A simple and compact algorithm for
SIDH with arbitrary degree isogenies

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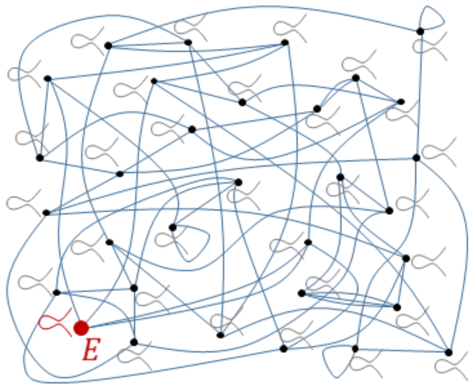
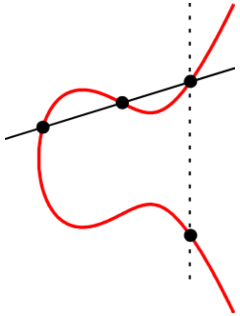
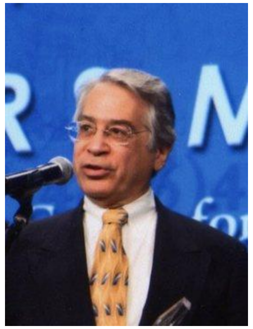
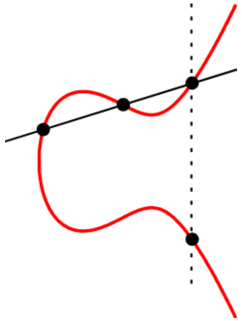
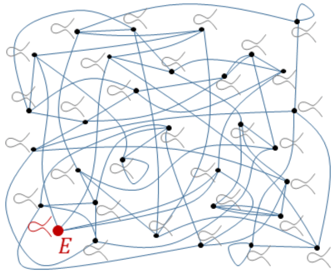
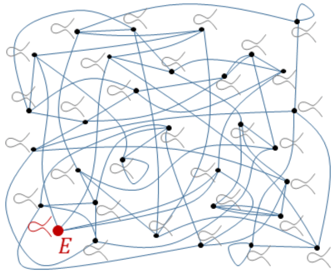


Diffie-Hellman instantiations

$\mathbb{Z}_q^*$  $g^a \bmod q$ / $g^b \bmod q$  $\mathbb{Z}_q^*$

  $[a]P$ / $[b]P$  



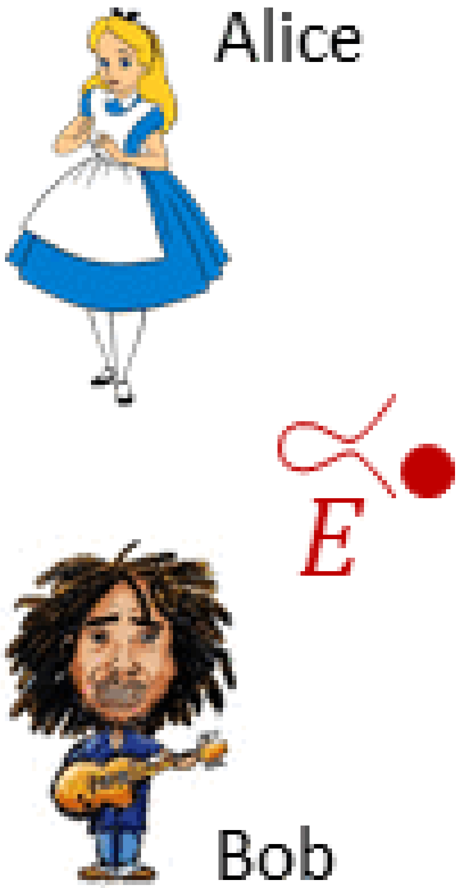
 $\phi_A(E)$ / $\phi_B(E)$  

Diffie-Hellman instantiations

	DH	ECDH	SIDH
Elements	integers g modulo prime	points P in curve group	curves E in isogeny class
Secrets	exponents x	scalars k	isogenies ϕ
computations	$g, x \mapsto g^x$	$P, k \mapsto [k]P$	$E, \phi \mapsto \phi(E)$
hard problem	given g, g^x find x	given $P, [k]P$ find k	given $E, \phi(E)$ find ϕ

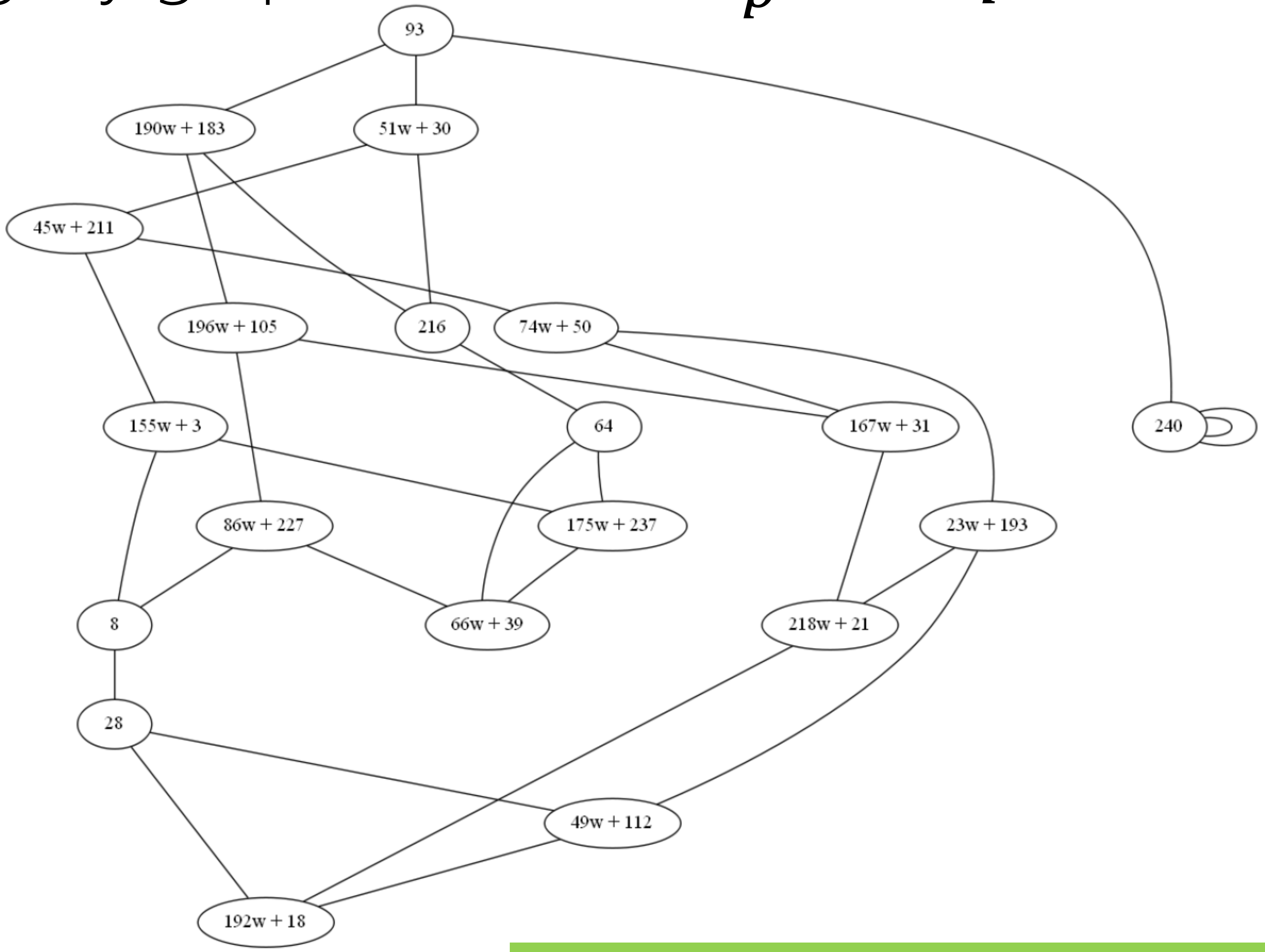
Setup: supersingular isogeny class over $\mathbb{F}_{p^2}$ …

roughly $p/12$ isomorphism classes within supersingular isogeny class…



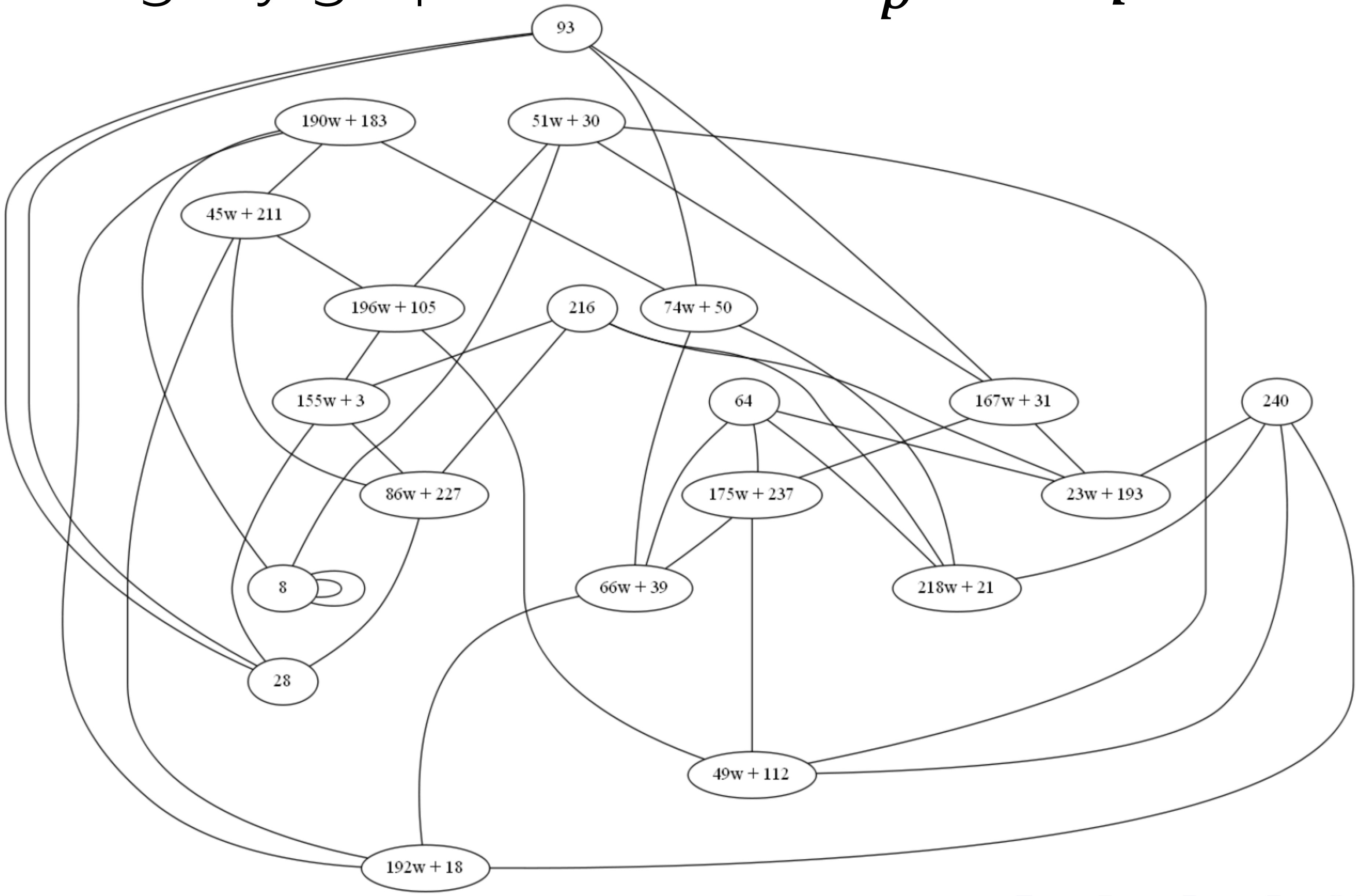
W. Castryck (GIF): "Elliptic curves are dead: long live elliptic curves" https://www.esat.kuleuven.be/cosic/?p=7404

Supersingular isogeny graph for $\ell = 2$: $\mathbb{F}_{p^2}$ with $p = 241$



Credit to Fre Vercauteren for example and pictures…

Supersingular isogeny graph for $\ell = 3$: $\mathbb{F}_{p^2}$ with $p = 241$



Credit to Fre Vercauteren for example and pictures…

(separable) isogenies ↔ subgroups

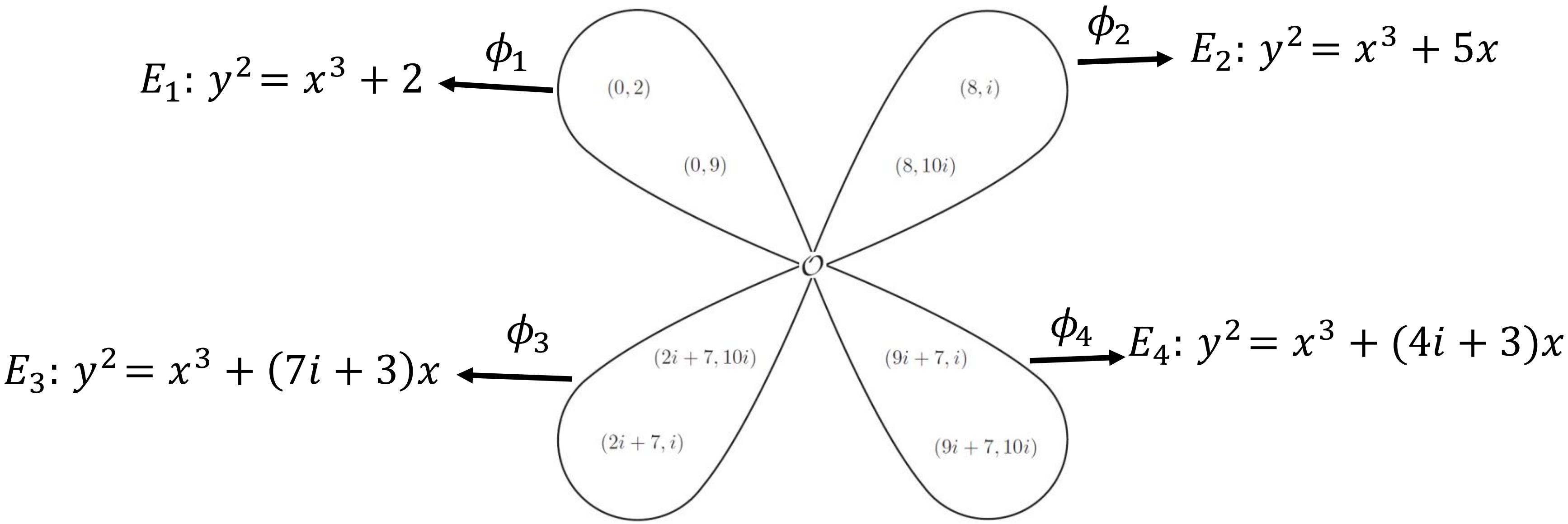
- An isogeny is a group homomorphism from E to E'

- Any finite subgroup $G \in E$, determines unique isogeny

$$\phi : \quad E \to E/G$$

- SIDH currently uses cyclic isogenies of degree $d = 2$ and $d = 3$
e.g.,

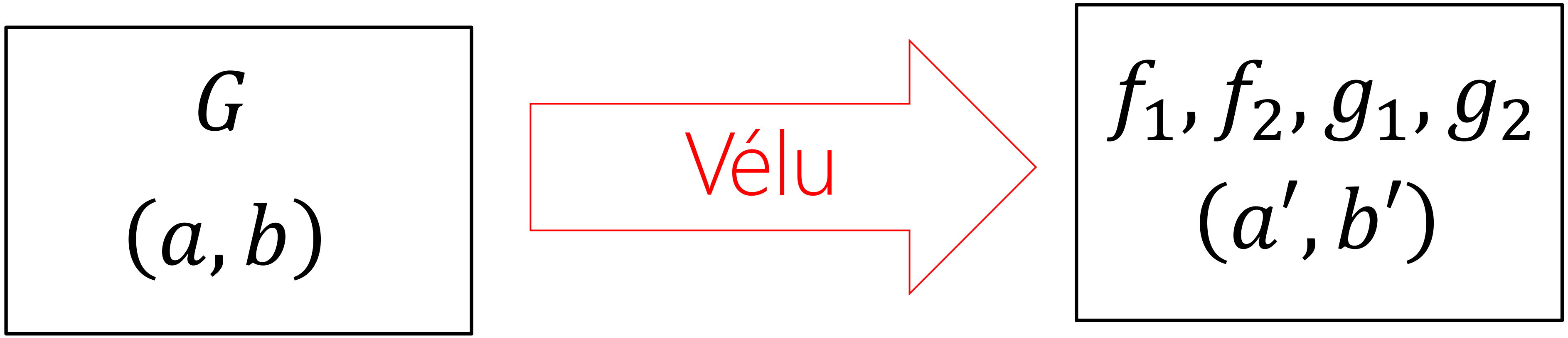
$$E/\mathbb{F}_{11^2}: y^2 = x^3 + 4$$
$$\#E(\mathbb{F}_{11^2}) = 12^2$$
$$d = 3$$



Computing isogenies with Vélu's formulas

- Consider the isogeny $\phi : E \to E/G, \quad (x,y) \mapsto \left(\frac{f_1(x,y)}{g_1(x,y)}, \frac{f_2(x,y)}{g_2(x,y)}\right)$

$$E : y^2 = x^3 + ax + b$$
$$E/G : y^2 = x^3 + a'x + b'$$



In SIDH: we need to compute the isogenous curve and evaluate isogenies at points

Point *and* isogeny arithmetic in $\mathbb{P}^1$

$$E_{a,b} : \quad by^2 = x^3 + ax^2 + x$$

$(x, y) \leftrightarrow (X : Y : Z)$

$(a, b) \leftrightarrow (A : B : C)$

$$E_{A/C\,,B/C}: \quad BY^2Z = CX^3 + AX^2Z + CXZ^2$$

$\mathbb{P}^1$ point arithmetic: $(X : Z) \mapsto (X' : Z')$

$\mathbb{P}^1$ isogeny arithmetic: $(A : C) \mapsto (A' : C')$

Motivation

$\mathbf{2}^{e}$ and $\mathbf{3}^{e}$ isogenies (on Montgomery curves) have been studied, but what about odd $\boldsymbol{\ell}^{e}$ for $\boldsymbol{\ell} \geq \mathbf{5}$?

Problems with Vélu's formulas on Montgomery curves…

- Let E be Montgomery. For the odd cyclic isogeny $\phi: \ E \to E/\langle P \rangle =: E'$, $\quad (x,y) \mapsto (X,Y)$, Vélu's formula says

$$X = x + \sum_{Q \in \langle P \rangle} 2 \cdot \frac{3x_Q^2 + 2Ax_Q + 1}{x - x_Q} + \frac{4y_Q^2}{(x - x_Q)^2}, \quad Y = y - \sum_{Q \in \langle P \rangle} \frac{8y_Q^2 y}{(x - x_Q)^3} + 2 \cdot (3x_Q^2 + 2Ax_Q + 1) \cdot \frac{(y + y_Q)}{(x - x_Q)^2}$$

- Vélu's formula also says that

$$E' : By^2 = x^3 + A_2 x^2 + A_4 x + A_6, \qquad A_4 \neq 1 \text{ and } A_6 \neq 0$$

(i.e., that the image curve is not Montgomery)

- Can (always) use isomorphism to convert E' to Montgomery form, but in general this requires root-finding

Theorem 1

Let P have odd order ℓ on Montgomery curve $E/K\colon By^2 = x^3 + Ax^2 + x$, and let $\phi : E \to E'$ with $E' = E/\langle P \rangle$. Then

$$\phi : (x, y) \mapsto \big(f(x), y \cdot f'(x)\big)$$

$$f(x) = x \cdot \prod_{1 \le i \le \ell - 1} \left(\frac{x \cdot x_{[i]P} - 1}{x - x_{[i]P}} \right)$$

$$E'\colon \quad B'y^2 = x^3 + A'x^2 + x$$

where $\quad A' = (6 \cdot \tilde{\sigma} - 6 \cdot \sigma + A) \cdot \pi^2$

$$B' = B \cdot \pi^2$$

with $\pi = \prod x_{[i]P}$, $\sigma = \sum x_{[i]P}$, $\tilde{\sigma} = \sum 1/x_{[i]P}$

Theorem 1 in the context of SIDH

Recall that in SIDH we only care about the $\boldsymbol{x}$-coordinate and $\boldsymbol{A}$ coefficient

$$\phi : E/\langle \ominus \rangle \to E'/\langle \ominus \rangle$$

$$x \mapsto x \cdot \prod_{1 \leq i \leq \ell - 1} \left(\frac{x \cdot x_{[i]P} - 1}{x - x_{[i]P}} \right)$$

$$A' = (6 \cdot \tilde{\sigma} - 6 \cdot \sigma + A) \cdot \pi^2$$

Theorem 1 in the context of SIDH

Recall that in SIDH we only care about the x-coordinate and $\boldsymbol{A}$ coefficient

$$\phi : E/\langle\ominus\rangle \to E'/\langle\ominus\rangle$$

$$x \mapsto x \cdot \prod_{\substack{1 \le i \le d \\ d=(\ell-1)/2}} \left(\frac{x \cdot x_{[i]P} - 1}{x - x_{[i]P}}\right)^2$$

$$x_{[i]P} = x_{[\ell-i]P}$$

$$A' = (6 \cdot \tilde{\sigma} - 6 \cdot \sigma + A) \cdot \pi^2$$

Theorem 1 in the context of SIDH

Recall that in SIDH we only care about the x-coordinate and A coefficient

$$\phi : E/\langle\ominus\rangle \to E'/\langle\ominus\rangle$$

$$(X : Z) \mapsto (X' : Z')$$

$$X' = X \cdot \left(\prod_i (X \cdot X_{[i]P} - Z_{[i]P} \cdot Z)\right)^2 \qquad Z' = Z \cdot \left(\prod_i (X \cdot Z_{[i]P} - X_{[i]P} \cdot Z)\right)^2$$

$$A' = (6 \cdot \tilde{\sigma} - 6 \cdot \sigma + A) \cdot \pi^2$$

with $\pi = \prod X_{[i]P}/Z_{[i]P}$, $\sigma = \sum X_{[i]P}/Z_{[i]P}$, $\tilde{\sigma} = \sum Z_{[i]P}/X_{[i]P}$

Theorem 1 in the context of SIDH

$$X' = X \cdot \left(\prod_i \Big((X - Z)\left(X_{[i]P} + Z_{[i]P}\right) + (X + Z)(X_{[i]P} - Z_{[i]P}) \Big) \right)^2$$

$$Z' = Z \cdot \left(\prod_i \Big((X - Z)\left(X_{[i]P} + Z_{[i]P}\right) - (X + Z)(X_{[i]P} - Z_{[i]P}) \Big) \right)^2$$

The simple and compact algorithm

Input: $\boldsymbol{x}(P) = (X_P : Z_P)$ and $\boldsymbol{x}(Q) = (X : Z)$ with $Q \notin \langle P \rangle$

Output: $\boldsymbol{x}(\phi(Q)) = (X_{\phi(Q)} : Z_{\phi(Q)})$ where $\ker(\phi) = \langle P \rangle$

Initialise: $T \leftarrow O_{E'}$, $X' \leftarrow 1$, $Z' \leftarrow 1$

for $i \in [1..d]$ do

$$(X_T : Z_T) = \boldsymbol{x}(T + P)$$
$$X' \leftarrow X' \cdot \big((X - Z) \cdot (X_T + Z_T) + (X + Z) \cdot (X_T - Z_T)\big)$$
$$Z' \leftarrow Z' \cdot \big((X - Z) \cdot (X_T + Z_T) - (X + Z) \cdot (X_T - Z_T)\big)$$

end for

return $(X \cdot X'^2 : Z \cdot Z'^2)$

$|\langle P \rangle| = 2d + 1$

What about computing the isogenous curve?

- Recall that the isogenous Montgomery curve has coefficient

$$A' = (6 \cdot \tilde{\sigma} - 6 \cdot \sigma + A) \cdot \pi^2$$

with $\pi = \prod X_{[i]P}/Z_{[i]P}$, $\sigma = \sum X_{[i]P}/Z_{[i]P}$, $\tilde{\sigma} = \sum Z_{[i]P}/X_{[i]P}$

- Relative to computing $x(P) \mapsto x(\phi(P))$, computing $A \mapsto A'$ becomes much more expensive as ℓ grows large…

- But for Montgomery curves, $A = -\alpha - 1/\alpha$ where $(\alpha, 0)$ is a point of order 2, so we can compute $(\alpha' : 0) = \phi((\alpha : 0))$ and recover $A' = -\alpha' - 1/\alpha'$ instead

- Now we only need one function for computing ℓ-isogenies on curves and points!

Upshot…

- Performance slowly degrades for odd ℓ-isogenies as ℓ increases, but not *too* bad…

- In traditional ECC, we are free to cherry-pick ***fastest*** prime characteristics, e.g.,

$$p = 2^{127} - 1, \qquad p = 2^{255} - 19, \qquad p = 2^{448} - 2^{224} - 1$$

- In SIDH, we are currently forced to choose much slower primes, like

$$p = 2^{250}3^{159} - 1, \qquad p = 2^{372}3^{239} - 1, \qquad p = 2^{486}3^{301} - 1$$

- Bos-Friedberger'17 get faster results for $p = 2^{391}19^{88} - 1$ than for $p = 2^{372}3^{239} - 1$, so the bottleneck party (e.g., server) computing 2-isogenies could be faster overall

- $p = 2^{448} - 2^{224} - 1$ and $p = 2^{480} - 2^{240} - 1$ are *almost** SIDH-friendly, e.g., $(p + 1) = 2^{224} \cdot \prod_i p_i^{e_i}$, but the larger p_i are just too big… is there some nice middle ground?

** Depends heavily on your definition of almost*

Some related stuff…

- Moody-Shumow had already figured this out in the case of (twisted) Edwards curves: see https://eprint.iacr.org/2011/430 👍

- Renes has, among several other things, recently solved the last piece of the Montgomery isogeny puzzle: efficient $\mathbf{2}$-isogenies 👍

- SIKE – supersingular isogeny key encapsulation was submitted to NIST last week. More work needed! 👍

Questions?