thus generates a high risk premium as well as high risk. (Jiang et al. (2019) quantify this puzzle.) But US government bonds have a very low expected return as well as low volatility.

The s-shaped surplus process solves the expected return puzzle as well, and the low average return of government bonds is one more piece of evidence for the s-shaped surplus process. Specifically, the present value expression

$$v_t = E_t \sum_{j=1}^{\infty} \rho^{j-1} s_{t+j}$$

and the observation that surpluses s_t are volatile and procyclical suggests that government bonds, a claim to surpluses, should have a stock-like expected return and variance. But with an s-shaped surplus response, government debt becomes like a security whose price rises every time its dividend declines, so even a volatile dividend stream has a steady return. From the debt accumulation equation (6) we can write the one-period real return,

$$r_{t+1} = i_t - \pi_{t+1} = \rho v_{t+1} - v_t + s_{t+1} = [a(\rho) - 1]\varepsilon_{s,t+1} + \varepsilon_{s,t+1}.$$

With $a(\rho) \geq 1$, the innovation in value v_{t+1} reinforces the surplus innovation, so that the rate of return is more volatile than surpluses. With $a(\rho) = 0$, however, a surprise deficit s_{t+1} is met by a rise in the value of debt v_{t+1} , as we have seen, so the overall rate of return is risk free.

In sum, an s-shaped surplus process solves the Jiang et al. (2019) puzzle, how a claim to a persistent, volatile, procyclical surplus can have a steady and low return. In turn, the low and quiet returns of government bonds are another piece of evidence for an s-shaped surplus process.

3.6 The conundrum of the standard specification

We can understand specification (27) into (28) more deeply by comparing it with the corresponding standard specification of fiscal policy. Write instead of (27)-(30),

$$s_{t+1} = \gamma v_{t+1} + u_{s,t+1} \quad (39)$$

$$\rho v_{t+1} = v_t - \Delta E_{t+1} \pi_{t+1} - s_{t+1} \quad (40)$$

$$u_{s,t+1} = \rho_s u_{s,t} + \varepsilon - s, t + 1. \quad (41)$$

Now surpluses respond to the value of the debt v_t , making no distinction between a rise in the value of debt coming from unexpected deflation and a rise coming from past deficits.

In this setup, any $\gamma > 0$ results in a passive fiscal policy (Canzoneri, Cumby, and Diba (2001)). To see this fact and what it means, substitute (39) into (40),

$$(\rho + \gamma) v_{t+1} = v_t - \Delta E_{t+1} \pi_{t+1} - u_{s,t+1}$$

and iterating forward

$$E_{t+1} v_{t+T} = \frac{1}{(\rho + \gamma)^T} (v_t - \Delta E_{t+1} \pi_{t+1}) - \sum_{j=0}^{T-1} \frac{1}{(\rho + \gamma)^{T-j}} E_{t+1} u_{s,t+1+j}.$$

The last term converges, since $u_{s,t}$ is stationary. Thus if $\gamma > 0$, the transversality condition $\lim_{T \rightarrow \infty} \rho^T E_{t+1} v_{t+T} = 0$ holds for any unexpected inflation $\Delta E_{t+1} \pi_{t+1}$. This is “passive” policy: The government debt valuation equation no longer determines unexpected inflation. Loosely, if you pay back any small fraction of outstanding debt, then debt grows at less than the interest rate. We usually impose a slightly more stringent condition that debt itself is stationary, or “local” equilibria. Now if $\gamma > 1 - \rho$, $\lim_{T \rightarrow \infty} E_{t+1} v_{t+T} = 0$. If you pay the interest, plus any amount, then debts eventually vanish.

So, to obtain active fiscal policy, the ability to use the government debt valuation equation to determine unexpected inflation, we set $\gamma = 0$. But if specify $\gamma > 0$, together with a surplus disturbance process $u_{s,t}$ that has a positive moving average representation like that of an AR(1), then the government never pays back *any* debts, including those accumulated from past deficits. We obtain the above strikingly counterfactual predictions, in particular that current deficits do not result in larger values of the debt.

Thus, using fiscal policy of the standard form (39)-(41), we are faced with an unpleasant choice: either abandon active fiscal policy, or sign on to a model that, at least in this simple

specification, starts out with dramatically counterfactual predictions, and also lacks the commonsense mechanism that some deficits are paid for by borrowing real resources, and promising future surpluses to pay back the debt.

The answer, of course, is that surpluses follow a process with an s-shaped moving average, not an AR(1). Then, we preserve active policy: (40) explodes forward at the rate ρ^{-1} , so only one value of unexpected inflation is an equilibrium.

4 Model

Now we are ready to digest the results of the main model. The model consists of

$$x_t = E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1}) \quad (42)$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t \quad (43)$$

$$i_t = \theta_{i\pi} \pi_t + \theta_{ix} x_t + u_{i,t} \quad (44)$$

$$s_t = \theta_{s\pi} \pi_t + \theta_{sx} x_t + \alpha v_t^* + u_{s,t} \quad (45)$$

$$\eta v_{t+1}^* = v_t^* + i_t - E_t \pi_{t+1} - s_{t+1} \quad (46)$$

$$\rho v_{t+1} = v_t + r_{t+1}^n - \pi_{t+1} - s_{t+1} \quad (47)$$

$$E_t r_{t+1}^n = i_t \quad (48)$$

$$r_{t+1}^n = \omega q_{t+1} - q_t \quad (49)$$

$$u_{i,t+1} = \rho_i u_{i,t} + \varepsilon_{i,t+1} \quad (50)$$

$$u_{s,t+1} = \rho_s u_{s,t} + \varepsilon_{s,t+1}. \quad (51)$$

These are the same equations as the standard new-Keynesian model, with a set of implicit fiscal equations spelled out. I parameterize and solve the model with active fiscal rather than active monetary policy. The resulting model can stand as a benchmark fiscal theory of monetary policy, parallel to the standard three-equation new-Keynesian model.

Equations (42) and (43) are the standard intertemporal IS and Phillips equations, with x_t denoting the output gap, i_t the nominal interest rate and π_t inflation. One can add disturbances to both equations. I leave disturbances out here as I only present responses to monetary and fiscal policy shocks below.

Equations (44) and (45) are monetary and fiscal policy rules, generalizing (21) and (22) to include output responses. Surpluses are likely to respond to output and inflation for both mechanical and policy reasons. Tax receipts are naturally procyclical as tax rate times income rises

with income. Spending is naturally countercyclical, due to entitlements such as unemployment insurance and deliberate stimulus programs. Imperfect indexation makes surpluses rise with inflation. Beyond fitting data, we are interested in fiscal policy rules that can stabilize inflation or avoid deflation, especially in a period of zero bounds or other constraints on monetary policy.

Following the important insight in Werning (2012), we never need to specify policy rules. We can always just model the path of interest rates and surplus that emerge in equilibrium, and are our only observables, as if they were a time-varying peg and fixed exogenous surplus. We will find the same inflation and output responses as occur if those interest rate and surplus paths are the result of rules and shocks. This observation also provides another deep non-identification (without more assumptions) theorem about policy rules, since multiple rules and shocks give rise to the same interest rate and surplus paths. Still, it is traditional in these models to think in terms of rules, and it is interesting in policy analysis to separate rules from disturbances. I continue that tradition here, following the same interests, and also to vary as little as possible other than the fiscal specification of the model, so the effects of the latter are clearest.

Equations (45)-(47) also generalize the simple fiscal policy specification (27)-(30) to include long-term debt and real interest rate variation. In (46), the state variable v^* responds to real interest rates. In (47), which is the debt accumulation identity (6), ex-post bond returns r_{t+1}^n raise or lower the value of debt. As detailed above, the state variable v^* allows us to conveniently write an s-shaped moving average in VAR(1) form, and it allows us to specify that surpluses respond to past deficits but not to changes in the value of debt that stem from arbitrary unanticipated inflation and deflation. One can generalize (46) to include a specific value of unexpected inflation, which becomes an additional interesting policy parameter. As above, the parameter $\eta > \rho$ in (46) allows for a s-shaped surplus process that does not fully repay debts. As $\eta \rightarrow \infty$, $v_{t+1}^* \rightarrow 0$, and we recover the policy rule plus AR(1) specification common, so far, in similar models.

Equation (48) is the bond pricing equation, which imposes the expectations hypothesis that expected returns on bonds of all maturities are the same. This simple model includes real interest rate variation but not risk premium variation.

Equation (49) relates the return on the government bond portfolio to its price q_t . Here I specify that nominal government debt has a geometric maturity structure with face value at maturity j that declines at the rate ω^j . This linearization is also derived in Cochrane (2019).

The IS and Phillips curves (42)-(43) leave two undetermined expectational errors, needing two forward-looking roots to give a unique equilibrium. As usual, they have one forward and one backward-looking root, so we need one extra forward-looking root. In active-money new-

Keynesian models $\theta_{\pi i} > 1$ (roughly speaking) generates the additional explosive root. I specify passive monetary policy with $\theta_{\pi i} < 1$.

With short-term debt, $\omega = 0$, we would have $r_{t+1}^n = i_t$, and the fiscal block (45)-(47) would provide the extra explosive or unit root. Long-term debt adds another expectational error, (48), but one more unstable root in (49). Together (48)-(49) solve forward to

$$q_t = -E_t \sum_{j=1}^{\infty} \omega^{j-1} i_{t+j-1},$$

the expectations hypothesis that long-term bond prices reflect an average of future short-term interest rates.

In section 2.2, we saw that $i_t = E_t \pi_{t+1}$ of that model gave the central bank, which sets the interest rate i_t but cannot affect fiscal policy, control over expected inflation. The bank retains a similar power in this model. Eliminating the output gap x_t from (42)-(43), we have

$$\beta E_t \pi_{t+2} - (1 + \beta - \sigma \kappa) E_t \pi_{t+1} + \pi_t = \sigma \kappa i_t. \quad (52)$$

We can write this equation that expected inflation $E_t \pi_{t+1}$ is a two-sided exponentially-weighted moving average of the interest rate i_t ,

$$\pi_{t+1} = \frac{\sigma \kappa}{\lambda_1 - \lambda_2} \left[i_t + \sum_{j=1}^{\infty} \lambda_1^{-j} i_{t-j} + \sum_{j=1}^{\infty} \lambda_2^j E_{t+1} i_{t+j} \right] + \sum_{j=0}^{\infty} \lambda_1^{-j} \Delta E_{t+1-j} \pi_{t+1-j} \quad (53)$$

with weights given by the roots of the lag polynomial (52) (Cochrane (2018) p. 165), plus an exponentially decaying transient response to shocks. This formula naturally generalizes the Fisher equation (13) $E_t \pi_{t+1} = i_t$ and hence $\pi_{t+1} = i_t + \Delta E_{t+1} \pi_{t+1}$. Therefore, monetary policy can still determine expected inflation. It just takes a more complex interest rate path to give any particular expected inflation path. The last term of (53) is an exponentially decaying transient. This term introduces persistent dynamics to responses that only last one period under flexible prices.

5 Responses

The Appendix documents the algebra for numerically solving the model in the standard way. I set the model up in VAR(1) form $y_{t+1} = Ay_t + B\varepsilon_{t+1} + C\delta_{t+1}$ with ε denoting structural shocks and δ denoting expectational errors, and then I solve unstable eigenvalues of A forward and stable eigenvalues backward.

I present four sets of response functions. I start with responses to fiscal u_s and monetary u_i shocks with no policy responses $\theta = 0$. Then I add the policy responses. The first set of calculations is less realistic, but it helps to understand the model mechanisms – what responses are due to the economics of the model, rather than to endogenous policy reactions. The second set of calculations then lets us see how systematic policy rules modify the effects of fiscal and monetary policy shocks.

Throughout I use parameters $\rho = 1$, $a(\rho) = 0.2$, $\sigma = 0.5$, $\kappa = 0.5$, $\alpha = 0.2$, and hence $\eta = \rho + \alpha(\rho)/(1 - \alpha(\rho)) = 1.05$. I pick the parameters to clearly illustrate mechanisms, not to match data. Parameters $\rho_i = 0.7$, $\rho_s = 0.4$ correspond to policy rule regressions presented in the Appendix.

5.1 Fiscal shocks, no policy rules

Figure 4 presents the responses of this model to a negative AR(1) fiscal policy disturbance $u_{s,t}$, in the case of no policy rules $\theta = 0$.

With neither monetary policy shock nor rule, the interest rate i and therefore long-term nominal bond return r^n do not move.

Inflation rises and decays with an AR(1) pattern. This is already important news: Fiscal shocks result in drawn-out inflation, not a one-period price-level jump. The drawn-out inflation here is entirely the effect of sticky prices. It reflects the last term of (53), the exponentially decaying response to a shock in a sticky-price model. As one reduces price stickiness, increasing κ , the inflation response approaches the one-period price level jump of the flexible price model. These responses are the same for any bond maturity ω . Since interest rates and bond returns do not move, the maturity structure of the debt cannot make a difference.

Inflation rises persistently, so why don't nominal long-term bond prices and the bond return r_1^n fall? The answer is that inflation and real rates exactly offset. The nominal interest rate does not move so the real rate falls exactly as inflation rises.

Output rises mirroring the path of inflation, following the forward-looking Phillips curve that output is high when inflation is declining. This deficit does stimulate, by provoking inflation.

The surplus s_t and the AR(1) surplus disturbance $u_{s,t}$ are not the same. The surplus initially declines, but deficits raise the v^* latent variable, which accumulates past deficits. A long string of small positive surplus responses on the right side of the graph then partially repays the incurred debt with an s-shaped response pattern.

This graph, like Figure 1, warns us of the empirical challenges ahead, and against many

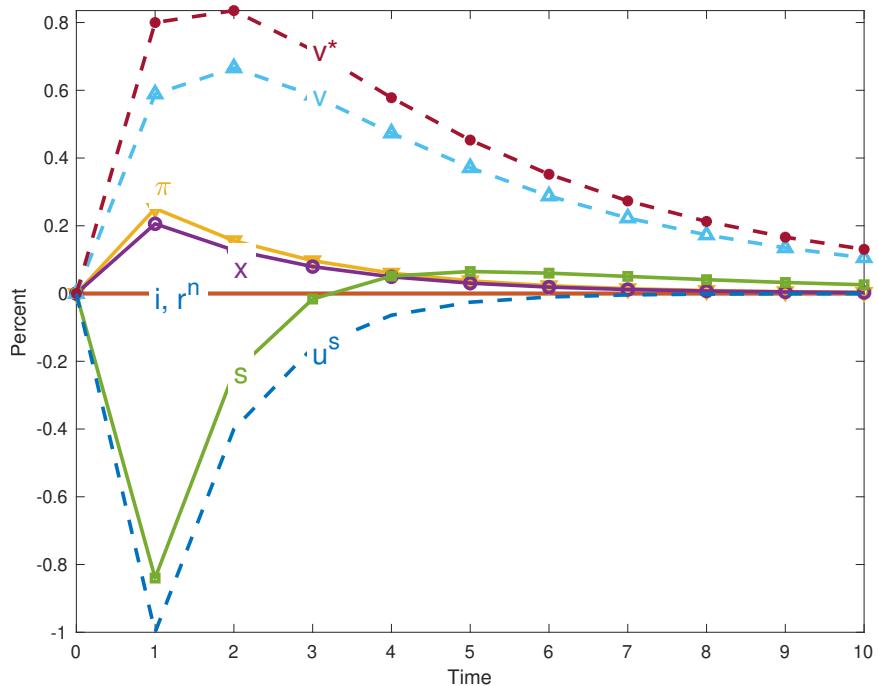


Figure 4: Responses of the sticky-price model to a fiscal shock with no policy rules.

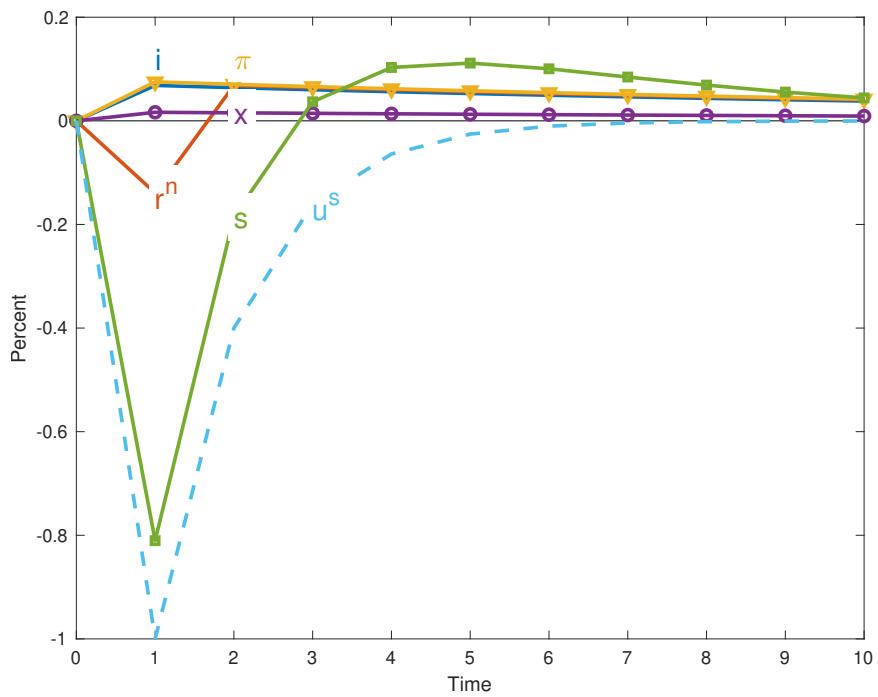


Figure 5: Responses of the sticky-price model to a fiscal shock, with policy rules.

apparently easy rejections of fiscal theory. It would be hard to distinguish the surplus s from the AR(1) disturbance u_s in the data, as they differ only in the long run and by a string of very small positive responses. Similarly, the value of debt v and state variable v^* follow much the same pattern. In a regression, the surplus would seem to respond to the value of debt v , indicating passive policy, though in reality it only responds to v^* and the model has active fiscal policy.

That inflation rises at all comes from the assumption $a(\rho) = 0.2$, not $a(\rho) = 0$. With $a(\rho) = 0$, the long run surplus response would be slightly higher, the sum of all future surpluses exactly zero, and there would be no inflation. With $a(\rho) \geq 1$, the long run surpluses would not rise at all above zero, and the deficit would be financed entirely by a much larger inflation. Thus $a(\rho) = 0.2$ models a deficit shock that is partially financed by devaluing outstanding bonds via inflation, though mostly financed by borrowing and repaying the resulting debt.

Shock and model	$\sum_{j=0}^{\infty} \omega^j \Delta E_1 \pi_{1+j}$	=	$-\sum_{j=0}^{\infty} \Delta E_1 s_{1+j} + \sum_{j=1}^{\infty} (1 - \omega^j) \Delta E_1 (r_{1+j}^n - \pi_{1+j})$
Fiscal, no θ rules	(0.44)	=	$-(-0.66) + (-0.22)$
Fiscal, yes θ rules	(0.22)	=	$-(-0.25) + (-0.03)$
Monetary, no θ rules	(-0.92)	=	$-(1.96) + (1.04)$
" , $\kappa = \infty$	(0)	=	$-(0) - (0)$
Monetary, yes θ rules	(-0.55)	=	$-(1.26) + (0.71)$
Shock and model	$\Delta E_1 \pi_1 - \Delta E_1 r_1^n$	=	$-\sum_{j=0}^{\infty} \Delta E_1 s_{1+j} + \sum_{j=1}^{\infty} \Delta E_1 (r_{1+j}^n - \pi_{1+j})$
Fiscal, no θ rules	(0.25)	$-(0.00)$	$= -(-0.66) + (-0.41)$
Fiscal, yes θ rules	(0.08)	$-(-0.14)$	$= -(-0.25) + (-0.03)$
Monetary, no θ rules	(-0.88)	$-(-1.37)$	$= -(1.96) + (2.45)$
" , $\kappa = \infty$	(-1.37)	$-(-1.37)$	$= -(0) + (0)$
Monetary, yes θ rules	(-0.58)	$-(-0.65)$	$= -(1.26) + (1.33)$
Shock and model	$\Delta E_1 r_1^n$	=	$-\sum_{j=1}^{\infty} \omega^j \Delta E_1 \pi_{1+j} - \sum_{j=1}^{\infty} \omega^j \Delta E_1 (r_{1+j}^n - \pi_{1+j})$
Fiscal, no θ rules	(0)	=	$-(0.19) - (-0.19)$
Fiscal, yes θ rules	(-0.14)	=	$-(0.14) - (0.004)$
Monetary, no θ rules	(-1.37)	=	$-(-0.04) - (1.41)$
" , $\kappa = \infty$	(-1.37)	=	$-(1.37) - (0)$
Monetary, yes θ rules	(-0.65)	=	$-(0.03) - (0.62)$

Table 3: Terms of the inflation decomposition and bond-return decomposition for sticky-price model response functions.

The “Fiscal, no θ rules” row of Table 3 presents the terms of the unexpected inflation decompositions for these responses. In the top panel of Table 3, the fiscal shock produces 0.44% total ω -weighted inflation, spread through time. That inflation comes from a 0.66% cumulative

fiscal shock offset by a 0.22% decline in discount rates.

We see two mechanisms that buffer fiscal shocks. First, the cumulative fiscal disturbance $\Delta E_1 \sum_{j=0}^{\infty} u_{s,j} = 1/(1 - \rho_s) = -1.67\%$. The endogenous response of surpluses to accumulated debt v^* , with the parameter $a(\rho) = 0.2$, pays off all but $\Delta E_1 \sum_{j=0}^{\infty} s_j = -0.66\%$ of this shock.

Second, higher inflation with sticky prices means a lower real interest rate, which raises the value of debt, a deflationary force. This discount rate effect offsets another 0.22% of the fiscal inflation. Sticky prices, by providing an endogenous real interest rate reaction, buffer the inflationary effects of fiscal shocks.

The second and third panels give the decomposition of one-period unexpected inflation, in which bond returns can soak up some changes in the present value of surpluses. The 0.25% inflation shock corresponds to the 0.66% decline in the sum of future surpluses. The endogenous discount rate effect (unweighted here) provides an 0.41% countervailing deflationary force. Ex-post bond returns do not move, as we saw in the Figure, and as the lower real discount rate (0.19) matches higher inflation (0.19) in the bond return decomposition shown of the last panel.

5.2 Response to a fiscal shock with policy rules

Now I add fiscal and monetary policy reaction functions. I use

$$i_t = 0.8 \pi_t + 0.5 x_t + u_{i,t} \quad (54)$$

$$s_t = 0.25 \pi_t + 1.0 x_t + 0.2 v_t^* + u_{s,t} \quad (55)$$

$$u_{i,t} = 0.7 u_{i,t-1} + \varepsilon_{i,t} \quad (56)$$

$$u_{s,t} = 0.4 u_{s,t-1} + \varepsilon_{s,t} \quad (57)$$

These parameters are also intended only as generally reasonable back of the envelope values that can illustrate mechanisms. Estimating policy rules is tricky, as the right hand variables are correlated with errors. The interest rate reaction to inflation $\theta_{i,\pi}$ is less than one, to easily generate a stationary passive-money model. I use a surplus response to output $\theta_{s,x} = 1.0$. The units of surplus are surplus/value of debt, or surplus/GDP divided by steady state value of debt/GDP, so one expects a value of about this magnitude. For example, real GDP fell 4 percentage points peak to trough in the 2008 recession, while the surplus/GDP ratio fell nearly 8 percentage points. Debt to GDP of 0.5 leads to a coefficient 1.0. Both a graph and an OLS regression in the Appendix suggest even larger values. Surpluses should react somewhat to inflation, as the tax code is less well indexed than spending. But it's hard to see that pattern in the data, as surpluses were low in the 1970s and an OLS regression, though surely biased, gives a negative coefficient. I use 0.25 to

explore what a small positive reaction can do.

Figure 5 presents the responses to a fiscal shock, holding constant the monetary policy disturbance $u_{i,t}$ but now allowing both fiscal and monetary policy rules to change surpluses and interest rates in response to endogenous inflation and output. Comparing the two figures, note the change in vertical scale.

The instantaneous inflation is less than half its previous value, but inflation is much more persistent. Endogenous policy responses smooth the inflationary effects of a shock, both directly and by smoothing inflation forward.

Monetary policy reacts to higher inflation and output by raising the nominal interest rate, which was constant before. Greater inflation and output also raise fiscal surpluses through the fiscal policy rule, offsetting more of the fiscal shock. Higher nominal interest rates also lower bond prices, which soak up some of the fiscal shock.

Table 3 quantifies these mechanisms. The weighted sum of inflation is cut in half from 0.44% to 0.22%. Instantaneous inflation is also cut by more than half, from 0.25% to 0.08%. The sum of future surpluses is -0.25%, not -0.66%. The offsetting discount rate terms are also much smaller, however. The policy rules here do most of their work by inducing greater surpluses. In the one-period accounting of the second and third panels, higher interest rates and inflation now reduce bond prices by 0.14%, soaking up half of the 0.25% surplus decline. This bond price decline comes entirely from future inflation. Monetary policy smooths an inflationary fiscal shock forward, producing a protracted inflation that slowly devalues long-term bonds rather than a sharp price level jump.

This example produces drawn-out inflation in response to a fiscal shock, not a price level jump. This is a suggestive story of the 1970s, but the model does not produce the lower output characteristic of stagflation. That failure is likely rooted in the simplistic nature of this Phillips curve.

5.3 Monetary shocks, no policy rules

Figure 6 presents the response of variables in this model to an AR(1) monetary policy shock $u_{i,t}$ with no policy rule response to endogenous variables $\theta = 0$. The nominal interest rate i_t just follows the AR(1) shock process $u_{i,t}$.

Inflation π declines, in contrast to the Fisherian model with one-period debt of section 2.2, and persistently, in contrast to the flexible-price long-term debt model of section 2.3. Sticky prices add persistence to the basic long-term debt mechanism outlined there.

Inflation does eventually rise to meet the higher nominal interest rate. This model re-

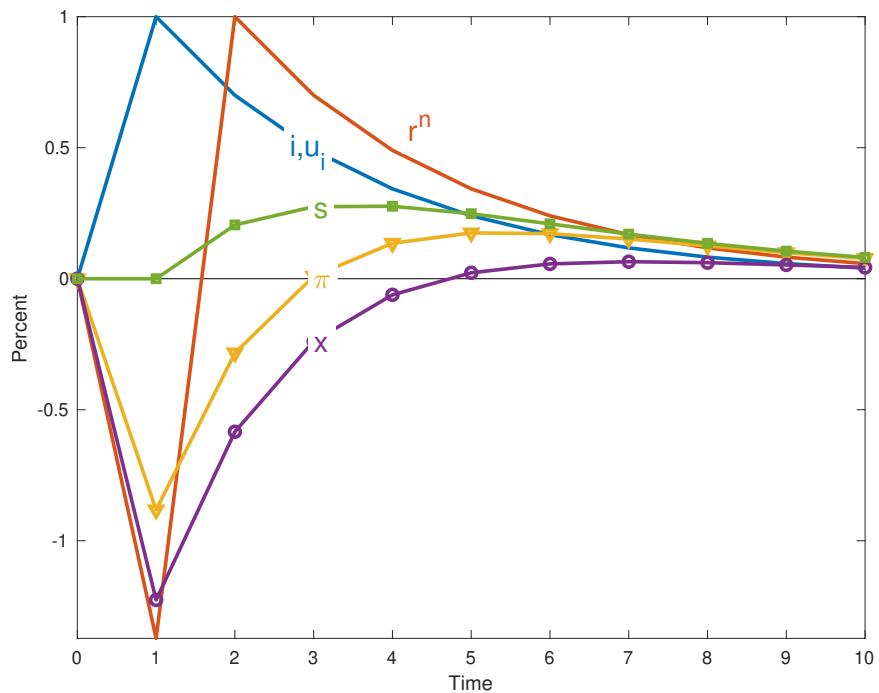


Figure 6: Responses of the sticky-price model to a monetary policy shock, with no policy rules.

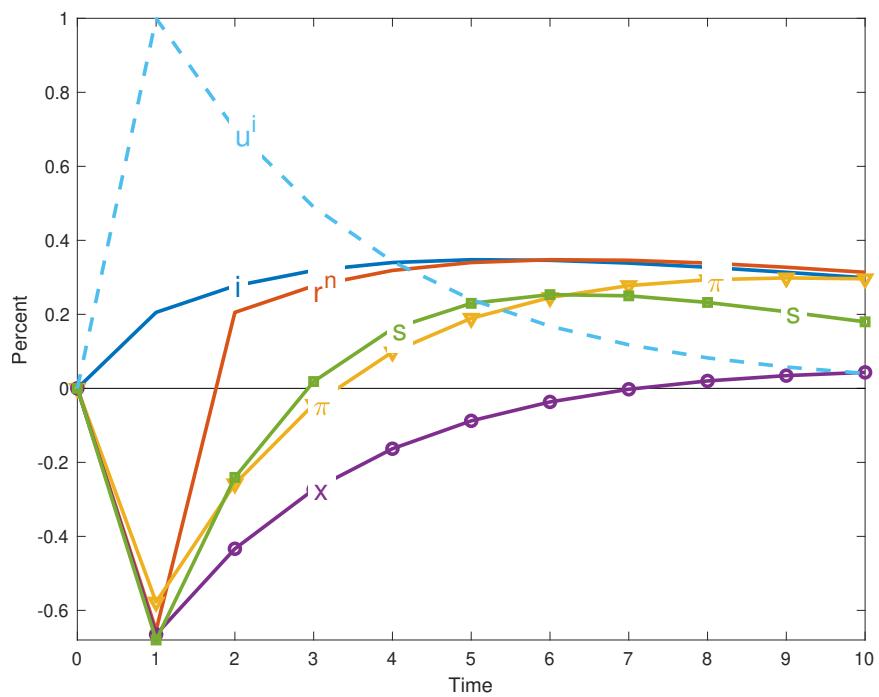


Figure 7: Responses of the sticky-price model to a monetary policy shock, with policy rules.

mains Fisherian in the long run, or to a permanent monetary policy shock. This is a robust prediction of rational expectations models. Such models are stable going forward, so $i = r + E\pi$ is a stable steady state, and persistently higher interest rates eventually attract inflation. But the rise in inflation is long delayed here, and would be hard to detect.

Output also declines, again following the new-Keynesian Phillips curve in which output declines when inflation is rising.

Expected nominal returns r_{j+1}^n follow the interest rate i_j , as this model uses the expectations hypothesis. That rise in expected returns and bond yields sends bond prices down, resulting in the sharply negative instantaneous bond return r_1^n . Subtracting inflation from these nominal bond returns, the expected real interest rate rate, expected real bond return, and discount rate rise persistently.

Surpluses are not constant. Here, I define a monetary policy shock that holds constant the fiscal policy *disturbance* $u_{s,t} = 0$, but not surpluses s_t themselves. Even though surpluses do not (yet) respond directly to inflation and output, surpluses respond to the increased value of the debt v^* that results from higher real returns on government bonds.

The terms of the unexpected inflation decompositions for these response functions are given by the “Monetary, no θ rules” rows of Table 3.

Start with the $\kappa = \infty$ flexible-price (but still long-term debt) row. In this case there is no overall inflation, surplus, or discount rate effect in the top panel. But there is a sharp decline in one-period inflation -1.37%, matched by a -1.37% decline in bond prices, in the middle panel. The bond price decline comes entirely from future expected inflation in the bottom panel. Absent price-stickiness, but in the presence of long-term debt, monetary policy $i_t = E_t\pi_{t+1}$ can *rearrange* the path of inflation, pushing the burden to long-term bondholders. This calculation reinforces the analysis of the flex-price long-term debt model in section 2.3.

In the model with price stickiness $\kappa < \infty$, we see a different pattern, as reflected in the Figure. In the top panel, the weighted sum of current and future inflation is -0.92%. This disinflation derives from a large 1.96% fiscal tightening, offset by a large 1.04% rise in discount rates. Both of these effects stem from higher real interest rates. The fiscal tightening comes from my assumption that surpluses rise following increases in the value of debt induced by higher real interest rates, providing a deflationary effect of higher real interest rates. Higher real interest rates have a direct inflationary effect, by raising the discount rate applied to future surpluses.

In the middle panel, we see the same 1.96% surplus shock, now mediated by a larger 2.45% unweighted discount rate effect. It results in -0.88% immediate inflation, and a 1.37% decline in bond prices. In the bottom panel, that decline in bond prices comes almost all from expected

future inflation. Here, by allowing persistent inflation, we see that monetary policy spreads more than half of the inflationary effect of the decline in present value of surpluses forward in time, and to lie on long-term bond holders.

In sum, the real interest rate effects of monetary policy with sticky prices deeply mediate the effects of monetary policy on inflation. Higher real interest rates discount surpluses at a higher rate, an inflationary force. Higher real interest rates lead here to higher surpluses, a deflationary force. Even in this simple example accounting for inflation involves multiple, and often countervailing fiscal forces.

5.4 Response to a monetary shock with policy rules

Figures 7 plots responses to the monetary policy shock, but now adding fiscal and monetary policy rules. Table 3 includes the inflation and bond return decompositions, in the rows marked “yes θ rules.”

In Figure 7 we see that the policy rule responses to lower inflation and growth push the interest rate i below its disturbance u_i . I held down the coefficient $\theta_{i\pi} = 0.8$, rather than a larger value, to keep the interest rate response from being negative, the opposite of the shock. Interest rates that go in the opposite direction from monetary policy shocks are a common feature in new-Keynesian models of this sort. (Cochrane (2018) p. 175 shows some examples.) The $\theta_s = 0.9$, $\theta_i = 0.9$ line of Figure 1 is an example here. But such responses are confusing, and my point here is to illustrate mechanisms. The interest rate then rises gradually, along with inflation, before settling down long past the right end shown in the figure. Long-term bonds again suffer a negative return on impact, and then follow interest rates with a one period lag, under the model’s assumption of an expectations hypothesis. The real rate, difference between interest rate and inflation, again rises persistently.

Comparing the case with and without policy rules, the surplus, responding to the output and inflation decline, now declines sharply on impact before recovering.

Output and inflation responses have broadly similar patterns, but about half the magnitude of the response, and somewhat more persistent dynamics. Again, these policy rules smooth and therefore help to buffer the inflation and output responses to shocks.

The “Monetary, yes θ rules” rows of Table 3 again quantify these offsetting effects on inflation. Inflation, weighted or not, is about half its value without policy rules. The overall surplus response is still positive though two thirds of its value without policy responses. The smoother interest rate path, lower than its disturbance $u_{s,t}$, implies a lower bond return shock $\Delta E_1 r_1^n$, and the discount rate effect is also less than half its previous value. The negative bond return, coming

from the largely positive response in nominal interest rates, now largely from real interest rate effects. The negative early and larger later inflation largely offset in their effects on bonds.

5.5 Shock definition and orthogonalization

Just how one defines and orthogonalizes monetary and fiscal policy is a crucial but subtle matter. Here I define a monetary policy shock that does not affect the fiscal shock $u_{s,t}$. But monetary policy nonetheless has fiscal consequences: The fiscal rule responds to output, (potentially) to inflation, and to real-interest-rate-induced rises in the value of debt. This is not passive fiscal policy in the traditional definition, since it does not respond to multiple-equilibrium unexpected-inflation induced variation in the value of the debt. But it is a likely fiscal response to a monetary policy shock.

Should an analysis of the effects of monetary policy include these systematic fiscal policy responses? I think yes. If one is advising Federal Reserve officials on the effects of monetary policy, they might have in mind the question, what if the Fed were to raise interest rates persistently $u_{i,t}$, but the Treasury took no unusual action? In answering that question the Fed officials would likely want one to include predictable endogenous fiscal responses, via output and inflation responses of the tax code and automatic stabilizers. They might even want predictable actions of fiscal authorities, as in stimulus programs, larger discretionary spending, or the removal of these in booms. But they might not want one to assume that fiscal authorities embark on a simultaneous deviation from standard practice, a $u_{s,t}$, which the “passive” fiscal assumption of new Keynesian models makes.

Perhaps not, however. Perhaps they would like us to keep fiscal surpluses constant in such calculations, so as not to think of “monetary policy” as having effects merely by manipulating fiscal authorities into austerity or largesse. An academic description of the effects of monetary policy might likewise want to turn off predictable fiscal reactions, again to describe the effects of monetary policy on the economy, not via manipulation of fiscal policy. It’s easy enough to calculate responses holding s constant, and even easier to estimate such responses.

There is no right and wrong in specifying policy questions, there is only interesting and uninteresting – and transparent vs. obscure. The issue is really just what do we – and the Federal Reserve – find an interesting question, and is the modeler clear on just what assumption has been made. The main point is that calculations of the effects of monetary policy must specify what parts of fiscal policy are held constant or allowed to move (an old, and frequently forgotten point), and that one can include such endogenous reactions or policy rules if it is interesting to do so.

Since fiscal shocks often have a more exogenous air about them, including the likely endogenous monetary policy response makes even more sense. Adding a likely Fed shock – a deliberate $u_{s,t+1}$ taken in response to the event causing a fiscal policy shock – might be sensible too. In the data, fiscal and monetary authorities likely respond with u disturbances to similar events not included in our model, like financial or political shocks. Again, since the issue is analyzing the effects of hypothetical policy actions and understanding the economy, not fitting data, we just need to be careful about what questions are interesting.

5.6 The way forward

This model is still simple and unrealistic. I advance it to show what *can* be done, and to build intuition for mechanisms that will appear in larger models, but clouded by the interaction of even more effects.

One hungers, of course, for a model that one can bring to data, estimate parameters, and formally match impulse-responses to structural and policy shocks. While everyone knows these IS and Phillips curves are wrong, a standard more successful alternative has not emerged. It is likely that the form of these curves that fits best will be different under a fiscal equilibrium than it is in a standard new-Keynesian model. One wishes, in the end, something like a Smets and Wouters (2007), or Christiano, Eichenbaum, and Evans (2005), adapted to fiscal theory as I adapted the textbook new-Keynesian model above. Eventually one wants a more ambitious model incorporating habits or other dynamic preferences and investment adjustment costs, heterogeneity, variation in risk premia, labor market and investment frictions, the latest in financial frictions, zero bounds, and so forth. The models surveyed in the literature section add many such ingredients, but (in my view) need the more realistic fiscal policy specification and long-term debt emphasized here. A major point of this paper is that one *can* construct such models, and quite easily from a technical standpoint. But finding the *right* model is not so easy, as that specification search has not been so easy for standard new-Keynesian models.

My monetary policy rule is simplistic, needing at least lags and a zero bound, plus matching policy rule regressions in data. Specifying and estimating the fiscal policy rule is a challenge of similar order, not yet started, and made even more challenging by the fact that any sensible rule, such as this one, has subtle but crucial long-run responses, or a latent state variable. On the other hand, much of the fiscal policy rule can be estimated from structural knowledge of the tax code, the nature of automatic stabilizers, and visible spending decisions such as stimulus programs in recessions, where the monetary policy rule consists only of modeling the human decisions of the Federal Reserve Board. Estimating the parameters θ of the fiscal policy may be

easier than running regressions with endlessly implausible instruments for right hand variables correlated with error terms.

Like the rest of the model, this surplus process can and should be generalized towards realism in many ways. In particular, the v^* process can respond to one particular value of unexpected inflation, rather than the strict zero-inflation target here. News about future surpluses and historical episodes are likely not well modeled by AR(1) shocks to the disturbance $u_{s,t}$. The choice to finance deficits by inflating existing debt vs. borrow against future surpluses is likely to change over time and in response to state variables as well.

There are many steps one should take. But each of these steps is also an unexplored opportunity.

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Online Appendix to “A Fiscal Theory of Monetary Policy with Partially Repaid Long-Term Debt”

A Simple model with policy responses

This section sets out the algebra to derive (25) and (26) from (20)-(24). Substituting out for i and s from (20)-(22), the equilibria are

$$E_t \pi_{t+1} = \theta_i \pi_t + u_{i,t} \quad (58)$$

$$\Delta E_{t+1} \pi_{t+1} = -\theta_s \sum_{j=0}^{\infty} \rho^j \Delta E_{t+1} \pi_{t+1+j} - \Delta E_{t+1} \sum_{j=0}^{\infty} \rho^j u_{s,t+1+j}. \quad (59)$$

Then we can evaluate the terms on the right hand side of (59). The first term:

$$\Delta E_{t+1} \pi_{t+1+1} = \theta_i \Delta E_{t+1} \pi_{t+1} + \varepsilon_{i,t+1}$$

$$\begin{aligned} \Delta E_{t+1} \pi_{t+1+2} &= \theta_i \Delta E_{t+1} \pi_{t+2} + \Delta E_{t+1} u_{i,t+2} \\ &= \theta_i^2 \Delta E_{t+1} \pi_{t+1} + (\theta_i + \rho_i) \varepsilon_{i,t+1} \end{aligned}$$

$$\begin{aligned} \Delta E_{t+1} \pi_{t+1+3} &= \theta_i \Delta E_{t+1} \pi_{t+3} + \Delta E_{t+1} u_{i,t+3} \\ &= \theta_i^3 \Delta E_{t+1} \pi_{t+1} + (\theta_i^2 + \rho_i \theta_i + \rho_i^2) \varepsilon_{i,t+1} \\ &= \theta_i^3 \Delta E_{t+1} \pi_{t+1} + \frac{\theta_i^3 - \rho_i^3}{\theta_i - \rho_i} \varepsilon_{i,t+1} \end{aligned}$$

$$\Delta E_{t+1} \pi_{t+1+j} = \theta_i^j \Delta E_{t+1} \pi_{t+1} + \frac{\theta_i^j - \rho_i^j}{\theta_i - \rho_i} \varepsilon_{i,t+1} \quad (60)$$

$$\begin{aligned} \sum_{j=1}^{\infty} \rho^j \Delta E_{t+1} \pi_{t+j+1} &= \sum_{j=1}^{\infty} \rho^j \left(\theta_i^j \Delta E_{t+1} \pi_{t+1} + \frac{\theta_i^j - \rho_i^j}{\theta_i - \rho_i} \varepsilon_{i,t+1} \right) \\ &= \frac{\rho \theta_i}{1 - \rho \theta_i} \Delta E_{t+1} \pi_{t+1} + \left(\sum_{j=1}^{\infty} \rho^j \frac{\theta_i^j - \rho_i^j}{\theta_i - \rho_i} \right) \varepsilon_{i,t+1} \\ &= \frac{\rho \theta_i}{1 - \rho \theta_i} \Delta E_{t+1} \pi_{t+1} + \frac{1}{\theta_i - \rho_i} \left(\frac{\rho \theta_i}{1 - \rho \theta_i} - \frac{\rho \rho_i}{1 - \rho \rho_i} \right) \varepsilon_{i,t+1} \\ &= \frac{\rho \theta_i}{1 - \rho \theta_i} \Delta E_{t+1} \pi_{t+1} + \frac{\rho}{(1 - \rho \theta_i)(1 - \rho \rho_i)} \varepsilon_{i,t+1} \end{aligned}$$

The second term:

$$\sum_{j=0}^{\infty} \rho^j \Delta E_{t+1} u_{s,t+j+1} = \sum_{j=0}^{\infty} \rho^j \rho_s^j \varepsilon_{s,t+1} = \frac{1}{1 - \rho \rho_s} \varepsilon_{s,t+1}.$$

Equation (59) then becomes

$$\Delta E_{t+1} \pi_{t+1} = -\theta_s \left(\left(1 + \frac{\rho \theta_i}{1 - \rho \theta_i} \right) \Delta E_{t+1} \pi_{t+1} + \frac{\rho}{(1 - \rho \theta_i)(1 - \rho \rho_i)} \varepsilon_{i,t+1} \right) - \frac{1}{1 - \rho \rho_s} \varepsilon_{s,t+1}$$

$$\Delta E_{t+1} \pi_{t+1} = -\frac{\rho \theta_s}{(1 - \rho \theta_i + \theta_s)(1 - \rho \rho_i)} \varepsilon_{i,t+1} - \frac{(1 - \rho \theta_i)}{(1 - \rho \theta_i + \theta_s)(1 - \rho \rho_s)} \varepsilon_{s,t+1}$$

From (60),

$$\begin{aligned} \Delta E_{t+1} \pi_{t+1+j} &= \theta_i^j \left[-\frac{\rho \theta_s}{(1 - \rho \theta_i + \theta_s)(1 - \rho \rho_i)} \varepsilon_{i,t+1} - \frac{(1 - \rho \theta_i)}{(1 - \rho \theta_i + \theta_s)(1 - \rho \rho_s)} \varepsilon_{s,t+1} \right] + \frac{\theta_i^j - \rho_i^j}{\theta_i - \rho_i} \varepsilon_{i,t+1} \\ \Delta E_{t+1} \pi_{t+1+j} &= \frac{1}{\theta_i - \rho_i} \left(\frac{(1 - \rho \theta_i)}{(1 - \rho \theta_i + \theta_s)} \frac{(1 - \rho \rho_i + \theta_s)}{(1 - \rho \rho_i)} \theta_i^j - \rho_i^j \right) \varepsilon_{i,t+1} - \frac{(1 - \rho \theta_i)}{(1 - \rho \theta_i + \theta_s)(1 - \rho \rho_s)} \theta_i^j \varepsilon_{s,t+1}. \end{aligned}$$

B Model solution algebra

This Appendix sets out the algebra to solve the model (42)-(51). I express the model in the form

$$Ay_{t+1} = By_t + C\varepsilon_{t+1} + D\delta_{t+1} \quad (61)$$

where y is a vector of variables, ε are the structural shocks, and δ are expectational errors in the equations that only tie down expectations. We eigenvalue decompose the transition matrix $A^{-1}B$, we solve unstable roots forward and stable roots backward to determine the expectational errors δ as a function of the structural shocks ε . Then, we can compute the impulse-response function. I use notation $\gamma \equiv 1 - a(\rho)$ and $\rho = 1$. Then we have $\eta = 1 + (1 - \gamma)/\gamma\alpha$.

First, eliminate redundant variables to write (42)-(51) as

$$\begin{aligned} x_{t+1} + \sigma \pi_{t+1} &= x_t + \sigma (\theta_{i\pi} \pi_t + \theta_{ix} x_t + u_{i,t}) + \delta_{t+1}^x + \sigma \delta_{t+1}^\pi \\ \pi_{t+1} &= \frac{1}{\beta} \pi_t - \frac{\kappa}{\beta} x_t + \delta_{t+1}^\pi \\ \left[1 + \left(\frac{1}{\gamma} - 1 \right) \alpha \right] v_{t+1}^* + (\theta_{s\pi} \pi_{t+1} + \theta_{sx} x_{t+1} + \alpha v_{t+1}^* + u_{s,t+1}) &= (\theta_{i\pi} \pi_t + \theta_{ix} x_t + u_{i,t}) - \left(\frac{1}{\beta} \pi_t - \frac{\kappa}{\beta} x_t \right) + v_t^* \\ v_{t+1} - (\omega q_{t+1} - q_t) + \pi_{t+1} + (\theta_{s\pi} \pi_{t+1} + \theta_{sx} x_{t+1} + \alpha v_{t+1}^* + u_{s,t+1}) &= v_t \end{aligned}$$

$$\omega q_{t+1} = (\theta_{i\pi}\pi_t + \theta_{ix}x_t + u_{i,t}) + q_t + \omega\delta_{t+1}^q,$$

or

$$\begin{aligned} x_{t+1} + \sigma\pi_{t+1} &= (1 + \sigma\theta_{ix})x_t + \sigma\theta_{i\pi}\pi_t + \sigma u_{i,t} + \delta_{t+1}^x + \sigma\delta_{t+1}^\pi \\ \pi_{t+1} &= -\frac{\kappa}{\beta}x_t + \frac{1}{\beta}\pi_t + \delta_{t+1}^\pi \\ \theta_{sx}x_{t+1} + \theta_{s\pi}\pi_{t+1} + \left(1 + \frac{\alpha}{\gamma}\right)v_{t+1}^* + u_{s,t+1} &= \left(\theta_{ix} + \frac{\kappa}{\beta}\right)x_t + \left(\theta_{i\pi} - \frac{1}{\beta}\right)\pi_t + v_t^* + u_{i,t} \\ \theta_{sx}x_{t+1} + (1 + \theta_{s\pi})\pi_{t+1} + \alpha v_{t+1}^* + v_{t+1} - \omega q_{t+1} + u_{s,t+1} &= v_t - q_t \\ \omega q_{t+1} &= \theta_{ix}x_t + \theta_{i\pi}\pi_t + q_t + u_{i,t} + \omega\delta_{t+1}^q. \end{aligned}$$

In matrix notation (61),

$$\begin{bmatrix} 1 & \sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \theta_{sx} & \theta_{s\pi} & 1 + \alpha/\gamma & 0 & 0 & 0 & 1 \\ \theta_{sx} & 1 + \theta_{s\pi} & \alpha & 1 & -\omega & 0 & 1 \\ 0 & 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ \pi_{t+1} \\ v_{t+1}^* \\ v_{t+1} \\ q_{t+1} \\ u_{i,t+1} \\ u_{s,t+1} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 + \sigma\theta_{ix} & \sigma\theta_{i\pi} & 0 & 0 & 0 & \sigma & 0 \\ -\kappa/\beta & 1/\beta & 0 & 0 & 0 & 0 & 0 \\ \theta_{ix} + \kappa/\beta & \theta_{i\pi} - 1/\beta & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ \theta_{ix} & \theta_{i\pi} & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_s \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \\ v_t^* \\ v_t \\ q_t \\ u_{i,t} \\ u_{s,t} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{i,t+1} \\ \varepsilon_{s,t+1} \end{bmatrix} + \begin{bmatrix} 1 & \sigma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_{t+1}^x \\ \delta_{t+1}^\pi \\ \delta_{t+1}^q \end{bmatrix}$$

Now, we solve the model as

$$Ay_{t+1} = By_t + C\varepsilon_{t+1} + D\delta_{t+1}$$

$$y_{t+1} = A^{-1}By_t + A^{-1}C\varepsilon_{t+1} + A^{-1}D\delta_{t+1}$$

$$\begin{aligned}y_{t+1} &= Q\Lambda Q^{-1}y_t + A^{-1}C\varepsilon_{t+1} + A^{-1}D\delta_{t+1} \\Q^{-1}y_{t+1} &= \Lambda Q^{-1}y_t + Q^{-1}A^{-1}C\varepsilon_{t+1} + Q^{-1}A^{-1}D\delta_{t+1} \\z_{t+1} &= \Lambda z_t + Q^{-1}A^{-1}C\varepsilon_{t+1} + Q^{-1}A^{-1}D\delta_{t+1}\end{aligned}$$

Let G_f select rows with eigenvalues greater than one, and G_b select rows with eigenvalues less than one. For example, if the first and third eigenvalues are greater than or equal to one,

$$G_f = \begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & \dots \end{bmatrix}.$$

Then, the z corresponding to eigenvalues greater than one must be zero so

$$\begin{aligned}0 &= G_f Q^{-1} A^{-1} C \varepsilon_{t+1} + G_f Q^{-1} A^{-1} D \delta_{t+1} \\ \delta_{t+1} &= - (G_f Q^{-1} A^{-1} D)^{-1} G_f Q^{-1} A^{-1} C \varepsilon_{t+1}\end{aligned}$$

For this to work there must be as many rows of G_f as columns of δ , i.e. as many eigenvalues greater or equal to one as there are expectational errors. Substituting, we have the evolution of the transformed z variables, i.e. the impulse response function,

$$z_{t+1} = \Lambda z_t + Q^{-1} A^{-1} \left[I - D (G_f Q^{-1} A^{-1} D)^{-1} G_f Q^{-1} A^{-1} \right] C \varepsilon_{t+1},$$

and then the original variables from

$$y_t = Q z_t.$$

I include two small refinements. First, for computation it is better to force the elements of z_t that should be zero to be exactly zero. Machine zeros ($1e-14$) multiplied by explosive eigenvalues eventually explode. Thus, I find the non-zero z only by simulating forward the nonzero elements of z ,

$$G_b z_{t+1} = G_b \Lambda z_t + G_b Q^{-1} A^{-1} \left[C - D (G_f Q^{-1} A^{-1} D)^{-1} G_f Q^{-1} A^{-1} C \right] \varepsilon_{t+1}.$$

Second, the consumer's transversality condition tells us that debt v_t cannot explode. There is no reason to impose that the latent state variable v_t^* cannot explode or have a unit root. In solving the model for some parameter values it is important not to unwittingly impose that condition. The most obvious example occurs for passive fiscal policy, if $s_t = \dots + \alpha v_t + \dots$, not $s_t = \dots + \alpha v_t^* + \dots$. Then $v_t^* + s_{t+1} + \dots = v_t^*$ has a unit root (or explosive in the usual model with

discounting), but the quantity v_t^* enters nowhere else in the model. We seem to get determinacy by adding a useless unit root variable.

Rather than $\lim_{T \rightarrow \infty} E_{t+1} y_{t+T} = 0$, we need to impose

$$\lim_{T \rightarrow \infty} R E_{t+1} y_{t+T} = 0$$

where R is of the form

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ddots \end{bmatrix},$$

i.e. omitting the row of y_t corresponding to v_t^* (or any other variable that can explode). Then, rather than simply setting to zero the z corresponding to unit and greater eigenvalues, we need to set only

$$\lim_{T \rightarrow \infty} R Q E_{t+1} z_{t+T} = \lim_{T \rightarrow \infty} R Q \Lambda^T z_{t+1} = 0.$$

Denote by $\lambda_{<1}$ the eigenvalues less than one and $\lambda_{>1}$ the eigenvalues greater than one, and similarly for the corresponding z . We want, for example,

$$\lim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} Q \begin{bmatrix} \lambda_{<1} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{<1} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{>1} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{>1} & 0 \\ 0 & 0 & 0 & 0 & \lambda_{>1} \end{bmatrix}^T \begin{bmatrix} z_{<1} \\ z_{<1} \\ z_{>1} \\ z_{>1} \\ z_{>1} \end{bmatrix} = 0.$$

(The actual system is larger.)

Let G_f^* denote a matrix with ones in the place of eigenvalues greater or equal to one and zeros elsewhere, for example,

$$G_f^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is the matrix G_f above with zero rows added back. A simple test whether this problem is occurring is whether the rank of RQG_f^* is the same as the rank of QG_f^* , i.e. of G_f^* itself since Q is full rank. If that test succeeds, then we are not using the false condition that v^* may not explode to set a linear combination of the z to zero.

If that test fails, then in place of setting $G_f z_{t+1} = 0$, we set $RQG_f^* z_{t+1} = 0$, i.e.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} Q \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_{<1} \\ z_{<1} \\ z_{>1} \\ z_{>1} \\ z_{>1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Express the matrix on the right hand side in row-echelon form, delete the rows with zeros, and proceed as before.

C Basic policy rule regressions

I report here basic policy rule regressions. I do not report them in the paper or use their values, since I do not attempt the hard topic of identification – whether the correlation between interest rates and inflation, say, represents the policy rule feedback or the response of inflation to an interest rate shock. I present them to show the data, and to give reassurance that a fiscal policy rule which loads on output and (to a lesser extent) on inflation is not unreasonable, and to examine the size of the correlations.

Table 4 presents regressions of the interest rate and surplus on inflation and the output gap. Figure 8 presents the interest rate, output gap and inflation data underlying the monetary policy rule regressions, and Figure 9 presents the surplus, output gap and inflation data underlying the surplus policy rule regressions. The data are annual, and the same as used in Cochrane (2019).

These are simple regressions, that make no effort to surmount the profound identification problems of estimating policy rules. In the model, inflation and output gap respond contemporaneously to a surplus and interest rate shocks, so the right hand variable and error terms of the policy rule regressions are correlated. I include the regressions to characterize the data, and to give some sense of reasonable magnitudes.

Though there are thousands of monetary policy rule regressions in the literature, I start

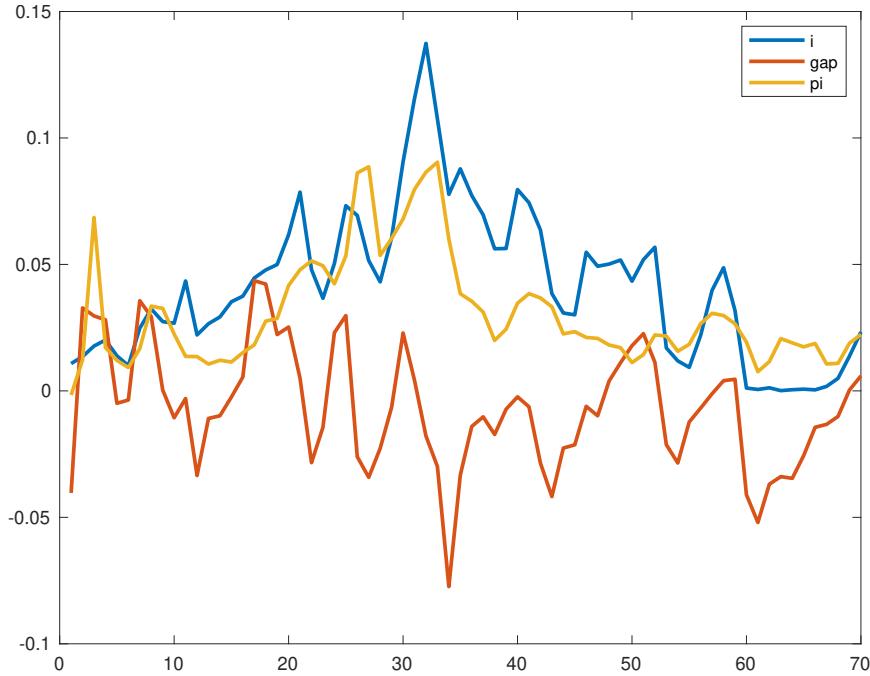


Figure 8: Interest rate, output gap, and inflation

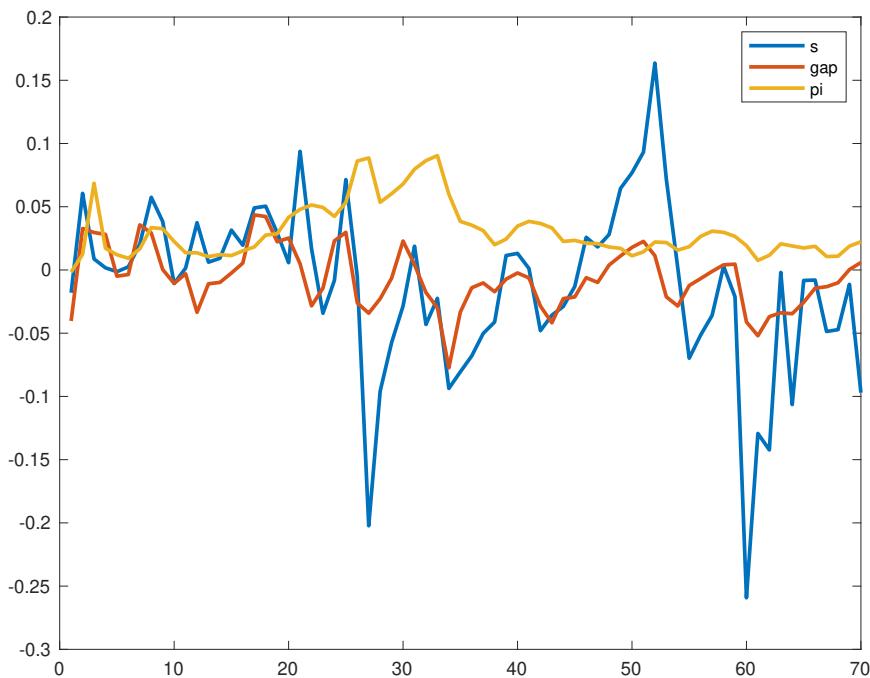


Figure 9: Surplus, output gap, and inflation

$i_t = a + \rho i_{t-1} + bx_t + c\pi_t + u_{i,t}$					
	ρ	b	c	ρ_u	R^2
OLS		0.19	0.98	0.72	0.52
s.e.		(0.16)	(0.21)		
Single OLS		0.13		0.90	0.01
s.e.		(0.23)			
Single OLS			0.97	0.69	0.50
s.e.			(0.21)		
$1 - \rho L$		0.29	0.48		0.29
s.e.		(0.10)	(0.24)		
With lag	0.81	0.30	0.22	0.16	0.86
s.e.	(0.06)	(0.06)	(0.11)		
$b/(1 - \rho)$		1.55	1.15		
$s_t = a + \rho s_{t-1} + bx_t + c\pi_t + u_{s,t}$					
	ρ	b	c	ρ_u	R^2
OLS		1.62	-0.38	0.37	0.37
s.e.		(0.34)	(0.35)		
Single OLS		1.64		0.38	0.35
s.e.		(0.32)			
Single OLS			-0.49	0.53	0.03
s.e.			(0.44)		
$1 - \rho L$		1.45	-0.24		0.26
s.e.		(0.33)	(0.37)		
With lag	0.39	1.27	-0.38	-0.06	0.50
s.e.	(0.10)	(0.27)	(0.27)		
$b/(1 - \rho)$		2.06	-0.62		

Table 4: Policy rule regressions. i = interest rate, x = GDP gap, s = surplus. Sample 1949-2018.

with that regression in part to frame the contrast with surplus policy rule regressions in the same data set. The OLS regressions show a small 0.19 output gap response and a large, nearly unit inflation response. The residual is strongly serially correlated. Figure 8 shows that the inflation response is, of course, driven by the rise and fall of inflation in the 1970s and 1980s. The single regression coefficients are just about the same as the multiple regression coefficients, with the output gap providing very little explanatory power.

One may wish to focus on the business cycle frequencies. The next two rows do that, and address serial correlation of the error, in two different ways. Using the error serial correlation ρ_i of the OLS regression, the regression labeled “ $1 - \rho L$ ” runs $(1 - \rho L)i_t$ on $(1 - \rho L)x_t + (1 - \rho L)\pi_t$. This specification mirrors that of the policy rule, (44) and (50). If the regression is correctly spec-

ified, the quasi-differenced coefficients should be the same. Here, the main effect is to lower the inflation response to about 0.5. The nearly unit inflation response does reflect the low frequency rise and fall of inflation rather than business cycle movement.

Adding a lagged interest rate, in the last regression estimates a partial adjustment model, common in the monetary policy shock literature. The coefficients are reduced, but the implied long run coefficients are much larger. One needs this sort of model to produce a coefficient greater than one on inflation, as Clarida, Galí, and Gertler (2000) famously found. Stationary data do not easily produce a coefficient that leads to explosive behavior.

Overall, these regressions reflect the great uncertainty and sensitivity to specification typical of the literature.

The OLS regression estimate of the surplus rule shows most of all a strong association with the output gap. This association stands out in Figure 9. It is clear at business cycle frequencies and also in the long dip of potential GDP (and, not reported, unemployment) in the 1970s and 1980s. The tables are turned. Here the output gap is the strong correlation, and the inflation coefficient is insignificant and results in very small R^2 .

This surplus is the ratio of surplus to value of the debt, or equivalently (surplus/GDP) to (value/GDP). Thus, the coefficient that a 1% rise in GDP gap results in a 1.62 percentage point rise in surplus means, if debt/GDP = 0.5, a 0.81 percentage point rise in surplus/GDP ratio.

One expects the coefficient of surplus on inflation to be positive, due to an imperfectly indexed tax code. The coefficient is -0.38, though insignificant. One can see in Figure 9 that the 1970s, with high inflation, had lower surpluses. This observation however reinforces the central weak point of such regressions. The negative correlation of surpluses with inflation is likely the response of inflation to surplus shocks, not the rule.

Any serious estimation of policy rules, which this is not, must take the identification problem seriously. To measure the interest rate or surplus policy rules, we must find movements in inflation and output gap which are not correlated with the interest rate or surplus disturbances $u_{i,t}$ and $u_{s,t}$. This is a different task than the usual one, of measuring directly the economy's response to monetary or fiscal policy shocks. There, one must find movements in $u_{i,t}$ and $u_{s,t}$ that are not correlated with changing expectations of future inflation, output, etc.

This is not a hopeless task. The Romer and Romer (1989) approach could look for such shocks. Romer and Romer looked for shocks that were a response to inflation, but not to output, in order to measure the response of output to such shocks. We need to measure the systematic part of policy, not the economy's response to policy. So, we either need narrative measurements of the systematic component, or we can use the monetary shock to measure the fiscal response

function. Likewise Ramey (2011) pioneered the use of military spending as an exogenous shock to $u_{s,t}$. We can use this to measure the monetary response function.

The structural or narrative approach may be much more fruitful for the fiscal response function than it is for the monetary response function. Much of the strong response of surpluses to output, and the response we wish to measure to inflation, are generated by the tax code and automatic stabilizers. Those can be modeled to generate θ_s parameters. Additional fiscal decisions are measurable too, in acts of congress.

Identification and estimation within the structure of a model may also be fruitful. The task is not as hopeless as it seems from the Cochrane (2011) critique of monetary policy rule estimation in new-Keynesian models. That paper concerned the difficulties of measuring off-equilibrium responses from data in an equilibrium, which really is hard. The θ responses here are all relations between variables that we do see in equilibrium.