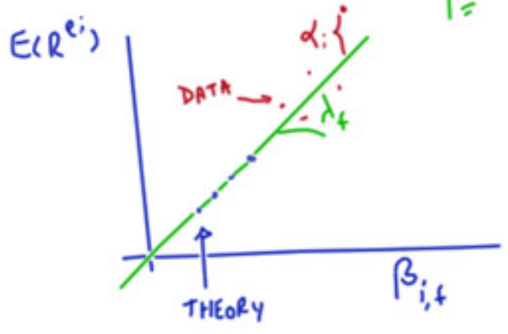
# FACTOR PRICING MODELS - CAPM, ICAPM, APT..

## 1. INTRODUCTION / OVERVIEW

$$E(R^{ei}) = \beta_{i,f}\,\lambda_f \quad [+\alpha_i]$$

$$R^{ei}_t = \alpha_i + \beta_{i,f}\cdot f_t + \varepsilon^i_t \qquad t=1\dots T \;\forall i$$

$\beta_{i,f}$




WHAT CAN WE USE FOR $f_t$?

"RULES OF THE GAME": AVOID EX-POST MVF?

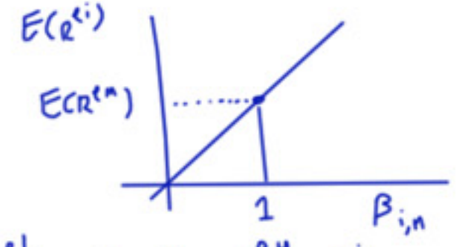
"EXPLAIN" $E(R^{ei})$?

SO FAR $f_t = \Delta C_t$ GOAL OTHER $f_t$

- CAPM $f_t$ = EXCESS RETURN ON MARKET PORTFOLIO $R^{eM}_t$

$$E(R^{ei}) = \beta_{i,M}\,\lambda_M$$

- NOTE $\beta_{M,M} = 1$, $E(R^{eM}) = 1\cdot\lambda_M \rightarrow E(R^{ei}) = \beta_{i,M}\,E(R^{eM})$




$$R^{ei}_t = \alpha_i + \beta_{iM} R^{eM}_t + \varepsilon^i_t$$

$$E(\cdot) \Rightarrow E(R^{ei}_t) = \alpha_i + \beta_{iM} E(R^{eM}_t)$$

- WHEN THE FACTOR IS A TRADED EXCESS RETURN, THE MEAN OF THE FACTOR SHOULD EQUAL THE FACTOR RISK PREMIUM $\lambda_f = E(f)$

B) THE TIME-SERIES INTERCEPT IS THE CROSS SECTIONAL ERROR

"CAPM: INTERCEPT SHOULD BE ZERO"

NOT IF THE FACTOR IS NOT TRADED, E.G. $f = \Delta C$

- ICAPM: $f_t$ = "INNOVATIONS TO STATE VARIABLES FOR INVESTMENT OPPORTUNITIES / OUTSIDE INCOME"
  MULTIFACTOR

- FAMA + FRENCH TABLE 1

$$R^{e_i}_t = \alpha_i + b_i\, rmrf_t + h_i\, hml_t + s_i\, smb_t + \varepsilon^i_t$$

$$E(R^{e_i}_t) = \alpha_i + b_i \lambda_m + h_i \lambda_h + s_i \lambda_s$$

"0?" "E(rmrf)" "E(hml)" = E(smb)

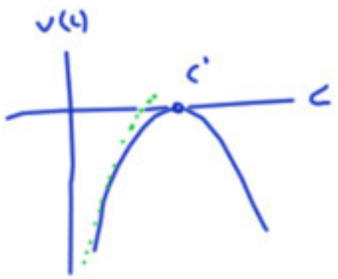
hml : VALUE - GROWTH
smb : SMALL - BIG
WHY?

- TODAY: DERIVE.
  A) EQUILIBRIUM
  B) APT

$\Delta c \leftarrow f_t$ { USE THEOREMS:
$m_{t+1} = a - b f_{t+1}$
$\Leftrightarrow$
$E(R^{e_i}) = \beta_{i,f} \lambda_f$

- WARNING. - NOT CLEAN!
  . CAPM → ICAPM → APT → $\Delta s$ c HISTORY. vs ←, WHY LESS CLEAR
  "AVOID c DATA"

## 2. CAPM / SIMPLE 2 PERIOD QUADRATIC




OBJECTIVE $m_{t+1} = a - b R^M_{t+1}$

A. UTILITY $U(c) = -\frac{1}{2}(c' - c_t)^2$

$$m_{t+1} = \beta \frac{c' - c_{t+1}}{c' - c_t}$$

B. LIVES 2 PERIODS, NO JOB OR OUTSIDE INCOME

$$c_{t+1} = w_{t+1} = R^w_{t+1}(w_t - c_t)$$

$$m_{t+1} = \beta \frac{c' - R^w_{t+1}(w_t - c_t)}{c' - c_t} = \underbrace{\left[\beta \frac{c'}{c' - c_t}\right]}_{a_t} - \left[\beta \frac{w_t - c_t}{c' - c_t}\right] R^w_{t+1}$$

$b_t$

$$\Leftrightarrow E_t(R^{e_i}_{t+1}) = \beta_{i,t} \cdot \lambda_{mt}$$

- POINT OF ASSUMPTIONS.
  B1: $R^w_{t+1}$ AND ONLY $R^w_{t+1} \Rightarrow c_{t+1}$ CAPM: .... AND NOTHING ELSE
  A): $m_{t+1}$ IS LINEAR

LOTS OF OTHER ASSUMPTIONS WORK:

# 3. CAPM. DERIVATION WITH LOG UTILITY OR I.I.D. CONSUMPTION GROWTH

$$U = E\int_{t=0}^{\infty} e^{-\delta t}\,\frac{C_t^{1-\gamma}}{1-\gamma}\,dt$$

this should be a -delta, not -r^f

SO FAR CONSUMPTION

$$\Lambda_t = e^{-\delta t} C_t^{-\gamma}$$

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t^f dt - \gamma\frac{dC_t}{C_t} + \frac{1}{2}(\gamma)(\gamma+1)\frac{dC_t^2}{C_t^2}$$

$$E_t(dR_t^i) = r_t^f dt + \gamma E_t\left(\frac{dC_t}{C_t}\cdot dR_t^i\right) \quad \left(\Leftarrow E_t\left(\frac{d\Lambda}{\Lambda}dR^i\right)\right)$$

WHAT IF

$$P_t^W = kC_t \rightarrow \frac{dP_t^W}{P_t^W} = \frac{dC_t}{C_t} \qquad dR_t^W = \frac{dP_t}{P_t} + \frac{X_t}{P_t}dt$$

$$\Rightarrow E_t(dR_t^i) = r_t^f dt + \gamma E_t(dR_t^W \cdot dR_t^i) \rightarrow \beta_{iw}\lambda_w$$

"CONSUMPTION CLAIM"

$$P_t^W = E_t\int_{s=0}^{\infty} e^{-\delta s}\left(\frac{C_{t+s}}{C_t}\right)^{-\gamma}\cdot C_{t+s}\cdot ds$$

E.G.

A) IID, $\frac{dC_t}{C_t} = \mu dt + \sigma dz_t \rightarrow f(C_{t+s}|C_t)$ THE SAME

$$\frac{P_t^W}{C_t} = E_t\int_{s=0}^{\infty} e^{-\delta s}\left(\frac{C_{t+s}}{C_t}\right)^{1-\gamma} ds = k$$

B) $\gamma=1$ EVEN IF NOT IID!

"FOR LOG UTILITY, INCOME (CASHFLOW) + SUBSTITUTION EFFECTS OFFSET"

$$\frac{P_t^W}{C_t} = E_t\int_{s=0}^{\infty} e^{-\delta s}\left(\frac{C_{t+s}}{C_t}\right)^{-1}\left(\frac{C_{t+s}}{C_t}\right) ds \qquad \text{DISCOUNT RATE}$$

$$= \int_{s=0}^{\infty} e^{-\delta s} = \frac{1}{\delta}$$

ASSUMPTIONS

- WHAT'S $P^W$? MARKET RETURN IF NO JOB, INCOME, REAL ESTATE.... "PROXY"
- NO JOB. LINK $\Delta C, M$ + $R^W$ – NOTHING ELSE
- $\gamma=1$ OR IID. LINK $\Delta C_t, M_t$ TO $R_t^W$ NOT NEWS,
- LINEARITY ← CONTINUOUS TIME. DISCRETE $M_{t+1} = 1/R_{t+1}^W \approx a - bR_{t+1}^W$

# 4. ICAPM / "STATE VARIABLES"

- NOT IID, $\gamma = 1$ . $R_{t,t+1} = a + b\underbrace{\left(\frac{D}{P_t}\right)}_{X_t} + \varepsilon_{t,t+1}$

$dR_{t,t+1} = \mu(X_t)\,dt + \sigma(X_t)\,dZ_t$

↳ "STATE VARIABLE FOR INVESTMENT OPPORTUNITIES"

MORE $X_t$ → GOODNEWS → $C_t$ ↑ → $u'(C_t)$ ↓

- $V(W_0, X_0) = \max_{\{\dots\}} E_0 \int_{s=0}^{\infty} e^{-\delta s} u(C_{t+s})\,ds$ s.t. … $W_0$ $X_t$ …

- "ENVELOPE THEOREM" $\frac{\partial V(W_0, X_0)}{\partial W_0} = \frac{\partial u(C_t)}{\partial C_t}$

$1 SAVED = $1 CONSUMED

USE TO SUBSTITUTE $C_t \leftarrow W_t, X_t$

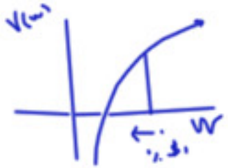
(CAPM) (NEW)

$e^{-\delta t} V_W(W_t, X_t) = e^{-\delta t} u'(C_t) = \Lambda_t$

$\frac{d\Lambda_t}{\Lambda_t} = -\delta\,dt + \frac{V_{WW} W_t}{V_W}\frac{dW_t}{W_t} + \frac{V_{WX}}{V_W} dX_t + (\text{Ito} - \text{de})$

$E_t(dR_t^i) - r_t^f dt = -E_t\left(\frac{d\Lambda_t}{\Lambda_t} dR_t^i\right)$

$= \left(1 - \frac{V_{WW} \cdot W}{V_W}\right) E_t\left(\frac{dW_t}{W_t} \cdot dR_t^i\right) - \frac{V_{WX}}{V_W} E_t(dX_t \cdot dR_t^i)$




RISK AVERSION! $V(W) = W^{1-\gamma}$ → $\gamma$

↳ "STATE VARIABLE AVERSION" [PROPERTY OF WHOLE ENVIRONMENT, NOT UTILITY]

$X \uparrow \rightarrow V_W = u'(C) \downarrow$ ?

$E_t(R_{t,t+1}^{Pi}) = \left(-\frac{V_{WW} W}{V_W}\right) \mathrm{Cov}_t(R_{t,t+1}^{Pi}, R_{t,t+1}^{W}) + \left(-\frac{V_{WX}}{V_W}\right) \mathrm{Cov}_t(R_{t,t+1}^{Pi}, X_{t,t+1})$ (SHOCKS)

$= \beta_{i,W} \lambda_M + \beta_{i,X} \lambda_X$

A TWO FACTOR MODEL, LIKE FF3F!

ASSUMPTIONS : STILL NO JOB …

## 5. MULTIFACTOR MODELS / OUTSIDE INCOME

### A. OUTSIDE INCOME

$$C_{t+1} = W_{t+1} + Y_{t+1}$$

$$W_{t+1} = R^W_{t+1}(W_t - C_t)$$

$$m_{t+1} = \beta \frac{c^{-\gamma} R^{W\,-\gamma}_{t+1}(W_t - C_t) - Y_{t+1}}{c^{-\gamma} - C_t}$$

$$= \left(\beta \frac{c^{-\gamma}}{c^{-\gamma} - C_t}\right) - \left(\beta \frac{W_t - C_t}{c^{-\gamma} - C_t}\right) R^W_{t+1} - \left(\frac{\beta}{c^{-\gamma} - C_t}\right) Y_{t+1}$$

$$m_{t+1} \approx a_t - b_t R^W_{t+1} - d_t Y_{t+1}$$

$$\Leftrightarrow E_t(R^i_{t+1}) = \beta_{i,W}\lambda_W + \beta_{i,Y}\lambda_Y$$

### B. STATE VARIABLES THAT FORECAST OUTSIDE INCOME

$$dY_t = M(X_t, Y_t)dt + \sigma(X_t, Y_t)dZ_t$$

… JUST LIKE ICAPM.

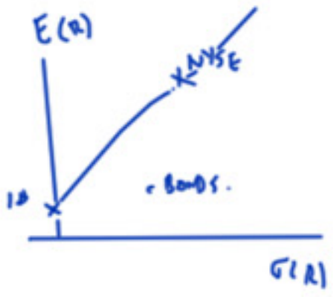
## 6. MULTIFACTOR MODELS PORTFOLIO INTUITION

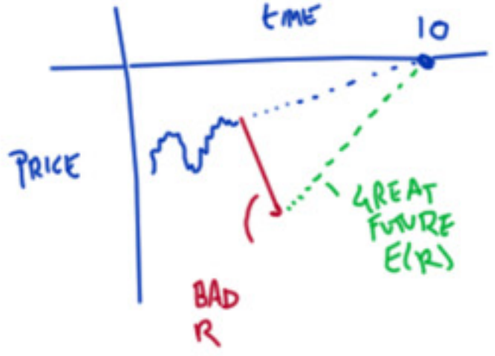
- "HEDGING DEMAND" PORTFOLIO LOGIC
- OUTSIDE INCOME.
  - A, B = $E, \sigma, \beta, \ldots$ $E(R^i) = \beta_i \lambda$ THEY ARE=
  - IN A RECESSION, YOU LOSE JOB. AP, BL $[R^i_t = a_i + \beta_{iM} R^M_t + \varepsilon^i_t \;;\; \Sigma^A \uparrow, \Sigma^B]$
  - BUY A! $P^A \uparrow \rightarrow E(R^A) \downarrow$ LOOKS LIKE $\lambda$!
    $P^B \downarrow \rightarrow E(R^B) \uparrow$
  - $E(R^i) = \beta_{i,M}\lambda_M + \underbrace{\beta_{i,UR}\lambda_{UR}}_{\text{WAS } d_i}$ $\left[R^i_t = \beta_{iM} R^M_t + \underbrace{\beta_{i,UR} \cdot UR_t + \varepsilon^i_t}_{\text{WAS } d_i + \varepsilon^i_t}\right]$
- STATE VARIABLES.
  - NEWS OF FUTURE JOB LOSS JUST AS BAD.
  - NEWS OF POOR INVESTMENT OPPORTUNITIES.
- MUST BE AGGREGATE HEDGE DEMAND → MOVE PRICES.

7. INTERTEMPORAL

EXAMPLE: LONG TERM BONDS.

| | E(R) | σ(R) |
|---|---|---|
| TBILL | 1% | 0% |
| STOCK | 7% | 16% |
| BONDS | 2% | 12% |




BOND PRICE ↓ → BAD NEWS TO CURRENT W

→ GOOD NEWS TO E(W...)

8 MULTIFACTOR MODELS Ü INTUITION, HALDO, MIMICKING PORTFOLIOS

$M_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$ ← "hunger", $C_{t+1}$ . $P^w_{t+1}$, NEWS, $Y_{t+1}$

MACRO $C_{t+1} =$ ← GDP, INVESTMENT, UR, INT. RATES .... ALL $f_{t+1}$

MIMICKING PORTFOLIOS $M_{t+1} = a + b\Delta y_{t+1}$

$x^0$ : $Proj(m_{t+1} | X)$ $\Delta y_{t+1} = \underbrace{\beta' R_{t+1}}_{\text{PORTFOLIO, } f} + \varepsilon_{t+1}$ $\Delta c_{t+1} = \beta' R_{t+1} + \varepsilon_{t+1}$

...

... ZOO! "FISHING EXPEDITION"

# 9 COMMENTS

- ALL REPLACE $\Delta C_{t+1}$ WITH DETERMINANTS.

  SPECIAL CASES OF $\Delta C$, NOT ALTERNATIVES TO BASIC IDEA

- ASSUMPTIONS: VERY SPECIAL
- PRACTICE: INSPIRATION, NOT CHECKING/TESTING.
  - CAPM: $\lambda = \gamma \sigma(\alpha)$? $\Delta C_{t+1} = R^W_{t+1}$ ! $16\% .6^2$
  - ICAPM: DO $X_t$ FORECAST? SOLVE $\frac{V_{Wx}}{V_W}$ ? NO
  - MACRO, MIMICKING -- "FACTORS MIGHT BE"
- POINT: PRACTICE, WORK. BUT... $X_t$? "RULES OF GAME"? ART. READFF?
- HOW MANY ASSUMPTIONS DO YOU USE? EXAMPLE

  $E(R^{ei})$, ANOMALY OR FUND. $E(R^{ei}) = \beta_{iM} E(R^{eM})$. / $R^{ei}_t = 0 + \beta_{iM} R^{eM}_t + \varepsilon^i_t$

  ✓ "CAN GET WITH INDEX" "NOT A NEW ANOMALY" "CAN HEDGE W. INDEX" "IS AS RATIONAL AS THE MARKET" × DEEP "EXPLAIN" "CAPM IS WRONG"?

# 10. APT

- GOAL $R^{ei}_t = \alpha_i + \beta_{i,1} f^1_t + \beta_{i,2} f^2_t + \varepsilon_{i,t}$ ($R^2$ // TRADED $R^*$)

$$\Rightarrow E(R^{ei}) = \beta_i E(f^1) + \beta_{i2} E(f^2)$$

($\alpha = 0$; $\beta_i \to \lambda_1$, $\beta_{i2} \to \lambda_2$)

APT DIFFERENT LOGIC. "SMALL" $\varepsilon \Rightarrow$ "SMALL" $\alpha$, NO VICE

- $R^{iP}_t = R^{ei}_t - \beta_{i2} f^1_t - \beta_{i2} f^2_t = \alpha^i + \varepsilon^i_t$ // PORTABLE $\alpha$

A PORTFOLIO. "OPTIMAL" HEDGE

MINIMUM VARIANCE

$\to E(R^{eP}_t) = \alpha_i$ $\quad \frac{E(R^{eP}_t)}{\sigma(R^{eP}_t)} = \frac{\alpha_i}{\sigma(\varepsilon_i)} = SR$

$\sigma^2(R^{eP}_t) = \sigma^2(\varepsilon^i)$

ASSUMPTION $|SR| < MAX = A$

"SMALL" $\varepsilon_i \to$ "SMALL" $\alpha_i$

IF $|SR| \leq A$ THEN AS $\sigma^2(\varepsilon) \to 0$ $|\alpha_i| \to 0$

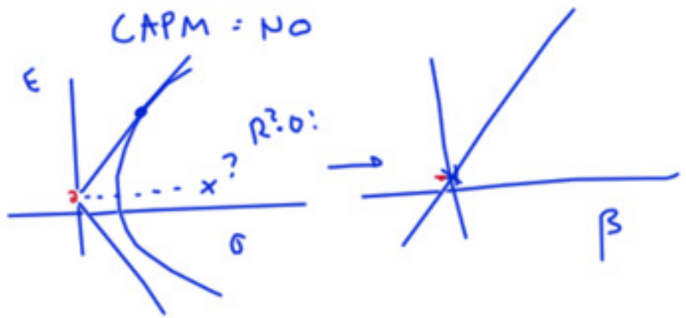
$\sigma^2(\varepsilon) < \delta \Rightarrow |\alpha_i| < \epsilon$

- ALPHAS SHOULD BE SMALL WHEN $R^2$ ARE LARGE

# 11 APT VS EQUILIBRIUM MODELS (CAPM)

- ABSOLUTE VS RELATIVE PRICING.

- DOES $R^2$ MATTER?




APT : YES!

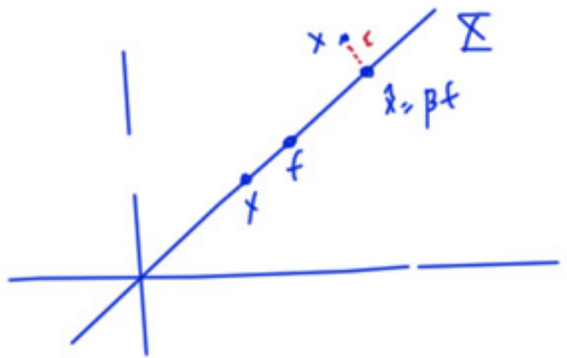
- DOWENEED FACTOR STRUCTURE FOR FACTOR PRICING

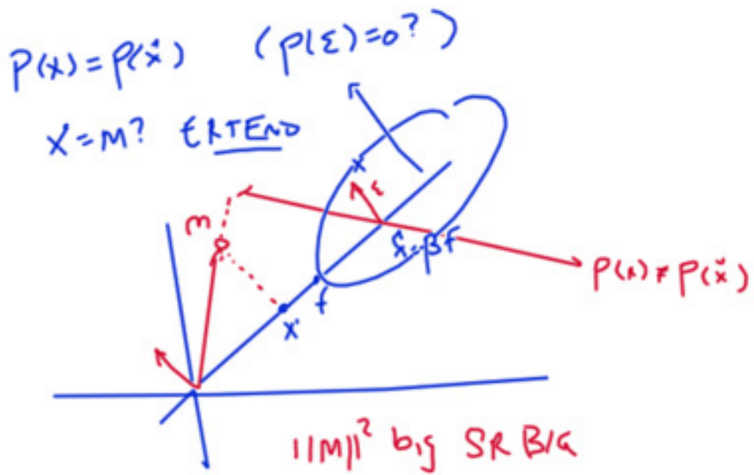
$$R^{ei}_{t+1} = \alpha_i + \beta_i f_{t+1} + \varepsilon^i_{t+1} \qquad Cov(R^i R^{i'}) = \underbrace{\beta\beta'\sigma_f^2}_{\text{BIG?}} + \Sigma$$

APT YES. CAPM NO

- CAPM ISOFTEN USEDASAPT!

$$E(R^{ei}) = \beta_{iM} E(R^{eM})$$




- SR PURE ARBITRAGE FAILED.

- APT ONLY FOR LARGE PORTFOLIOS | SMALL $\varepsilon_i$