The Fiscal Roots of Inflation

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Abstract

Unexpected inflation devalues nominal government bonds. It must therefore correspond to a decline in expected future surpluses, or a rise in their discount rates, so that the real value of debt equals the present value of surpluses. I measure each component using a vector autoregression, via responses to inflation, recession, surplus and discount rate shocks. Discount rates account for much inflation variation, for the cyclical pattern of inflation, and why persistent deficits often do not cause inflation. Long-term debt is important. In response to a fiscal shock, smooth inflation slowly devalues outstanding long-term bonds.

Keywords: Fiscal theory of the price level, monetary policy, fiscal policy, inflation.

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1. Introduction

The real value of nominal debt equals the present value of real primary surpluses. Higher inflation devalues nominal government debt. Higher inflation must therefore correspond to lower surplus/GDP ratios, lower GDP growth, or higher discount rates for government debt. I develop a set of linearized identities that expresses this identity. I measure the components via impulse-response functions of a simple vector autoregression (VAR).

I look first at an unanticipated movement in inflation. Two thirds of the total inflationary effect of that shock corresponds to a change in discount rates, one

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third from a change in growth, and essentially none to a change in surplus/GDP ratios.

I look next at a shock in which both inflation and growth move unexpectedly and together. This exercise is motivated by events such as 2008-2009. There is a big recession, with large and persistent deficits. Yet inflation falls, raising the real value of nominal debt. How can this be? Well, perhaps people expect higher subsequent primary surpluses to pay back the cumulated deficits, and more. Aside from its implausibility, I do not find this pattern in the data. But nominal and real interest rates on government debt fall sharply, which raise the value of government debt, a deflationary force. I find that the decline in expected returns is large and persistent enough quantitatively to account for inflation shocks in a recession, and vice versa in a boom.

I also examine persistent shocks to surpluses and shocks to discount rates. These shocks come with essentially no inflation. Shocks to surpluses are highly correlated with shocks to discount rates, so the surplus and discount rate terms of the present value formula largely offset. Viewed in ex-post terms, persistent deficits come at the same time as low returns. Low returns bring back the value of debt, without needing repayment via later surpluses, or devaluation via an initial inflation. The strong correlation between discount rates and deficits provide fiscal roots of the absence of inflation in the presence of large variation in surpluses and discount rates.

The first and third observations are not contradictory. There are multiple sources of variation in the data. Not all business cycles are alike. When we isolate a shock to inflation, we see events in which discount rates and deficits do not offset. When we isolate a shock to discount rates or deficits, we see a different slice of data, in which they do offset and there is not much inflation or deflation.

I also find an important role for long-term debt. Simple models focus on one-period debt, and price-level jumps devalue such debt. With long-term debt, a slow inflation can devalue long-term bonds when they come due. Expectation of such future inflation lowers nominal bond prices, restoring present value balance in place of a price-level jump. This mechanism is evident in the data, with expected future inflation accounting for large fractions of changes in the present value of debt.

I interpret the results through the lens of the fiscal theory of monetary policy: models with interest rate targets, fiscal theory of the price level, and potentially sticky prices, as described in Cochrane (2020a), Cochrane (2020b). (More literature below.) In this interpretation, changes in expected surpluses and discount rates *cause* unexpected inflation. In this interpretation, we study the fiscal roots rather than the fiscal consequences of inflation. This paper establishes a set of facts that will be useful for constructing such models. My causal language below refers to this interpretation.

But the identities whose terms I measure hold in almost all macroeconomic models used to quantitatively address inflation, and therefore form a widely useful set of stylized facts for monetary and fiscal interaction. The computations of this paper are deliberately "measurement without theory." I do not estimate any structural parameters, identify any structural shocks, or test one model vs. another. A "shock" only means a movement in a variable that is not forecast by the VAR, without structural interpretation.

In particular, standard new-Keynesian / DSGE models posit an opposite causality. Equilibrium-selection policy by the central bank determines unexpected inflation. Fiscal policy reacts "passively," raising or lowering surpluses to validate inflation-induced changes in the value of government debt. These fiscal underpinnings are not often examined, but they should be as they are also important parts of the model, just as monetary-fiscal coordination is important to classic monetarist thought. The results of this paper can also be interpreted as measures of the fiscal adjustments to inflation that a standard new-Keynesian model must envision. The fact that discount rates do much of the adjusting, and the measured time-path of surpluses following inflation shocks, are important fiscal underpinnings of such models.

Since the analysis is based on identities, and since I make no effort to identify structural shocks of a model or exogenous policy shocks, the empirical results do nothing to establish one or another causal story. But which element in an identity moves – whether surpluses or discount rates account for inflation-induced variation in the value of government debt – is still an interesting measurement, that bears on the construction of any theory.

More narrowly, this paper addresses a common attempt at armchair refutation of fiscal theory: We have huge debt and deficits, and no inflation. Debt and deficits increase in recessions, where inflation declines. The theory must be wrong. No. First, a low real interest rates quantitatively account for the disinflation and rise in the value of government debt in recessions. Second, since the government debt valuation equation holds equally in conventional monetary theories, if there is a puzzle in the fiscal foundations of inflation, it applies equally to conventional theories. It does not reject fiscal theory in favor of those other theories.

As a paper on pure facts, I do not offer here theory or evidence on *why* surpluses or expected returns on government bonds vary as they do. Given their variation, inflation makes fiscal sense.

2. Literature

The technique in this paper is adapted from asset pricing. The general approach to linearizing the valuation identity follows Campbell and Shiller (1988). The summary of this literature in Cochrane (2011b) and the treatment of identities in Cochrane (2008) are obvious precursors to this work. The uniting theme in the former is that asset price and return variation corresponds in great measure to variation in discount rates.

The analysis of government finances, how debt is paid off, grown out of, or inflated away, is a huge literature. Hall and Sargent (1997), Hall and Sargent (2011) are the most important recent precursors. Hall and Sargent focus on the market value of debt, as I do, not the face value reported by the Treasury, and consequent proper accounting for interest costs. Cochrane (1998) constructs a linearized present value equation similar to that used here, and uses it to decompose the value of government debt. Cochrane (2019) improves on that calculation, using the value identity (2). Both papers find that variation in expected primary surpluses is an important determinant of the value of debt.

The main methodological novelty is that this paper uses the innovation identities, (3) and (5) below, to focus on inflation rather than the value of government debt, paralleling VAR-based return (rather than price-dividend ratio) decompositions such as Campbell and Ammer (1993). I find a greater role for discount rates in this inflation accounting, where varying expected surpluses are more important in accounting for variation in the level of the value of debt.

The fiscal theory of monetary policy is the latest step in a long literature on the fiscal theory of the price level, starting with Leeper (1991), that integrates fiscal theory with sticky-price models and interest rate targets. Leeper and Leith (2016) offers an excellent example and literature review. Cochrane (2020b) offers an extensive literature review.

Cochrane (2020a) works a fiscal theory of monetary policy model with the s-shaped surplus processes I find here, and calculates inflation decompositions and response functions from the model. It is not quite the theory paper corresponding to this work. I do not here identify the structural monetary and fiscal policy shocks studied there. I do not there extend the model with the dynamic embellishments and multiple shocks necessary to match responses of this paper. Bringing theory and data closer together is obviously an important goal for future work.

3. Identities

Start with a linearized version of the government debt flow identity,

$$\rho v_{t+1} = v_t + r^n_{t+1} - \pi_{t+1} - g_{t+1} - s_{t+1}. \tag{1}$$

I derive this identity in Online Appendix. The quantity v_t is the log of the ratio of the market value of debt to GDP, henceforth just "debt." Debt at the end of period $t+1$, v_{t+1}, is equal to debt at the end of period t, v_t, increased by the log nominal return on the portfolio of government bonds r^n_{t+1}, less inflation π_{t+1}, less log GDP growth g_{t+1}, and less the scaled real primary surplus to GDP ratio s_{t+1}. The measured return r^n_{t+1} includes any effects of maturity structure or of liquidity premiums accruing to government debt. The parameter ρ is a constant of linearization, which I take to be $\rho = 1$ in the numerical results. We can express ρ in terms of return r and growth g values around which we take the linearization as $\rho = e^{-(r-g)}$.

All variables in (1) are logs, except the surplus. I Taylor expand the level of the surplus, to allow the surplus to be negative. As a result the surplus is scaled to generate percentage units: The variable s_t is ρ times the ratio of primary surplus to GDP scaled by the debt to GDP ratio at the linearization point. With $\rho = 1$, s_t can also represent the real primary surplus divided by the previous period's debt. Either definition leads to the same linearization. In the data, I impute the surplus from the other terms of (1), so its definition only matters when one wishes to assess an independent data source on surpluses. For brevity, I refer to s_t simply as the "surplus," or when necessary for clarity as "surplus to GDP ratio." With $\rho < 1$ there is also a constant in the linearization, or the variables are deviations from steady state.

Iterating forward, we have a present value identity,

$$v_t = \sum_{j=1}^{\infty} \rho^{j-1} s_{t+j} + \sum_{j=1}^{\infty} \rho^{j-1} g_{t+j} - \sum_{j=1}^{\infty} \rho^{j-1} \left(r^n_{t+j} - \pi_{t+j} \right). \tag{2}$$

Taking expected values, the debt to GDP ratio is the present value of future surplus to GDP ratios, discounted at the ex-post real return, and adjusted for growth. Higher GDP growth, with the same surplus to GDP ratio, gives rise to greater surpluses.

Taking time $t+1$ innovations

$$\Delta E_{t+1} \equiv E_{t+1} - E_t$$

and rearranging, we have an unexpected inflation identity,

$$\begin{aligned}\Delta E_{t+1}\pi_{t+1}-\Delta E_{t+1}r^n_{t+1}=-\sum_{j=0}^{\infty}\rho^j\Delta E_{t+1}s_{t+1+j}-\\ -\sum_{j=0}^{\infty}\rho^j\Delta E_{t+1}g_{t+1+j}+\sum_{j=1}^{\infty}\rho^j\Delta E_{t+1}r_{t+1+j},\end{aligned}\tag{3}$$

where

$$r_{t+1}\equiv r^n_{t+1}-\pi_{t+1}$$

denotes the ex-post real return on the portfolio of government debt. A decline in the present value of surpluses, coming either from a decline in surplus to GDP ratios, a decline in GDP growth, or a rise in discount rates, must correspond to a lower real value of the debt. This reduction can come about by unexpected inflation, or by a decline in nominal long-term bond prices and hence a negative ex-post return. I use time $t+1$ to denote unexpected events, and time 1 as the date of a shock in the impulse-response functions.

What determines the bond return r^n_{t+1}? I linearize the return of the government bond portfolio around a geometric maturity structure, in which the face value of maturity j debt declines at rate ω^j, yielding

$$\Delta E_{t+1}r^n_{t+1}=-\sum_{j=1}^{\infty}\omega^j\Delta E_{t+1}r^n_{t+1+j}=-\sum_{j=1}^{\infty}\omega^j\Delta E_{t+1}\left(r_{t+1+j}+\pi_{t+1+j}\right).\tag{4}$$

Lower nominal bond prices, and a lower ex-post bond return, mechanically correspond to higher bond expected nominal returns, which in turn are composed of real returns and inflation. The Online Appendix presents the algebra.

We can then eliminate the bond return in (3)-(4) to focus on inflation and fiscal affairs alone,

$$\begin{aligned}\sum_{j=0}^{\infty}\omega^j\Delta E_{t+1}\pi_{t+1+j}=-\sum_{j=0}^{\infty}\rho^j\Delta E_{t+1}s_{t+1+j}-\sum_{j=0}^{\infty}\rho^j\Delta E_{t+1}g_{t+1+j}+\\ +\sum_{j=1}^{\infty}(\rho^j-\omega^j)\Delta E_{t+1}r_{t+1+j}.\end{aligned}\tag{5}$$

I focus on this decomposition. Each of the terms is, directly, a sum of the elements of an impulse-response function.

Identity (5) highlights several interesting mechanisms which we can look for in the data and models. Consider the simple case with constant expected returns $E_t r^n_{t+1} = E_t \pi_{t+1}$. With one-period debt, $\omega = 0$, there is only one term on the left-hand side of (5), $\Delta E_{t+1}\pi_{t+1}$. Shocks to the present value of surpluses must be soaked up by a price-level jump.

With long-term debt, $\omega > 0$, however, a shock to the present value of surpluses can result in a drawn-out period of inflation, which slowly devalues outstanding long-term bonds. In the identity (3), the term r^n_{t+1} marks the future inflation to market, as future inflation in (4) lowers that return. In fact, equation (5) allows the entire effect of the fiscal shock to show up in expected future inflation with no movement in current inflation $\Delta E_{t+1}\pi_{t+1} = 0$. Drawn-out inflation accompanying fiscal problems is more realistic than one-time price-level jumps. So, we can productively look for fiscal roots of drawn-out inflation.

With one-period debt, expected inflation may continue to be high after an initial inflation shock, but this fact has no impact on one-period unexpected inflation or this fiscal accounting. With $\omega = 0$, $\Delta E_1 \pi_j$ for $j > 1$ is irrelevant in (5). With long-term debt, the weighted sum of changes in expected inflation substitutes for inflation at time 1, but only the ω-weighted sum. Additional persistence in inflation, though interesting for matching data, has no fiscal consequence or consequence for understanding unexpected inflation.

Higher discount rates, a higher expected real bond return, are an inflationary force exactly parallel to low surpluses. They lower the value of government debt, and thus require current or future inflation.

As the maturity structure of government debt lengthens, ω increases, and the discount rate terms in the last part of (5) get smaller. When $\omega = \rho$, almost a perpetuity, the discount rate term drops out. Intuitively, a government that funds itself with near-perpetuities can pay off its current debt while ignoring real interest rate variation, just as a household that takes out a fixed-rate mortgage is immune from interest rate variation.

The identity (5) is also useful for understanding the operation of models and model predictions for responses to policy shocks. For example Cochrane

(2020a) uses the identity to understand how long-term debt is useful to generate a negative inflation response to a monetary policy shock, and how monetary policy by controlling expected inflation can smooth the effects of a fiscal shock over time.

3.1. What about $r < g$?

To get to (2), I iterate forward the flow identity (1) to

$$v_t = \sum_{j=1}^{T} \rho^{j-1} s_{t+j} + \sum_{j=1}^{T} \rho^{j-1} g_{t+j} - \sum_{j=1}^{T} \rho^{j-1} r_{t+j} + \rho^T v_{t+T}. \tag{6}$$

Then I assume that the expected value of the terminal condition vanishes, and the sums converge. With all variables stationary, this assumption requires $\rho \leq 1$, i.e. $r \geq g$. What about $r < g$?

First, the parameter $\rho = e^{-(r-g)}$ represents a point of linearization. It does not have to be calculated from the sample or population mean of the government bond return and growth rate. So take $\rho \leq 1$ even if $E(r) < E(g)$. The return linearization is not sensitive to the linearization point, since the variables vary by so much relative to their means. Then, with stationary variables, the terms of the linearized identities all converge. Indeed, convergence depends more on the stationarity of the variables than it does on ρ. The limit $\lim_{T\to\infty} \rho^T E_t(v_{t+T}) = 0$ not because $\rho < 1$ but because v_t is stationary, and the formula applies to deviations from the mean, so $\lim_{T\to\infty} E_t(v_{t+T}) = 0$. The point estimate of 0.98 autocorrelation would allow ρ as large as 1.02. The terms converge in the estimates, which is really all that matters for these decompositions.

The worry, then, is that the configuration of the economy which produces $r < g$ is one in which the true present value does not operate. In a perfect-certainty frictionless economy with $r < g$, the economy will always grow out of debt, so deficits ($s_t < 0$) need not require later surpluses. Debt is the accumulation of past deficits, but does not require future surpluses. A linearization with $\rho \leq 1$ misses this crucial fact. In this circumstance, we should linearize with $\rho > 1$ and also solve backwards.

However, our economy does not feature perfect certainty and no frictions. Even if it is true that $E(r) < E(g)$ in our economy, that fact does *not* imply that debts do not have to be repaid in present value, that fiscal expansion has "no fiscal cost" in the provocative analysis of Blanchard (2019). The present value of debt can be well defined, properly using the stochastic discount factor, contingent claim price, or marginal utility to discount, and large deficits still need to be repaid by subsequent surpluses, yet the economy displays $E(r) < E(g)$. In this case, the linearized present value formula remains valid as long as its terms converge, which they do. For small $r < g$ or for liquidity premiums or seignorage which do not scale, one may also apply the linearized identities to deviations about the mean, even if the mean causes trouble, i.e. due to a small perpetual deficit.

Bohn (1995) gives an early classic example. Bassetto and Cui (2018) analyze the issue with a specific eye to fiscal theory, and Bassetto and Sargent (2020) give a general discussion. Reis (2021) gives a detailed example emphasizing discounting at the marginal product of capital, which is higher than the growth rate. Cochrane (2021) offers a short summary of the issues. The Online Appendix gives a fuller analysis in this context.

4. Data

I use data on the market value of government debt held by the public and the nominal rate of return of the government debt portfolio from Hall, Payne, and Sargent (2018). I use standard BEA data for GDP and total consumption. I use the GDP deflator to measure inflation. I use CRSP data for the three-month Treasury rate. I use the 10-year constant maturity government bond yield from 1953 on and the yield on long-term United States bonds before 1953 to measure a long-term bond yield.

I measure the debt to GDP and surplus to GDP ratios by the ratios of debt and surplus to consumption, times the average consumption to GDP ratio. For brevity, I still refer to ratios as ratios to GDP. We conventionally reference debt to GDP, but there is no fundamental economic reason to divide debt by GDP rather than another macroeconomic aggregate. For most uses, we want some measure of debt relative to the government's long-term taxing power and spending habits. In time-series work we want a divisor that renders debt stationary. But dividing debt by GDP introduces additional dynamics due to GDP dynamics. Is it interesting, for example, to say that the debt to GDP ratio declines in a recovery because GDP grows, mechanically raising the denominator? Dividing by a measure of trend or permanent GDP produces a stationary series in which variation in the debt ratios comes more from fiscal affairs than from predictable dynamics in the denominator. Potential GDP has a severe look-ahead bias for a VAR. Consumption, being close to a random walk, is a good stochastic trend for GDP and divisor for debt for this time-series analysis.

I infer the primary surplus from the flow identities. This calculation measures how much money the government actually borrows. NIPA surplus data, though broadly similar, does not obey the flow identity.

I infer the surplus/GDP s_t for the VAR from the linearized identity (1), at an annual frequency. By doing so, the data obey the identity exactly. Therefore VAR estimates of the decompositions add up exactly with no approximation error. The approximation errors are much smaller than sampling errors, so this choice just produces clearer tables.

I approximate around $r = g$ or $\rho = 1$. The variables are all stationary, impulse-responses and expected values converge, so weighting higher-order terms by, say, 0.99^j vs. 1.0 makes little difference to the results. One can also view the unweighted $\rho = 1$ identities as $r \to g$ limits.

Figure 1 presents the surplus/GDP and compares three measures. The "Linear, s_t" line imputes the surplus/GDP from the linearized flow identity (1) directly at the one-year horizon, which is the measure I use below.

There *are* primary surpluses. One's impression of endless deficits comes from the deficit including interest payments on the debt. NIPA measures (not shown) also show regular positive primary surpluses. Steady primary surpluses from 1947 to 1975 helped to pay off WWII debt. The year 1975 started an era

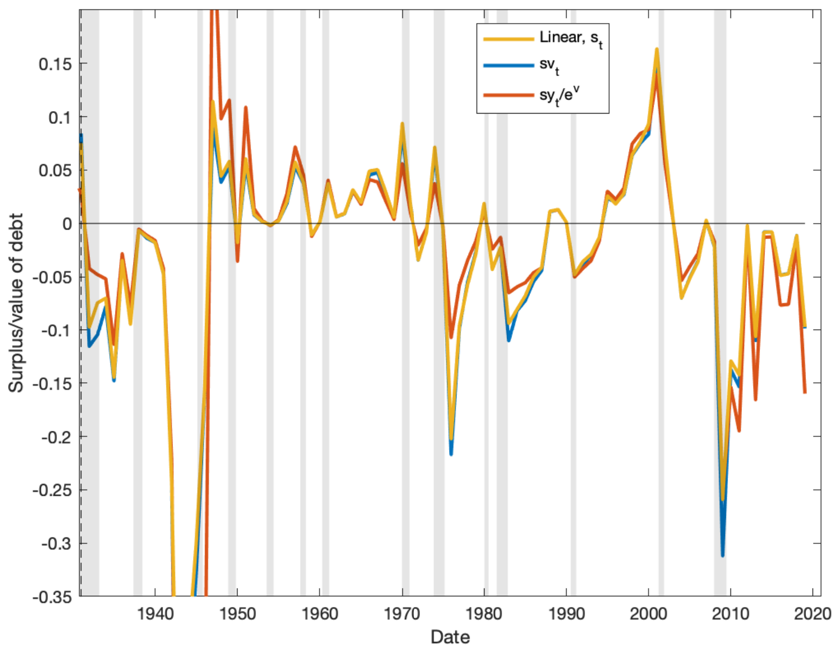

Figure 1: Surplus/GDP. “Linear” is inferred from the linearized flow identity, and is the definition used in VAR analysis. “sv ” is the exact ratio of the primary surplus to the previous year's market value of the debt. “sy” is the exact ratio of surplus to consumption, scaled by the average consumption to GDP ratio and the average value of debt. Vertical shading denotes NBER recessions.

of large primary deficits, interrupted by the strong surpluses of the late 1990s. Postwar primary surpluses also have a clear cyclical pattern. The primary surplus correlates very well with the unemployment rate (not shown), a natural result of procyclical tax revenues, automatic spending such as unemployment insurance, disability and food stamps, and regular discretionary countercyclical “stimulus” spending.

To measure the accuracy of the linear approximation, I also infer the real primary surplus from the exact nonlinear flow identity, as detailed in the Online Appendix. The “sv_t” line presents the ratio of the exact surplus to the previous year's value of the debt. The “sy_t/e^v” line presents the exact surplus to

GDP ratio – actually, the ratio of surplus to consumption, times the average consumption to GDP ratio – scaled by the average value to GDP ratio $e^{E(v_t)}$. The linearization applies equally to either concept.

The three surplus measures in Figure 1 are close. The linearization is less accurate when the value of debt is far from its mean, both in WWII and in the 1970s.

I use a postwar data sample 1947-2018 for the main VAR analysis, as is conventional in empirical macroeconomics. Financing that war, and expectations and reality of paying off war debt, clearly follow a different pattern than fiscal-monetary policy in the subsequent decades of largely cyclical deficits. The WWII deficits come with very low unemployment and high output, contrary to the postwar pattern, and the war featured extensive price controls.

The Online Appendix includes results from 1930-2018, including the great depression and WWII. The results are quite different, in ways traceable to a few influential data points. That analysis suggests that using full sample results to characterize the post-WWII regime is not a good idea.

5. Vector autoregression

Table 1 presents OLS estimates of the VAR coefficients. Each column is a separate regression. The order of variables has no significance. The VAR includes the central variables for the inflation identity – nominal return on the government bond portfolio r^n, consumption growth rate g, inflation π, surplus s and value v. I include the three-month interest rate i and the 10 year bond yield y as they are important forecasting variables for growth, inflation, and long-term bond returns.

It is important to include the value of debt v_t in the VAR, even if we are calculating terms of the innovation identity (3) that does not reference that variable. When we deduce from the present value identity (2) expressions $v_t = E(\cdot|I_t)$, we must include v_t in the information set I_t that takes the expectation. Moreover, the surplus typically follows an s-shaped process, in which deficits

	r^n_{t+1}	g_{t+1}	π_{t+1}	s_{t+1}	v_{t+1}	i_{t+1}	y_{t+1}
r^n_t	-0.17**	-0.02	-0.10**	-0.32*	0.28*	-0.08*	0.04*
g_t	-0.27*	0.20*	0.16*	1.37**	-2.00**	0.28**	0.06
π_t	-0.15	-0.14*	0.53**	-0.25	-0.29	0.09	0.04
s_t	0.12**	0.03	-0.03*	0.35**	-0.24*	-0.04*	-0.04**
v_t	0.01	-0.00	-0.02**	0.04*	0.98**	-0.01	-0.00
i_t	-0.32*	-0.40*	0.29*	0.50	-0.72	0.73**	0.36**
y_t	1.93**	0.54**	-0.17	-0.04	1.60*	0.11	0.46**
$100 \times std(\varepsilon_{t+1})$	2.18	1.53	1.12	4.75	6.55	1.27	0.82
Corr $\varepsilon, \varepsilon_\pi$	-0.29	-0.24	1.00	-0.14	-0.11	0.21	0.31
R^2	0.71*	0.17*	0.73*	0.48*	0.97*	0.82*	0.90*
$100 \times std(x)$	4.08	1.68	2.16	6.61	37.00	2.96	2.63

Table 1: OLS VAR estimate. Sample 1947-2018. One (two) stars means the estimate is one (two) Monte Carlo standard errors away from zero.

today are followed by surpluses in the future. The process is not properly recovered by VARs that exclude the value of debt. Leaving out the value of debt is simply a mistake.

I use a single lag. Adding the last variable, the long-term rate, already introduces slight wiggles in the impulse-response function indicative of overfitting. The results depend on long-run forecasts, which are controlled by the most persistent combination of variables. Fast-moving variables that improve short-term forecasts have little effect on long-term forecasts.

I compute standard errors from a Monte Carlo, described in the Online Appendix. The stars in Table 1 represent one or two standard errors above zero. Since we aren't testing anything, stars are just a visual way to show standard errors without another table.

In the first column, the long-term bond yield y_t forecasts the government bond portfolio return r^n_{t+1} (1.93). The negative coefficient on the three-month rate i_t means that the long-short spread also forecasts those returns. Since the y_t

and i_t coefficients are not repeated in forecasting inflation and growth, the long rate and long-short spread forecast real, growth-adjusted, and excess returns on government bonds, as we expect from the long literature in which yield spreads forecast bond risk premia (Fama and Bliss (1987), Campbell and Shiller (1991), Cochrane and Piazzesi (2005)). The long rate y_t is thus an important state variable for measuring expected bond returns, the relevant discount rate for our present value computations.

Growth g_t is only very slightly persistent (0.20). The term spread $y_t - i_t$ also predicts economic growth, and reinforcing the importance of the interest rates as state variables.

Inflation π_t is moderately persistent (0.53). The interest rate and growth help a bit to predict inflation, but not much else does. We will see inflation responses that mostly look like AR(1) decay.

The surplus/GDP is somewhat persistent (0.35). Growth g_t predicts higher surplus/GDP, an important and realistic feedback mechanism. Inflation forecasts deficits (-0.25), so we expect that to some extent inflation may be related to subsequent deficits.

Debt also forecasts surplus/GDP (0.04), which is important to the following dynamics. Deficits raise debt, and then larger debts lead to surpluses which slowly pay off some of the debt accumulated from the deficits. The 0.04 VAR coefficient of surplus/GDP on debt does not mean that the estimates measure a passive fiscal policy. Surpluses that follow a completely exogenous s-shaped process will produce this coefficient, and surpluses may respond to past deficits but not to off-equilibrium inflation. See Leeper and Li (2017), an extensive counterexample in Cochrane (2020a) and long discussion in Cochrane (2020b).

The value of the debt is very persistent (0.98). It thus becomes the most important state variable for long-run calculations. A larger surplus/GDP s_t forecasts lower debt, v_{t+1}, (-0.24), as one expects. The long-run yield y_t forecasts a rise in the value of debt v_{t+1}, as we expect given its effect on the expected return r^n_{t+1}.

The short rate i_t and long yield y_t are also persistent (0.73, 0.46) and the interest rate forecasts the long yield, again reflecting standard yield curve dynamics. The combination of interest rate and long yield form the second most persistent state variable, which drives medium term responses that differ from those responding to the value of debt.

For calculations reported below, I use the standard notation

$$x_{t+1} = Ax_t + \varepsilon_{t+1} \tag{7}$$

to denote this VAR.

6. Responses and decompositions

I start by examining the fiscal roots of a simple inflation shock, an unexpected movement in inflation $\Delta E_1 \pi_1 = \varepsilon_1^{\pi} = 1$. I allow all other variables to move contemporaneously to the inflation shock. In either reading of causality, we want to measure simultaneous movements of inflation and other variables. To measure how much other variables typically move conditional on seeing an inflation shock, I fill in shocks to the other variables by running regressions of their shocks on the inflation shock. For each variable z, I run

$$\varepsilon_{t+1}^{z} = b_{z,\pi} \varepsilon_{t+1}^{\pi} + \eta_{t+1}^{z}.$$

Then I start the VAR (7) at

$$\varepsilon_1 = \left[\begin{array}{cccccc} b_{r^n,\pi} & b_{g,\pi} & \varepsilon_1^{\pi} = 1 & b_{s,\pi} & \ldots \end{array} \right]'.$$

This procedure is equivalent to the usual orthogonalization of the shock covariance matrix with inflation last, but it is more transparent and it generalizes more easily later. I denote the VAR innovations as the change in expectations at time 1, so the response of variable x, j periods in the future is $\Delta E_1 x_j$.

Figure 2 plots responses to this inflation shock. The "Inflation" rows of Table 2 present the terms of the inflation and bond return decompositions for these responses. (I discuss the remaining rows of Table 2 later.) Figure 2 also presents some of the main terms in the decomposition identities, (3), (4), (5).

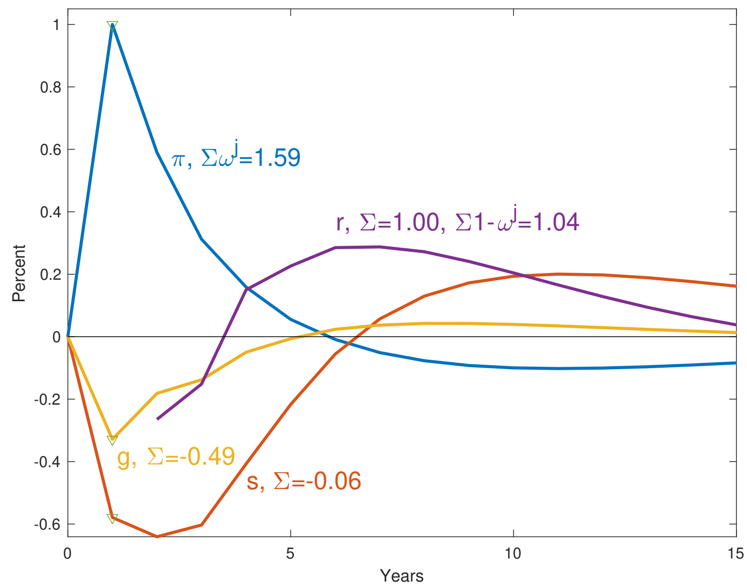

Figure 2: Responses to a 1% inflation shock.

In any interpretation, these responses and calculations answer the question, "if we see an unexpected 1% inflation, how should we revise our forecasts of other variables?" In a fiscal-theoretic interpretation, they answer "what changes in expectations caused the 1% inflation?" As shown in the Online Appendix, the inflation decompositions are also decompositions of the variance of unexpected inflation: They answer the question, "What fraction of the variance of unexpected inflation is due to each component?"

Table 3 presents Monte Carlo quantiles of the sampling distributions of the terms of the inflation decompositions in Table 2. Figure 9, below, plots quantiles of the impulse-response functions. I discuss sampling variation below, after seeing the message in point estimates.

In Figure 2, the inflation shock is moderately persistent, largely following the AR(1) dynamics we noticed in the VAR coefficients. As result, the weighted sum $\sum_{j=0}^{\infty} \omega^j \Delta E_1 \pi_{1+j} = 1.59\%$, greater than the 1% initial shock.

The inflation shock coincides with a deficit s_1, which builds with a hump shape. That shape largely reflects the -0.25 coefficient by which inflation fore-

$$\sum_{j=0}^{\infty} \omega^j \Delta E_1 \pi_{1+j} = -\sum_{j=0}^{\infty} \Delta E_1 s_{1+j} - \sum_{j=0}^{\infty} \Delta E_1 g_{1+j} + \sum_{j=1}^{\infty} (1-\omega^j) \Delta E_1 r_{1+j}$$

	π	$=$	s	g	r
Inflation	1.59	=	-(-0.06)	-(-0.49)	+(1.04)
Recession	-2.36	=	-(-1.15)	-(-1.46)	+(-4.96)
Surplus	-0.10	=	-(-0.66)	-(-0.34)	+(-1.10)
Disc. Rate	-0.18	=	-(-0.54)	-(-0.28)	+(-1.00)
Surplus, no i	0.38	=	-(-0.52)	-(-0.48)	+(-0.62)

$$\Delta E_1 \pi_1 - \Delta E_1 r_1^n = -\sum_{j=0}^{\infty} \Delta E_1 s_{1+j} - \sum_{j=0}^{\infty} \Delta E_1 g_{1+j} + \sum_{j=1}^{\infty} \Delta E_1 r_{1+j}$$

	π	r^n	$=$	s	g	r
Inflation	1.00	-(-0.56)	=	-(-0.06)	-(-0.49)	+(1.00)
Recession	-1.00	-(1.19)	=	-(-1.15)	-(-1.46)	+(-4.79)
Surplus	0.02	-(0.27)	=	-(-0.66)	-(-0.34)	+(-1.25)
Disc. Rate	-0.03	-(0.28)	=	-(-0.54)	-(-0.28)	+(-1.13)
Surplus, no i	0.36	-(0.03)	=	-(-0.52)	-(-0.48)	+(-0.67)

$$\Delta E_1 r_1^n = -\sum_{j=1}^{\infty} \omega^j \Delta E_1 r_{1+j} - \sum_{j=1}^{\infty} \omega^j \Delta E_1 \pi_{1+j}$$

	r^n	$=$	r	π
Inflation	-0.56	=	-(-0.03)	-(0.59)
Recession	1.19	=	-(0.17)	-(-1.36)
Surplus	0.27	=	-(-0.15)	-(-0.12)
Disc. Rate	0.28	=	-(-0.13)	-(-0.15)
Surplus, no i	0.03	=	-(-0.05)	-(0.02)

Table 2: Terms of the inflation and bond return identities. Each entry is the indicated sum of response functions.

casts surplus/GDP. One might think these persistent deficits account for inflation. But surplus/GDP eventually rises to offset almost all of the incurred debt. The sum of all surplus/GDP s responses is -0.06%, essentially zero.

$$\sum_{j=0}^{\infty}\omega^j\Delta E_1\pi_{1+j} = -\sum_{j=0}^{\infty}\Delta E_1 s_{1+j} - \sum_{j=0}^{\infty}\Delta E_1 g_{1+j} + \sum_{j=1}^{\infty}(1-\omega^j)\Delta E_1 r_{1+j}$$

	π	=	s	g	r
Inflation 25 %	1.38	=	-(-0.69)	-(-0.72)	+(0.16)
Inflation 75 %	1.64	=	-(0.23)	-(-0.22)	+(1.46)
Recession 25 %	-2.41	=	-(-1.28)	-(-1.45)	+(-4.84)
Recession 75 %	-2.05	=	-(0.49)	-(-0.57)	+(-2.43)
Surplus 25 %	-0.11	=	-(-0.78)	-(-0.39)	+(-1.11)
Surplus 75 %	0.02	=	-(-0.61)	-(-0.22)	+(-0.98)
Disc. Rate 25 %	-0.26	=	-(-0.63)	-(-0.34)	+(-1.00)
Disc. Rate 75 %	-0.13	=	-(-0.46)	-(-0.18)	+(-1.00)
Surplus, no i 25 %	0.21	=	-(-0.78)	-(-0.48)	+(-0.76)
Surplus, no i 75 %	0.45	=	-(-0.52)	-(-0.22)	+(-0.50)

$$\Delta E_1\pi_1 - \Delta E_1 r_1^n = -\sum_{j=0}^{\infty}\Delta E_1 s_{1+j} - \sum_{j=0}^{\infty}\Delta E_1 g_{1+j} + \sum_{j=1}^{\infty}\Delta E_1 r_{1+j}$$

	π	r^n	=	s	g	r
Inflation 25%	1.00	-(-0.71)	=	-(-0.69)	-(-0.72)	+(0.16)
Inflation 75%	1.00	-(-0.39)	=	-(0.23)	-(-0.22)	+(1.55)
Recession 25%	-1.00	-(0.96)	=	-(-1.28)	-(-1.45)	+(-4.84)
Recession 75%	-1.00	-(1.40)	=	-(0.49)	-(-0.57)	+(-2.35)
Surplus 25%	0.00	-(0.21)	=	-(-0.78)	-(-0.39)	+(-1.30)
Surplus 75%	0.09	-(0.34)	=	-(-0.61)	-(-0.22)	+(-1.15)
Disc. Rate 25%	-0.07	-(0.25)	=	-(-0.63)	-(-0.34)	+(-1.24)
Disc. Rate 75%	-0.01	-(0.42)	=	-(-0.46)	-(-0.18)	+(-1.10)
Surplus, no i 25%	0.18	-(-0.08)	=	-(-0.78)	-(-0.48)	+(-0.86)
Surplus, no i 75%	0.38	-(0.07)	=	-(-0.52)	-(-0.22)	+(-0.57)

$$\Delta E_1 r_1^n = -\sum_{j=1}^{\infty}\omega^j\Delta E_1 r_{1+j} - \sum_{j=1}^{\infty}\omega^j\Delta E_1\pi_{1+j}$$

	r^n	=	r	π
Inflation 25%	-0.71	=	-(-0.12)	-(0.38)
Inflation 75%	-0.39	=	-(0.19)	-(0.64)
Recession 25%	0.96	=	-(-0.17)	-(-1.41)
Recession 75%	1.40	=	-(0.28)	-(-1.05)
Surplus 25%	0.21	=	-(-0.24)	-(-0.13)
Surplus 75%	0.34	=	-(-0.12)	-(-0.05)
Disc. Rate 25%	0.25	=	-(-0.24)	-(-0.20)
Disc. Rate 75%	0.42	=	-(-0.11)	-(-0.11)
Surplus, no i 25%	-0.08	=	-(-0.18)	-(0.00)
Surplus, no i 75%	0.07	=	-(-0.00)	-(0.10)

Table 3: Monte Carlo quantiles of the inflation and bond return identities.

Inflation also is also correlated with a persistent decline in economic growth g. The stagflationary episodes of the 1970s drive this result. The growth decline contributes 0.49% to the inflation decompositions.

Lower growth means a lower actual surplus for given surplus/GDP. Thus, of the total 1.59% inflation on the left hand side, a decline in surpluses, accounts for 0.49+0.06 = 0.55%, about one third, and almost all of that via lower growth not lower surplus/GDP.

The line marked r plots the response of the real discount rate, $\Delta E_1 r_{1+j} = \Delta E_1 (r^n_{1+j} - \pi_{1+j})$. These points are plotted at the time of the ex-post return, $1+j$, so they are the expected return one period earlier, at time j. The line starts at time 2, where the terms of the discount-rate sums in the inflation decompositions start, and representing the time-1 expected return. After two periods, this discount rate rises and stays persistently positive. The weighted sum of discount rate terms is 1.04% while the unweighted sum is 1.00% (really 1.004%). I choose the weight $\omega = 0.69$ to make the identity (4) hold exactly for this response function. The value 0.69^j declines rapidly, so weighting by 1 vs. $1 - \omega^j$ makes little difference in the face of this persistent response.

Weighted or unweighted, the discount rate terms account for 1% inflation, and about 2/3 of the overall inflation. A higher discount rate lowers the value of government debt, an inflationary force.

Overall, then,

- *A 1% shock to inflation corresponds to a 1.6% decline in the present value of surpluses. A rise in discount rate contributes about 1%, and a decline in growth accounts for about 0.6% of that decline. Changes in the surplus/GDP ratio account for nearly nothing. The additional 0.6% fiscal shock corresponds to a persistent rise in expected inflation, which slowly devalues outstanding long-term bonds, and produces a 1.6% overall rise in inflation weighted by the maturity structure of debt.*

This is an important finding for matching the fiscal theory to data, for understanding the fiscal side of standard passive-fiscal models, or for questions of cyclical fiscal policy and debt sustainability in general. Thinking in all contexts has focused on the presence or absence of surpluses, or surplus to GDP ratios, not discount rate effects, time-varying returns. Thinking in all contexts has considered one-period unexpected inflation, to devalue one-period bonds, not a rise in expected inflation that slowly devalues outstanding long-term bonds.

Turn to Table 2 for a more systematic view of the inflation decompositions, and to see the role of one-period bond returns $\Delta E_1 r_1^n$. The top row of the top panel presents the just-discussed overall decomposition (5) of current and expected future inflation in terms of surplus, growth and discount rate shocks. The second and third panels express the decomposition of one-period inflation, using the bond return r_1^n. The sum of surplus and growth rate terms are the same in this second panel as in the top panel, but I repeat them so one can see the terms of each identity more clearly. In the first row of the second panel, the 1% inflation shock corresponds to a roughly 1.56% overall fiscal shock. That shock comes similarly from the same tiny 0.06% decline in surplus/GDP, a 1.004% rise in discount rate and 0.49% reduction in growth. In this decomposition, the extra 0.56% fiscal shock is absorbed by a 0.56% decline in the value of government debt, r_1^n. Turning to the last panel, we see that -0.56% return on government debt comes almost entirely from expected inflation (0.59%) not a higher real discount rate (0.03%). That fact ties together the decompositions of the first and second panels. The government bond return essentially marks to market the expected future inflation of the top panel.

Discount rates matter in the inflation decompositions of the top two panels but not in this return decomposition because the former have weights that emphasize long-term movements (1 and $1-\omega^j$), while the ω^j weights of the bottom panel emphasize a short-run movement in discount rate.

In sum,

- *The 1.6% fiscal shock that comes with 1% unexpected inflation is buffered by an 0.5% decline in bond prices, which corresponds to 0.5% additional expected future inflation. The additional expected inflation slowly devalues* *long-term bonds as they come due, a loss in value marked to market in the initial fall in bond prices.*

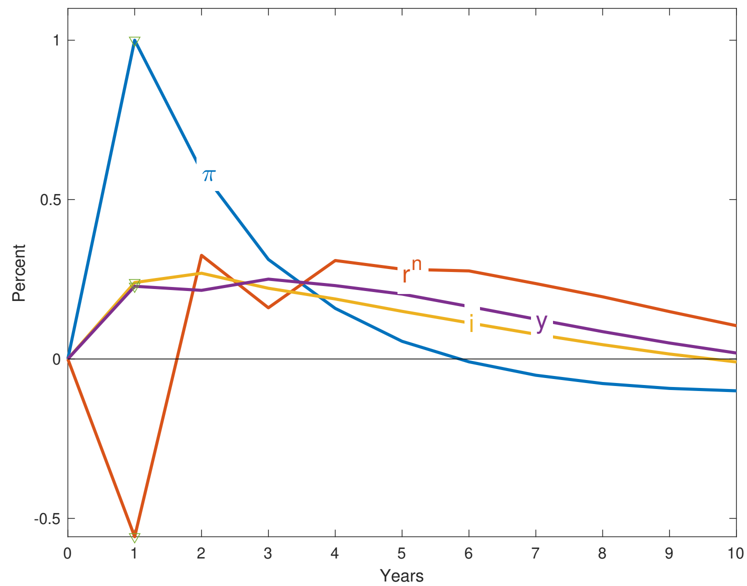

Figure 3: Responses to 1% inflation shock

Figure 3 adds detail and intuition to the interest rate and return responses. The interest rate i, bond yield y, and expected return r^n all move together and persistently. (The sawtooth pattern in r^n at time 3 comes from a slightly negative eigenvalue of the VAR, which is far below statistical significance.) The return shock r_1^n moves down sharply as expected subsequent returns rise. Bond prices decline when yields rise. The rise in expected return is largely driven by the rise in the interest rate, with smaller contribution from a larger risk premium.

In turn, the rise in real discount rates we saw in Figure 2 stems from the apparent disconnect between nominal returns and inflation that we see in Figure 3. Inflation is initially above nominal rates, giving a few periods of lower real rates. When inflation declines below the more persistent nominal rates, implied real interest rates rise on the right hand side of the graphs, and persistently. In the VAR, interest rates do not forecast inflation as strongly as they forecast interest rates, which generates the high real rates from persistently high nominal rates.

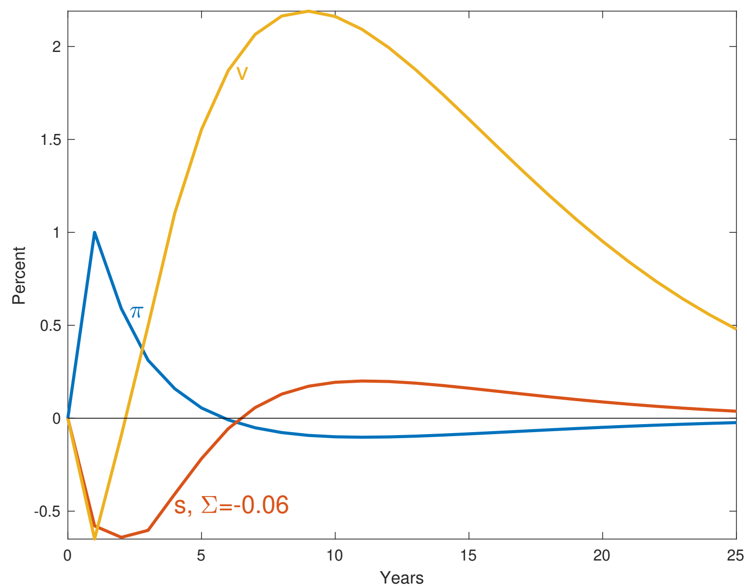

Figure 4: Responses to 1% inflation shock

Figure 4 plots the response of surplus/GDP and value of debt to the unexpected inflation shock. The debt-to-GDP ratio v_1 declines -0.65% on impact, reflecting the offsetting forces of deficits, inflation, bond returns, and growth in the innovation version of the flow identity (1). The long string of deficits then raises the value of debt. But, crucially, higher debt leads to higher surpluses. Eventually, therefore, surpluses rise and pay down the debt.

The s-shaped surplus/GDP response is a crucial lesson. It means that early debts are repaid, at least in part, by following surpluses. The surplus/GDP does *not* follow an AR(1)-like process. Mechanically, this pattern is a result of the VAR coefficient of surplus/GDP on lagged debt, and the persistence of debt. Thus, the finding is econometrically robust; it does not rely on a tenuous measurement of high-order surplus autocorrelations.

However, this analysis illustrates the vital practical importance of including debt in the VAR. Without debt in the VAR, the surplus/GDP is positively autocorrelated throughout, and surplus/GDP never rises to pay off deficits.

I use the words "shock," and "response," which are conventional in the VAR literature, and compactly describe the calculations for those familiar with VARs. The calculations do not imply or require a causal structure, nor do they make any pretense to measure structural shocks. A "shock" here is only an "innovation," a movement in a variable not forecast by the VAR. A "response" is a change in VAR expectations of a future variable coincident with such a movement.

In fact, my fiscal theory interpretation offers a reverse causal story: News about future surpluses and discount rates causes inflation to move today. That news in turn reflects news about future productivity, fiscal and monetary policy and other truly exogenous or structural disturbances. Many VAR exercises attempt to find an "exogenous" movement in a variable by careful construction of shocks, or they attempt to measure structural shocks, and they attempt to measure responses as causal effects of such identified structural or exogenous policy shocks. I do not.

I do not assume that people use only the VAR information set to form expectations. Since we start with an identity (1) that holds ex-post, or under people's information sets, the identity holds using any coarser information set that includes the value of debt. The model $v_t = E(x_{t+1}|\Omega_t)$ implies $v_t = E(x_{t+1}|I_t \subset \Omega_t)$, so long as $v_t \in I_t$. But "unexpected" here means relative to the VAR information set. People may see a lot more. The VAR forecasts are correct *on average*, but they integrate out other variables which people may see.

6.1. Recession or aggregate demand shocks

We can use the same procedure to understand the fiscal underpinnings or correlates of other shocks. For any interesting ε_1, we can compute impulse-response functions, and thereby the terms of the inflation decompositions. I show in the Online Appendix that we can consider these calculations as a decomposition of the covariance of unexpected inflation with the shock ε_1, rather the decomposition of the variance of unexpected inflation.

I start with a shock that moves inflation and growth in the same direction. The inflation shock in Figure 2 is stagflationary, in that growth falls when inflation rises. Unexpected inflation is, in this sample, negatively correlated with unexpected growth. The stagflationary 1970s drive this correlation.

However, it is interesting to examine the response to disinflations which come in recessions, and inflations that come in expansions, following a conventional Phillips curve. Such events are common, as in the recession following the 2008 financial crisis. But they pose a fiscal puzzle. In such a recession, deficits soar, yet inflation declines. How is this possible? As I outlined in the introduction, future surpluses or lower discount rates could give that deflationary force, needed whether fiscal policy is active or passive. Can we see these effects in the data, and which one is it?

To answer that question, I simply specify $\varepsilon_1^\pi = -1$, $\varepsilon_1^g = -1$. The model is linear, so the sign doesn't matter, but the story is clearer for a recession. To give it a name, I call this a "recession shock" in the tables. We could also call it an "aggregate demand" shock, because output and inflation move in the same direction, as opposed to "aggregate supply" shocks which move output and inflation in opposite directions.

Again, we want shocks to other variables to have whatever value they have, on average, conditional on the inflation and output shock. To initialize the other shocks of the VAR, then, I run a multiple regression

$$\varepsilon_{t+1}^z = b_{z,\pi}\varepsilon_{t+1}^\pi + b_{z,g}\varepsilon_{t+1}^g + \eta_{t+1}^z$$

for each variable z. I fill in the other shocks at time 1 from their predicted variables given $\varepsilon_1^\pi = -1$ and $\varepsilon_1^g = -1$, i.e. I start the VAR at

$$\varepsilon_1 = -\left[\begin{array}{ccccc} b_{r^n,\pi} + b_{r^n,g} & \varepsilon_1^g = 1 & \varepsilon_1^\pi = 1 & b_{s,\pi} + b_{s,g} & \ldots \end{array}\right]'.$$

Figure 5 presents responses to this shock, and Table 2 collects the inflation decomposition elements in the "Recession" rows.

Both inflation π and growth g responses start at -1%, by construction. Inflation is once again persistent, with a ω-weighted sum of current and expected

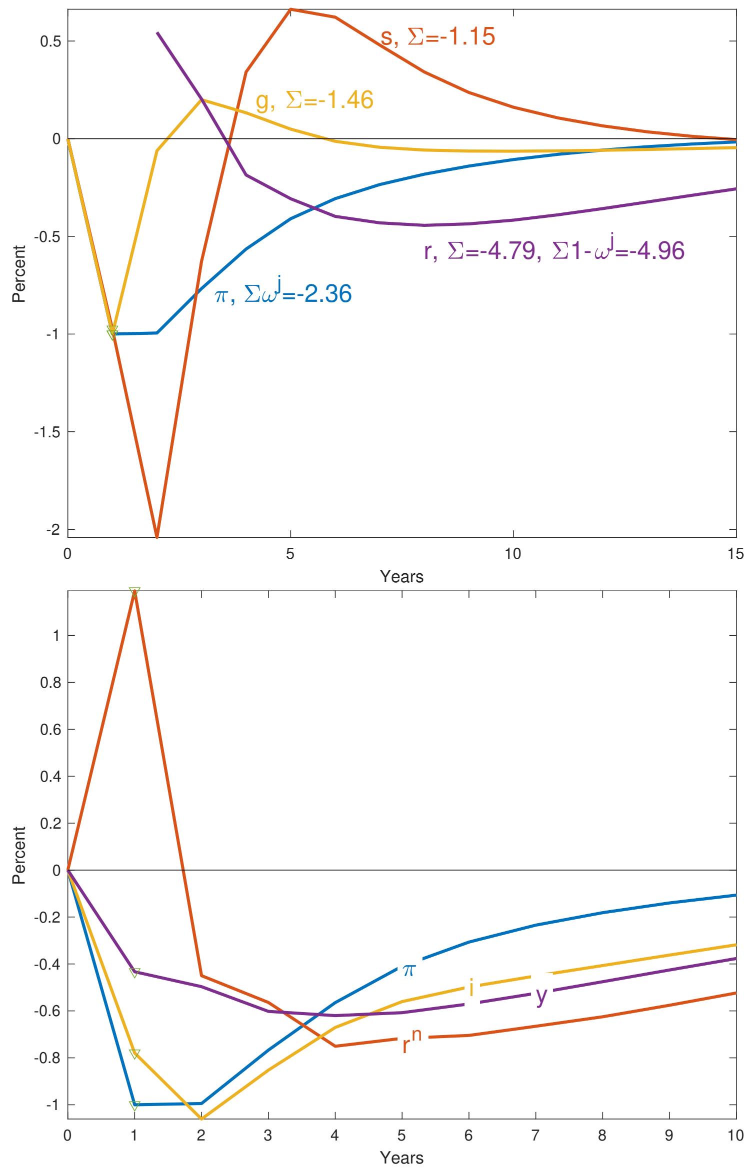

Figure 5: Responses to a recession or aggregate demand shock, $\varepsilon_1^\pi = \varepsilon_1^g = -1$.

future inflation equal to -2.36%. Growth g returns rapidly, but does not much overshoot zero, so the level of consumption does not recover much at all. Consumption is roughly a random walk in response to this shock. The nominal interest rate i falls in the recession, and recovers a bit more slowly than inflation. Long-term bond yields y also fall, but not as much as the short-term rate, for about 4 years. We see here the upward-sloping yield curve of a recession. The expected bond return follows the long-term yield. The persistent fall in expected return corresponds to a large positive ex-post bond return $\Delta E_1 r_1^n$. The recession includes a large deficit s, which continues for three years. In short, we see a standard picture of a recession similar to 2008-2009.

Why do we not see inflation at times with such large deficits? Surplus/GDP subsequently turns positive, paying down some of the debt. But the total surplus/GDP response is still -1.15. Left to their own devices, surplus/GDP would produce a 1.15% *inflation* during the recession. Growth also adds an inflationary force. The decline in consumption is essentially permanent, and would lead on its own to another 1.46% *in*flation.

Discount rates are the central story for disinflation in recessions. After one period, expected real returns ri decline persistently, accounting for 4.96% cumulative deflation.

In terms of the unexpected inflation accounting in the second and third panels of Table 2, again surplus/GDP and growth provide a total 1.15% + 1.46% = 2.61% fiscal loosening, an inflationary force. The unweighted sum of future discount rates provides a 4.79% deflationary force, for an overall fiscal shock of 2.19% deflation. Of that, 1% results in unexpected deflation and 1.19% is soaked up by lower long-term bond prices. In the bottom panel, that 1.19% overwhelmingly represents lower expected inflation, essentially marking it to market for a one-period accounting.

In sum, rounding the numbers,

- *Disinflation in a recession is driven by a lower discount rate, reflected in lower interest rates and bond yields. For each 1% disinflation and growth* *shock, the expected return on bonds falls so much that the present value of debt rises by nearly 5%. This discount rate shock overcomes a 1.1% inflationary shock coming from persistent deficits, and 1.5% inflationary shock coming from lower growth. The overall fiscal shock is 1.6%, with the extra 0.6% spread to future disinflation and soaked up by long-term bond prices.*

The opposite conclusions hold of inflationary shocks in a boom. Discount rate variation gives us a fiscal Phillips curve, accounting for the otherwise puzzling correlation of deficits with disinflation and surpluses with inflation.

The relative magnitudes of the inflation and growth shocks that I use in this calculation are obviously arbitrary. The plots and calculations correctly report the answer to the question, "If we see a -1% growth shock and a -1% inflation shock together, how does that observation change our forecasts of all variables?" The labels "aggregate demand" or "recession" are just suggestive to give the exercise a label, with no pretense to identify structural shocks. The only question is whether that combination of shocks is interesting, or whether some other combination of growth, inflation, and other shocks might present a more interesting calculation.

To produce a better-named shock one should write a model and find an identification in the data. One might separate "aggregate supply" shocks or "stagflationary" "Phillips-curve shift" shocks from "aggregate demand" shocks or "movement along the Phillips curve" shocks. Those restrictions might include the other variables as well. Even these concepts refer to ideas from the 1970s. Today's intertemporal models specify objects such as technology shocks, financial friction shocks, marginal cost shocks, and so forth. Rather than belabor the point with such calculations, or fill the paper with multiple graphs, I choose a simple and transparent value, consistent with the measurement without theory philosophy of the rest of this paper.

The Online Appendix includes plots of GDP growth and CPI inflation, as well as the growth and inflation VAR residuals. The 1970s show the opposite sign: inflation and growth move in opposite directions, stagflation, and basically one for one. These large events drive the negative full sample correlation. By contrast, 1982 saw a sharp decline in inflation along with the comparably sized recession. Inflation moved a bit less than growth in the 2000 recession, but again moved about one for one with growth in 2008. The late 1940s and early 1950s also show roughly one for one positive comovement. Of course, no historical event is the pure result of a single external or policy shock. The point is only, some recessions involve inflation that moves roughly one for one with growth, and some involve inflation that moves in the opposite direction. It's interesting to plot responses to the former kind of event and one for one is in the range of experience.

6.2. Surplus and discount rate shocks

We have studied what happens to surpluses and to discount rates given that we see unexpected inflation. What happens to inflation if we see changes in surpluses or discount rates? These are not the same questions. An inflation shock may come, on average, with a discount rate shock, but a discount rate shock may not come on average with inflation. The average person who gets hit by a bus has tried to cross the street, but the average person who crosses a street does not get hit by a bus.

I calculate here how the variables in the VAR react to an unexpected change in current and expected future primary surpluses including growth, $\Delta E_1 \sum_{j=0}^{\infty}(s_{t+j}+g_{t+j}) = -1$, and other shocks to the VAR take their average values given this innovation. I call this a "surplus shock." This is, by construction, a shock that is not repaid by subsequent surpluses, so it either must correspond to inflation or to a change in discount rate. The results are almost the same with or without the growth term in the shock definition. Then I calculate how the variables in the VAR react to an unexpected change in discount rates, $\Delta E_1 \sum_{j=1}^{\infty}(1-\omega^j)(r^n_{t+1}-\pi_{t+1}) = 1$, again letting all other variables take their average values given this innovation. I call this a "discount rate shock."

These are not monetary and fiscal policy shocks, as studied in Cochrane

(2020a), most models, and VAR literature. The fiscal shock may be, and is, correlated with a change in interest rates, and the discount rate shock may be, and is, correlated with fiscal changes, where “policy shocks” are more interestingly defined to be orthogonal to each other. I also make no attempt to orthogonalize these shocks relative to forecasts of inflation, growth, or other variables, to call them “exogenous.” In fact, they are the opposite, deliberately reflecting news to the economy as a whole, changing expectations of true and future structural and policy disturbances.

The response of the sum of future surpluses and growth to any shock ε_1 is

$$\Delta E_1 \sum_{j=0}^{\infty} (s_{1+j} + g_{t+j}) = (a_s + a_g)' (I - A)^{-1} \varepsilon_1.$$

To calculate how VAR shocks respond to a surplus shock, I run for each variable z a regression

$$\varepsilon^z_{t+1} = b_z \left[(a_s + a_g)' (I - A)^{-1} \varepsilon_{t+1} \right] + \eta^z_{t+1} \tag{8}$$

where a_z pulls variable z from the VAR, $a_z' x_t = z_t$. Then, I start the surplus-shock response function at

$$\varepsilon_1 = - \begin{bmatrix} b_{r^n} & b_g & b_\pi & ... \end{bmatrix}'.$$

I plot a negative surplus shock, i.e. a deficit shock, as that sign tells an easier story.

Similarly, to calculate responses to a discount-rate shock, I run

$$\varepsilon^z_{t+1} = b_z \left\{ (a_{r^n} - a_\pi)' \left[A(I - A)^{-1} - \omega A (I - \omega A)^{-1} \right] \varepsilon_{t+1} \right\} + \eta^z_{t+1}.$$

I start the discount-rate response function with the negative of these regression coefficients as well, capturing the response to a discount rate decline.

Figure 6 presents the responses to the deficit shock, and Figure 7 presents the responses to the discount rate shock. Table 2 collects relevant contributions to the inflation decompositions.

The sum of surplus/GDP and growth responses to the deficit shock are -0.66 -0.34 = -1.00 by construction. Surplus/GDP still has an s-shaped response, but the initial deficits are not matched by subsequent surpluses.

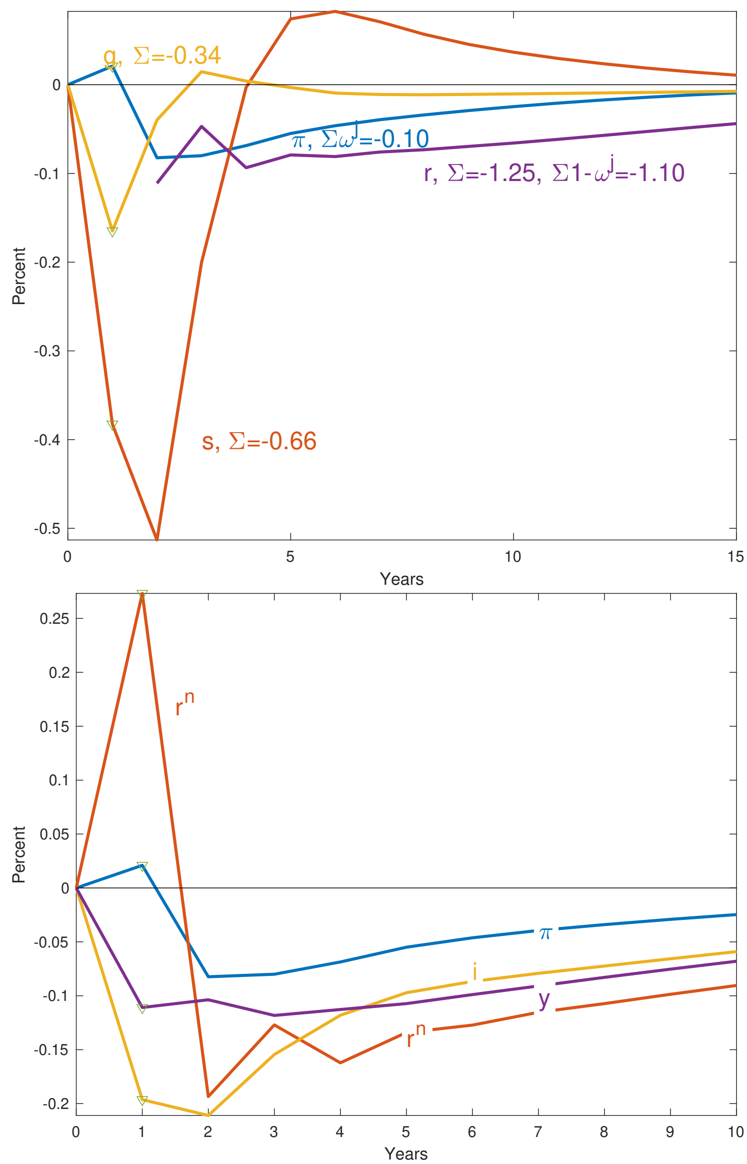

Figure 6: Responses to a surplus and growth shock, $\Delta E_1 \sum_{j=0}^{\infty} (s_{1+j} + g_{1+j}) = -1$.

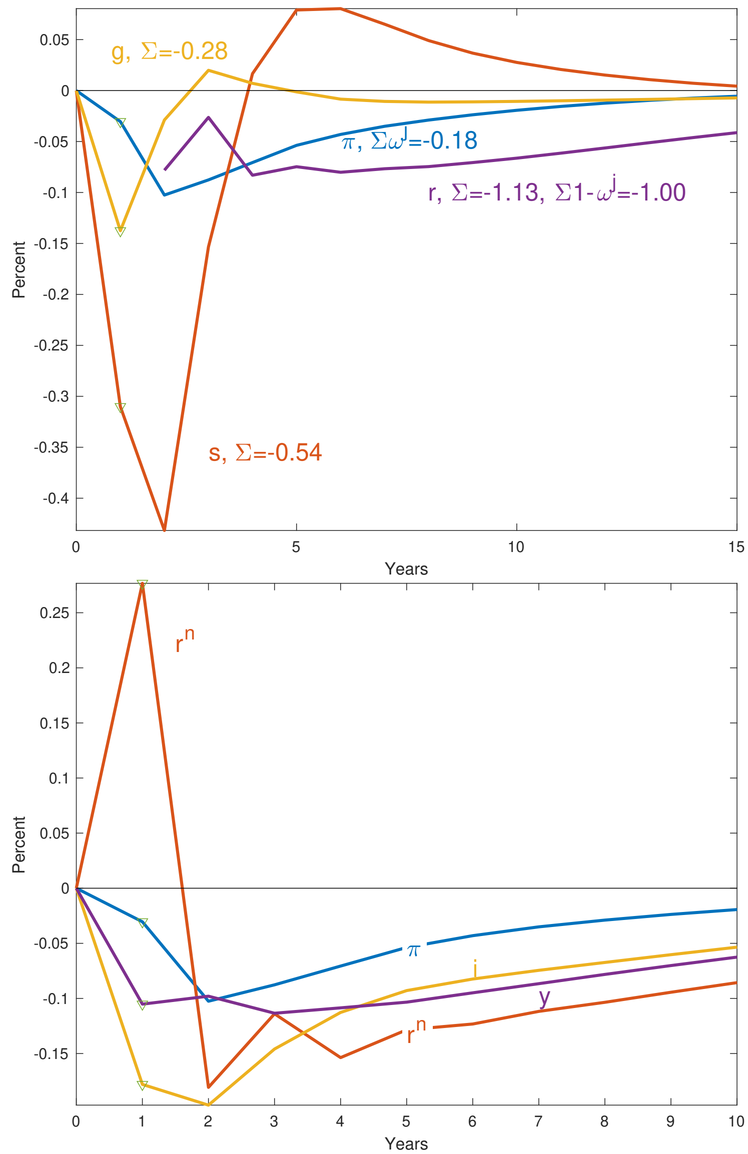

Figure 7: Responses to a discount-rate shock $\Delta E_1 \sum_{j=1}^{\infty}(1-\omega^j)\left(r^n_{1+j}-\pi_{1+j}\right)=1$.

This decline in surpluses and growth has essentially no effect on inflation. Starting in year 2, inflation declines – the "wrong" direction given deficits and lower growth – by less then a tenth of a percent, and the overall weighted sum of inflation declines by a tenth of a percent. Why is there no inflation? Because discount rates also decline, with a weighted sum of 1.10%, almost exactly matching the surplus decline. The lower panel of Figure 6 adds insight. We see a sharp and persistent decline in the interest rate, long-term bond yield, and expected bond return, along with deficits and the growth decline.

This figure captures the event of a widening deficit, accompanied by a decline in growth and interest rates, i.e. a recession. These deficits are on average not directly repaid by subsequent surpluses or growth. Instead, real interest rates decline persistently in the recession and its aftermath. This decline in real returns essentially pays for the deficits. Ex-post, a low real return brings the value of debt back rather than larger taxes or lower spending. There is, on average, very little inflation or deflation. The opposite sign occurs for positive shocks.

The response to the discount rate shock in Figure 7 is, surprisingly, almost exactly the same. The weighted discount rate response ($\sum 1-\omega^j$) is -1.00 here by construction. This discount rate decline should be deflationary, and it is – but the disinflation peaks at -0.1% and the weighted sum is only -0.18%. A sharp growth and surplus decline accompanies this discount rate decline, with a pattern almost exactly the same as we found from the growth and surplus shock. In the bottom panel, the expected return decline comes with a decline in interest rates and bond yields, as we would expect.

Clearly, the surplus + growth shock and the expected return shock have isolated essentially the same events – recessions in which growth falls, deficits rise persistently, interest rates fall, and, on average in this sample, inflation *doesn't* move much, and the converse pattern of expansions. The correlation of the surplus+growth and discount rate shocks is 0.96.

The responses to a one-period surplus shock, $\Delta E_1 s_1 = 1$, a pure growth shock $\Delta E_1 g_1 = 1$ and a one-period discount rate shock $\Delta E_1 r_2^n = 1$ are all quite similar as well.

The fiscal roots of the *absence* of inflation, in the end, characterize these business-cycle movements in the data. Since well-run fiscal and monetary policies borrow and repay without excessive inflation, that outcome is just as interesting.

In sum,

- *Surplus and discount rate shocks paint the same picture: Persistent deficits that are not repaid by subsequent growth or surpluses do not produce inflation. Instead, such deficits come with periods of extended low expected returns. Discount rate declines come with offsetting deficits and do not produce much deflation.*

6.3. A surplus shock without accommodation

The fact that interest rates move in opposition to the surplus shock is obviously key to the noninflationary result. What if there is a surplus shock and the Federal Reserve does not accommodate the shock, or its economic correlates, with the prominent interest decline seen in Figure 6? To answer this question, I modify the surplus+growth shock so that the short-term interest rate remains constant for two years. I now run

$$\varepsilon_{t+1}^{z} = b_{z,s}\left[(a_s + a_g)'(I - A)^{-1}\varepsilon_{t+1}\right] + b_{z,i0}\varepsilon_{t+1}^{i} + b_{z,i1}\left(a_i' A \varepsilon_{t+1}\right) + \eta_{t+1}^{z}.$$

The last term before the error is the expected interest rate one year forward. Then, I initialize the VAR at

$$\varepsilon_1 = -\left[\begin{array}{cccc} b_{r^n,s} & b_{g,s} & b_{\pi,s} & ... \end{array}\right]'.$$

Figure 8 presents the responses, and Table 2 collects the terms of the identities. Starting in the bottom panel of Figure 8, verify that the interest rate i now stays constant for two years, by construction. This behavior contrasts with the strong interest rate decline in the bottom panel of Figure 6. Except for the one-period expected return decline in year two, the long-term bond yields and expected returns follow the interest rate. All decline eventually.

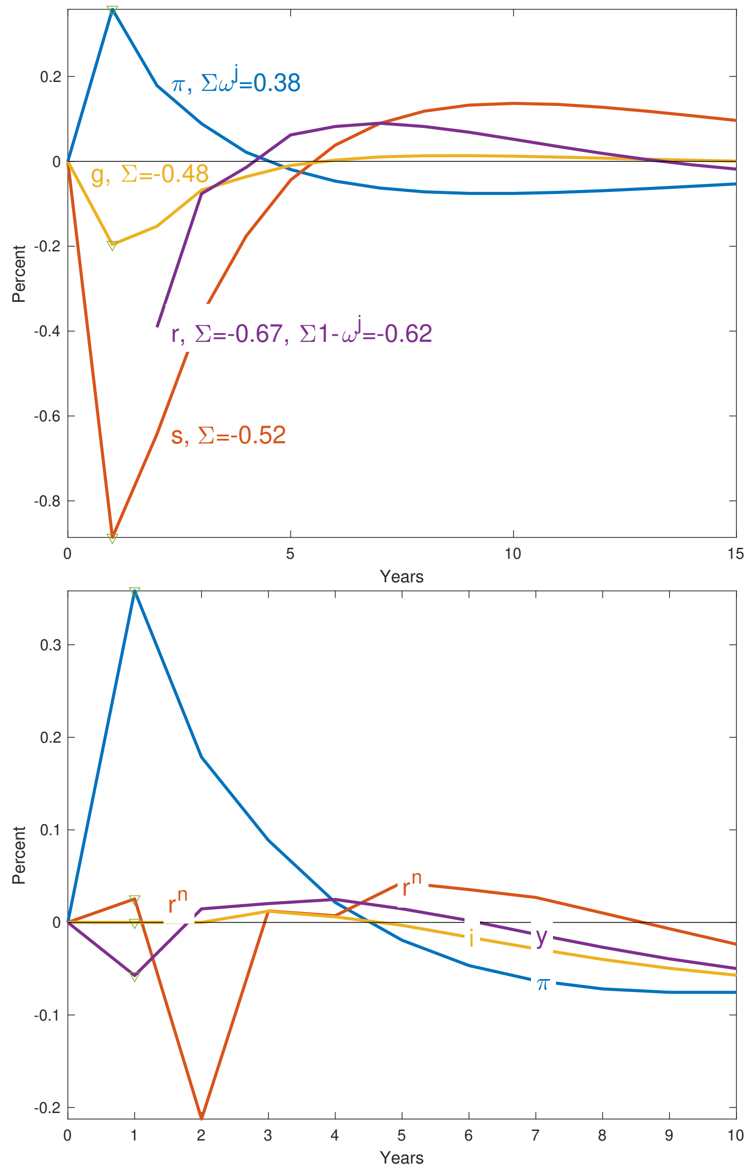

Figure 8: Responses to a surplus and growth shock with no interest rate movement for two years, $\Delta E_1 \sum_{j=0}^{\infty}(s_{1+j}+g_{1+j})=1$, $\Delta E_1 i_1=0$, $\Delta E_1 i_2=0$.

Turning to the upper panel, the sum of surplus (-0.52) and growth (-0.48) shocks remains -1.00% by construction. Deficits are initially much larger than 0.52%, but much of this immediate deficit is repaid by higher long-term surpluses, so in the end the fiscal shock is split equally between surpluses and growth. The discount rate term is now reduced to 0.62% - 0.67%, however, so the surplus shock now produces 0.36% immediate and 0.38% weighted sum inflation.

In sum, without the interest rate response, the fiscal shock does result in unexpected inflation. We see here a parallel of the theoretical analysis that central bank accommodation of shocks, via the interest rate target, smooth forward and thereby reduce unexpected inflation, even though the bank cannot control fiscal policy.

Will the real recession please stand up? How do we have by one calculation recessions with disinflations, and by another recessions with no change in inflation? Alas, our macroeconomy is not a one-factor model, with all time-series moving in lockstep. Different (true, structural) shocks dominate different events. The recessions of the 1970s featured stagflation, those since 1990 did not. All recessions are not the same. Sometimes inflation falls, sometimes it doesn't. I have examined five, hopefully interesting, slices of the full covariance matrix of shocks. They are different.

7. Standard errors

I have delayed a discussion of standard errors because there is nothing important to test. Identities are identities. If $x = y + z$ and x moves, y or z must move, and all we can do is to measure which one moves. Standard errors only serve to give us a sense of how accurate the measurement is. In addition, unlike the case in asset pricing, no important economic hypothesis rests on whether one of surpluses or discount rates do not move. Asset pricing finds the hypothesis that expected returns are constant over time interesting to test.

I run a Monte Carlo to evaluate sampling distributions. The Online Appendix gives details. Most of the interesting statistics – variance decompositions, impulse-response functions, $(I - A)^{-1}$, etc. – are nonlinear functions of the underlying data, and the near-unit root in value v_t also induces non-normal distributions. For these reasons, I largely characterize the sampling distribution by its 25% and 75% percentiles.

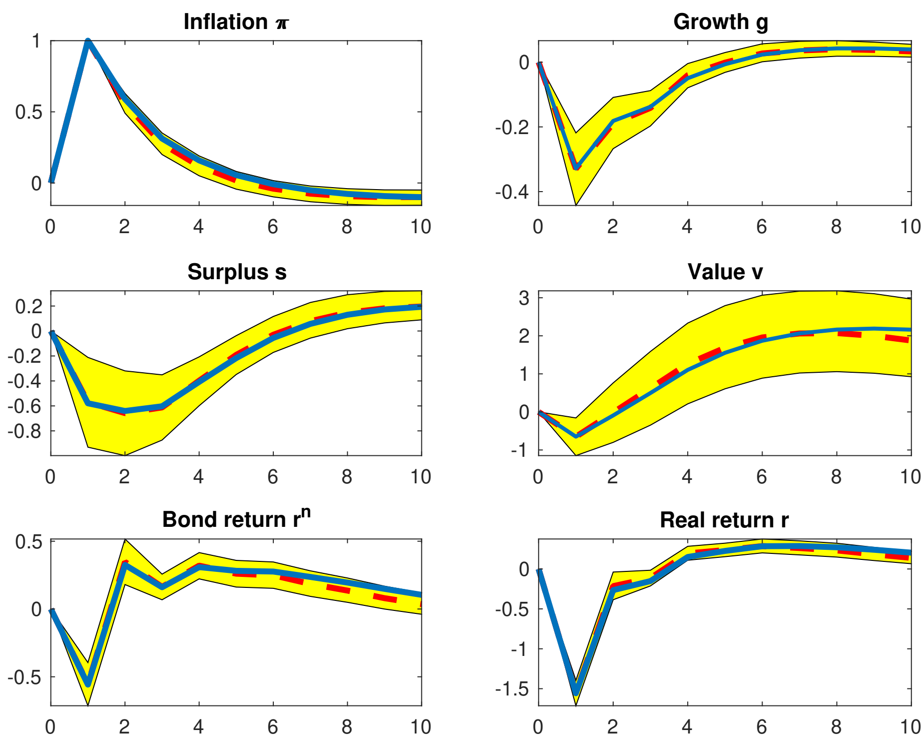

Figure 9: Distribution of the responses to an inflation shock. The bands are 25% and 75% points of the sampling distribution, the dashed line is the median, and the solid line is the estimate.

Table 3 collects the sampling quantiles for the variance decompositions of Table 2. Figure 9 presents the main components of the impulse-response function relevant to the inflation variance decomposition presented in Figure 2. The bands are 25% and 75% points of the sampling distribution, the dashed line is the median, and the solid line is the estimate.

Start with the "Inflation" shock in Table 3. In the second panel, inflation quantiles are 1.00 because the shock is defined as a 1% movement in inflation in every sample. Likewise, there is no sampling variation in the first inflation response of the top left panel of Figure 9. The 1.59% weighted sum of inflation has 1.38% to 1.64% quantiles in the top panel of Table 3. The -0.06% sum of future surpluses has quartiles -0.69% to 0.23%. The -0.49% sum of growth rates has quantiles -0.72% to -0.22% The 1.04% (weighted) and 1.00% (unweighted) discount rate term has quantiles 0.16% to 1.46% and 0.16% to 1.55%. That discount rates matter is a pretty solid conclusion, but deficits may contribute more to unexpected inflation than the point estimate suggests.

There are several sources of this rather large sampling variation. First, the shocks are large. As shown in Table 1, the surplus innovation has a 4.75 percentage point standard deviation, and value 6.55 percentage points, compared to 1.12 percentage points for inflation. Our friend $\sigma/\sqrt{T}$ starts off badly.

Second, the shocks are imperfectly correlated. This matters, because in each case I find movements in other variables contemporaneous with the shock of interest by running a regression of the other shocks on the shock of interest. The sampling uncertainty of this orthogonalization adds to that of the VAR coefficient estimates. We see a correspondingly wide band around the initial surplus and growth responses in Figure 9. Higher frequency data may better measure shock correlations, at the cost that one must model the strong seasonal in primary surpluses. Other shock identifications may have better-measured correlations.

Third, we measure sums of future surpluses and discount rates. The value of the debt v_t is the main long-run state variable, and uncertainty about its evolution adds to the uncertainty about the sum of surpluses. The coefficient of debt v_t on its own lag is 0.98 in Table 1, so small variations in that value lead to large variation in $(I-A)^{-1}$ sums. The Online Appendix shows that the last two sources of variation contribute about equally. A larger or smaller value of this coefficient raises and lowers all the long-run responses together, so the apparently reasonably-measured individual responses of Figure 9 all add up to larger sampling variation of the decomposition terms.

Table 3 also presents 25% and 75% quantiles for the recession, surplus and discount rate shocks of Table 2. The -1.15% total surplus response to a recession shock has quantiles -1.28% to 0.49%, spanning zero, while the -4.96% and -4.79% discount rate response has quantiles from -4.84% to -2.43% and -4.84% to -2.35%. The conclusion that discount rate variation is a central part of the story for understanding aggregate-demand inflation is fairly solid. The small inflation and offsetting surplus and discount rate responses to surplus and discount rate shocks are similarly measured.

It would be nice if the elements of the identities were more precisely measured. But there is nothing one can do within the framework of this VAR to improve on them, so it's worth examining point estimates while awaiting more data or other approaches such as model-based estimates that impose prior structure. The rather large sampling variation should, however, discourage one from the inevitable temptation to split up the sample or add complexity to the specification.

8. Concluding comments

One can apply these decompositions to any VAR, or to the impulse-responses of theoretical models. Such calculations beckon.

In particular, it is interesting to apply the inflation decompositions to model predictions or empirical estimates of monetary and fiscal policy shocks. Cochrane (2020a) presents such calculations from a simple fiscal theory of monetary policy model. Making such calculations in data require one to solve the formidable identification problems of estimating policy rules, given that the right-hand variables react to the disturbances, and identifying sufficiently orthogonal shocks. The state of the art goes well beyond the simple recursive and long-run strategies available in the annual VAR here, to include instruments, high frequency data, narrative approaches, and other devices. The literature still does does not offer a robustly successful approach (Ramey (2016), Cochrane (2011a)). Teasing out monetary policy shocks that are also orthogonal to fiscal policy shocks requires some thought. I attempted monetary and fiscal policy shocks by recursive identification in this data, but one-year interest-rate, inflation, and growth shocks are all highly correlated. Assuming all of that correlation flows from interest rates to inflation and growth results in positive effects of interest rates on inflation and growth. Assuming all correlation reflects rule-like responses of interest rates to inflation and growth eliminates the unexpected inflation response we wish to measure. Obviously, reality lies in between.

Additional measurements beckon. Quarterly or monthly data are attractive, offering potentially better measurement of correlations and shock orthogonalization but requiring us to model the strong seasonality in surpluses, and not to let seasonal adjustment, which uses ex-post data, influence forecasts. Debt data go back centuries, allowing and requiring us to think what is the same and different across different periods of history. Inflation through wars and under the gold standard may well have different fiscal foundations than in the postwar environment. The Online Appendix finds quite different behavior in 1930-1947, though that sample is dominated by a few influential data points and does not offer by itself enough evidence to measure a different regime. A narrative counterpart, especially for big episodes such as the 1970s and 1980s, awaits. Different countries under different monetary and exchange rate regimes and different fiscal constraints will behave differently. US interest rates go down, and the dollar rises in recessions, from a flight to quality. Other countries' interest rates go up, and their currencies fall. A parallel investigation of exchange rates beckons, following Jiang (2019a), Jiang (2019b). One could define shocks in many additional interesting ways.

I omitted analysis of the remaining shocks in the VAR. A shock to any other variable, orthogonal to the inflation shock, can move all of the other terms of the inflation identities. Such movements must offset: In (5), if a shock does not move the inflation term, but does move the sum of future surpluses, then it must also move the sum of growth rates or discount rates. These additional effects are large. The variation in $\Delta E_1 \sum_{j=0}^{\infty} s_{1+j}$ when other shocks move is large; the corresponding movement in the discount rate term is also large, and the two movements are negatively correlated. We get a hint of that behavior in the surplus+growth and discount rate shock responses. I do not pursue this question because it is much more interesting if one can give some structural or economic interpretation to the shocks to other variables.

Perhaps most of all, linking these theory-free characterizations to explicit models is an obviously important step. Why do discount rates vary as they do? What fiscal policies generate the observed pattern of surpluses? One needs economic models to answer such questions, but the identities help to summarize and characterize the forces at work in models.

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Online Appendix to "The Fiscal Roots of Inflation"

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Appendix A. Derivation of the linearized identities

In this appendix I derive the linearized identities (1), (2), and (3),

$$\rho v_{t+1} = v_t + r^n_{t+1} - \pi_{t+1} - g_{t+1} - s_{t+1} \tag{A.1}$$

$$v_t = \sum_{j=1}^{\infty} \rho^{j-1} s_{t+j} + \sum_{j=1}^{\infty} \rho^{j-1} g_{t+j} - \sum_{j=1}^{\infty} \rho^{j-1} \left(r^n_{t+j} - \pi_{t+j} \right)$$

and

$$\begin{aligned} \Delta E_{t+1}\pi_{t+1} - \Delta E_{t+1} r^n_{t+1} = &-\sum_{j=0}^{\infty} \rho^j \Delta E_{t+1} s_{t+1+j} - \\ &- \sum_{j=0}^{\infty} \rho^j \Delta E_{t+1} g_{t+1+j} + \sum_{j=1}^{\infty} \rho^j \Delta E_{t+1} \left(r^n_{t+1+j} - \pi_{t+1+j} \right). \end{aligned} \tag{A.2}$$

I also define the variables more carefully.

The symbols are as follows:

$$V_t = M_t + \sum_{j=0}^{\infty} Q_t^{(t+1+j)} B_t^{(t+1+j)}$$

is the nominal end-of-period market value of debt, where M_t is non-interest-bearing money, $B_t^{(t+j)}$ is zero-coupon nominal debt outstanding at the end of period t and due at the beginning of period $t+j$, and $Q_t^{(t+j)}$ is the time t price of that bond, with $Q_t^{(t)} = 1$. Taking logs,

$$v_t \equiv \log \left(\frac{V_t}{Y_t P_t} \right)$$

is log market value of the debt divided by GDP, where P_t is the price level and Y_t is real GDP or another stationarity-inducing divisor such as consumption, potential GDP, etc. I use consumption times the average GDP to consumption ratio in the empirical work, but I will call Y and ratios to Y "GDP" for brevity. The quantity

$$R^n_{t+1} \equiv \frac{M_t + \sum_{j=1}^{\infty} Q_{t+1}^{(t+j)} B_t^{(t+j)}}{M_t + \sum_{j=1}^{\infty} Q_t^{(t+j)} B_t^{(t+j)}} \tag{A.3}$$

is the nominal return on the portfolio of government debt, i.e. how the change in prices from the end of t to the beginning of $t+1$ affects the value of debt held between periods. The quantity

$$r^n_{t+1} \equiv \log(R^n_{t+1})$$

is the log nominal return on that portfolio. The symbols

$$\pi_t \equiv \log\left(\frac{P_t}{P_{t-1}}\right), \; g_t \equiv \log\left(\frac{Y_t}{Y_{t-1}}\right)$$

are log inflation and GDP growth rate.

We can accommodate explicit default, so the formulas can also apply to countries that borrow in foreign currency such as the members of the Euro. An explicit default is a reduction in the nominal quantity of debt between periods. The $B_t^{(t+j)}$ in the numerator of (A.3) represents the post-default number of bonds outstanding, i.e. at the beginning of period $t+1$, while the $B_t^{(t+j)}$ in the denominator represents the pre-default number of bonds outstanding, i.e. at the end of period t. A partial default then shows up as a low return. To handle default one would, of course, add notation distinguishing the pre- and post-default quantity of debt in the definition of return.

We start with the nonlinear flow identity,

$$M_t + \sum_{j=1}^{\infty} Q_{t+1}^{(t+j)} B_t^{(t+j)} = P_{t+1} sp_{t+1} + M_{t+1} + \sum_{j=1}^{\infty} Q_{t+1}^{(t+1+j)} B_{t+1}^{(t+1+j)}. \tag{A.4}$$

Here, sp_{t+1} denotes the real primary (not including interest payments) surplus or deficit. At the beginning of period $t+1$, money M_t and bonds $B_t^{(t+1+j)}$ are outstanding. Money M_{t+1} at the end of period $t+1$ and beginning of period $t+2$ then equals money M_t, money printed up to redeem bonds $B_{t+1}^{(t+1)}$, less money soaked up by a primary surplus $P_{t+1} sp_{t+1}$, or conversely printed to finance a primary deficit, and less money soaked up by net new bond sales, or printed to finance long-term bond purchases, $\sum_{j=1}^{\infty} Q_{t+1}^{t+1+j}(B_{t+1}^{(t+1+j)} - B_t^{(t+1+j)})$.

Using the definition of return, (A.4) becomes

$$\left(M_t + \sum_{j=1}^{\infty} Q_t^{(t+j)} B_t^{(t+j)}\right) R_{t+1}^n = P_{t+1} sp_{t+1} + \left(M_{t+1} + \sum_{j=1}^{\infty} Q_{t+1}^{(t+1+j)} B_{t+1}^{(t+1+j)}\right),$$

or,

$$V_t R_{t+1}^n = P_{t+1} sp_{t+1} + V_{t+1}.$$

The nominal value of government debt is increased by the nominal rate of return, and decreased by primary surpluses. This seems easy. The algebra all comes from properly defining the return on the portfolio of government debt.

Expressing the result as ratios to GDP, we have a flow identity

$$\frac{V_t}{P_t Y_t} \times \frac{R_{t+1}^n}{G_{t+1}} \frac{P_t}{P_{t+1}} = \frac{sp_{t+1}}{Y_{t+1}} + \frac{V_{t+1}}{P_{t+1} Y_{t+1}}, \tag{A.5}$$

where $G_{t+1} \equiv Y_{t+1}/Y_t$.

We can iterate this flow identity (A.5) forward to express the nonlinear government debt valuation identity as

$$\frac{V_t}{P_t Y_t} = \sum_{j=1}^{\infty} \prod_{k=1}^{j} \frac{1}{R_{t+k}^n / (\Pi_{t+k} G_{t+k})} \frac{sp_{t+j}}{Y_{t+j}}. \tag{A.6}$$

where $\Pi_{t+1} \equiv P_{t+1}/P_t$. The market value of government debt at the end of period t, as a fraction of GDP, equals the present value of primary surplus to GDP ratios, discounted at the government debt rate of return less the GDP growth rate.

The nonlinear present value identities (A.5) and (A.6) are cumbersome, and as explained below they may not converge even when the true present value and linearized identities do converge. I linearize the flow equation (A.5) and then iterate forward to obtain a linearized version of (A.6). Taking logs of (A.5), we have

$$v_t + r_{t+1}^n - \pi_{t+1} - g_{t+1} = \log\left(\frac{sp_{t+1}}{Y_{t+1}} + \frac{V_{t+1}}{P_{t+1} Y_{t+1}}\right). \tag{A.7}$$

I linearize this equation in the *level* of the surplus, not its log as one conventionally does in asset pricing, since the surplus is often negative. To linearize in terms of the surplus/GDP ratio, Taylor expand the last term,

$$v_t + r^n_{t+1} - \pi_{t+1} - g_{t+1} = \log(e^v + sy) + \frac{e^v}{e^v + sy}(v_{t+1} - v) + \frac{1}{e^v + sy}(sy_{t+1} - sy)$$

where

$$sy_{t+1} \equiv \frac{sp_{t+1}}{Y_{t+1}} \tag{A.8}$$

denotes the surplus to GDP ratio, and variables without subscripts denote a steady state of (A.7). With $r \equiv r^n - \pi$,

$$r - g = \log \frac{e^v + sy}{e^v}.$$

Then,

$$v_t + r^n_{t+1} - \pi_{t+1} - g_{t+1} = \left[\log(e^v + sy) - \frac{e^v}{e^v + sy}\left(v + \frac{sy}{e^v}\right)\right] + \frac{e^v}{e^v + sy} v_{t+1} + \frac{e^v}{e^v + sy}\frac{sy_{t+1}}{e^v}$$

$$v_t + r^n_{t+1} - \pi_{t+1} - g_{t+1} = \left[v + r - g - \frac{e^v}{e^v + sy}\left(v + \frac{e^v + sy}{e^v} - 1\right)\right] + \rho v_{t+1} + \rho\frac{sy_{t+1}}{e^v}$$

$$v_t + r^n_{t+1} - \pi_{t+1} - g_{t+1} = [r - g + (1 - \rho)(v - 1)] + \rho v_{t+1} + \rho\frac{sy_{t+1}}{e^v} \tag{A.9}$$

where

$$\rho \equiv e^{-(r-g)}. \tag{A.10}$$

Suppressing the small constant, and thus interpreting variables as deviations from means, the linearized flow identity is

$$v_t + r^n_{t+1} - \pi_{t+1} - g_{t+1} = \rho\frac{sy_{t+1}}{e^v} + \rho v_{t+1}. \tag{A.11}$$

Iterating forward, the present value identity is

$$v_t = \sum_{j=1}^{T} \rho^{j-1}\left[\rho\frac{sy_{t+j}}{e^v} - \left(r^n_{t+j} - \pi_{t+j} - g_{t+j}\right)\right] + \rho^T v_T. \tag{A.12}$$

If we linearize around $r - g = 0$, then the constant in (A.11) is zero ($sy = 0$), and we obtain the linearized flow and present value identities (A.1) and (A.2), with the symbol s_t representing sy_t/e^v. There is nothing wrong with expanding about $r = g$. The point of expansion need not be the sample mean.

To approximate in terms of the surplus to value ratio, write (A.7) as

$$v_t + r^n_{t+1} - \pi_{t+1} - g_{t+1} = \log\left(\frac{V_t}{P_tY_t}\frac{\frac{sp_{t+1}}{Y_{t+1}}}{\frac{V_t}{P_tY_t}} + \frac{V_{t+1}}{P_{t+1}Y_{t+1}}\right)$$

$$r^n_{t+1} - \pi_{t+1} - g_{t+1} = \log\left(\frac{\frac{sp_{t+1}}{Y_{t+1}}}{\frac{V_t}{P_tY_t}} + \frac{\frac{V_{t+1}}{P_{t+1}Y_{t+1}}}{\frac{V_t}{P_tY_t}}\right)$$

$$r^n_{t+1} - \pi_{t+1} - g_{t+1} = \log\left(sv_{t+1} + e^{v_{t+1}-v_t}\right).$$

At a steady state

$$r - g = \log\left(1 + sv\right). \tag{A.13}$$

$$e^{r-g} = 1 + sv.$$

Taylor expanding around a steady state,

$$r^n_{t+1} - \pi_{t+1} - g_{t+1} = \log\left(1 + sv\right) + \frac{1}{(1+sv)}\left(sv_{t+1} - sv + v_{t+1} - v_t\right)$$

$$v_t + (1+sv)\left[r^n_{t+1} - \pi_{t+1} - g_{t+1}\right] = \left[(1+sv)\log\left(1+sv\right) - sv\right] + sv_{t+1} + v_{t+1}. \tag{A.14}$$

The linearized flow identity (A.1) follows, with the symbol s_t representing the surplus to value ratio $s_t = sv_t$, if we suppress the constant, using deviations from means in the analysis, or if we use $r = g$ or $sv = 0$, as a point of expansion.

The linearizations in terms of the surplus to value ratio sv_t are more accurate. The units of the flow identities (A.1), (A.11) are rates of return. Dividing the surplus by the previous period's value gives a better approximation to the growth in value, when the value of debt is far from the steady state.

A constant ratio of surplus to market value of debt for any price level path leads to a passive fiscal policy: An unexpected deflation raises the real value of debt. If surpluses always rise in response, they validate the lower price level. Thus, although on the equilibrium path one can describe dynamics via either linearization, if one wants to think about how fiscal-theory equilibria are formed, it is better to describe a surplus that does not react to price level changes, so only one value v_t emerges, as is the case in (A.12). For such purposes, the surplus to GDP definition is appropriate, as well as adopting a linearization point $r > g$ and $\rho < 1$. It's also better to use the nonlinear versions of the identities for determinacy issues. The analysis of this paper is about what happens in equilibrium, and does not require an active-fiscal assumption, so the difference is irrelevant here.

I infer the surplus from the linearized flow identity (A.1) so which concept the surplus corresponds to makes no difference to the analysis. The difference is only the accuracy of approximation, how close the surplus recovered from the linearized flow identity corresponds to a surplus recovered from the nonlinear exact identity (A.7).

Appendix B. Linearizing the bond return formula

Here I derive the linearized identity

$$r^n_{t+1} \approx \omega q_{t+1} - q_t,$$

which leads to (4),

$$\Delta E_{t+1} r^n_{t+1} = -\sum_{j=1}^{\infty} \omega^j \Delta E_{t+1} \left[(r^n_{t+1+j} - \pi_{t+1+j}) + \pi_{t+1+j} \right].$$

I also derive expectations-hypothesis bond-pricing equations.

$$E_t r^n_{t+1} = i_t$$

$$\omega E_t q_{t+1} - q_t = i_t.$$

These equations are used in the sticky-price model Cochrane (2020a).

Denote the maturity structure by

$$\omega_{j,t} \equiv \frac{B_t^{(t+j)}}{B_t^{(t+1)}}$$

and $B_t \equiv B_t^{(t+1)}$. Then the end of period t nominal market value of debt is

$$\sum_{j=1}^{\infty} B_t^{(t+j)} Q_t^{(t+j)} = B_t \sum_{j=1}^{\infty} \omega_{j,t} Q_t^{(t+j)}.$$

(I ignore money to keep the formulas simple.) Define the price of the government debt portfolio

$$Q_t = \sum_{j=1}^{\infty} \omega_{j,t} Q_t^{(t+j)}.$$

The return on the government debt portfolio is then

$$R_{t+1}^n = \frac{\sum_{j=1}^{\infty} B_t^{(t+j)} Q_{t+1}^{(t+j)}}{\sum_{j=1}^{\infty} B_t^{(t+j)} Q_t^{(t+j)}} = \frac{\sum_{j=1}^{\infty} \omega_{j,t} Q_{t+1}^{(t+j)}}{\sum_{j=1}^{\infty} \omega_{j,t} Q_t^{(t+j)}} = \frac{1 + \sum_{j=1}^{\infty} \omega_{j+1,t} Q_{t+1}^{(t+1+j)}}{Q_t}. \tag{B.1}$$

I loglinearize around a geometric maturity structure, $B_t^{(t+j)} = B_t \omega^{j-1}$, or equivalently $\omega_{j,t} = \omega^{j-1}$. I use variables with no subscripts to denote the linearization points, and tildes to denote deviations from those points.

When we linearize, we move bond prices holding the maturity structure at its steady-state, geometric value, and then we move the maturity structure while holding bond prices at their steady-state value. As a result, changes in maturity structure have no first-order effect on the linearized bond return. At the steady state $Q_t^{t+j} = 1/(1+i)^j$,

$$R_{t+1}^n = \frac{\sum_{j=1}^{\infty} \omega_{j,t}/(1+i)^{j-1}}{\sum_{j=1}^{\infty} \omega_{j,t}/(1+i)^j} = (1+i)$$

independently of $\{w_{j,t}\}$. Intuitively, at the steady state bond prices, all bonds give the same return, so all portfolios of bonds give the same return. Moreover, maturity structure is a time-t variable in the definition of return R_{t+1}^n. The return from t to $t+1$ is not affected by the time $t+1$ maturity structure. (Changes in maturity structure might affect returns if there is price pressure in bond markets. These are formulas for measurement, however, and such effects would show up as changes in measured prices coincident with changes in quantities.)

Maturity structure only has a second-order *interaction* effect on the bond portfolio return. For example, a longer maturity structure at t raises the bond portfolio return at $t+1$ if there is also a level shock, raising long-maturity bond returns at $t+1$. A longer maturity structure at t raises the expected return if the yield curve at t is also temporarily upward sloping. But a linear VAR and a linear decomposition do not include interaction effects.

To be clear, I measure the bond portfolio return r^n_{t+1} directly, and exactly, and this measure includes effects of maturity structure, as well as liquidity and other effects. With a longer maturity structure, the same movement in interest rates will give a larger return. The linearization only affects the decomposition of the bond portfolio return to future inflation and future expected returns. A second-order approximation would effectively use a different ω in the decomposition formula for different dates, as well as estimate a VAR with parameters that depend on the maturity structure or interaction terms. But variation in the geometric maturity structure parameter ω makes little difference to the results. And the sample is too short to add more variables, interaction terms, or time-varying parameters.

The term of the linearization with steady-state bond prices and shocks to maturity thus adds nothing. The linearization only includes a linearization with steady-state, geometric maturity structure and changing bond prices. Linearizing (B.1) then, we have

$$r^n_{t+1} = \log\left(1+\omega e^{q_{t+1}}\right) - q_t \approx \log\left(\frac{1+\omega Q}{Q}\right) + \frac{\omega Q}{1+\omega Q}\tilde{q}_{t+1} - \tilde{q}_t \tag{B.2}$$

where as usual variables without subscripts are steady state values and tildes are deviations from steady state. In a steady state,

$$Q = \sum_{j=1}^{\infty} \omega^{j-1}\frac{1}{(1+i)^j} = \left(\frac{1}{1+i}\right)\left(\frac{1}{1-\frac{\omega}{1+i}}\right) = \frac{1}{1+i-\omega}. \tag{B.3}$$

The limits are $\omega = 0$ for one-period bonds, which gives $Q = 1/(1+i)$, and $\omega = 1$ for perpetuities, which gives $Q = 1/i$. The terms of the approximation (B.2) are then

$$\begin{aligned} \frac{1+\omega Q}{Q} &= 1+i \\ \frac{\omega Q}{1+\omega Q} &= \frac{\omega}{1+i} \end{aligned}$$

so we can write (B.2) as

$$r^n_{t+1} \approx i + \frac{\omega}{1+i}\tilde{q}_{t+1} - \tilde{q}_t.$$

Since $i < 0.05$ and $\omega \approx 0.7$, I further approximate to

$$r^n_{t+1} \approx i + \omega \tilde{q}_{t+1} - \tilde{q}_t. \tag{B.4}$$

I find the value of ω that best fits the return identity, rather than measure the maturity structure directly, so the difference between ω and $\omega/(1+i)$ makes no practical difference.

To derive the bond return identity (4), iterate (B.4) forward to express the bond price in terms of future returns,

$$\tilde{q}_t = -\sum_{j=1}^{\infty} \omega^j \tilde{r}^n_{t+j}.$$

Take innovations, move the first term to the left hand side, and divide by ω,

$$\Delta E_{t+1} \tilde{r}^n_{t+1} = -\sum_{j=1}^{\infty} \omega^j \Delta E_{t+1} \tilde{r}^n_{t+1+j}. \tag{B.5}$$

Then add and subtract inflation to get (4),

$$\Delta E_{t+1} \tilde{r}^n_{t+1} = -\sum_{j=1}^{\infty} \omega^j \Delta E_{t+1} \left[(\tilde{r}^n_{t+1+j} - \tilde{\pi}_{t+1+j}) + \tilde{\pi}_{t+1+j} \right]. \tag{B.6}$$

The expectations hypothesis states that expected returns on bonds of all maturities are the same,

$$\begin{aligned} E_t r^n_{t+1} &= i_t \\ i + \omega E_t \tilde{q}_{t+1} - \tilde{q}_t &= i_t \\ \omega E_t \tilde{q}_{t+1} - \tilde{q}_t &= \tilde{\imath}_t. \end{aligned}$$

In the text, all variables are deviations from steady state, so I drop the tilde notation.

Appendix C. What if r is less than g?

Does $r < g$ imply that the present value of debt is infinite, so deficits do not have to be repaid by subsequent surpluses, and the forward-looking present value is fundamentally wrong? I give here a deeper treatment of the issue.

Again, the present value of debt can be well defined, properly using the stochastic discount factor, contingent claim price, or marginal utility to discount, and large deficits still need to be repaid by subsequent surpluses, yet the economy displays $E(r) < E(g)$.

In essence, though debt is grown out of on many paths and on average, there are paths with low growth and high contingent claim value where growing out of debt fails. The lesson is that one should not take the mean values of r and g from a stochastic economy and use them in perfect certainty discounting formulas. (Bohn (1995), Bassetto and Cui (2018), Bassetto and Sargent (2020), Reis (2021), Cochrane (2021))

The approach here, which discounts using the stochastic ex-post return, is still not out of the woods. While one can always use the ex-post return to discount a finite stream of payoffs – $1 = E_t(R_{t+1}^{-1} R_{t+1})$ – that assurance does not extend to an infinite stream. Discounting using returns can fail, even stochastic returns, while the true present value formula holds.

Here is what can happen. Suppose the value of debt is well-defined using the stochastic discount factor, contingent claims price, or marginal utility Λ_t. Suppose

$$\frac{V_t}{Y_t} = E_t \sum_{j=0}^{T} \frac{\Lambda_{t+j}}{\Lambda_t} \frac{Y_{t+j}}{Y_t} \frac{s_{t+j}}{Y_{t+j}} + E_t \frac{\Lambda_{t+T}}{\Lambda_t} \frac{Y_{t+T}}{Y_t} \frac{V_{t+T}}{Y_{t+T}} \tag{C.1}$$

is well defined, and the right hand term tends to zero. I include GDP terms which complicate the formula but you can see they just multiply and divide the basic present value formula.

Now, try to discount using ex post returns rather than the discount factor. Start with the identity

$$V_{t+1} = R_{t+1} \left(V_t - s_t \right).$$

Rearrange to

$$\frac{V_t}{Y_t} = \frac{s_t}{Y_t} + \frac{1}{R_{t+1}} \frac{Y_{t+1}}{Y_t} \frac{V_{t+1}}{Y_{t+1}}$$

and iterate forward

$$V_t = \sum_{j=0}^{T} \prod_{k=0}^{j} \frac{1}{R_{t+k}} \frac{Y_{t+k}}{Y_{t+k-1}} s_{t+j} + \prod_{k=0}^{T} \frac{1}{R_{t+k}} \frac{Y_{t+k}}{Y_{t+k-1}} \frac{V_{t+T}}{Y_{t+T}}. \tag{C.2}$$

This is the exact nonlinear version of the present value identity (2). It can happen that the expected value of *this* limiting term explodes, and the present value term explodes in the opposite direction, even though the true present value formula (C.1) is well-behaved. And, with stationary debt to GDP ratio, you can see that a nonlinear stochastic version of $r < g$ is exactly the condition for this pathological case.

In sum, by $1 = E_t(R_{t+1}^{-1} R_{t+1})$, the inverse return is a discount factor under mild conditions, basically that moments exist. The inverse return is only an infinite period discount factor if in addition the sums and terminal condition converge.

This result seems to put the whole project of discounting by returns in jeapordy. But that is not necessarily so. For the terms of the *linearized* identity (6) and those of the true present value formula (C.1) may converge, while the terms of the the *nonlinear* identity (C.2) do not converge. I linearize the return formula and then iterate forward. I do not take a direct Taylor approximation of the return-based identity (C.2). The condition for a substitute discount factor to work is, first, that it prices one-period claims, and second, that the sums converge. The linearized return-based present value formula can work when the nonlinear one does not. "Work" means that the formula gives a good approximation to the true, discount-factor, based result.

For example, if $E(r^n - \pi) < E(g)$, the linearized identity (2) indicates that the government can finance a small steady deficit/GDP ratio $E(s) < 0$, while maintaining a steady debt/GDP ratio. But any large additional deficits must still be financed by subsequent surpluses.

If the government issues only non-interest-bearing money, then $r^n = 0$. This is the familiar case of seignorage, which can finance a small steady deficit, but large additional spending must still be financed by borrowing and repayment. The identity indicates that the same principle applies for other sources of $r < g$.

The linearized identity captures this fact of the true present value formula, that the nonlinear return-based present value formula may not capture.

Appendix D. A variance decomposition

I use the elements of the impulse response function and their sums to calculate the terms of the unexpected inflation identity (3). We can interpret this calculation as an decomposition of the variance of unexpected inflation. Multiply both sides of (3) by $\Delta E_{t+1}\pi_{t+1}$ and take expectations, giving

$$\begin{aligned}
& var\left(\Delta E_{t+1}\pi_{t+1}\right) - cov\left[\Delta E_{t+1}\pi_{t+1}, \Delta E_{t+1}\left(r^n_{t+1} - g_{t+1}\right)\right] = \\
& = -\sum_{j=0}^{\infty} cov\left[\Delta E_{t+1}\pi_{t+1}, \Delta E_{t+1}s_{t+1+j}\right] + \\
& + \sum_{j=1}^{\infty} cov\left[\Delta E_{t+1}\pi_{t+1}, \Delta E_{t+1}\left(r^n_{t+1+j} - \pi_{t+1+j} - g_{t+1+j}\right)\right].
\end{aligned} \tag{D.1}$$

Unexpected inflation may only vary to the extent that it covaries with current bond returns, or if it forecasts surpluses or real discount rates.

Dividing by $var\left(\Delta E_{t+1}\pi_{t+1}\right)$, we can express each term as a fraction of the variance of unexpected inflation coming from that term. This decomposition adds up to 100%, within the accuracy of approximation, but it is not an orthogonal decomposition, nor are all the elements necessarily positive. Each term is also a regression coefficient of future long-run variables on unexpected inflation.

The two approaches give exactly the same result – the terms of (D.1) are exactly the terms of the impulse-response function, to an inflation shock orthogonalized last, i.e. a shock that moves all variables at time 1 including $\Delta E_1\pi_1$.

To see this fact, write the VAR in standard notation

$$x_{t+1} = Ax_t + \varepsilon_{t+1} \tag{D.2}$$

so

$$\Delta E_{t+1}\sum_{j=1}^{\infty} x_{t+j} = (I-A)^{-1}\varepsilon_{t+1}.$$

Let a denote vectors which pull out each variable, i.e.

$$\pi_t = a'_\pi x_t,\ s_t = a'_s x_t, \tag{D.3}$$

etc. Then the present value identity (3) reads and may be calculated as

$$a'_\pi\varepsilon_{t+1} - (a_{r^n} - a_g)'\varepsilon_{t+1} = -a'_s(I-A)^{-1}\varepsilon_{t+1} + a'_{rg}(I-A)^{-1}A\varepsilon_{t+1} \tag{D.4}$$

where

$$a_{rg} \equiv a_{r^n} - a_\pi - a_g.$$

We can calculate the variance decomposition (D.1) by

$$a'_\pi \Omega a_\pi - (a_{r^n} - a_g)' \Omega a_\pi = -a'_s (I-A)^{-1} \Omega a_\pi + a'_{rg}(I-A)^{-1} A \Omega a_\pi$$

where $\Omega = cov(\varepsilon_{t+1}, \varepsilon'_{t+1})$, and then divide by $a'_\pi \Omega a_\pi$ to express the result as a fraction,

$$1 - (a_{r^n} - a_g)' \frac{\Omega a_\pi}{a'_\pi \Omega a_\pi} = -a'_s (I-A)^{-1} \frac{\Omega a_\pi}{a'_\pi \Omega a_\pi} + a'_{rg}(I-A)^{-1} A \frac{\Omega a_\pi}{a'_\pi \Omega a_\pi}. \quad \text{(D.5)}$$

To show that this variance decomposition is the same as the elements and sum of elements of the impulse-response function to an inflation shock, orthogonalized last, note that the regression coefficient of any other shock ε^z on the inflation shock is

$$b_{\varepsilon^z, \varepsilon^\pi} = \frac{cov(\varepsilon^z_{t+1}, \varepsilon^\pi_{t+1})}{var(\varepsilon^\pi_{t+1})} = \frac{a'_z \Omega a_\pi}{a'_\pi \Omega a_\pi},$$

so the VAR shock, consisting of a unit movement in inflation $\varepsilon^\pi_1 = 1$ and movements $\varepsilon^z_1 = b_{\varepsilon^z, \varepsilon^\pi}$ in each of the other variables is given by

$$\varepsilon_1 = \frac{\Omega a_\pi}{a'_\pi \Omega a_\pi}.$$

We recognize in (D.5) the responses and sums of responses to this shock. Dividing (D.1) by the variance of unexpected inflation, or examining the terms of (D.5), we recognize that each term is also the coefficient in a single regression of each quantity on unexpected inflation.

In an analogous way, we can interpret the responses to other shocks as a decomposition of the *co*variance of unexpected inflation with that shock, based on

$$cov\left(\Delta E_{t+1} \pi_{t+1} \varepsilon_{t+1}\right) - cov\left[\varepsilon_{t+1}, \Delta E_{t+1}\left(r^n_{t+1} - g_{t+1}\right)\right]$$
$$= -\sum_{j=0}^{\infty} cov\left[\varepsilon_{t+1}, \Delta E_{t+1} s_{t+1+j}\right] + \sum_{j=1}^{\infty} cov\left[\varepsilon_{t+1}, \Delta E_{t+1}\left(r^n_{t+1+j} - \pi_{t+1+j} - g_{t+1+j}\right)\right].$$

This variance decomposition is similar in style to the decomposition of return variance in Campbell and Ammer (1993). To avoid covariance terms, however, it follows the philosophy of the price/dividend variance decomposition in Cochrane (1992), extended to a multivariate context. With $x = y + z$, I explore $var(x) = cov(x, y) + cov(x, z)$ rather than $var(x) = var(y) + var(z) + 2cov(y, z)$.

Appendix E. Monte Carlo details

To evaluate sampling distributions I run a simple Monte Carlo. I start with the estimated VAR. I find the covariance matrix of the residuals ε_{t+1}. The identity (1) implies

$$\varepsilon_{s,t+1} = \varepsilon_{r^n,t+1} - \varepsilon_{g,t+1} - \varepsilon_{\pi,t+1} - \varepsilon_{v,t+1}. \tag{E.1}$$

Since I infer the surplus data s_t from (1), the data obey this identity and the covariance matrix of residuals is singular. Thus I simulate iid shocks from the covariance matrix of all shocks except the surplus, and then I infer the surplus shock from the identity (E.1).

I initialize the VAR at the first data point, thereby generating the conditional sampling distribution. I simulate forward 50,000 artificial data samples using the estimated VAR parameters. I re-estimate the VAR and I calculate impulse responses and inflation decompositions in each artificial sample. I tabulate the sampling distribution of these quantities and report quantiles.

In a very few artificial samples, the VAR estimate has eigenvalues greater than or equal to one, so $(I - A)^{-1}$ cannot be computed. I omit these 38 out of 50,000 samples. As a result the reported quantiles are slightly smaller than actual quantiles. Avoiding these infinities and beyond is one reason that I report quantiles rather than standard errors. More generally, the distribution of statistics is not normal.

It is also not always possible to find $\omega \in [0, 1]$ to satisfy the return identity, so many Monte Carlo draws use a best fit value of ω in which the return identity does not hold. Weights have little effect on the results however, so this fact seems to have little effect. Since this is what I would have done in sample had I not been able to find an $\omega \in [0, 1]$ that satisfied the return identity, this fact just fills out the correct sampling distribution.

I run the Monte Carlo using sample estimates, and in particular the estimated 0.98 coefficient of debt on lagged debt. Near unit roots are biased down, and one might wish also to run a Monte Carlo with a bias-corrected estimate with eigenvalues closer to one. That procedure would likely lead to somewhat larger sampling distributions.

Between the conditional Monte Carlo – starting at the first data point – the problem of draws with A eigenvalues greater than one, near-unit roots, and non-normal error distributions, one could likely find sampling experiments that produce even larger distributions.

But remember, I am not testing anything, so the point is simply to give a sense of the sampling error of the measurements. My main conclusion is that the sampling distribution of the response functions and decompositions, though narrow enough that the qualitative results are reasonably reliable, is still pretty wide already, steering me away from model complications. Sampling exercises that produce even wider distributions would only emphasize that point.

Appendix F. Sources of sampling variation

Table F.4 includes the regression of other shocks on inflation shock that starts off the main inflation decomposition, and thus determines the instantaneous response in Figures 2 and 9. The table also includes the correlation matrix of the shocks.

To measure the relative contribution of the shock correlation and the long-run response function given the shock identification as sources of variation, Table F.5 includes two other sampling calculations. The “no b” columns resample data using the original regression of shocks ε^z_{t+1} on inflation shocks ε^π_{t+1}, the top row of Table F.4, in each sample. The VAR coefficients still vary across samples, but the identification of the inflation shock does not. The “no A” columns likewise keep constant the VAR regression coefficients, but reestimate the shock regression in each sample. Turning off either source of sampling variation reduces that variation, but not as much as you might think. Sampling

	r^n	g	π	s	v	i	y
	Regression of other shocks on inflation shock						
Coefficient	-0.56	-0.33	1.00	-0.58	-0.65	0.24	0.23
Std. err.	(0.24)	(0.17)	(0.00)	(0.53)	(0.74)	(0.14)	(0.24)
	Correlation matrix of VAR shocks						
r^n	1.00	-0.25	-0.29	-0.27	0.63	-0.74	-0.93
g	-0.25	1.00	-0.24	0.39	-0.56	0.41	0.20
π	-0.29	-0.24	1.00	-0.14	-0.11	0.21	0.31
s	-0.27	0.39	-0.14	1.00	-0.88	0.35	0.26
v	0.63	-0.56	-0.11	-0.88	1.00	-0.63	-0.60
i	-0.74	0.41	0.21	0.35	-0.63	1.00	0.75
y	-0.93	0.20	0.31	0.26	-0.60	0.75	1.00

Table F.4: Regression of other shocks on inflation shock, and correlation matrix of VAR shocks

	Fraction			No b		No A	
Component	Estimate	25%	75%	25%	75%	25%	75%
Inflation π_1	1.00						
Bond return $(r_1^n - g_1)$	-0.23	-0.45	0.00	-0.23	-0.23	-0.45	0.00
Future Σs	-0.06	-0.69	0.23	-0.60	0.14	-0.69	0.23
Future $\Sigma r - g$	1.17	0.42	1.57	0.63	1.37	0.42	1.57

Table F.5: Decomposition of unexpected inflation variance – distribution quantiles. No b holds the initial response constant across trials. No A holds the VAR regression coefficients constant across trials variation is still large in either case, and variances add, not standard deviations. Moreover the sampling variation associated with shock orthogonalization – the "no A" exercise – does not go away no matter how small the shocks. Both left and right hand sides of the shock on shock regressions get smaller at the same rate.

Appendix G. 1980-2018 subsample results

This section presents results using the 1980-2018 subsample. Much monetary macroeconomics isolates this period as having a different set of correlations that the earlier 1970s inflation, 1960s under Bretton woods, etc. Breaking the sample also allows us to see if the results are stable across subsamples.

	r^n_{t+1}	g_{t+1}	π_{t+1}	s_{t+1}	v_{t+1}	i_{t+1}	y_{t+1}
r^n_t	-0.22*	0.05	-0.10**	-0.25*	0.08	-0.04	0.07*
g_t	-0.11	0.13	0.06	0.76	-1.06	0.20*	-0.04
π_t	-0.04	-0.57*	0.67**	-1.55	1.41	-0.08	-0.12
s_t	0.10*	0.07*	-0.02	0.38**	-0.34*	-0.02	-0.04*
v_t	-0.00	0.00	-0.00	0.05	0.95**	0.01	0.00
i_t	-0.12	-0.27*	0.20*	1.14*	-1.19	0.61**	0.31*
y_t	1.61**	0.67**	-0.08	0.05	0.98	0.32*	0.57**
$100 \times std(\varepsilon_{t+1})$	2.44	1.10	0.51	5.17	7.00	1.15	0.93
Corr $\varepsilon, \varepsilon_\pi$	-0.40	0.14	1.00	0.11	-0.31	0.27	0.42
R^2	0.73*	0.54*	0.88*	0.50*	0.94*	0.85*	0.89*
$100 \times std(x)$	4.74	1.63	1.48	7.30	28.88	2.97	2.84

Table G.6: OLS VAR estimate. Sample 1980-2018. One (two) stars means the estimate is one (two) Monte Carlo standard errors away from zero.

Table G.6 presents OLS VAR regression coefficients, parallel to Table 1. Table G.7 compiles inflation decompositions, parallel to Table 2. Figures G.10, G.11 and G.12 plot responses to inflation shocks, paralleling Figures 2, 3, and 4.

The broad pattern of Figure G.10 is similar to the full postwar sample. There are some differences. The surplus and growth shocks are now positively correlated with the inflation shock, seen in the period 1 responses. There is less

$$\sum_{j=0}^{\infty} \omega^j \Delta E_1 \pi_{1+j} = -\sum_{j=0}^{\infty} \Delta E_1 s_{1+j} - \sum_{j=0}^{\infty} \Delta E_1 g_{1+j} + \sum_{j=1}^{\infty} (1-\omega^j) \Delta E_1 r_{1+j}$$

	π	$=$	s	g	r
Inflation	2.32	=	-(0.19)	-(0.52)	+(3.03)
Recession	-2.50	=	-(-0.31)	-(-1.67)	+(-4.49)
Surplus	-0.08	=	-(-0.46)	-(-0.54)	+(-1.08)
Disc. Rate	-0.12	=	-(-0.40)	-(-0.49)	+(-1.00)
Surplus, no i	0.07	=	-(-0.70)	-(-0.30)	+(-0.92)

$$\Delta E_1 \pi_1 - \Delta E_1 r_1^n = -\sum_{j=0}^{\infty} \Delta E_1 s_{1+j} - \sum_{j=0}^{\infty} \Delta E_1 g_{1+j} + \sum_{j=1}^{\infty} \Delta E_1 r_{1+j}$$

	π	r^n	$=$	s	g	r
Inflation	1.00	-(-1.92)	=	-(0.19)	-(0.52)	+(3.63)
Recession	-1.00	-(2.44)	=	-(-0.31)	-(-1.67)	+(-5.42)
Surplus	-0.01	-(0.32)	=	-(-0.46)	-(-0.54)	+(-1.33)
Disc. Rate	-0.02	-(0.35)	=	-(-0.40)	-(-0.49)	+(-1.25)
Surplus, no i	0.07	-(-0.01)	=	-(-0.70)	-(-0.30)	+(-0.92)

$$\Delta E_1 r_1^n = -\sum_{j=1}^{\infty} \omega^j \Delta E_1 r_{1+j} - \sum_{j=1}^{\infty} \omega^j \Delta E_1 \pi_{1+j}$$

	r^n	$=$	r	π
Inflation	-1.92	=	-(0.60)	-(1.32)
Recession	2.44	=	-(-0.94)	-(-1.50)
Surplus	0.32	=	-(-0.25)	-(-0.07)
Disc. Rate	0.35	=	-(-0.25)	-(-0.09)
Surplus, no i	-0.01	=	-(-0.00)	-(0.00)

Table G.7: Terms of the inflation and bond return identities. Sample 1930-2018.

need to isolate a separate growth+inflation shock in this period, dominated by "aggregate demand" rather than "stagflation" episodes.

However, the surplus and growth responses turn negative after one period, as they are in the full sample. Higher inflation strongly forecasts a lower surplus, -1.55 in Table G.6 rather than -0.25 in Table 1, and similarly higher inflation forecasts lower growth -0.57 rather than -0.14. The overall responses are then similar to the full period.

Surpluses then recover and turn positive as before. The sum of the surplus response remains small, 0.19 rather than -0.06.

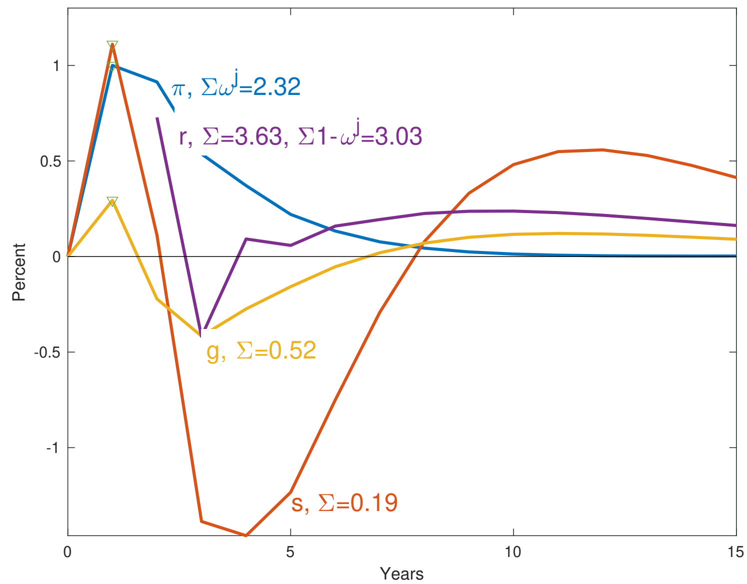

Figure G.10: Response to inflation shocks, sample 1980-2018.

Figure G.11 explores the long-run surplus response, and you can see the same dynamics playing out. Inflation forecasts a rise in debt (1.41 in Table G.6), and the period of deficits also raises debt (-0.34). But the rise in debt leads to a rise in surpluses, which slowly pay down much of that debt.

The expected return also rises in Figure G.10, and accounts for all the inflation and more in this subsample as it does in the main estimate.

Figure G.12 shows the interest rate response in more detail. The wiggly response, which I pointed out in the postwar sample and is a result of slight overfitting there, is even more pronounced here. However, wiggles aside, the basic picture is similar. Interest rates and the expected bond return rise together, and almost permanently in response to the inflation shock. They do not rise as much as inflation, giving a few periods of negative expected returns, but their rise is so much more persistent than that of inflation that we see a very long period of high expected returns on the right side of the graph. As in the full sample, the much greater persistence of yield-curve changes than of inflation generates the long-term discount rate rise which accounts for most of

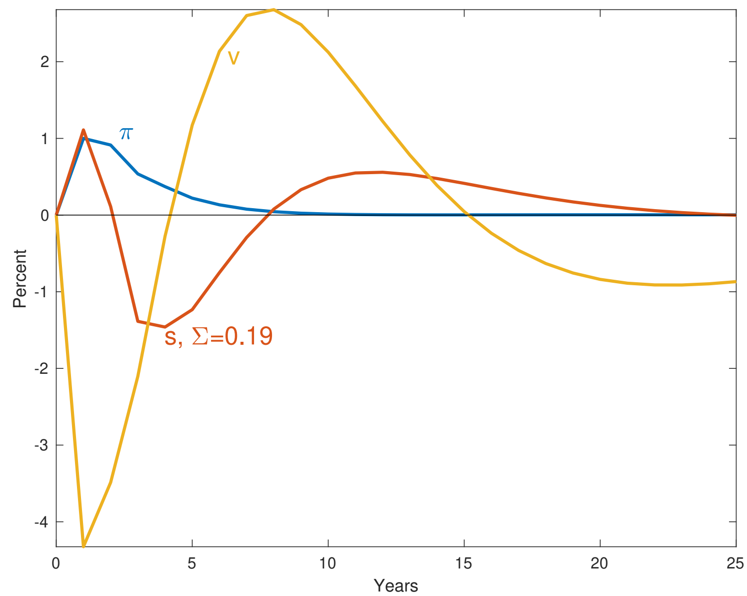

Figure G.11: Response to inflation shocks, sample 1980-2018.

the inflation shock.

The impulse-response quantiles, plotted in Figure G.13, are even larger than those of the full sample, but not so large that the results are meaningless.

Overall, we see a comfortingly similar picture, and many signs of weak estimation in a short sample. At least it is comforting not to see the point estimates paint a much different picture, as they do in the prewar sample studied in the next section.

I do not present results for the 1947-1980 subsample to save space, since it too paints about the same picture. The near-term (5 years) response functions are similar. However the point estimate has an eigenvalue of the transition matrix greater than one, so one must either reduce that or make calculations based on the first few responses only, not $(I - A)^{-1}$ calculations.

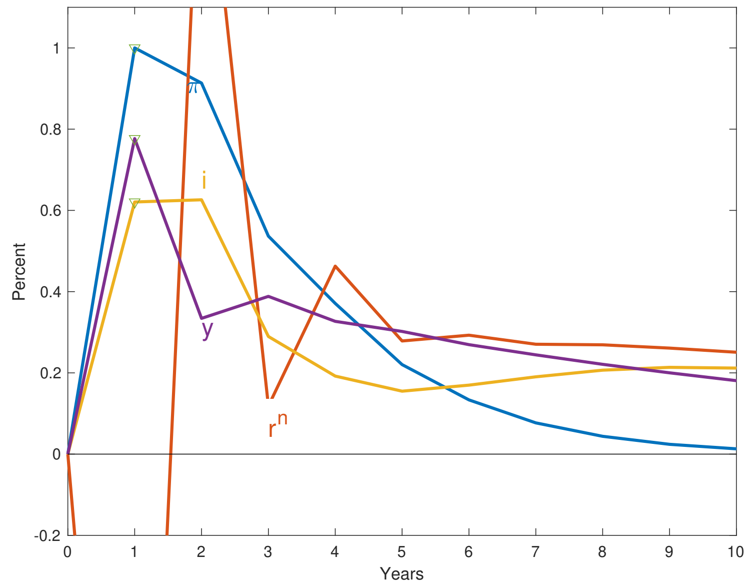

Figure G.12: Response to inflation shocks, sample 1980-2018.

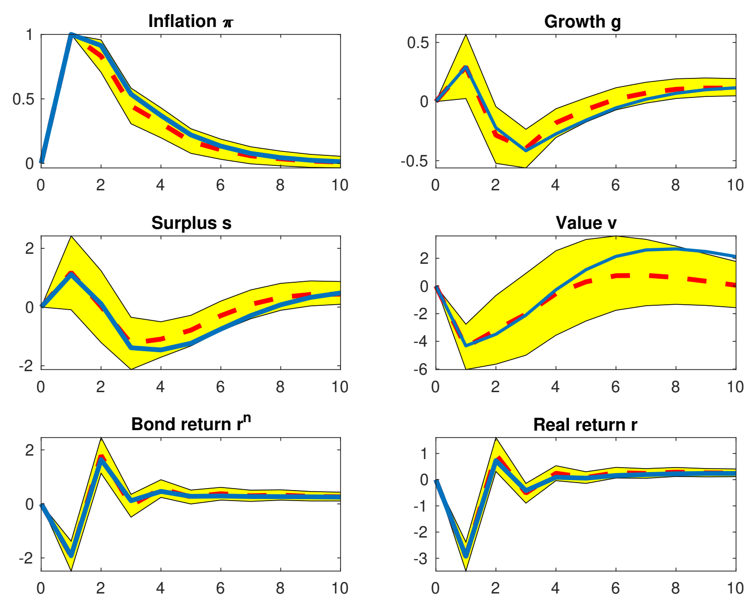

Figure G.13: Inflation shock response quantiles, sample 1980-2018.

Appendix H. Full sample results

This section presents results using the full sample of data that I have been able to collect, 1930-2018.

	r^n_{t+1}	g_{t+1}	π_{t+1}	s_{t+1}	v_{t+1}	i_{t+1}	y_{t+1}
r^n_t	-0.23^{**}	0.06	-0.02	-0.12	-0.14	-0.06^{*}	0.05^{*}
g_t	0.02	0.42^{**}	0.25^{**}	0.52^{*}	-1.17^{**}	0.07^{*}	-0.01
π_t	-0.11^{*}	0.05	0.53^{**}	-0.75^{**}	0.05	0.02	0.02
s_t	0.01	-0.02	-0.02	0.65^{**}	-0.61^{**}	0.00	-0.01^{*}
v_t	0.01	0.01^{*}	0.01	0.08^{**}	0.91^{**}	-0.00	-0.00^{*}
i_t	-0.32^{*}	-0.35^{*}	0.26	0.63	-0.87	0.79^{**}	0.31^{**}
y_t	1.85^{**}	0.40^{*}	-0.05	0.59	0.90	0.14	0.52^{**}
$100 \times std(\varepsilon_{t+1})$	2.22	2.15	2.28	7.34	9.04	1.24	0.77
Corr $\varepsilon, \varepsilon_\pi$	-0.14	0.21	1.00	-0.07	-0.28	0.15	0.17
R^2	0.68^{*}	0.32^{*}	0.56^{*}	0.54^{*}	0.96^{*}	0.84^{*}	0.91^{*}
$100 \times std(x)$	3.92	2.61	3.44	10.80	42.76	3.05	2.60

Table H.8: OLS VAR estimate. Sample 1930-2018. One (two) stars means the estimate is one (two) Monte Carlo standard errors away from zero.

Table H.8 presents OLS VAR regression coefficients, parallel to Table 1. Table H.9 compiles inflation decompositions, parallel to Table 2. Figures H.14, H.15, and H.16 plot responses to inflation shocks, paralleling Figures 2, 3, and 4. Figure H.17 presents sampling quantiles, paralleling Figure 9.

Start with the impulse response function for the inflation shock, Figure H.14, paralleling Figure 2. The general pattern is similar. But the magnitudes are completely different. The 1% inflation shock still corresponds to a prolonged deficit, and the deficit eventually turns to surplus. But the deficit is larger and longer, and following surpluses no longer pay off the accumulated debts. The sum of the surplus responses is -2.59, not -0.06, accounting for more than all of the 1.83% weighted sum of inflation.

Discount rates follow the same general pattern as well. But the decline in discount rate is longer lasting, and the subsequent rise much smaller, so discount rates now account for -0.52% inflation, not +1.004% inflation.

$$\sum_{j=0}^{\infty} \omega^j \Delta E_1 \pi_{1+j} = -\sum_{j=0}^{\infty} \Delta E_1 s_{1+j} - \sum_{j=0}^{\infty} \Delta E_1 g_{1+j} + \sum_{j=1}^{\infty} (1-\omega^j) \Delta E_1 r_{1+j}$$

	π	=	s	g	r
Inflation	1.83	=	-(-2.59)	-(0.93)	+(0.17)
Recession	-2.00	=	-(2.59)	-(-2.13)	+(-1.54)
Surplus	0.09	=	-(-1.04)	-(0.04)	+(-0.91)
Disc. Rate	-0.05	=	-(-0.89)	-(-0.05)	+(-1.00)
Surplus, no i	0.30	=	-(-1.27)	-(0.27)	+(-0.70)

$$\Delta E_1 \pi_1 - \Delta E_1 r_1^n = -\sum_{j=0}^{\infty} \Delta E_1 s_{1+j} - \sum_{j=0}^{\infty} \Delta E_1 g_{1+j} + \sum_{j=1}^{\infty} \Delta E_1 r_{1+j}$$

	π	r^n	=	s	g	r
Inflation	1.00	-(-0.14)	=	-(-2.59)	-(0.93)	+(-0.52)
Recession	-1.00	-(0.17)	=	-(2.59)	-(-2.13)	+(-0.72)
Surplus	0.07	-(0.13)	=	-(-1.04)	-(0.04)	+(-1.05)
Disc. Rate	-0.01	-(0.16)	=	-(-0.89)	-(-0.05)	+(-1.11)
Surplus, no i	0.26	-(0.01)	=	-(-1.27)	-(0.27)	+(-0.75)

$$\Delta E_1 r_1^n = -\sum_{j=1}^{\infty} \omega^j \Delta E_1 r_{1+j} - \sum_{j=1}^{\infty} \omega^j \Delta E_1 \pi_{1+j}$$

	r^n	=	r	π
Inflation	-0.14	=	-(-0.69)	-(0.83)
Recession	0.17	=	-(0.82)	-(-1.00)
Surplus	0.13	=	-(-0.14)	-(0.02)
Disc. Rate	0.16	=	-(-0.11)	-(-0.05)
Surplus, no i	0.01	=	-(-0.05)	-(0.04)

Table H.9: Terms of the inflation and bond return identities. Sample 1930-2018.

The growth response goes the other way, now rising with inflation rather than declining, and therefore contributes -0.93% inflation rather than +0.49%.

In sum, the full sample data paint a picture more than diametrically opposite. A 1% inflation shock, drawn out to 1.83% cumulative weighted inflation, is more than accounted for by 2.53% cumulative deficits, and buffered by an 0.52% disinflationary decline in discount rates, and 0.93% disinflationary rise in growth.

The full-sample results appear to support a simple fiscal theory, which would be convenient – inflation comes from persistent deficits. Discount rates only

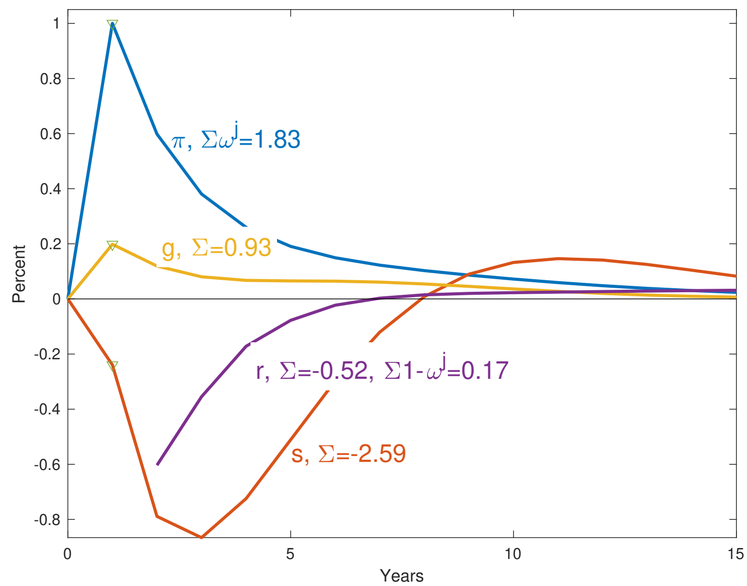

Figure H.14: Response to inflation shocks, sample 1930-2018.

mitigate that result.

Why then do I emphasize the postwar sample in the text, and relegate these to an online appendix? Clearly, the full sample results do not carry through the postwar period to the present. As in essentially all macroeconomics and monetary economics, which studies the post-1947 sample, the post-1959 sample, or, increasingly, the post-1980 sample, the war and prewar data behave differently. My interest in this paper is to characterize the behavior of inflation in postwar recessions, and the peacetime inflation of the 1970s and 1980s. Making an inference about that behavior from war and prewar data, when the central results switch in a postwar-only sample would be hugely misleading.

The nature of the prewar and war regime is interesting. Alas, the 1930-1947 sample is too short for these VAR methods. An investigation of the prewar regime with a long historical time series beckons.

What are the stylized facts and influential data points behind this switch in behavior? As before, long-run forecasts are driven by slow-moving state

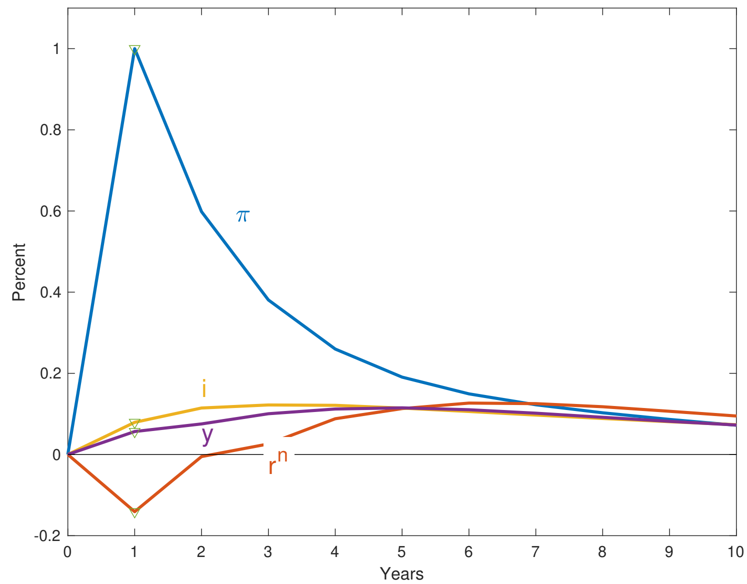

Figure H.15: Response to inflation shocks, sample 1930-2018.

variables. Think of a system

$$x_{t+1} = \alpha y_t + \varepsilon_{x,t+1}$$
$$y_{t+1} = \rho y_t + \beta \varepsilon_{x,t+1} + \varepsilon_{y,t+1}.$$

In the second equation, I express the y shock in terms of a component correlated with the x shock and an orthogonal component. In this system, the variable y is the persistent state variable for long-run responses. The long-response of x to the ε_x shock depends on how much the state variable y moves, β, and the persistence of the y variable. In response to $\varepsilon_{x,1} = 1$, the long-run x response is

$$\Delta E_1 \sum_{j=0}^{\infty} x_{1+j} = 1 + \frac{\alpha\beta}{1-\rho}.$$

With this insight, let us understand the responses of Figure H.14. Three state variables matter most. From Table H.8, inflation basically follows its own AR(1), unaffected by other variables, with a a persistence of 0.53, the same value as the postwar sample. The value of debt is the most important state variable for long-run responses with an 0.91 coefficient on its own lag. However, this

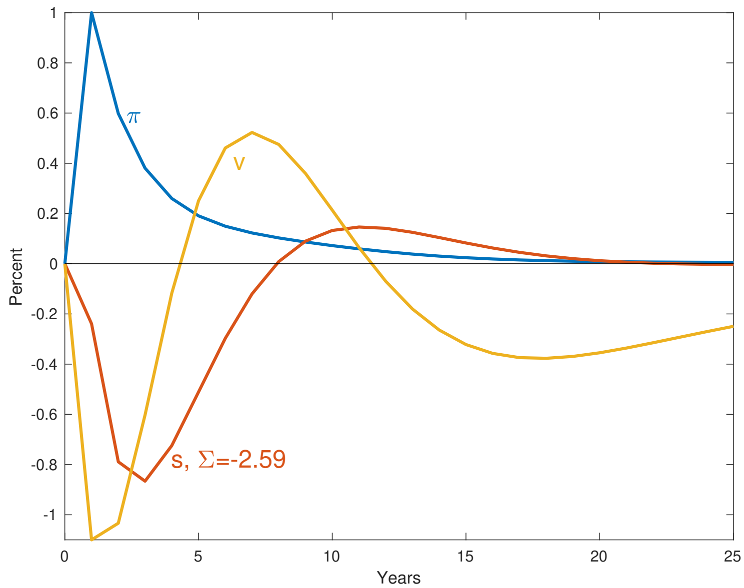

Figure H.16: Response to inflation shocks, sample 1930-2018.

debt to GDP ratio does respond strongly (-0.61) to surpluses, and to lagged growth (-1.1) as we would expect, so at medium runs it evolves jointly with these other variables. The surplus has a strong coefficient on its lag, 0.65, so in part any shock to surpluses coincident with the inflation shock will persist. The surplus also responds positively though with a small value 0.08 to the debt. This coefficient does not account for much of the short run dynamics, as the movements of surplus and debt are roughly the same size, but is the dominant force behind very long run surpluses which repay debts. The surplus responds and negatively -0.75 to inflation. This key coefficient is only -0.25 in the postwar sample. Interest rates also have a persistent response, but they move so little in this estimate that they are not an important state variable.

So, what accounts for the long deficits in Figure H.14? The surplus does not jump down by a large amount with the shock, declining only 0.25, so the surplus' autocorrelation is not a big part of the story. The big decline in surplus follows from its -0.75 coefficient on inflation, and the inflation AR(1) response. If inflation this year forecasts deficits next year, then a very simple fiscal theory

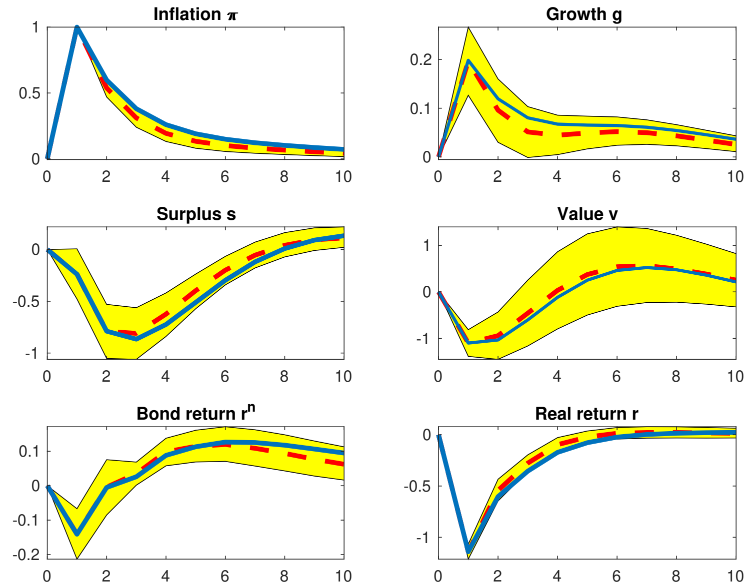

Figure H.17: Inflation shock response quantiles, sample 1930-2018.

story that inflation is accounted for by deficits follows swiftly.

But deficits should raise the value of debt, and the rise in the value of debt, which is very persistent, should pull deficits back to surplus, no? Here, another difference in the full sample is key. In the full sample, the value of debt v jumps down by 1.10% when inflation jumps up 1%, where in the postwar sample the value of debt jumps down half as much, 0.65%. Now, a low value of debt does not put into motion additional surpluses. So, the effect seen in the postwar sample of Figure 3, that deficits quickly give rise to higher debt which then triggers surpluses, is absent here because so much debt was wiped out by the inflation shock.

Contrasting Figures H.15 and 3 help to explain the differing behavior of the discount rate. In both cases, the behavior of nominal interest rates is disturbingly disconnected from the behavior of inflation. In the postwar sample, nominal rates rise immediately and very persistently. When inflation declines and passes by the higher nominal rates, real rates are higher. In the full sample, nominal rates move much less, reflecting the zero bound in the great depression and interest rate targets in WWII and the early postwar period. The resulting real rate the inverse of the inflation AR(1), and mostly negative.

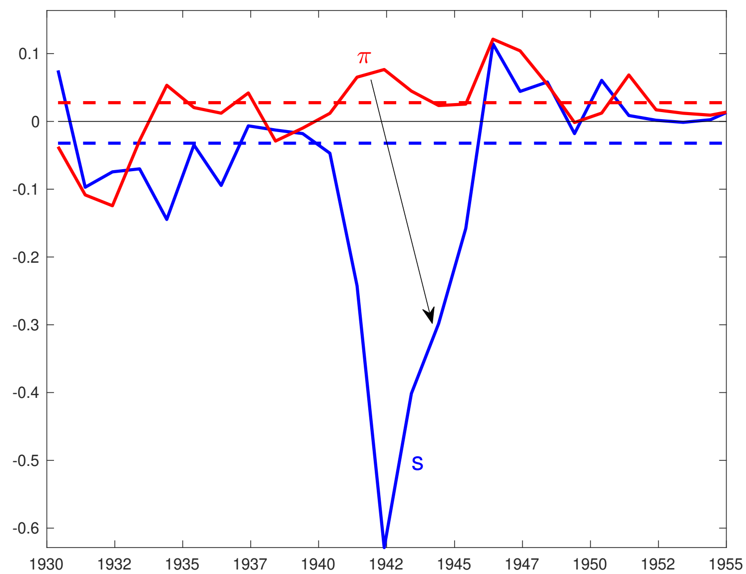

Figure H.18: Surplus and inflation during WWII

The massive deficits of 1943 and 1944 are key influential data points that account for the shift in behavior of the full sample. Estimates from the 1940-2018 sample, not shown, are similar. Figure H.18 plots inflation and surplus during WWII. The WWII deficits are immense. Inflation, more volatile in the pre-1947 period, was above its mean in the years prior to these immense deficits. Thus, this inflation preceding deficits of 1943 and 1944 drives the result that inflation forecasts deficits in the full sample, and thus the result that inflation shocks are accounted for by deficits. This is clearly not a robust result, or one that should be taken as evidence that inflation today is due to deficits.

The strong negative correlation between shocks to inflation and to the value of debt in the full sample comes from a different set of influential observations. The inflation of 1943 and 1944 was largely expected, according to the VAR, and preceded rather than coincided with increased debt. Instead, the sharp and unexpected (by the VAR) postwar inflation of 1947 coincided with a sharp decline in the real value of debt, and the sharp deflation of 1932 coincided with a sharp rise in the real value of debt. These events are conventionally regarded as times in which deflation raised the value of debt, in the first, and inflated it away, in the second. But again, one is loath to let these two observations double our estimate of the correlation between shocks to inflation and the value of debt for the postwar period.

The inflation shock is already positively correlated with a growth shock in the full sample, due to a strong positive correlation in the 1930s. As a result, the response to the inflation + growth shock (not shown) is not much different from the response to the inflation shock. Again, the 1% deflation and 2% cumulative inflation corresponds to 2.6% cumulative rise in surpluses. This time a long-run decline in discount rate contributes to deflation, but an equally large decline in growth contributes to inflation.

Appendix I. Growth and inflation plots

This section plots growth and inflation, to document that responses to a shock that combines 1% lower growth and 1% lower inflation is interesting. Figure I.19 presents GDP growth and CPI inflation.

Growth and inflation move in opposite directions during the 1970s stagflation episodes. By contrast, inflation declined in 1982 along with the comparably sized recession. This decline was permanent, not a u shape, but nonetheless coincident with the recession. Inflation moved a bit less than growth in the 2000 recession, but again moved about one for one with growth in 2008. The late 1940s and early 1950s also show roughly one for one positive comovement.

Figure I.20 presents the growth and inflation VAR residuals. These are not as clear, being annual data, consumption growth, GDP deflator, but tell roughly the same story.

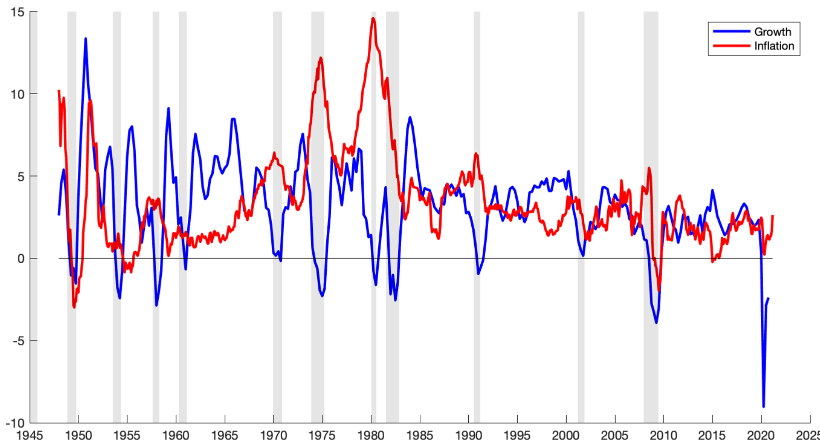

Figure I.19: CPI inflation and real GDP growth; percent changes from a year earlier.

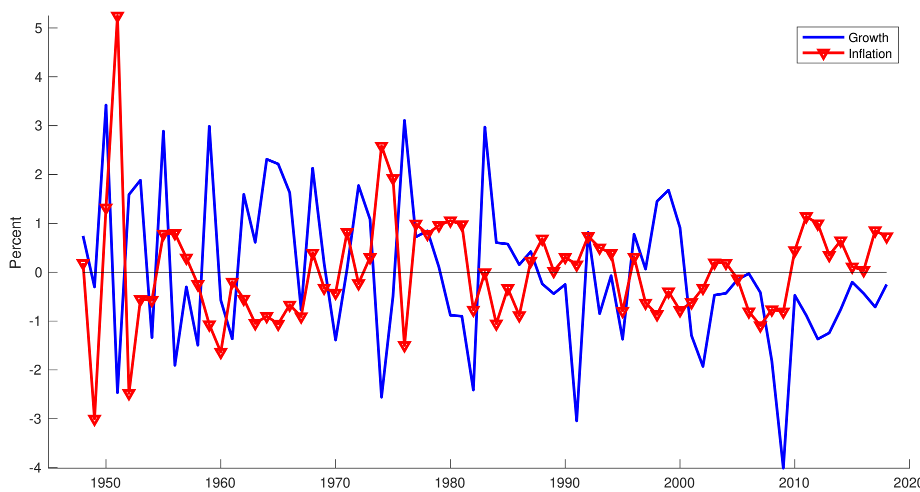

Figure I.20: VAR residuals to growth (consumption growth) and inflation (GDP deflator). Annual data 1948-2018.