Computing Cryptographic Pairings: the State of the Art

Craig Costello

craig.costello@qut.edu.au
QUT and UCI

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University of California, Irvine

Pairing computation speeds: then and now

- **Then**:
  - 1993 Menezes' elliptic curve book (post MOV attack) : few minutes

...BIG GAP...

- **Now**:
  - -2009 Hankerson, Menezes, Scott: 4.01ms
  - -April 2010 Naehrig, Niederhagen, Schwabe: 1.80ms
  - -June 2010 Beuchat *et al.*: 0.94ms
  - -October 2010 Aranha *et al.*: 0.65ms

So what happened in the big gap?

- Heaps of exciting protocol stuff has happened...
  ID-based encryption (IBE), ID-based key agreement, short signatures, group signatures, ring signatures, certificateless encryption, hierarchical encryption, predicate-based encryption, attribute-based encryption, ... and many many more!!!

- Heaps of cool pairing optimizations have 'followed'...
  - *Tate pairing instead of Weil pairing*
  - *denominator elimination*
  - *group choices and twisted curves*
  - *endomorphism rings and loop shortening*
  - *low rho-valued curves*
  - *pairing and towering-friendly fields*
  - *quick explicit formulas*
  - *... and many more!!!*

Pairings

A mapping $e : \mathbb{G}_1 \times \mathbb{G}_2 \rightarrow \mathbb{G}_T$:

- $P \in \mathbb{G}_1$, $Q \in \mathbb{G}_2$ and $e(P, Q) \in \mathbb{G}_T$: (cyclic) groups are all of prime order r (usually)

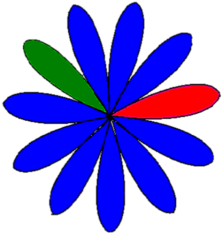
- **Bilinear** : $\mathbf{e(aP, bQ) = e(P, Q)^{ab} = e(bP, aQ)}$

- *Note:* $\mathbb{G}_1$ and $\mathbb{G}_2$ must be linearly independent

- $e(P, Q) = f(x_P, y_P, x_Q, y_Q) \in \mathbb{F}_{q^k}$

Groups involved: the r-torsion and Frobenius eigenspaces

- The points P and Q in the pairing come from the r-torsion $E(\bar{\mathbb{F}}_q)[r] = \mathbb{Z}_r \times \mathbb{Z}_r$.



- $\mathbb{F}_q$ must be extended $(\mathbb{F}_{q^k})$ to contain the entire r torsion
- $P \in \mathbb{G}_1 = E(\mathbb{F}_q)[r]$ $\qquad$ $Q \in \mathbb{G}_2 \subset E(\mathbb{F}_{q^k})[r]$
- Frobenius endomorphism $\qquad$ $\pi_q(x, y) \mapsto (x^q, y^q)$
- $G_1 = E[r] \cap \mathrm{Ker}(\pi_q - [1])$ $\qquad$ $G_2 = E[r] \cap \mathrm{Ker}(\pi_q - [q])$
- Both eigenspaces are very (computationally) convenient

The embedding degree k and pairing-friendly curves

- $\#E(\mathbb{F}_q) = q + 1 - t \approx q$ and $\#E(\mathbb{F}_q) = hr$ (h small, r big prime)
- To contain entire r-torsion (both $\mathbb{G}_1$ and $\mathbb{G}_2$), must extend $\mathbb{F}_q$ to $\mathbb{F}_{q^k}$
- $k \in \mathbb{N}$ is smallest s.t. $r \mid q^k - 1$
- In general, $k \approx r$ (Balasubramanian and Koblitz)
- Let's be modest: $q = 160$ bits, $r = 160$ bits $\rightarrow$ $\mathbb{F}_{q^k} \approx \mathbb{F}_{2^{160(2^{160})}}$
- Need to find 'pairing-friendly' elliptic curves where k is small enough $k < 50$
- Finding pairing-friendly curves is an art in itself...

Pairing-friendly curves

- Attacker can target either discrete log problem: $E(\mathbb{F}_q)$ or $\mathbb{F}_{q^k}$
- We aim to balance their difficulty to optimize implementation
- Define $\rho = \log q / \log r$ (closer to 1 the better)

(AES) Security level (bits)	Subgroup size r (bits)	Extension field q^k (bits)	Embedding degree k	
			$\rho \approx 1$	$\rho \approx 2$
80	160	960-1280	6-8	2-4
112	224	2200-3600	10-16	5-8
128	256	3000-5000	12-20	6-10
192	384	8000-10000	20-26	10-13
256	512	14000-18000	28-36	14-18

Table: I stole this table from the "taxonomy" paper (Freeman, Scott, Teske)

A good example: BN curves

- Barreto and Naehrig found a family of really nice curves for $k = 12$
  $q(x) = 36x^4 - 36x^3 + 24x^2 - 6x + 1$
  $\#E(\mathbb{F}_q)(x) = 36x^4 - 36x^3 + 18x^2 - 6x + 1$
  $t(x) = 6x^2 + 1$
- Find x s.t. $q(x)$ is prime and $\#E(x)$ is also prime and you have a BN curve $y^2 = x^3 + b$
- In fact, almost all constructions (r prime) result in a curve $y^2 = x^3 + b$ or $y^2 = x^3 + ax$ (no CM needed)
- The "bible": Freeman-Scott-Teske - "A taxonomy of pairing-friendly elliptic curves"

The elements of $\mathbb{G}_2$ are much bigger than the elements of $\mathbb{G}_1$ (e.g. $k = 12$)

$$\mathbb{F}_{q^{12}} = \mathbb{F}_{q^4}(\alpha) = \mathbb{F}_{q^2}(\gamma) = \mathbb{F}_q(\beta)$$

$P \in \mathbb{G}_1$: [341746248540, 710032105147]

$Q \in \mathbb{G}_2$:

$[((502478767360 \cdot \beta + 1034075074191) \cdot \gamma + 342970860051 \cdot \beta + 225764301423) \cdot \alpha^2 + ((205398279920 \cdot \beta + 182600014119) \cdot \gamma + 860891557473 \cdot \beta + 435210764901) \cdot \alpha + (1043922075477 \cdot \beta + 566889113793) \cdot \gamma + 150949917087 \cdot \beta + 21392569319,$

$((654337640030 \cdot \beta + 744622505639) \cdot \gamma + 1092264803801 \cdot \beta + 895826335783) \cdot \alpha^2 + ((529466169391 \cdot \beta + 550511036767) \cdot \gamma + 985244799144 \cdot \beta + 554170865706) \cdot \alpha + (194564971321 \cdot \beta + 969736450831) \cdot \gamma + (579122687888 \cdot \beta + 581111086076)]$

The twisted curve

- Original curve is $E(\mathbb{F}_q) : y^2 = x^3 + ax + b$
- Twisted curve is $E'(\mathbb{F}_{q^{k/d}})$: $y^2 = x^3 + a\omega^4 x + b\omega^6$, $\omega \in \mathbb{F}_{q^k}$
- Possible degrees of twists are $d \in \{2, 3, 4, 6\}$: the bigger the better!
- Twist $\Psi : E' \to E : (x', y') \to (x'/\omega^2, y'/\omega^3)$ induces $\mathbb{G}_2' = E'(\mathbb{F}_{q^{k/d}})[r]$ so that $\Psi : \mathbb{G}_2' \to \mathbb{G}_2$
- Instead of working with $Q \in \mathbb{G}_2$, a lot of work can be done with $Q' \in \mathbb{G}_2'$ defined over subfield $\mathbb{F}_{q^e} = \mathbb{F}_{q^{k/d}}$

$P \in \mathbb{G}_1$: $(341746248540, 710032105147)$

$Q' \in \mathbb{G}_2' = \Psi^{-1}(\mathbb{G}_2)$:

$((917087150949\beta + 25693192139) \cdot \omega^2, (878885791226\beta + 860765811110) \cdot \omega^3)$

Achieving a bilinear pairing

- On elliptic curves, group homomorphism from points to divisor classes
$$P \mapsto (P) - (\mathcal{O}) = D_P$$
- Let D be the divisor $D = \sum_P n_P(P)$ on E and $f \in \mathbb{F}_{q^k}(E)$:
$$f(D) = \prod_P f(P)^{n_P}$$
- $f, g \in \mathbb{F}_{q^k}(E)$: **Weil reciprocity:** $f(\mathrm{div}(g)) = g(\mathrm{div}(f))$
- Achieve bilinearity (and other necessary properties) by finding a function f_P whose divisor is some (linear) multiple of $D_P = (P) - (\mathcal{O})$...

Achieving a bilinear pairing (cont.)

- Let $P \in E[r]$, (assume) we can construct the function $f_{v,P}$ such that

$$\mathrm{div}(f_{v,P}) = v(P) - ([v]P) - (v-1)(\mathcal{O})$$

- When $v = r$, we have

$$\begin{aligned}\mathrm{div}(f_{r,P}) &= r(P) - ([r]P) - (r-1)(\mathcal{O}) \\ &= r(P) - r(\mathcal{O}) \\ &= rD_P\end{aligned}$$

- $f_P = f_{r,P}$ is a degree r function (has zero of degree r at P)...
- Remember r has to be large $> 2^{160}$ for ECDLP to be hard

Weil vs. Tate pairings

Weil pairing

$$e : \mathbb{G}_1 \times \mathbb{G}_2 \rightarrow \mu_r \in \mathbb{F}_{q^k}, \qquad (P, Q) \mapsto f_{r,P}(Q)/f_{r,Q}(P)$$

Tate(-Lichtenbaum) pairing

$$e : \mathbb{G}_1 \times \mathbb{G}_2 \rightarrow \mu_r \in \mathbb{F}_{q^k}, \qquad (P, Q) \mapsto f_{r,P}(Q)^{\frac{q^k-1}{r}}.$$

- ~~Weil pairing: compute two degree r functions~~
- Tate pairing: compute one degree r function and exponentiate (much faster)
- Exponentiation is somewhat standard, so how to compute $f_{r,P}(Q)$ efficiently
- 1986: Miller proposes efficient algorithm for $f_{r,P}(Q)$ ("The Weil pairing, and it's efficient calculation")

Miller's algorithm

Miller's algorithm to compute $f_{r,P}(Q)$

$r = (r_{l-1}, \ldots, r_1, r_0)_2$ initialize: $U = P$, $f = 1$

for $i = l - 2$ to 0 do

a. i. Compute $f_{\mathrm{DBL}(U)}$ in the doubling of U

ii. $U \leftarrow [2]U$ //(DBL)

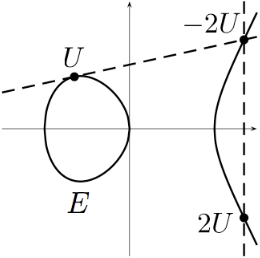
iii. $f \leftarrow f^2 \cdot f_{\mathrm{DBL}(U)}(Q)$

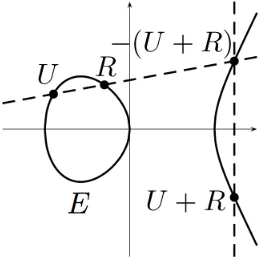
b. if $m_i = 1$ then

i. Compute $f_{\mathrm{ADD}(U,P)}$ in the addition of $U + P$

ii. $U \leftarrow U + P$ //(ADD)

iii. $f \leftarrow f \cdot f_{\mathrm{ADD}(U,P)}(S)$





Optimization: force $r(x)$ to have low Hamming-weight

$r = (r_{l-1}, \ldots, r_1, r_0)_2$ initialize: $U = P$, $f = 1$

for $i = l - 2$ to 0 do

a. i. Compute $f_{\mathrm{DBL}(U)}$ in the doubling of U

ii. $U \leftarrow [2]U$ //(DBL)

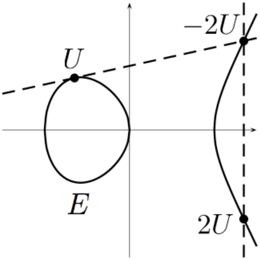
iii. $f \leftarrow f^2 \cdot f_{\mathrm{DBL}(U)}(Q)$

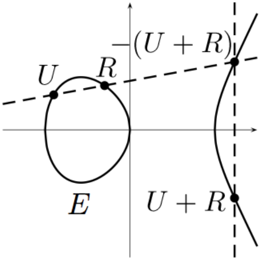
b. ~~if $m_i = 1$ then~~

i. ~~Compute $f_{\mathrm{ADD}(U,P)}$ in the addition of $U + P$~~

ii. ~~$U \leftarrow U + P$ //(ADD)~~

iii. ~~$f \leftarrow f \cdot f_{\mathrm{ADD}(U,P)}(S)$~~





Optimization: avoid costly inversions and exploit exponentiation

$r = (r_{l-1}, \ldots, r_1, r_0)_2$ initialize: $U = P$, $f = 1$
for $i = l - 2$ to 0 do

i. Compute $f_{\mathrm{DBL}(U)}$ in the doubling of U
ii. $U \leftarrow [2]U$ //(DBL)
iii. $f \leftarrow f^2 \cdot f_{\mathrm{DBL}(U)}(Q)$

- **Irrelevant factors:** Because the final value of f is exponentiated to $(q^k - 1)/r$, any subfield factors accumulated in f can be ignored!
- **Projective coordinates:** Affine coordinates require inversions: use $(X : Y : Z)$ to represent $(x, y) = (X/Z, Y/Z)$ or some other projection

Optimization: lower degree Miller functions (loop shortening)

- Exploit the fact that since $Q \in \mathbb{G}_2 = E[r] \cap \mathrm{Ker}(\pi_q - [q])$, a bilinear pairing with a much smaller degree (than r) if Q is the first argument
- $e(Q, P) = f_{\lambda,Q}(P)^{(q^k-1)/r}$ where $\lambda \equiv q \bmod r$
- Vercauteren ("Optimal pairings") and Hess ("Pairing lattices") prove that λ can be achieved as small as $r^{1/\varphi(k)}$
- Most of the computations are performed on the first argument (now $Q \in E(\mathbb{F}_{q^k})$), but many less iterations required for the lower degree function
- Dubbed the (optimal) "ate" pairing (since it reverses the arguments of the "eta" pairing, and it is (generally) faster than the Tate pairing

Optimization: pairing and towering-friendly fields

- Koblitz-Menezes 2005: Build extension fields as towers of extensions (using irreducible binomials)
- e.g. $k = 24$ build $\mathbb{F}_{q^k}$ as

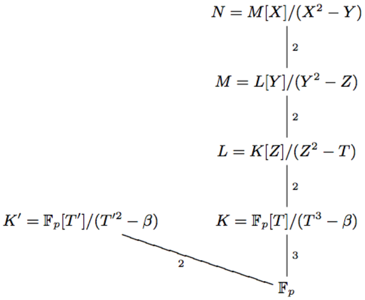
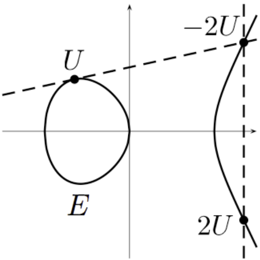
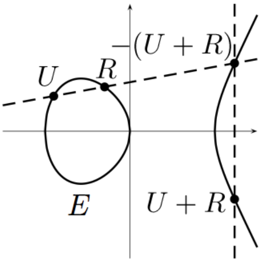


Fig. 1. Tower of pairing-friendly fields

- Arithmetic and implementation much easier $k = 2^i 3^j$ means $\mathbf{m}_k = 3^i 5^j \mathbf{m}_1$ (e.g. $\mathbf{m}_{24} = 135\mathbf{m}_1$)
- Best way to tower: Benger-Scott WAIFI2010 paper

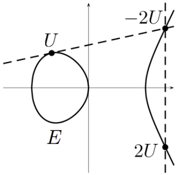
Optimization: quick explicit formulas

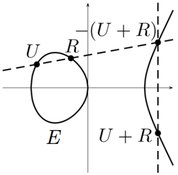




- In the Tate pairing, point operations and line computations were performed on $P \in E(\mathbb{F}_q)$ (somewhat negligible compared to the dominant operations in $\mathbb{F}_{q^k}$ for larger k)
- In the ate pairing, these operations are now performed in $\mathbb{F}_{q^{k/d}}$
- Important to optimize the combination of a point doubling $U \mapsto [2]U$ (resp. additions) and the line computations that contribute to $f_{\lambda,Q}$

Optimization: quick explicit formulas (cont.)

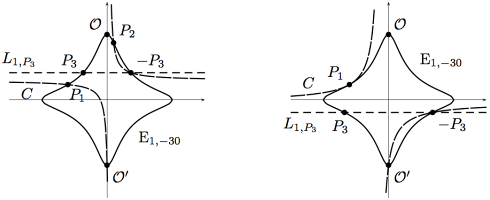




- C-Hisil-Boyd-Gonzalez-Wong (Pairing09): fastest pairings for $y^2 = x^3 + c^2$ (special Weierstrass): homogenous projective coordinates achieve 8 subfield multiplications
- C-Lange-Naehrig (PKC2010): "Faster pairings on curves with high-degree twists":
  - i. $y^2 = x^3 + ax$ ($j = 1728$ or $D = 1$): weight-(1,2) coordinates achieve 10 subfield multiplications
  - ii. $y^2 = x^3 + b$ ($j = 0$ or $D = 3$): Projective coordinates achieve 9 subfield multiplications (used in recent record 0.65ms)

Other curve models

- Weierstrass curves are nice for pairings since the line computations are inherent in the point addition formulas
- Edwards curves (also Jacobi-Quartics, Hessian etc) are far superior in standard ECC because of fast addition formulas



(a) $P_1 \neq P_2$, $P_1, P_2 \neq \mathcal{O}'$, $P_3 = P_1 + P_2$ (b) $P_1 = P_2 \neq \mathcal{O}'$, $P_3 = 2P_1$

Figure: Picture taken from Arene *et al.* Edward's pairing paper

- Pairing-based cryptosystems need more than just pairings
- Galbraith showed E and E' can't both be written in Edwards form ("Edwards curves aren't likely candidates for ate pairing which requires computations")...

Ate pairing on Edwards curves

- C-Lange-Naehrig (PKC2010): a bilinear pairing can be computed entirely on the twist E'
- Choose E so that E' can be written in Edwards form (it doesn't matter that E can't)
- C-Lange-Naehrig: "The ate pairing on twisted Edwards curves" (work in progress)

Some recent results

i. Compute $f_{\mathrm{DBL}(U)}$ in the doubling of U

ii. $U \leftarrow [2]U$ //(DBL)

iii. $f \leftarrow f^2 \cdot f_{\mathrm{DBL}(U)}(S)$

$$\text{(DBL) } [2](x_1, y_1) = (x_3, y_3)$$

$$f_{\mathrm{DBL}(U)}(x, y) = y - \lambda \cdot x - (y_1 - \lambda \cdot x_1)$$
$$f_{\mathrm{DBL}(U)}(S) = y_S - \lambda \cdot x_S - (y_1 - \lambda \cdot x_1)$$

- Perhaps it isn't optimal to evaluate indeterminate function $f_{\mathrm{DBL}(U)}(x, y)$ yet
- Leave as an indeterminate function for n-iterations (CBGW - AfricaCrypt2010 paper, CBGW - WAIFI 2010 paper)
- Even more advantageous in the case of a fixed pairing argument (C-Stebila - "Fixed argument pairings" - LatinCrypt 2010)

$e(R,S)$: R-dependent vs. S-dependent computations

a. i. Compute $f_{\mathrm{DBL}(U)}$ in the doubling of U

ii. $U \leftarrow [2]U$ //(DBL)

iii. $f \leftarrow f^2 \cdot f_{\mathrm{DBL}(U)}(S)$

b. if $m_i = 1$ then

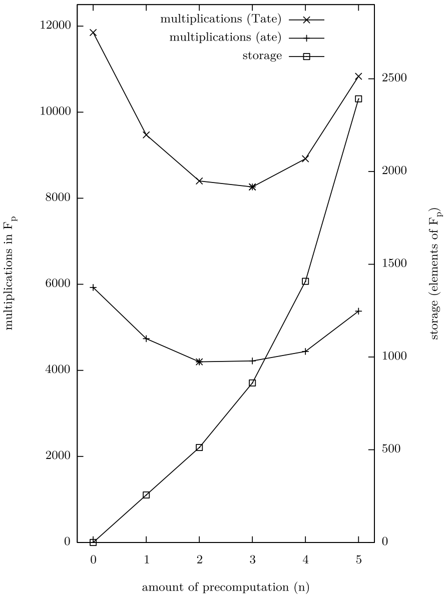
i. Compute $f_{\mathrm{ADD}(U,R)}$ in the addition of $U + R$

ii. $U \leftarrow U + R$ //(ADD)

iii. $f \leftarrow f \cdot f_{\mathrm{ADD}(U,R)}(S)$

- All the point operations and line coefficient computations are completely R-dependent ($U = vR$ throughout)
- If R is a fixed argument, we can pre-compute all of this before we input (or know) S
- Pre-compute and store all the $(\lambda, x_{U_i}, y_{U_i})$ tuples (Scott 2006)
- **C-Stebila: do much more with all of the f_{ADD} functions before S is known (or input)**

Tate and ate $\mathbb{F}_p$-muls vs. storage cost ($k = 12$, $r = 256$)



Current/future work: genus 2 pairings

- Working in the Jacobian $\mathrm{Jac}_C(\mathbb{F}_q)$
- The general belief is that genus 2 pairings won't be competitive with pairings on elliptic curves
- I'm naive in this arena and am therefore not yet convinced
- Holding genus 2 implementations back: ρ-values are currently very bad in comparison

$$\rho = g \log q / \log r$$

- **At the top of my wish list**: pairing-friendly genus 2 curves $k \leq 50$ and $\rho << 4$

Thanks for your attention...

Questions?